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INFERENCE FOR UNSTABLE LONG-MEMORY PROCESSES WITH APPLICATIONS TO FRACTIONAL UNIT ROOT AUTOREGRESSIONS[1]

BY NGAI HANG CHAN AND NORMA TERRIN

Hong Kong University of Science and Technology and Carnegie Mellon University

An autoregressive time series is said to be unstable if all of its characteristic roots lie on or outside the unit circle, with at least one on the unit circle. This paper aims at developing asymptotic inferential schemes for an unstable autoregressive model generated by long-memory innovations. This setting allows both nonstationarity and long-memory behavior in the modeling of low-frequency phenomena. In developing these procedures, a novel weak convergence result for a sequence of long-memory random variables to a stochastic integral of fractional Brownian motions is established. Results of this paper can be used to test for unit roots in a fractional AR model.

1. Introduction. Statistical analysis of time series data which exhibit high power at very low frequencies has received considerable attention in the literature recently. On the one hand, unstable time series with roots on the unit circle are an example of this phenomenon. General results for unstable time series are given in Chan and Wei (1988). These results involve nonstandard asymptotics which have a significant impact on inference for so-called unit root econometrics. The latter subject has been under intensive study for the last decade. Empirical work in this field can be found in Stock and Watson (1988). For a recent comprehensive review on this topic, see Banerjee, Dolado, Galbraith and Hendry (1993) and the references therein.

On the other hand, processes displaying long-memory behavior possess spectra with high power at low frequencies. Long-memory behavior can be found in diverse disciplines ranging from hydrology to economics. The recent survey article by Beran (1992) provides an interesting and comprehensive discussion on the statistical aspect of this subject. Among different long-memory models, the so-called fractional ARMA models play a particular role in econometrics. Illustrations on economic applications of fractional ARMA models can be found in Robinson (1994) and Diebold (1988).

Although both unstable time series and fractional ARMA are used to model the phenomenon of high power at low frequencies, their underlying

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motivations are quite different. The first attempt to connect these two concepts is given by Sowell (1990), who considered the fractional unit root distribution for a simple AR(1) model. In this paper, a comprehensive study is conducted to link these two approaches for a general unstable AR(p) model. Instead of assuming the innovations of the model to be a martingale difference sequence, as in Chan and Wei (1988), we allow it to be generated by some long-memory process such as a fractional ARMA so that both long-memory behavior and unit roots are allowed in the model. There are two basic motivations for studying this general model.

First, as shown in Sowell (1990), the asymptotics of a unit root fractional AR(1) model differ significantly from a short-memory AR case. It is thus important to study the theoretical properties of the least squares estimates (LSE) in this general case. A weak convergence result involving integration of fractional Brownian motions is established and used to develop the asymptotic theory of the model.

Second, the differencing parameter d of the fractional AR model $(1-B)^d y_t = \varepsilon_t$, $\varepsilon_t \sim \text{iid}(0, \sigma^2)$, must satisfy $|d| < \frac{1}{2}$. If we extend this model to the case $(1-B)y_t = x_t$, where $\{x_t\}$ itself is a fractional AR process, $(1-B)^d x_t = \varepsilon_t$, then $(1-B)^{d+1} y_t = (1-B)^d x_t = \varepsilon_t$. Thus, $\{y_t\}$ satisfies a more general fractional AR with $d^* = d + 1 \in (\frac{1}{2}, \frac{3}{2})$. Such a case seems to have special interest in economics [see Diebold (1988)]. More generally, if we consider $(1-B)^a y_t = x_t$, where $\{x_t\}$ satisfies $(1-B)^d x_t = \varepsilon_t$, and a is an integer, then $(1-B)^{a+d} y_t = \varepsilon_t$. Thus, $\{y_t\}$ can be viewed as a general fractional AR with order $a + d \in (a - \frac{1}{2}, a + \frac{1}{2})$.

This paper is organized as follows. Preliminaries on unstable AR(p) model are given in Section 2, and the convergence of certain functionals of long-memory processes to stochastic integrals is proved in Section 3. Asymptotic distributions of the LSE by componentwise arguments are given in Sections 4–6. Section 7 consists of concluding remarks.

