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THE ASYMPTOTIC ACCURACY OF THE BOOTSTRAP OF U-QUANTILES[1]

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The order of the Kolmogorov–Smirnov distance for the bootstrap of U-quantiles is considered. We observe that the order of the bootstrap of U-quantiles depends on the order of the local variance of the first term of the Hoeffding decomposition at the U-quantile. This order can be smaller than the order of the bootstrap of quantiles: U-quantiles can be smoother than quantiles.

1. Introduction. Let $X_1, \dots, X_n$ be independent identically distributed random variables (i.i.d. r.v.'s) with common distribution function (d.f.) F. Denote by F_n the corresponding empirical d.f. The bootstrap [term coined by Efron (1979)] consists in doing Monte Carlo approximation from the sample, that is, take $X^*_{n,1}, \dots, X^*_{n,n}$ i.i.d. r.v.'s with distribution function F_n. So, conditionally on the sample, we have a bootstrap probability Pr*. The bootstrap has proved to be a very versatile statistical method. We refer to Hall (1992) for general facts about the bootstrap, in particular, its Appendix 6 on the bootstrap of quantiles.

We will consider the bootstrap of U-quantiles. Next, we will describe the framework we are going to work in. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. r.v.'s with values in a measurable space $(S, \mathscr{S})$. We define for a measurable function $f\colon S^m \to \mathbb{R}$, the U-statistic with kernel f by

$$U_n(f) = \frac{(n-m)!}{n!} \sum_{(i_1, \dots, i_m) \in I_m^n} f(X_{i_1}, \dots, X_{i_m}), \tag{1.1}$$

where $I_m^n = \{(i_1, \dots, i_m)\colon 1 \le i_j \le n,\ i_j \ne i_k \text{ for } j \ne k\}$. We refer to Serfling (1980), Lee (1990) and Koroljuk and Borovskich (1994) for more on U-statistics. The basic fact about U-statistics we will use is the Hoeffding decomposition, which we state next. We define

$$\pi_{k,m} f(x_1, \dots, x_k) = (\delta_{x_1} - P) \cdots (\delta_{x_k} - P) P^{m-k} f, \tag{1.2}$$

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where $Q_1 \cdots Q_m f = \int f(x_1, \dots, x_m)\, dQ_1(x_1) \cdots dQ_m(x_m)$. Then, the Hoeffding decomposition can be written as

$$U_n(f) = \sum_{k=0}^{m} \binom{m}{k} U_n(\pi_{k,m} f). \tag{1.3}$$

Observe that the variance of $U_n(\pi_{k,m} f)$ is of the order n^{-k}. Fix a measurable function h in S^m. Let $H_n(t) = U_n(I_{h \le t})$ and let $H(t) = \Pr\{h(X_1, \dots, X_m) \le t\}$; $H_n(t)$ is called the empirical d.f. of U-statistic structure. It takes the role of the empirical d.f. in some problems involving U-statistics. Let $0 < p < 1$. Suppose that there is a $\xi_0 \in \mathbb{R}$ such that $H(\xi_0) = p$; ξ_0 is called a U-quantile. Define

$$\xi_n := H_n^{-1}(p) = \inf\{t \in \mathbb{R}: H_n(t) \ge p\}; \tag{1.4}$$

ξ_n is the sample U-quantile. By Theorem 3.1 in Serfling (1984),

$$n^{1/2}(\xi_n - \xi_0) \to_d N\big(0, (\sigma_1/H'(\xi_0))^2\big), \tag{1.5}$$

where $\sigma_1^2 := \operatorname{Var}(mg(X_1, \xi_0))$ and $g(x, t) = \Pr\{h(x, X_2, \dots, X_m) \le t\}$.

Several common estimators are U-quantiles. For example, one very often used alternative to the median as a center of symmetry is the Hodges–Lehmann estimator: the median of $2^{-1}(X_i + X_j)$, $1 \le i < j \le n$ [see Hodges and Lehmann (1963)]. This is the U-quantile (with respect to $p = 1/2$) of the kernel $h(x_1, x_2) = 2^{-1}(x_1 + x_2)$. Another interesting example is the U-quantile of the kernel $h(x_1, x_2) = |x_1 - x_2|$ with respect to $p = 1/2$. This U-quantile is a measure of the spread of the distribution. It was introduced by Bickel and Lehmann (1979). Choudhury and Serfling (1988) introduced a U-quantile which estimates the regression slope. Consider the linear regression model $Y_i = \alpha + \beta X_i + \delta_i$; α and β are constants; δ_i is independent of X_i. The U-quantile of the kernel $h((x_1, y_1), (x_2, y_2)) = (y_2 - y_1)/(x_2 - x_1)$, with respect to $p = 1/2$, is a natural estimator of the parameter β. This estimator is the median of the values $(Y_j - Y_i)/(X_j - X_i)$, $1 \le i < j \le n$.

In Section 2, we will mention that

$$\limsup_{n \to \infty} n^{1/2} \sup_t \left|\Pr\{n^{1/2}(\xi_n - \xi_0) \le t\} - \Phi(tH'(\xi_0)\sigma_1^{-1})\right| < \infty, \tag{1.6}$$

where Φ is the d.f. of a Studentized normal distribution. We will see that, in general, the limit superior in (1.6) is not zero.

In Section 3, we will consider the bootstrap of U-quantiles. For a measurable function f in S^m we define

$$U_n^*(f) = \frac{(n-m)!}{n!} \sum_{(i_1, \dots, i_m) \in I_m^n} f(X_{n,i_1}^*, \dots, X_{n,i_m}^*).$$

Let $H_n^*(t) = U_n^*(I_{h \le t})$. We will study the asymptotic order of

$$\sup_t \left|\Pr{}^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\right|, \tag{1.7}$$

where $\xi_n^* = (H_n^*)^{-1}(p)$. Helmers, Janssen and Veraverbeke (1992) show that

$$\sup_t \left|\Pr^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\right| = O\big(n^{-1/4}(\log n)^{3/4}\big) \quad \text{a.s.}$$

We will study the order of (1.7), both a.s. and in probability. It is remarkable that the order in (1.7) can be different from the order for the bootstrap of i.i.d. quantiles. This order (as the order of the Bahadur representation of U-quantiles) depends on $E[|g(X, t) - g(X, \xi_0)|^2]$. If $m = 1$, we have that

$$\limsup_{n\to\infty} \frac{n^{1/4}}{(\log\log n)^{1/2}} \times \sup_t \left|\Pr^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\right| = c \quad \text{a.s.,} \tag{1.8}$$

where c is a finite constant [Singh (1981), Theorem 2] and that

$$n^{1/4} \sup_t \left|\Pr^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\right| \tag{1.9}$$

converges in distribution to a nondegenerate distribution function [Falk and Reiss (1989) and Falk (1990)]. Here, we will see that

$$n^{1/4} \sup_t \left|\Pr^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\right| = O_{\Pr}(1) \tag{1.10}$$

and

$$\frac{n^{1/4}}{(\log\log n)^{1/2}} \sup_t \left|\Pr^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\right| = O(1) \quad \text{a.s.} \tag{1.11}$$

These rates are the exact rates of convergence for some kernels. If the function $g(x, t)$ is differentiable enough in t, then the following exact rates are obtained:

$$n^{1/2} \sup_t \left|\Pr^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\right| = O_{\Pr}(1) \tag{1.12}$$

and

$$\frac{n^{1/2}}{(\log\log n)^{1/2}} \sup_t \left|\Pr^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\right| = O(1) \quad \text{a.s.} \tag{1.13}$$

This is the order of the sample mean [see Theorem 1 in Singh (1981)] and, therefore, the order of smooth enough statistics.

The main ingredients in the proofs are several limit theorems for empirical processes and U-processes indexed by VC classes and a Berry–Eseen-type inequality for U-statistics in van Zwet [(1984), Corollary 4.1]. Given a set S and a collection of subsets $\mathscr{C}$, for $A \subset S$, let $\Delta^{\mathscr{C}}(A) = \text{card}\{A \cap C: C \in \mathscr{C}\}$, let $m^{\mathscr{C}}(n) = \max\{\Delta^{\mathscr{C}}(A): \text{card}(A) = n\}$ and let $s(\mathscr{C}) = \inf\{n: m^{\mathscr{C}}(n) < 2^n\}$; $\mathscr{C}$ is said to be a VC class of sets if $s(\mathscr{C}) < \infty$. General properties of VC classes of sets can be found in Chapters 9 and 11 in Dudley (1984). Given a function f: $S \to \mathbb{R}$, the subgraph of f is the set $\{(x, t) \in S \times \mathbb{R}: 0 \le t \le f(x)$ or $f(x) \le t \le 0\}$. A class of functions $\mathscr{F}$ is a VC-subgraph class if the collection of subgraphs of $\mathscr{F}$ is a VC class. The interest of these classes of functions lies in their good properties with respect to covering numbers. Given a pseudometric space (T, d), the ε-covering number $N(\varepsilon, T, d)$ is defined as

$$(1.14)\qquad \begin{aligned} &N(\varepsilon, T, d)\\ &\qquad = \min\{n: \text{there exists a covering of } T \text{ by } n \text{ balls of radius} \le \varepsilon\}. \end{aligned}$$

Given a positive measure μ on $(S, \mathscr{S})$ we define $N_2(\varepsilon, \mathscr{F}, \mu) = N(\varepsilon, \mathscr{F}, \|\cdot\|_{L_2(\mu)})$. If $\mathscr{F}$ is a VC-subgraph class [Pollard (1984), Proposition II. 25], there are finite constants A and v such that, for each probability measure μ with $\mu F^2 < \infty$,

$$(1.15)\qquad N_2(\varepsilon, \mathscr{F}, \mu) \le A\Big(\mu(F^2)^{1/2}/\varepsilon\Big)^v,$$

where $F(x) = \sup_{f \in \mathscr{F}}|f(x)|$ and A and v can be choosen depending only on $s(\mathscr{F})$, that is, uniformly over all the classes of functions with the same number $s(\mathscr{F})$. By Pisier's maximal inequality [see Theorem 3.1 in Marcus and Pisier (1981); see also Dudley (1967)] there is constant c depending only on A and v such that, for any class of functions satisfying (1.15),

$$(1.16)\qquad E\left[\sup_{f\in\mathscr{F}}\left|n^{-1/2}\sum_{i=1}^{n}\varepsilon_i f(X_i)\right|^2\right] \le cE\big[F^2(X)\big],$$

where $\{\varepsilon_i\}_{i=1}^{\infty}$ is a Rademacher sequence independent of the sequence $\{X_i\}_{i=1}^{\infty}$ [see, e.g., (7.8) in Pollard (1990)].

2. Accuracy of the normal approximation for U-quantiles. The following extends Theorem 1.1 in Reiss (1974) on the accuracy of the normal approximation for quantiles to U-quantiles.

PROPOSITION 1. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. r.v.'s with values in a measurable space $(S, \mathscr{S})$, let h: $S^m \to \mathbb{R}$ be a measurable function and let $0 < p < 1$. Define $H(t) = \Pr\{h(X_1, \dots, X_m) \le t\}$. Suppose that there exists a*

$\xi_0 \in \mathbb{R}$ *such that* $H(\xi_0) = p$. *Let* $g(x, t) = \Pr\{h(x, X_2, \dots, X_m) \le t\}$. *Suppose that* $H'(\xi_0) > 0$, H'' *exists and is uniformly bounded in neighborhood of* ξ_0 *and* $\sigma_1^2 = m^2 \operatorname{Var}(g(X, \xi_0)) > 0$. *Then*

$$(2.1)\quad \limsup_{n\to\infty} n^{1/2} \sup_t \left|\Pr\{n^{1/2}(\xi_n - \xi_0) \le t\} - \Phi(H'(\xi_0)t\sigma_1^{-1})\right| < \infty.$$

Given the literature on Berry–Esseen bounds, the last proposition is expected, so its proof is omitted.

REMARK 2. In general the lim sup in (2.1) is different from zero. Setting $t = 0$ in (2.1), we get

$$\begin{aligned} &n^{1/2}\left|\Pr\{n^{1/2}(H_n^{-1}(p) - \xi_0) \le 0\} - 2^{-1}\right| \\ &\quad = n^{1/2}\left|\Pr\{0 \le n^{1/2}(H_n(\xi_0) - H(\xi_0))\} - 2^{-1}\right|. \end{aligned}$$

If $m = 2$, under mild conditions, we can apply Theorem 1.1 in Bickel, Götze and van Zwet (1986) to get that the last expression converges to

$$(2.2)\quad 6^{-1}\sigma_1^{-3}\left|E[g^3(X, \xi_0)] + 3E[g(X_1, \xi_0)g(X_2, \xi_0)I_{h(X_1, X_2) \le \xi_0}]\right|,$$

which is, in general, different from zero. For example, if $p = 1/2$, X has the uniform distribution on [0, 1] and $h(x, y) = x + y$, the expression in (2.2) is $2\sqrt{3}/3$.

3. Accuracy of the bootstrap of *U*-quantiles. We will see that the accuracy of the bootstrap of U-quantiles depends on the order of $E[|g(X, t) - g(X, \xi_0)|^2]$. First, we will consider the a.s. behavior of (1.7).

We will need the following lemma:

LEMMA 3. *Let* $h\colon S^m \to \mathbb{R}$ *be a measurable function and let* $0 < p < 1$. *Suppose that there is a* ξ_0 *such that* $H(\xi_0) = p$, $H'(\xi_0) > 0$ *and* $\sigma_1^2 = m^2 \operatorname{Var}(g(X, \xi_0)) > 0$. *Then there is a finite constant c such that*

$$(3.1)\quad \begin{aligned} &\limsup_{n\to\infty} \sup_{t\in\mathbb{R}} n^{3/4}(\log\log n)^{-1/2} \\ &\quad \times\left|H_n(\xi_n + tn^{-1/2}) - H_n(\xi_n) - H(\xi_n + tn^{-1/2}) + H(\xi_n)\right|(1 + t^2)^{-1} \\ &\quad \le c \quad \text{a.s.} \end{aligned}$$

PROOF. We will denote by c a finite constant which may vary from line to line. By Theorem 4.1 in Arcones (1993), with probability 1,

$$(3.2)\quad \left\{(n/2\log\log n)^{1/2}(\xi_n - \xi_0)\right\}_{n=1}^{\infty}$$

is almost sure relatively compact and its limit set is $[-\sigma_1(H'(\xi_0))^{-1}, \sigma_1(H'(\xi_0))^{-1}]$. So, it suffices to show that, for each $M < \infty$,

$$\begin{aligned}\limsup_{n\to\infty} \sup_{\substack{t\in\mathbb{R}\\ |s|\le M}} n^{3/4}(\log\log n)^{-1/2}\Big|&H_n\big(\xi_0 + s(\log\log n/n)^{-1/2} + tn^{-1/2}\big)\\ &-H_n\big(\xi_0 + s(\log\log n/n)^{1/2}\big)\\ &-H\big(\xi_0 + s(\log\log n/n)^{1/2} + tn^{-1/2}\big)\\ &+H\big(\xi_0 + s(\log\log n/n)^{1/2}\big)\Big|(1+t^2)^{-1}\\ &\le c \quad \text{a.s.}\end{aligned}$$