2. Preliminaries. Consider the unstable AR(p) model

$$\alpha(B) y_t = e_t, \tag{2.1}$$

where $\alpha(B)$ has the general unstable form as in Chan and Wei (1988),

$$\alpha(B) = (1-B)^a (1+B)^b \prod_{k=1}^{l} \left(1 - 2\cos\theta_k B + B^2\right)^{d_k} \varphi(B), \tag{2.2}$$

$\varphi(B)$ is a qth-order polynomial in B with roots outside the unit circle and $q = p - (a + b + 2\sum_{k=1}^{l} d_k)$. The innovation $\{e_t\}$ in (2.1), assumed to be generated by a long-memory process, will be defined more precisely in the next section. As shown by Chan and Wei (1988), abbreviated as CW herein, we can transform $\{y_t\}$ into various components corresponding to the location of their roots and proceed to analyze each individual component. Specifically, let

$u_t = \alpha(B)(1-B)^{-a}y_t$, $v_t = \alpha(B)(1+B)^{-b}y_t$, $z_t = \alpha(B)\varphi^{-1}(B)y_t$ and $x_t(k) = \alpha(B)(1-2\cos\theta_k B + B^2)^{-d_k}y_t$, $k = 1,\dots,l$. Then $(1-B)^a u_t = e_t$, $(1+B)^b v_t = e_t$, $\varphi(B)z_t = e_t$ and $(1-2\cos\theta_k B + B^2)^{d_k}x_t(k) = e_t$. Define

$$\text{(2.3)}\qquad \mathbf{u}_t = (u_t,\dots,u_{t-a+1})', \qquad \mathbf{v}_t = (v_t,\dots,v_{t-b+1})',$$

$$\text{(2.4)}\qquad \mathbf{x}_t(k) = \big(x_t(k),\dots,x_{t-2d_k+1}(k)\big)', \qquad \mathbf{y}_t = (y_t,\dots,y_{t-p})'$$

and

$$\text{(2.5)}\qquad \mathbf{z}_t = (z_t,\dots,z_{t-q+1})'.$$

As shown in (3.2) of CW, there exists a nonsingular matrix Q such that $Q\mathbf{y}_t = (\mathbf{u}'_t, \mathbf{v}'_t, \mathbf{x}'_t(1),\dots,\mathbf{x}'_t(l), \mathbf{z}'_t)'$. Further, there exists a normalization matrix $G_n = \operatorname{diag}(J_n, K_n, L_n(1),\dots,L_n(l), M_n)$ such that

$$\begin{aligned} &G_n Q \sum \mathbf{y}_{t-1}\mathbf{y}'_{t-1} Q' G'_n \\ \text{(2.6)}\qquad &\sim_p \operatorname{diag}\Big(J_n \sum \mathbf{u}_{t-1}\mathbf{u}'_{t-1} J'_n, \dots, L_n(l) \sum \mathbf{x}_{t-1}(l)\mathbf{x}'_{t-1}(l) L'_n(l), \\ &\qquad\qquad M_n \sum \mathbf{z}_{t-1}\mathbf{z}'_{t-1} M'_n\Big), \end{aligned}$$

and

$$\text{(2.7)}\qquad (Q'G'_n)^{-1}(\hat{\boldsymbol{\alpha}}_n - \boldsymbol{\alpha}) \sim_p \begin{pmatrix} (J'_n)^{-1}\big(\sum \mathbf{u}_{t-1}\mathbf{u}'_{t-1}\big)^{-1} \sum \mathbf{u}_{t-1}e_t \\ \vdots \\ \big(L'_n(l)^{-1}\big)\big(\sum \mathbf{x}_{t-1}(l)\mathbf{x}'_{t-1}(l)\big)^{-1} \sum \mathbf{x}_{t-1}(l)e_t \\ (M'_n)^{-1}\big(\sum \mathbf{z}_{t-1}\mathbf{z}'_{t-1}\big)^{-1} \sum \mathbf{z}_{t-1}e_t \end{pmatrix},$$

where $\hat{\boldsymbol{\alpha}}_n$ is the least squares estimate of the parameter vector $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_p)'$.