By the LIL for canonical U-processes indexed by VC-subgraph classes [Theorem 2.5 in Arcones and Giné (1996)],

$$\limsup_{n\to\infty} \sup_{t\in\mathbb{R}} (n/\log\log n)^{k/2}|U_n(\pi_{k,m} I_h \le t)| \le c \quad \text{a.s.},$$

for $2 \le k \le m$. So, by the Hoeffding decomposition, it suffices to show that

$$\limsup_{n\to\infty} \sup_{\substack{t\in\mathbb{R}\\ |s|\le M}} n^{3/4}(\log\log n)^{-1/2}|(P_n - P)f_{n,s,t}|(1+t^2)^{-1} \le c \quad \text{a.s.},$$

where

$$\begin{aligned} f_{n,s,t}(x) &= g\big(x, \xi_0 + s(\log\log n/n)^{1/2} + tn^{-1/2}\big)\\ &\quad - g\big(x, \xi_0 + s(\log\log n/n)^{1/2}\big).\end{aligned}$$

The first term of the Hoeffding expansion is the more difficult to handle, since it is the one to give the rate. Next, we see that

$$\limsup_{n\to\infty} E\left[\sup_{\substack{t\in\mathbb{R}\\ |s|\le M}} n^{3/4}(\log\log n)^{-1/4}|(P_n - P)f_{n,s,t}|(1+t^2)^{-1}\right] \le c. \tag{3.3}$$

It is easy to see that the class

$$\mathscr{F}_\infty := \Big\{(I_{h\le u} - I_{h\le v})(1+t^2)^{-1} : u, v, t \in \mathbb{R}\Big\}$$

is a VC-subgraph class. Hence, so is the class

$$\begin{aligned}\mathscr{F}_n := \Big\{\big(&I_{h\le \xi_0 + s(\log\log n/n)^{1/2} + tn^{-1/2}}\\ &-I_{h\le \xi_0 + s(\log\log n/n)^{1/2}}\big)(1+t^2)^{-1} : |s| \le M, t\in\mathbb{R}\Big\}.\end{aligned}$$

Moreover, $s(\mathscr{F}_n) \le s(\mathscr{F}_\infty)$ for each n. Therefore, there are two constants a and v, independent of n, such that

$$N(u, \mathscr{F}_n, L_2(\mu)) \le a\Big(\big(\mu(H_n^{\text{env}})^2\big)^{1/2}/u\Big)^v,$$

for each $u > 0$ and each measure μ on S^m, where

$$H_n^{\mathrm{env}}(x_1, \dots, x_m) = \sup_{\substack{t \in \mathbb{R} \\ |s| \le M}} \left| I_{h \le \xi_0 + s(\log\log n/n)^{1/2} + tn^{-1/2}} - I_{h \le \xi_0 + s(\log\log n/n)^{1/2}} \right| (1 + t^2)^{-1}.$$

The argument in Arcones and Giné [(1993), page 1524] implies that

$$N\left(u, \bar{\mathscr{F}}_n, L_2(\mu)\right) \le a\left(\left(\mu \left| \bar{H}_n^{\mathrm{env}} \right|^2\right)^{1/2} u\right)^v,$$

for each $u > 0$ and each measure μ on S^m, where $\bar{\mathscr{F}}_n := \{P^{m-1}f \colon f \in \mathscr{F}_n\}$ and $\bar{H}_n^{\mathrm{env}}(x) = P^{m-1}H_n^{\mathrm{env}}$. So, by symmetrization [see, e.g., Lemma 6.3 in Ledoux and Talagrand (1991)] and the maximal inequality for VC classes [see, e.g., (1.16) above],

$$\begin{aligned} &E\left[\sup_{\substack{t \in \mathbb{R} \\ |s| \le M}} n^{3/4}(\log\log n)^{-1/4} \left|(P_n - P) f_{n,s,t}\right| (1 + t^2)^{-1}\right] \\ &\quad \le c \frac{n^{1/4}}{(\log\log n)^{1/4}} \left(E\left[\sup_{\substack{t \in \mathbb{R} \\ |s| \le M}} \left| I_{h \le \xi_0 + s(\log\log n/n)^{1/2} + tn^{-1/2}} \right.\right.\right. \\ &\qquad\qquad \left.\left.\left. - I_{h \le \xi_0 + s(\log\log n/n)^{1/2}} \right| (1 + t^2)^{-2} \right]\right)^{1/2}. \end{aligned}$$

We have that

$$\begin{aligned} &(n/\log\log n)^{1/2} E\left[\sup_{\substack{t \in \mathbb{R} \\ |s| \le M}} \left| I_{h \le \xi_0 + s(\log\log n/n)^{1/2} + tn^{-1/2}} \right.\right. \\ &\qquad\qquad \left.\left. - I_{h \le \xi_0 + s(\log\log n/n)^{1/2}} \right| (1 + t^2)^{-2}\right] \\ &\quad \le (n/\log\log n)^{1/2} E\left[I_{|h - \xi_0| \le 2M(\log\log n/n)^{1/2}} \right. \\ &\qquad\qquad \left. + I_{2M(\log\log n/n)^{1/2} < |h - \xi_0|} 2^{-4} n^{-2} |h - \xi_0|^{-4} \right] = O(1). \end{aligned}$$

By (3.3) and the Lemma 7.1 in Ledoux and Talagrand (1991), it suffices to show that, with probability 1,

$$\limsup_{n \to \infty} \sup_{\substack{t \in \mathbb{R} \\ |s| \le M}} n^{-1/4}(\log\log n)^{-1/2} \left| \sum_{i=1}^n \varepsilon_i f_{n,s,t}(X_i) \right| (1 + t^2)^{-1} \le c \quad \text{a.s.} \tag{3.4}$$

Let $n_j = 2^j$. For $n_j \le n \le n_{j+1}$,

$$\sup_{\substack{t\in\mathbb{R}\\|s|\le M}} n^{-1/4}(\log\log n)^{-1/2}\left|\sum_{i=1}^{n}\varepsilon_i f_{n,s,t}(X_i)\right|(1+t^2)^{-1}$$

$$\le \sup_{\substack{t\in\mathbb{R}\\|s|\le M}} n_j^{-1/4}(\log\log n_j)^{-1/2}$$

$$\times\left|\sum_{i=1}^{n}\varepsilon_i\Big(g\Big(X_i,\xi_0+s(\log\log n_j/n_j)^{1/2}i+tn_j^{-1/2}\Big)\right. \tag{3.5}$$

$$\left.-g\Big(X_i,\xi_0+s(\log\log n_j/n_j)^{1/2}\Big)\Big)\right|\Big(1+t^2nn_j^{-1}\Big)^{-1}$$

$$\le \sup_{\substack{t\in\mathbb{R}\\|s|\le M}} n_j^{-1/4}(\log\log n_j)^{-1/2}\left|\sum_{i=1}^{n}\varepsilon_i f_{n_j,s,t}(X_i)\right|(1+t^2)^{-1}$$

By the Lévy inequality and the Talagrand isoperimetric inequality [see, e.g., Theorem 6.17 in Ledoux and Talagrand (1991)],

$$\sum_{j=1}^{\infty}\Pr\left\{\sup_{\substack{n_j\le n\le n_j+1\\t\in\mathbb{R}\\|s|\le M}}\left|\sum_{i=1}^{n}\varepsilon_i f_{n_j,s,t}(X_i)\right|(1+t^2)^{-1}\ge cn_j^{1/4}(\log\log n_j)^{1/2}\right\}$$

$$\le 2\sum_{j=1}^{\infty}\Pr\left\{\sup_{\substack{t\in\mathbb{R}\\|s|\le M}}\left|\sum_{i=1}^{n_{j+1}}\varepsilon_i f_{n_j,s,t}(X_i)\right|(1+t^2)^{-1}\ge cn_j^{1/4}(\log\log n_j)^{1/2}\right\}$$

$$\le \sum_{j=1}^{\infty}\left(2j^{-2}+2e^{-c\log j}\right)<\infty.$$

[Observe that $\sup_{t\in\mathbb{R},|s|\le M} n\operatorname{Var}(f_{n_j,s,t}(X))(1+t^2)^{-2}<\infty$.] So (3.4) follows. □

Next, we will detect the term of greater order in the Edgeworth expansion of the bootstrap of U-quantiles.

LEMMA 4. *Let $h\colon S^m\to\mathbb{R}$ be a measurable function and let $0<p<1$. We define $V(t)=\Pr\{h(X_1,\dots,X_m)\le t, h(X_1,X_{m+1},\dots,X_{2m-1})\le t\}$. Suppose that there is a ξ_0 such that $H(\xi_0)=p$, $H'(\xi_0)>0$, $\sigma_1^2=m^2\operatorname{Var}(g(X,\xi_0))>0$ and H'' exists and is uniformly bounded in a neighborhood of ξ_0. Then*

$$(n/\log\log n)^{1/2}\sup_{t\in\mathbb{R}}\Big|\Pr{}^*\{n^{1/2}(\xi_n^*-\xi_n)\le t\}$$

$$-\Pr\{n^{1/2}(\xi_n-\xi_0)\le t\} \tag{3.6}$$

$$-\phi\big(H'(\xi_0)t\sigma_1^{-1}\big)Z_n(t)\Big|\to 0\quad a.s.,$$

where

$$\begin{aligned}Z_n(t) = &-m^2 t H'(\xi_0) 2^{-1}\sigma_1^{-3}(V_n(\xi_0) - V(\xi_0))\\ &- m^2 t H'(\xi_0) 2^{-1}\sigma_1^{-3}(V(\xi_n) - V(\xi_0))\\ &+ \sigma_1^{-1} n^{1/2}\big(H_n(\xi_n + tn^{-1/2})\\ &\qquad -H_n(\xi_n) - H(\xi_n + tn^{-1/2}) + H(\xi_n)\big)\\ &- H''(\xi_0) t (H'(\xi_0))^{-1}\sigma_1^{-1}(H_n(\xi_0) - H(\xi_0))\end{aligned}$$

and

$$V_n(t) = n^{1-2m} \sum_{i_1,\dots,i_{2m-1}=1}^{n} I_{h(X_{i_1},\dots,X_{i_m}) \le t} I_{h(X_{i_1}, X_{i_{m+1}},\dots,X_{i_{2m-1}}) \le t}.$$

PROOF. By Proposition 1,

$$(n/\log\log n)^{1/2} \sup_t \left|\Pr\{n^{1/2}(\xi_n - \xi_0) \le t\} - \Phi(H'(\xi_0) t \sigma_1^{-1})\right| \to 0. \tag{3.7}$$

Take $c = 2m^2(p(1-p))^{1/2}(H'(\xi_0))^{-1}$. We have that

$$\sup_{|t| \ge c(\log n)^{1/2}} (n/\log\log n)^{1/2} |Z_n(t)| \phi(H'(\xi_0) t \sigma_1^{-1}) \to 0 \quad \text{a.s.}$$

Let

$$\overline{H}_n(t) = n^{-m} \sum_{i_1,\dots,i_m=1}^{n} I_{h(X_{i_1},\dots,X_{i_m}) \le t}.$$

By the Bernstein inequality for U-statistics [see, e.g., Serfling (1980)]

$$\begin{aligned}&n^{1/2}\Pr{}^*\{n^{1/2}(\xi_n' - \xi_n) \le -c(\log n)^{1/2}\}\\ &\quad = n^{1/2}\Pr{}^*\left\{p \le H_n^*\left(\xi_n - c\left(\frac{\log n}{n}\right)^{1/2}\right)\right\}\\ &\quad \le n^{1/2}\exp\left\{\frac{[n/m](p_n - p)^2}{2p_n(1-p_n) + (2/3)|p_n - p|}\right\},\end{aligned}$$

where $p_n = \overline{H}_n(\xi_n - c(\log n/n)^{1/2})$. By the law of the large numbers for U-processes indexed by VC classes [see Corollary 3.3 and Theorem 3.11 in Arcones and Giné (1993)],

$$\sup_{t \in \mathbb{R}} \left|\overline{H}_n(t) - H(t)\right| \to 0 \quad \text{a.s.}$$

Hence,

$$p_n = \overline{H}_n\left(\xi_n - c(\log n/n)^{1/2}\right) \to p \quad \text{a.s.} \tag{3.8}$$

We have that

$$
\begin{aligned}
(3.9)\qquad & (n/\log n)^{1/2}(p_n - p) \\
&= (n/\log n)^{1/2} \\
&\quad \times\Big(\overline{H}_n\big(\xi_n - c(\log n/n)^{1/2} - H_n\big(\xi_n - c(\log n/n)^{1/2}\big)\big)\Big) \\
&\quad + (n/\log n)^{1/2}(H_n(\xi_n) - p) \\
&\quad + (n/\log n)^{1/2}\Big(H_n\big(\xi_n - c(\log n/n)^{1/2}\big) - H_n(\xi_n) \\
&\qquad\qquad - H\big(\xi_n - c(\log n/n)^{1/2}\big) + H(\xi_n)\Big) \\
&\quad + (n/\log n)^{1/2}\Big(H\big(\xi_n - c(\log n/n)^{1/2}\big) - H(\xi_n)\Big) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}
$$

We have that

$$
(3.10)\qquad |\overline{H}_n(t) - H_n(t)| \le \tau n^{-1},
$$

where τ is a finite constant independent on t. Hence, $I \to 0$ a.s. By hypothesis, there is a $\delta_0 > 0$ such that $H(t)$ is continuous and increasing in $(\xi_0 - \delta_0, \xi_0 + \delta_0)$. Hence, for $\{i_1, \dots, i_m\} \cap \{j_1, \dots, j_m\} = \varnothing$, $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$,

$$
\Pr\big\{h(X_{i_1}, \dots, X_{i_m}) = h(X_{j_1}, \dots, X_{j_m}) \in (\xi_0 - \delta_0, \xi_0 + \delta_0)\big\} = 0.
$$

This implies that, for all $|s - \xi_0| \le \delta_0$,

$$
|H_n(s) - H_n(s-)| \le \binom{n}{m}^{-1}\left|\binom{n}{m} - \binom{n-m}{m}\right| \le cn^{-1} \quad \text{a.s.}
$$

Therefore, eventually

$$
(3.11)\qquad |H_n(\xi_n) - p| \le cn^{-1} \quad \text{a.s.}
$$

So, II $\to 0$ a.s. By the LIL for U-processes indexed by VC classes [Theorem 2.2 in Arcones (1993)],

$$
\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{\rho(t,s) \le \delta} \left(\frac{n}{\log\log n}\right)^{1/2} |H_n(s) - H_n(t) - H(s) + H(t)| = 0 \quad \text{a.s.},
$$

where $\rho^2(s,t) = \operatorname{Var}(g(X,t) - g(X,s))$. So, III $\to 0$ a.s. Since H is differentiable at ξ_0 and $\xi_n \to \xi_0$ a.s.,

$$
(3.12)\qquad \lim_{n \to \infty} (n/\log n)^{1/2}\left|H\big(\xi_n - c(\log n/n)^{1/2}\big) - H(\xi_n)\right| = cH'(\xi_0) \quad \text{a.s.}
$$

From all these estimations,

$$
(3.13)\qquad \lim_{n \to \infty} (n/\log n)^{1/2}(p_n - p) = cH'(\xi_0) \quad \text{a.s.}
$$

Hence,

$$n^{1/2}\,\Pr^*\left\{n^{1/2}(\xi_n'-\xi_n)\le -c(\log n)^{1/2}\right\}\to 0\quad\text{a.s.}$$