Unless otherwise stated, summations are always taken from 1 to n over t and limits are always taken as n tends to infinity in this paper. The normalization matrices $J_n, K_n, \dots, L_n(l), M_n$ will be defined in subsequent sections. Observe that the reparametrizations (2.3)–(2.5) translate the original LSE of (2.1) into individual components in (2.6) and (2.7). We further assume that $y_i = 0$ for $i = 1,\dots,-p+1$, which in turn implies that $u_i = 0$, $i = 0,\dots,-a+1$, $v_i = 0$, $i = 0,\dots,-b+1$, and $x_i(k) = 0$, $i = 0,\dots,-2d_k+1$, $k = 1,\dots,l$ and $z_i = 0$, $i = 0,\dots,-q+1$.

3. Probability results for long-memory processes. In order to study the asymptotic behavior of the components described in the preceding section, three limit theorems involving long-memory processes are derived in this section. The results will be applied to the long-memory errors $\{e_t\}$ in the model (2.1). In this section, however, the notation X_t is used in place of e_t. The results pertain to a very general class of stationary, Gaussian processes with regularly varying spectral density $f(\lambda)$; that is,

$$f(\lambda) = |\lambda|^{1-2H} L(|\lambda|^{-1}), \tag{3.1}$$

where $0 < H < 1$, L is slowly varying at ∞ [i.e., $L(ta)/L(t) \to 1$ as $t \to \infty$ for any $a > 0$] and f is integrable on $[-\pi, \pi]$. Two special cases of the model (3.1) are fractional Gaussian noise and fractional ARIMA with differencing parameter $d = H - \frac{1}{2}$. If $\frac{1}{2} < H < 1$, then the spectrum has a singularity at the origin, and the process is said to exhibit long memory. When $H = \frac{1}{2}$ and L is constant, the process is white noise.

If X_n is the Gaussian process defined by its spectrum (3.1), then X_n has spectral representation

$$X_n = \int_{-\pi}^{\pi} \exp(in\lambda) f^{1/2}(\lambda) W(d\lambda), \tag{3.2}$$

$n = 1, 2, \dots,$ where $W(\cdot)$ is the complex-valued, Gaussian random measure satisfying

$$W(d\lambda) = \overline{W(-d\lambda)},$$

$$EW(d\lambda) = 0 \quad \text{and} \quad EW(d\lambda)\overline{W(d\mu)} = \begin{cases} 0, & \text{if } \lambda \neq \mu, \\ d\lambda, & \text{if } \lambda = \mu. \end{cases}$$

When $H > \frac{1}{2}$, the covariance function r_k of $\{X_n\}$ satisfies the relation

$$r_k \sim 2k^{2H-2} L(k) \Gamma(2 - 2H) \sin\left(\pi\left(H - \tfrac{1}{2}\right)\right),$$

as $k \to \infty$, where Γ is the gamma function [see, e.g., Zygmund (1988), Theorem 2.24].

The following theorem describes the asymptotic behavior of the stationary component $\{z_t\}$ of the model given in Section 2.

THEOREM 3.1. *Let X_t be the sequence defined by* (3.2), *with* $\frac{1}{2} < H < 1$. *Suppose $\mathbf{z}_t$ is a sequence of random vectors in $\mathscr{R}^q$ which satisfies*

$$\mathbf{z}_t = A\mathbf{z}_{t-1} + \mathbf{X}_t,$$

where $\mathbf{z}_0 = \mathbf{0}$, $\mathbf{X}'_t = (X_t, 0, \dots, 0)$ and A is a $q \times q$ invertible constant matrix with all eigenvalues lying inside the unit circle. Then

$$\left(\sum_{t=1}^{n} \mathbf{z}_{t-1}\mathbf{z}'_{t-1}\right)^{-1} \sum_{t=1}^{n} \mathbf{z}_{t-1} X_t \to_p \Sigma^{-1}\left\{\int_{-\pi}^{\pi} \left[I_q \exp(i\lambda) - A\right]^{-1} f(\lambda)\, d\lambda\right\}\mathbf{1}, \tag{3.3}$$

where

$$\Sigma = \int_{-\pi}^{\pi} \left[I_q \exp(i\lambda) - A\right]^{-1} \left[I_q \exp(-i\lambda) - A'\right]^{-1} f(\lambda)\, d\lambda,$$