Therefore, it suffices to deal with the part where $|t|\le c(\log n)^{1/2}$. By the Berry–Esseen inequality for U-statistics,

$$\begin{aligned}(n/\log\log n)^{1/2}&\sup_{|t|\le c(\log n)^{1/2}}\Big|\Pr^*\{n^{1/2}(\xi_n'-\xi_n)\le t\}\\ &\qquad-\Phi\Big((\operatorname{Var}^*(H_n'(\zeta_{n,t})))^{1/2}(\overline{H}_n(\zeta_{n,t})-p)\Big)\Big|\\ &\le K_0(\log\log n)^{-1/2}\Big(\Big(V_n(\zeta_{n,t})-(\overline{H}_n(\zeta_{n,t}))^2\Big)^{-3/2}\\ &\qquad+\Big(V_n(\zeta_{n,t})-(\overline{H}_n(\zeta_{n,t}))^2\Big)^{-1}\Big),\end{aligned}\tag{3.14}$$

where $\zeta_{n,t}=\xi_n+tn^{-1/2}$. By the LLN for U-processes,

$$\sup_{t\in\mathbb{R}}\left|V_n(t)-E\left[g^2(X,t)\right]\right|\to 0\quad\text{a.s.}$$

and

$$\sup_{t\in\mathbb{R}}\left|\overline{H}_n(t)-H(t)\right|\to 0\quad\text{a.s.}$$

Hence,

$$\sup_{|t|\le c(\log n)^{1/2}}\left|V_n(\zeta_{n,t})-E\left[g^2(X,\xi_0)\right]\right|\to 0\quad\text{a.s.}\tag{3.15}$$

and

$$\sup_{|t|\le c(\log n)^{1/2}}\left|\overline{H}_n(\zeta_{n,t})-H(\xi_0)\right|\to 0\quad\text{a.s.}\tag{3.16}$$

So, (3.14) goes to zero a.s.

Next, we will show that

$$\begin{aligned}\left(\frac{n}{\log\log n}\right)^{1/2}&\sup_{|t|\le c(\log n)^{1/2}}\Big|\Phi\Big((\operatorname{Var}^*(H_n'(\zeta_n,t)))^{-1/2}(\overline{H}_n(\zeta_{n,t})-p)\Big)\\ &-\Phi(H'(\xi_0)t\sigma_1^{-1})\\ &-\Big((\operatorname{Var}^*(H_n'(\zeta_{n,t})))^{-1/2}(\overline{H}_n(\zeta_{n,t})-p)\\ &-H'(\xi_0)t\sigma_1^{-1}-Z_n(t)\Big)\phi(H'(\xi_0)t\sigma_1^{-1})\Big|\to 0\text{ a.s.}\end{aligned}\tag{3.17}$$

By the Taylor theorem, if suffices to show that

$$\begin{aligned}\left(\frac{n}{\log\log n}\right)^{1/4}&\sup_{|t|\le(\log n)^{1/2}}\left|\frac{\overline{H}_n(\zeta_{n,t})-p}{(\operatorname{Var}^*(H_n'(\zeta_{n,t})))^{1/2}}-\frac{H'(\xi_0)t}{\sigma_1}\right|\\ &\times(1+t^2)^{-1}\to 0\quad\text{a.s.}\end{aligned}\tag{3.18}$$

and

$$\left(\frac{n}{\log\log n}\right)^{1/2} \sup_{|t|\le c(\log n)^{1/2}} \left| \frac{\overline{H}_n(\zeta_{n,t}) - p}{(\mathrm{Var}^*(H_n'(\zeta_{n,t})))^{1/2}} - \frac{H'(\xi_0)t}{\sigma_1} - Z_n(t) \right| \times (1+t^2)^{-1} \to 0 \quad \text{a.s.} \tag{3.19}$$

We have that

$$\begin{aligned} V(t) - V(\xi_0) &= E\big[I_{h(X_1,\dots,X_m)\le t}\big(I_{h(X_1,X_{m+1},\dots,X_{2m-1})\le t} - I_{h(X_1,X_{m+1},\dots,X_{2m-1})\le \xi_0}\big)\big] \\ &\quad + E\big[I_{h(X_1,\dots,X_m)\le t}\big(I_{h(X_1,\dots,X_m)\le t} - I_{h(X_1,\dots,X_m)\le \xi_0}\big)\big], \end{aligned}$$

so

$$|V(t) - V(\xi_0)| \le 2|H(t) - H(\xi_0)|. \tag{3.20}$$

Hence,

$$(n/\log\log n)^{1/2} \sup_{|t|\le c(\log n)^{1/2}} |Z_n(t)|(1+t^2)^{-1} = O(1) \quad \text{a.s.} \tag{3.21}$$

Therefore, (3.19) implies (3.18). So, it suffices to prove (3.19). We have that

$$\begin{aligned} &\left(\frac{\overline{H}_n(\zeta_{n,t}) - p}{(\mathrm{Var}^*(H_n'(\zeta_{n,t})))^{1/2}} - \frac{H'(\xi_0)t}{\sigma_1} \right) \\ &\quad = n^{1/2}\big(\overline{H}_n(\zeta_{n,t}) - p\big) \\ &\qquad \times \frac{\big(n\,\mathrm{Var}^*(H_n^*(\zeta_{n,t})) - \sigma_1^2\big)}{\big(n\,\mathrm{Var}^*(H_n^*(\zeta_{n,t}))\big)^{1/2}\sigma_1\Big(\big(n\,\mathrm{Var}^*(H_n^*(\zeta_{n,t}))\big)^{1/2} + \sigma_1\Big)} \\ &\qquad + \sigma_1^{-1}\Big(n^{1/2}\big(\overline{H}_n(\zeta_{n,t}) - p\big) - tH'(\xi_0)\Big) =: \mathrm{I}_n'(t) + \mathrm{II}_n'(t). \end{aligned} \tag{3.22}$$

In order to deal with the last decomposition, we need some estimations in the variance of the empirical d.f. of U-statistic structure. We have that

$$\mathrm{Var}\big(H_n(\xi_0 + tn^{-1/2})\big) = \sum_{j=1}^{m} \binom{m}{j}\binom{n-m}{m-j}\binom{n}{m}^{-1} \sigma_j^2\big(I_{h\le \xi_0 + tn^{-1/2}}\big),$$

where

$$\sigma_j^2(f) = Ef(X_1,\dots,X_m)f(X_1,\dots,X_j,X_{j+1}',\dots,X_m') - (Ef(X_1,\dots,X_m))^2$$

and $\{X_j'\}_{j=1}^{\infty}$ is an independent copy of $\{X_j\}_{j=1}^{\infty}$ [see, e.g., Serfling (1980), Lemma A, page 183]. Since $\sigma_j(I_{h\le t}) \le 1$, we have that

$$\begin{aligned} nm\binom{n-m}{m-1}\binom{n}{m}^{-1} \sigma_1^2\big(I_{h\le\xi_0+tn^{-1/2}}\big) &\le n\,\mathrm{Var}\big(H_n(\xi_0 + tn^{-1/2})\big) \\ &\le nm\binom{n-m}{m-1}\binom{n}{m}^{-1} \sigma_1^2\big(I_{h\le\xi_0+tn^{-1/2}}\big) + n\sum_{j=2}^{m}\binom{m}{j}\binom{n-m}{m-j}\binom{n}{m}^{-1} \end{aligned}$$

and

$$\sup_{t\in\mathbb{R}}\left|n\operatorname{Var}\left(H_n(\zeta_0+tn^{-1/2})\right)-m^2\sigma_1^2\left(I_{h\le\xi_0+tn^{-1/2}}\right)\right|\le\tau n^{-1},$$

where τ is a finite constant. So, we have that

$$\sup_{t\in\mathbb{R}}\left|n\operatorname{Var}^*(H_n^*(\zeta_{n\,t}))-m^2\left(V_n(\zeta_{n\,t})-\overline{H}_n^2(\zeta_{n,t})\right)\right|\le\tau n^{-1}, \tag{3.23}$$

where τ is a finite constant. By this, (3.15) and (3.16),

$$\sup_{|t|\le c(\log n)^{1/2}}\left|n\operatorname{Var}^*(H_n^*(\zeta_{n,t}))-m^2\left(V_n(\zeta_{n,t})-\overline{H}_n^2(\zeta_{n,t})\right)\right|\to 0\quad\text{a.s.}$$

We have that

$$\begin{aligned}
&(n/\log\log n)^{1/2}(V_n(\zeta_{n,t})-V(\xi_0))\\
&\quad=(n/\log\log n)^{1/2}(V_n(\xi_0)-V(\xi_0))\\
&\qquad+(n/\log\log n)^{1/2}\big(V_n(\xi_n+tn^{-1/2})-V_n(\xi_0)\\
&\qquad\qquad-V_n(\xi_n+tn^{-1/2})+V(\xi_0)\big)\\
&\qquad+(n/\log\log n)^{1/2}\big(V(\xi_n+tn^{-1/2})-V(\xi_n)\big)\\
&\qquad+(n/\log\log n)^{1/2}(V(\xi_n)-V(\xi_0)).
\end{aligned}$$

By the LIL for U-processes indexed by VC classes,

$$\begin{aligned}
(n/\log\log n)^{1/2}\sup_{|t|\le c(\log n)^{1/2}}&\left|V_n(\xi_n+tn^{-1/2})-V_n(\xi_0)\right.\\
&\left.-V_n(\xi_n+tn^{-1/2})+V(\xi_0)\right|\to 0\quad\text{a.s.}
\end{aligned}$$

We also have that

$$\begin{aligned}
\sup_{|t|\le c(\log n)^{1/2}}(n/\log\log n)^{1/2}&\left|V(\xi_n+tn^{-1/2})\right.\\
&\left.-V(\xi_n)\right|(1+t^2)^{-1}\to 0\quad\text{a.s.}
\end{aligned}$$

Therefore, we get that

$$\begin{aligned}
&(n/\log\log n)^{1/2}(V_n(\zeta_{n,t})-V(\xi_0))\\
&\quad\approx(n/\log\log n)^{1/2}(V_n(\xi_0)-V(\xi_0))\\
&\qquad+(n/\log\log n)^{1/2}(V(\xi_n)-V(\xi_0)).
\end{aligned}\tag{3.24}$$

We also have that

$$\begin{aligned}
&(n/\log\log n)^{1/2}\left(\overline{H}_n^2(\zeta_{n,t})-H_n^2(\xi_0)\right)\\
&\quad=(n/\log\log n)^{1/2}\left(\overline{H}_n(\zeta_{n,t})-H_n(\zeta_{n\,t})\right)\left(\overline{H}_n(\zeta_{n,t})+H_n(\zeta_{n,t})\right)\\
&\qquad+(n/\log\log n)^{1/2}(H_n(\zeta_{n,t})-H(\zeta_0))(H_n(\zeta_{n\,t})+H(\xi_0)).
\end{aligned}$$

So, from this, (3.10) and (3.16),

$$(3.25)\quad \begin{aligned}&(n/\log\log n)^{1/2}\big(\overline{H}_n^2(\zeta_{n,t}) - H_n^2(\zeta_{n,t})\big)\\ &\qquad \approx 2p(n/\log\log n)^{1/2}\big(H_n(\zeta_{n,t}) - H(\xi_0)\big).\end{aligned}$$

We have that

$$\begin{aligned}&(n/\log\log n)^{1/2}(H_n(\zeta_{n,t}) - H(\xi_0))\\ &\quad = (n/\log\log n)^{1/2}(H_n(\xi_0) - H(\xi_0))\\ &\qquad + (n/\log\log n)^{1/2}\big(H_n(\xi_n + tn^{-1/2}) - H(\xi_0)\\ &\qquad\qquad - H(\xi_n + tn^{-1/2}) + H(\xi_0)\big)\\ &\qquad + (n/\log\log n)^{1/2}\big(H(\xi_n + tn^{-1/2}) - H(\xi_0)\big).\end{aligned}$$

By an argument similar to those before,

$$(3.26)\quad \begin{aligned}&(n/\log\log n)^{1/2}(H_n(\zeta_{n,t}) - H(\xi_0))\\ &\qquad \approx (n/\log\log n)^{1/2}\big(H_n(\xi_0) - H(\xi_0) + H'(\xi_0)(\xi_n - \xi_0)\big).\end{aligned}$$

By the Bahadur–Kiefer representation for U-quantiles [see Theorem 8 in Arcones (1992)], (3.26) goes to zero a.s.

We have that

$$(3.27)\quad \begin{aligned}&\big(n^{1/2}\big(\overline{H}_n(\zeta_{n,t}) - p\big) - tH'(\xi_0)\big)(1 + t^2)^{-1}\\ &\quad = n^{1/2}\big(\overline{H}_n(\zeta_{n,t}) - H_n(\zeta_{n,t})\big)(1 + t^2)^{-1}\\ &\qquad + n^{1/2}(H_n(\xi_0) - p)(1 + t^2)^{-1}\\ &\qquad + n^{1/2}\big(H_n(\xi_n + tn^{-1/2}) - H_n(\xi_n)\\ &\qquad\qquad - H(\xi_n + tn^{-1/2}) + H(\xi_n)\big)(1 + t^2)^{-1}\\ &\qquad + n^{1/2}\big(H(\xi_n + tn^{-1/2}) + H(\xi_n) - tn^{-1/2}H'(\xi_n)\big)(1 + t^2)^{-1}\\ &\qquad + t\big(H'(\xi_n) - H'(\xi_0)\big)(1 + t^2)^{-1},\end{aligned}$$

which goes to 0 a.s., uniformly in $|t| \le c(\log n)^{1/2}$. So we get that

$$(3.28)\quad \begin{aligned}&\sup_{|t|\le c(\log n)^{1/2}} \left(\frac{n}{\log\log n}\right)^{1/2}\\ &\qquad \times \big|\Gamma_n'(t) + m^2 tH'(\xi_0)2^{-1}\sigma_1^{-3}\\ &\qquad\quad \times (V_n(\xi_0) - V(\xi_0) + V(\xi_n) - V(\xi_0))\big| \to 0 \quad \text{a.s.}\end{aligned}$$

As to II,

$$
\begin{aligned}
&(n/\log\log n)^{1/2}\big(n^{1/2}\big(\overline{H}_n(\zeta_{n,t}) - p\big) - tH'(\xi_0)\big)\\
&\quad = n(\log\log n)^{-1/2}\big(\overline{H}_n\big(\xi_n + tn^{-1/2}\big) - H_n\big(\xi_n + tn^{-1/2}\big)\big)\\
&\qquad + n(\log\log n)^{-1/2}\\
&\qquad\quad \times\big(H_n\big(\xi_n + tn^{-1/2}\big) - H(\xi_n) - H_n\big(\xi_n + tn^{-1/2}\big) + H(\xi_n)\big)\\
&\qquad + n(\log\log n)^{-1/2}(H(\xi_n) - p)\\
&\qquad + n(\log\log n)^{-1/2}\big(H\big(\xi_n + tn^{-1/2}\big) - H(\xi_n) - tn^{-1/2}H'(\xi_n)\big)\\
&\qquad + (n/\log\log n)^{1/2} t(H'(\xi_n) - H'(\xi_0))\\
&\quad \approx n(\log\log n)^{-1/2}\big(H_n\big(\xi_n + tn^{-1/2}\big) - H(\xi_n) - H_n\big(\xi_n + tn^{-1/2}\big) + H(\xi_n)\big)\\
&\qquad + (n/\log\log n)^{1/2} tH''(\xi_0)(\xi_n - \xi_0).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\sup_{|t|\le c(\log n)^{1/2}} (n/\log\log n)^{1/2}\\
&\quad\times\Big|\mathrm{II}'_n(t) - \sigma_1^{-1}n^{1/2}\big(H_n\big(\xi_n + tn^{-1/2}\big) - H_n(\xi_n)\\
&\qquad\qquad - H_n\big(\xi_n + tn^{-1/2}\big) + H(\xi_n)\big)\\
&\qquad + H''(\xi_0)t(H'(\xi_0))^{-1}\sigma_1^{-1}(H_n(\xi_0) - H(\xi_0))\Big| \to 0 \quad \text{a.s.}
\end{aligned}
\tag{3.29}
$$