I_q is the q-dimensional identity matrix, $\mathbf{1}' \equiv (1\,0 \dots 0)$ and, for any matrix M, $[\int M]_{ij} = \int M_{ij}$.

REMARK. Note that $\mathbf{z}_t$ in Theorem 3.1 is expressed in the companion form of the AR(q) model $\varphi(B)z_t = X_t$. The left-hand side of (3.3) then simply equals the normalized LSE $\hat{\boldsymbol{\varphi}}_n - \boldsymbol{\varphi}$, where $\hat{\boldsymbol{\varphi}}_n = (\hat{\varphi}_1, \dots, \hat{\varphi}_q)'$ is the LSE of the parameter vector $\boldsymbol{\varphi} = \{\varphi_1, \dots, \varphi_q\}'$ of the stationary AR(q) polynomial $\varphi(B)$. In contrast to the case where X_t is a martingale difference sequence, the limit in Theorem 3.1 is a constant. Thus, for a stationary AR model with long-memory innovations, the LSE is inconsistent.

PROOF OF THEOREM 3.1. For simplicity, the proof is carried out with $L \equiv 1$. With slight modifications, the proof is valid for any slowly varying function. It is easy to check that

$$\mathbf{z}_t = \sum_{k=1}^{t} A^{t-k} \mathbf{X}_k.$$

Hence,

$$\begin{aligned} \sum_{t=1}^{n} \mathbf{z}_{t-1} X_t &= \left\{ \sum_{t=1}^{n} \sum_{k=1}^{t-1} A^{t-1-k} X_k X_t \right\} \mathbf{1} \\ &= \left\{ \sum_{t=1}^{n} \sum_{k=1}^{t-1} A^{t-1-k} \int_{-\pi}^{\pi} \exp(ik\lambda_1) |\lambda_1|^{1/2-H} W(d\lambda_1) \right. \\ &\qquad \left. \times \int_{-\pi}^{\pi} \exp(it\lambda_2) |\lambda_2|^{1/2-H} W(d\lambda_2) \right\} \mathbf{1} \qquad (3.4) \\ &= \left\{ \sum_{t=1}^{n} \sum_{k=1}^{t-1} A^{t-1-k} \left[2 \int''_{[-\pi,\pi]^2} \exp(ik\lambda_1) \exp(it\lambda_2) \right. \right. \\ &\qquad \times |\lambda_1|^{1/2-H} |\lambda_2|^{1/2-H} W(d\lambda_1) W(d\lambda_2) \\ &\qquad \left. \left. + \int_{-\pi}^{\pi} \exp[i(k-t)\lambda] |\lambda|^{1-2H}\, d\lambda \right] \right\} \mathbf{1}, \end{aligned}$$

where $\int''$ indicates that integration on the diagonals $\lambda_1 = \pm\lambda_2$ is excluded. The double integral is a Wiener–Itô integral defined by Major (1981). The right-hand side of (3.4) consists of a random term plus the mean of the left-hand side of (3.4). We will show that the random term is $o_p(n)$ and that the mean, divided by n, converges to

$$\left\{\int_{-\pi}^{\pi}\left[I_q \exp(i\lambda) - A\right]^{-1}|\lambda|^{1-2H}\, d\lambda\right\}\mathbf{1}. \tag{3.5}$$