From (3.28) and (3.29), (3.19) follows. □

THEOREM 5. *Let $h\colon S^m \to \mathbb{R}$ be a measurable function and let $0 < p < 1$. Suppose that there is a ξ_0 such that $H(\xi_0) = p$, $H'(\xi_0) > 0$, $\sigma_1^2 = m^2\,\mathrm{Var}(g(X, \xi_0)) < 0$ and H'' exists and is uniformly bounded in a neighborhood of ξ_0. Then there is a finite constant c such that*

$$
\limsup_{n\to\infty} n^{1/4}(\log\log n)^{-1/2} \sup_{t\in\mathbb{R}}\big|\Pr{}^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\big| \le c \quad \text{a.s.}
$$

PROOF. We apply Lemmas 3 and 4. Here, c will denote a finite constant which may vary from line to line. By the LIL for U-statistics [Serfling (1971)],

$$
\limsup_{n\to\infty} (n/\log\log n)^{1/2}|V_n(\xi_0) - V(\xi_0)| \le c \quad \text{a.s.} \tag{3.30}
$$

and

$$
\limsup_{n\to\infty} (n/\log\log n)^{1/2}|H_n(\xi_0) - H(\xi_0)| \le c \quad \text{a.s.} \tag{3.31}
$$

By (3.2) and (3.20),

$$\limsup_{n\to\infty} (n/\log\log n)^{1/2} |V(\xi_n) - V(\xi_0)| \le c \quad \text{a.s.} \tag{3.32}$$

so the result follows. □

The next proposition shows that the order in the last theorem is attainable in some situations.

PROPOSITION 6. *Let $h: S^m \to \mathbb{R}$ be a measurable function and let $0 < p < 1$. Suppose that there is a ξ_0 such that $H(\xi_0) = p$, $H'(\xi_0) > 0$, $\sigma_1^2 = m^2 \operatorname{Var}(g(X, \xi_0)) > 0$ and H'' exists and is uniformly bounded in a neighborhood of ξ_0. Assume also that there is a number β such that*

$$n^{1/2} E\Big[\Big(g\big(X, \xi_0 + (\log\log n/n)^{1/2} + tn^{-1/2}\big) - g\big(X, \xi_0 + (\log\log n/n)^{1/2}\big)\Big)^2\Big] \to \beta^2 |t|,$$

for each $t \in \mathbb{R}$, and

$$n^{1/2} E\Big[\Big(g\big(X, \xi_0(\log\log n/n)^{1/2} + t_1 n^{-1/2}\big) - g\big(X, \xi_0(\log\log n/n)^{1/2} + t_2 n^{-1/2}\big)\Big)^2\Big] \to \beta^2 |t_1 - t_2|,$$

for each $t_1, t_2 \in \mathbb{R}$. Then with probability 1,

$$\Big\{n^{1/4}(2\log\log n)^{-1/2}\big(\Pr^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\big): t \in \mathbb{R}\Big\}$$

is relatively compact and its limit set is

$$\Big\{\Big(m\sigma_1^{-1}\beta\phi(\sigma_1^{-1}H'(\xi_0)t)\int_0^t \alpha(z)\,dz\Big)_{t\in\mathbb{R}} : \int_{-\infty}^{\infty} \alpha^2(z)\,dz \le 1\Big\}.$$

PROOF. By Lemma 4 and (3.30)–(3.32),

$$\begin{aligned} n^{1/4}(2\log\log n)^{-1/2} \sup_{t\in\mathbb{R}} \Big|\Big(&\Pr^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} \\ &- \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\Big) - \sigma_1^{-1}\phi(\sigma_1^{-1}H'(\xi_0)t)n^{1/2} \\ &\times \big(H_n(\xi_n + tn^{-1/2}) - H_n(\xi_n) \\ &- H(\xi_n + tn^{-1/2}) + H(\xi_n)\big)\Big| \to 0 \quad \text{a.s.} \end{aligned}$$

It is easy (and quite tedious) to see that the method in the proof of Theorem 1 in Arcones (1994) and the estimations used in Lemma 4, give that, with probability 1,

$$\begin{aligned}
\Big\{\Big(n^{3/4}(2\log\log n)^{-1/2}\sigma_1^{-1}\phi\big(\sigma_1^{-1}H'(\xi_0)t\big) \\
\times\Big(H_n\big(\xi_0+s(2\log\log n/n)^{1/2}+tn^{-1/2}\big) \\
-H_n\big(\xi_0+s(2\log\log n/n)^{1/2}\big) \\
-H\big(\xi_0+s(2\log\log n/n)^{1/2}+tn^{-1/2}\big) \\
+H\big(\xi_0+s(2\log\log n/n)^{1/2}\big)\Big), \\
(n/2\log\log n)^{1/2}(\xi_n-\xi_0)\Big): s,t\in\mathbb{R}\Big\}
\end{aligned}\tag{3.33}$$

is relatively compact and its limit set is

$$\left\{\left(m\beta\sigma_1^{-1}\phi\big(\sigma_1^{-1}H'(\xi_0)t\big)\int_0^t\alpha(z)\,dz,\ \sigma_1(H'(\xi_0))^{-1}u\right)_{s,t\in\mathbb{R}}:\ \int_{-\infty}^{\infty}\alpha^2(z)\,dz+u^2\le 1\right\}.$$

Taking a convenient composition in (3.33), the result follows. □

The last proposition applies to $h(x_1,\dots,x_m)=\max(x_1,\dots,x_m)$.

THEOREM 7. *Let $h\colon S^m\to\mathbb{R}$ be a measurable function and let $0<p<1$. Suppose that there is a ξ_0 such that $H(\xi_0)=p$, $H'(\xi_0)>0$, $\sigma_1^2=m^2\operatorname{Var}(g(X,\xi_0))>0$ and H'' exists and is uniformly bounded in a neighborhood of ξ_0. Suppose also that there is a nonnegative function $\lambda(x)$ and a $\delta_0>0$ such that $E[\lambda^2(X)]<\infty$ and*

$$|g(x,t)-g(x,s)|\le\lambda(x)|t-s|,\tag{3.34}$$

for all $|t-\xi_0|,|s-\xi_0|\le\delta_0$. Then there is finite constant c such that

$$\begin{aligned}
\limsup_{n\to\infty}\frac{n^{1/2}}{(\log\log n)^{1/2}}\sup_{t\in\mathbb{R}}\big|\Pr{}^*\{n^{1/2}(\xi_n^*-\xi_n)\le t\} \\
-\Pr\{n^{1/2}(\xi_n-\xi_0)\le t\}\big|\le c\quad\text{a.s.}
\end{aligned}\tag{3.35}$$

PROOF. By the argument in Lemma 3, it suffices to show that

$$\sup_{\substack{t\in\mathbb{R}\\|s|\le M}} n(\log\log n)^{-1/2}|(P_n-P)f_{n,s,t}|\phi(at)=O(1)\quad\text{a.s.},$$

where, as defined in the proof of Lemma 3,

$$\begin{aligned}
f_{n,s,t}(x)=g\big(x,\xi_0+s(\log\log n/n)^{1/2}+tn^{-1/2}\big) \\
-g\big(x,\xi_0+s(\log\log n/n)^{1/2}\big)
\end{aligned}$$

and $a = H'(\xi_0)t\sigma^{-1}$; $\phi(at)$ is related to the coefficient of $Z_n(t)$ in (3.6). By symmetrization [see, e.g., Ledoux and Talagrand (1991), Lemma 6.6]

$$E\left[\sup_{\substack{t\in\mathbb{R}\\|s|\le M}} n(\log\log n)^{-1/2}|(P_n - P)f_{n,s,t}|\phi(at)\right]$$
$$\le 2E\left[\sup_{\substack{t\in\mathbb{R}\\|s|\le M}} (\log\log n)^{-1/2}\left|\sum_{i=1}^n \varepsilon_i f_{n,s,t}(X_i)\right|\phi(at)\right].$$

We have that

$$E\left[\sup_{|t|\ge 2a^{-1}(\log n)^{1/2}} (\log\log n)^{-1/2}\left|\sum_{i=1}^n \varepsilon_i f_{n,s,t}(X_i)\right|\phi(at)\right] \to 0.$$

We also have that

$$E\left[\sup_{2a^{-1}(\log\log n)^{1/2}|t|\le 2a^{-1}(\log n)^{1/2}} (\log\log n)^{-1/2}\left|\sum_{i=1}^n \varepsilon_i f_{n,s,t}(X_i)\right|\phi(at)\right]$$
$$\le E\left[\sup_{2a^{-1}(\log\log n)^{1/2}|t|\le 2a^{-1}(\log n)^{1/2}} (\log n)^{-2}(\log\log n)^{-1/2}\left|\sum_{i=1}^n \varepsilon_i f_{n,s,t}(X_i)\right|\right]$$
$$\le cn^{1/2}(\log n)^{-2}(\log\log n)^{-1/2}$$
$$\times\left(E\left[\sup_{2a^{-1}(\log n)^{1/2}|t|\le 2a^{-1}(\log n)^{1/2}} |f_{n,s,t}(X)|^2\right]\right)^{1/2}$$
$$\le cn^{1/2}(\log n)^{-2}(\log\log n)^{-1/2}$$
$$\times\left(E\left[|g\left(X, \xi_0 + 3a^{-1}(\log n/n)^{1/2}\right) - g(X, \xi_0)|^2\right]\right)^{1/2}$$
$$+cn^{1/2}(\log n)^{-2}(\log\log n)^{-1/2}$$
$$\times\left(E\left[|g\left(X, \xi_0 + 3a^{-1}(\log n/n)^{1/2}\right) - g(X, \xi_0)|^2\right]\right)^{1/2} \to 0$$

and that

$$E\left[\sup_{|t|\le 2a^{-1}(\log\log n)^{1/2}} (\log\log n)^{-1/2}\left|\sum_{i=1}^n \varepsilon_i f_{n,s,t}(X_i)\right|\phi(at)\right]$$
$$\le 2E\left[\sup_{|t|\le 3a^{-1}(\log\log n/n)^{1/2}} (\log\log n)^{-1/2}\right.$$
$$\left.\times\left|\sum_{i=1}^n \varepsilon_i(g(X_i, \xi_0 + t) - g(X_i, \xi_0))\right|\right] = O(1).$$

So

$$E\left[\sup_{\substack{t\in\mathbb{R}\\|s|\le M}} n(\log\log n)^{-1/2}|(P_n-P)f_{n,s,t}|\phi(at)\right]=O(1).$$

Hence, by the Lemma 7.1 in Ledoux and Talagrand (1991), it suffices to show that there is a finite constant c such that

$$\limsup_{n\to\infty}\ \sup_{\substack{t\in\mathbb{R}\\|s|\le M}} (\log\log n)^{-1/2}\left|\sum_{i=1}^{n}\varepsilon_i f_{n,s,t}(X_i)\right|\phi(at)\Bigg] \le c \quad \text{a.s.}$$

Let $n_j = 2^j$. For $n_n \le n \le n_{j+1}$, by the argument in (3.5),

$$\sup_{\substack{t\in\mathbb{R}\\|s|\le M}} (\log\log n)^{-1/2}\left|\sum_{i=1}^{n}\varepsilon_i f_{n,s,t}(X_i)\right|\phi(at)$$

$$\le \sup_{\substack{t\in\mathbb{R}\\|s|\le M}} \left(\log\log n_j\right)^{-1/2}\left|\sum_{i=1}^{n}\varepsilon_i f_{n_j,s,t}(X_i)\right|\phi(at).$$

By the Lévy inequality and the Talagrand isoperimetric inequality [see, e.g., Ledoux and Talagrand (1991), Theorem 6.17],

$$\sum_{j=1}^{\infty}\Pr\left\{\sup_{n_j\le n\le n_{j+1}}\left|\sum_{i=1}^{n}\varepsilon_i f_{n_j,s,t}(X_i)\right|\phi(at)\ge c\left(\log\log n_j\right)^{1/2}\right\}$$

$$\le 2\sum_{j=1}^{\infty}\Pr\left\{\sup_{\substack{t\in\mathbb{R}\\|s|\le M}}\left|\sum_{i=1}^{n_{j+1}}\varepsilon_i f_{n_j,s,t}(X_i)\right|\phi(at)\ge c\left(\log\log n_j\right)^{1/2}\right\}$$

$$\le 2\sum_{j=1}^{\infty}\left(2^{1-2j}+2\Pr\left\{\max_{1\le i\le n_j}\sup_{\substack{t\in\mathbb{R}\\|s|\le M}}\left|f_{n_j,s,t}(X_i)\right|\phi(at)\right.\right.$$

$$\left.\left.\ge 2\left(\log\log n_j\right)^{-1/2}\right\}+2e^{-c\log j}\right).$$

We have that

$$\sum_{j=1}^{\infty}\Pr\left\{\max_{1\le i\le n_j}\sup_{\substack{t\in\mathbb{R}\\|s|\le M}}\left|f_{n_j,s,t}(X_i)\right|\phi(at)\ge 2\left(\log\log n_j\right)^{-1/2}\right\}$$

$$\le \sum_{j=1}^{\infty} n_j\Pr\left\{\lambda(X)\ge c n_j^{1/2}\right\}<\infty.$$

Therefore, the result follows. □

In general, the constant c in the previous theorem is different from zero. This follows from the fact that the a.s. order of the terms in Z_t is

$n^{1/2}(\log\log n)^{-1/2}$, as is easy to see. Next, we give two examples of U-statistics which satisfy (3.34).

Suppose that $h(x_1, x_2) = x_1 + x_2$. Then $g(x, t) = F(x - t)$. So if F is a Lipschitz function with Lipschitz constant λ, then

$$E\left[|g(X,t) - g(X,s)|^2\right] \le \lambda^2 |s - t|^2.$$

Suppose that $h(x_1, x_2) = |x_1 - x_2|^r$ for some $r > 0$. Then $g(x, t) = F(x + t^{1/r}) - F(x - t^{1/r})$. So if F is a Lipschitz function with Lipschitz constant λ, then

$$E\left[g(x,t) - g(X,s)|^2\right] \le 4\lambda^2 |s^{1/r} - t^{1/r}|^2 \le c|t - s|^2,$$

for s and t close to ξ_0.

Similarly to the a.s. case, we have the following bounds in probability:

THEOREM 8. *Under the assumptions in Theorem* 5,

$$n^{1/4} \sup_{t\in\mathbb{R}} \left|\Pr{}^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\right| = O_{\Pr}(1). \tag{3.36}$$

THEOREM 9. *Under the assumptions in Theorem* 7,

$$n^{1/2} \sup_{t\in\mathbb{R}} \left|\Pr{}^*\{n^{1/2}(\xi_n^* - \xi_n) \le t\} - \Pr\{n^{1/2}(\xi_n - \xi_0) \le t\}\right| = O_{\Pr}(1). \tag{3.37}$$

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