We have

$$\begin{aligned}
&\sum_{t=1}^{n}\sum_{k=1}^{t-1} A^{t-1-k}\int''_{[-\pi,\pi]^2}\exp(ik\lambda_1)\exp(it\lambda_2)\\
&\qquad\times|\lambda_1|^{1/2-H}|\lambda_2|^{1/2-H}W(d\lambda_1)W(d\lambda_2)\\
&\quad=\sum_{t=1}^{n}A^{t-1}\int''_{[-\pi,\pi]^2}\left\{\sum_{k=1}^{t-1}A^{-k}\exp(ik\lambda_1)\right\}\\
&\qquad\times\exp(it\lambda_2)|\lambda_1|^{1/2-H}|\lambda_2|^{1/2-H}W(d\lambda_1)W(d\lambda_2)\\
&\quad=\sum_{t=1}^{n}A^{t-1}\int''_{[-\pi,\pi]^2}\left[A^{-1}\exp(i\lambda_1)-I_q\right]^{-1}\\
&\qquad\times\left[A^{-t}\exp(i\lambda_1 t)-A^{-1}\exp(i\lambda_1)\right]\\
&\qquad\times\exp(i\lambda_2 t)|\lambda_1|^{1/2-H}|\lambda_2|^{1/2-H}W(d\lambda_1)W(d\lambda_2)\\
&\quad=\int''_{[-\pi,\pi]^2}\left[I_q\exp(i\lambda_1)-A\right]^{-1}\\
&\qquad\times\sum_{t=1}^{n}\exp[i(\lambda_1+\lambda_2)t]|\lambda_1|^{1/2-H}|\lambda_2|^{1/2-H}W(d\lambda_1)W(d\lambda_2)\\
&\qquad-\int''_{[-\pi,\pi]^2}\left[I_q\exp(i\lambda_1)-A\right]^{-1}A^{-1}\exp(i\lambda_1)\\
&\qquad\times\sum_{t=1}^{n}A^{t}\exp(i\lambda_2 t)|\lambda_1|^{1/2-H}|\lambda_2|^{1/2-H}W(d\lambda_1)W(d\lambda_2).
\end{aligned} \tag{3.6}$$

The first term on the right-hand side of (3.6) is

$$\begin{aligned}
&\int''_{[-\pi,\pi]^2}\frac{\exp[i(\lambda_1+\lambda_2)(n+1)]-\exp[i(\lambda_1+\lambda_2)]}{\exp[i(\lambda_1+\lambda_2)]-1}\\
&\qquad\times\left[I_q\exp(i\lambda_1)-A\right]^{-1}|\lambda_1|^{1/2-H}|\lambda_2|^{1/2-H}W(d\lambda_1)W(d\lambda_2).
\end{aligned} \tag{3.7}$$

After making the change of variables $y_i = n\lambda_i$, $i = 1, 2$, it can be seen, by Lemma 3 of Dobrushin and Major (1979), that the integral (3.7), divided by

n^{2H-1}, converges in distribution to

$$[I_q - A]^{-1} \int''_{\mathscr{R}^2} \frac{\exp[i(\lambda_1 + \lambda_2)] - 1}{i(\lambda_1 + \lambda_2)} |\lambda_1|^{1/2-H} |\lambda_2|^{1/2-H} W(d\lambda_1) W(d\lambda_2),$$

since $\frac{1}{2} < H < 1$. Hence it is $o_p(n)$, since $2H - 1 < 1$. The second term on the right-hand side of (3.6) is

$$\begin{aligned} -\int''_{[-\pi,\pi]^2} & [I_q \exp(i\lambda_1) - A]^{-1} A^{-1} \exp(i\lambda_1)[A\exp(i\lambda_2) - I_q]^{-1} \\ & \times [A^{n+1}\exp[i\lambda_2(n+1)] - A\exp(i\lambda_2)] \\ & \times |\lambda_1|^{1/2-H} |\lambda_2|^{1/2-H}\, dW(\lambda_1)\, dW(\lambda_2), \end{aligned}$$

which is $O_p(1)$, since the entries of

$$\begin{aligned} & [I_q \exp(i\lambda_1) - A]^{-1}[A\exp(i\lambda_2) - I_q]^{-1} \\ & \quad \times [A^{n+1}\exp[i\lambda_2(n+1)] - A\exp(i\lambda_2)], \end{aligned}$$

are bounded. Thus the random term on the right-hand side of (3.4) is $o_p(n)$. The second term on the right-hand side of (3.4) is

$$\begin{aligned} & \left\{ \sum_{t=1}^{n} \sum_{k=1}^{t-1} A^{t-1-k} \int_{-\pi}^{\pi} \exp(ik\lambda)\exp(-it\lambda)|\lambda|^{1-2H}\, d\lambda \right\} \mathbf{1} \\ & \quad = \left\{ \int_{-\pi}^{\pi} \sum_{t=1}^{n} A^{t-1}\exp(-it\lambda)[A^{-1}\exp(i\lambda) - I_q]^{-1} \right. \\ & \qquad \left. \times [A^{-t}\exp(it\lambda) - A^{-1}\exp(i\lambda)]|\lambda|^{1-2H}\, d\lambda \right\} \mathbf{1} \\ & \quad = \left\{ \sum_{t=1}^{n} \int_{-\pi}^{\pi} [I_q\exp(i\lambda) - A]^{-1} |\lambda|^{1-2H}\, d\lambda \right. \\ & \qquad - \int_{-\pi}^{\pi} [I_q \exp(i\lambda) - A]^{-1} A^{-1}\exp(i\lambda) \\ & \qquad \left. \times \sum_{t=1}^{n} A^t \exp(-it\lambda)|\lambda|^{1-2H}\, d\lambda \right\} \mathbf{1}. \end{aligned} \tag{3.8}$$

The first term on the right-hand side of (3.8), when divided by n, converges (indeed is equal) to (3.5), and the second term of (3.8) is $O(1)$. Hence $n^{-1}\sum_{t=1}^{n} \mathbf{z}_{t-1} X_t$ converges in probability to (3.5). It can be seen by a similar calculation that

$$\sum_{t=1}^{n} \mathbf{z}_{t-1}\mathbf{z}'_{t-1} - E\left(\sum_{t=1}^{n} \mathbf{z}_{t-1}\mathbf{z}'_{t-1} \right) = o_p(n),$$

and

$$\lim_n \frac{1}{n} E\left(\sum_{t=1}^{n} \mathbf{z}_{t-1}\mathbf{z}'_{t-1}\right) = \int_{-\pi}^{\pi} \left[I_q \exp(i\lambda) - A\right]^{-1} \times \left[I_q \exp(-i\lambda) - A'\right]^{-1} |\lambda|^{1-2H}\, d\lambda.$$

This establishes (3.3). □

The following theorem gives the joint limiting distribution for the "building blocks" of those components of $\hat{\alpha}$ which correspond to roots on the unit circle.

THEOREM 3.2. *Let X_n be the sequence defined by* (3.2), *with* $0 < H < 1$. *Let $\theta_i \in (0, \pi)$ such that $\theta_i \neq \theta_j$ if $i \neq j$, for $i, j = 1, 2, \dots, l$, and let*

$$Y'_n(u, v, t_1, \dots, t_{2l}) = \Bigg(a_n \sum_{k=1}^{[nu]} X_k,\ b_n \sum_{k=1}^{[nv]} (-1)^k X_k,\ b_n \sum_{k=1}^{[nt_1]} \sqrt{2} \sin k\theta_1 X_k,\ b_n \sum_{k=1}^{[nt_2]} \sqrt{2} \cos k\theta_1 X_k, \dots,\ b_n \sum_{k=1}^{[nt_{2l-1}]} \sqrt{2} \sin k\theta_l X_k,\ b_n \sum_{k=1}^{[nt_{2l}]} \sqrt{2} \cos k\theta_l X_k\Bigg), \tag{3.9}$$

where $a_n = n^{-H} L^{-1/2}(n)$ and $b_n = [nL(n)]^{-1/2}$. Then there exist constants C_1, C_2, $C_1(\theta_i)$ and $C_2(\theta_i)$, $i = 1, \dots, l$, such that Y'_n converges weakly to

$$\left(C_1 B_H, C_2 B^{(1)}, C_1(\theta_1) B^{(2)}, C_2(\theta_1) B^{(3)}, \dots, C_1(\theta_l) B^{(2l-1)}, C_2(\theta_l) B^{(2l)}\right),$$

where B_H is a fractional Brownian motion with parameter H, the $B^{(i)}$ are standard Brownian motions, $i = 1, \dots, 2l$, and the components of the limiting vector are independent.

REMARK. It follows from Theorem 3.3 that $C_1 = K_H^{-1}$, where $K_H = \pi^{-1} H \Gamma(2H) \sin \pi H$ and Γ is the gamma function.

PROOF OF THEOREM 3.2. Let $r(k)$ be the covariance function of $\{X_k\}$, $r(k) = EX_1 X_{k+1}$. To prove convergence of the components of (3.9), it suffices to show the following, for some constants C_1, C_2, $C_1(\theta)$, and $C_2(\theta)$:

(i) $\sum_{k=1}^{n} \sum_{l=1}^{n} r(k-l) \sim C_1^2 n^{2H} L(n)$;
(ii) $\sum_{k=1}^{n} \sum_{l=1}^{n} (-1)^{k-l} r(k-l) \sim C_2^2 n L(n)$;
(iii) $\sum_{k=1}^{n} \sum_{l=1}^{n} \exp[i(k-l)\theta] r(k-l) \sim (C_1^2(\theta) + C_2^2(\theta)) n L(n)$.

[See Taqqu (1975), Lemma 5.1.] We have

$$a_n^2 \sum_{k=1}^{n} \sum_{l=1}^{n} r(k-l)$$
$$= a_n^2 \sum_{k=1}^{n} \sum_{l=1}^{n} \int_{-\pi}^{\pi} \exp[i(k-l)\lambda] |\lambda|^{1-2H} L(|\lambda|^{-1})\, d\lambda$$
$$= a_n^2 \int_{-\pi}^{\pi} \left| \sum_{k=1}^{n} \exp(ik\lambda) \right|^2 |\lambda|^{1-2H} L(|\lambda|^{-1})\, d\lambda$$
$$= a_n^2 \int_{-\pi}^{\pi} \left| \frac{\exp[i(n+1)\lambda] - \exp(i\lambda)}{\exp(i\lambda) - 1} \right|^2 |\lambda|^{1-2H} L(|\lambda|^{-1})\, d\lambda$$
$$= \frac{1}{L(n)} \int_{-n\pi}^{n\pi} \left| \frac{\exp[i(n+1)\lambda/n] - \exp(i\lambda/n)}{n(\exp(i\lambda/n) - 1)} \right|^2 |\lambda|^{1-2H} L(|\lambda|^{-1})\, d\lambda$$
$$\to \int_{\mathscr{R}} \left| \frac{\exp(i\lambda) - 1}{i\lambda} \right|^2 |\lambda|^{1-2H}\, d\lambda,$$

which establishes (i). To see that (iii) holds, note that

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \exp[i(k-l)\theta] r(k-l)$$
$$= \int_{-\pi}^{\pi} \left| \sum_{k=1}^{n} \exp[ik(\lambda+\theta)] \right|^2 |\lambda|^{1-2H} L(|\lambda|^{-1})\, d\lambda$$
$$= n \int_{n(-\pi+\theta)}^{n(\pi+\theta)} \left| \frac{\exp[i\mu(n+1)/n] - \exp(iu/n)}{n(\exp(i\mu/n) - 1)} \right|^2$$
$$\times |n^{-1}\mu - \theta|^{1-2H} L(|n^{-1}\mu - \theta|^{-1})\, d\mu,$$

which converges to

$$|\theta|^{1-2H} \int_{-\infty}^{\infty} \left| \frac{\exp(i\mu) - 1}{i\mu} \right|^2 d\mu.$$

Hence, (iii) holds. Condition (ii) is established by similar arguments.

It remains only to prove asymptotic independence. We have

$$b_n^2 E \sum_{k=1}^{[nt_1]} \exp(ik\theta_1) X_k \sum_{l=1}^{[nt_2]} \exp(-il\theta_2) X_l$$
$$= b_n^2 \int_{-\pi}^{\pi} \sum_{k=1}^{[nt_1]} \exp[ik(\theta_1+\lambda)]$$
$$\times \sum_{l-1}^{[nt_2]} \exp[-il(\theta_2+\lambda)] |\lambda|^{1-2H} L(|\lambda|^{-1})\, d\lambda,$$

which is bounded in magnitude by a constant multiple of

$$\frac{n^{2\delta}}{nL(n)} t_1^{\delta} t_2^{\delta} \int_{-\pi}^{\pi} |\theta_1 + \lambda|^{\delta-1} |\theta_2 + \lambda|^{\delta-1} |\lambda|^{1-2H} L(|\lambda|^{-1})\, d\lambda \to 0,$$

for any $0 < \delta < \frac{1}{2}$. Likewise, the remaining covariances tend to zero. □

The first component of $\hat{\alpha}$ involves a functional of the first component of the right-hand side of (3.9). If X_t were a sequence of martingale differences, the limit of that functional would be an integral of Brownian motions, as shown by CW. However, in this case X_t has long memory, and hence the limit involves fractional Brownian motions rather than Brownian motions, as is demonstrated in Theorem 3.3. Fractional Brownian motion with parameter H, $0 < H < 1$, has spectral representation

$$B_H(t) = K_H \int_{\mathscr{R}} \frac{\exp(it\lambda) - 1}{i\lambda} |\lambda|^{1/2-H} W(d\lambda), \tag{3.10}$$

where $K_H = \pi^{-1} H \Gamma(2H) \sin \pi H$ and Γ is the gamma function. Although fractional Brownian motion is nowhere differentiable, formally,

$$\frac{dB_H(t)}{dt} = K_H \int_{\mathscr{R}} \exp(it\lambda) |\lambda|^{1/2-H} W(d\lambda).$$

This heuristic serves to justify the definition

$$\begin{aligned} \int_0^s B_{H_1}(t)\, dB_{H_2}(t) = {} & K_{H_1} K_{H_2} \int_{\mathscr{R}_2}'' \left[\int_0^s \exp(it\lambda) \frac{\exp(it\mu) - 1}{i\mu}\, dt \right] \\ & \times |\mu|^{1/2-H_1} |\lambda|^{1/2-H_2} W(d\mu) W(d\lambda) \\ & + K_{H_1} K_{H_2} \int_{\mathscr{R}} \left[\int_0^s \frac{1 - \exp(-it\mu)}{i\mu}\, dt \right] |\mu|^{1-(H_1+H_2)}\, d\mu. \end{aligned} \tag{3.11}$$

The first term on the right-hand side is a double Wiener–Itô integral, which exists in the L_2 sense for $H_1 + H_2 > 1$, $0 < H_i < 1$, $i = 1, 2$. The presence of the second term, a nonzero constant, reflects the fact that the increments of B_{H_1} and B_{H_2} are dependent. Note that $\int_0^t B_H\, dB_H$ cannot be defined in the usual Itô sense, because fractional Brownian motion is not a semimartingale.

Let X_n and Y_n be defined by (3.2) with parameters $H = H_1$ and $L = L_1$ for X_n, and $H = H_2$ and $L = L_2$ for Y_n, and suppose also that the slowly varying functions are of bounded variation on bounded intervals.

THEOREM 3.3. *Let* $U_n(t) = \sum_{k=1}^{[nt]} X_k$, $V_n(t) = \sum_{k=1}^{[nt]} Y_k$ *and* $\frac{1}{2} < H_i < 1$, $i = 1, 2$. *Then*

$$\left(a_n U_n, b_n V_n, a_n b_n \sum_{k=1}^{n-1} U_n\left(\frac{k}{n}\right) Y_{k+1} \right) \to_{\mathscr{L}} \left(K_{H_1}^{-1} B_{H_1}, K_{H_2}^{-1} B_{H_2}, (K_{H_1} K_{H_2})^{-1} \int_0^1 B_{H_1}\, dB_{H_2} \right),$$

where $a_n = n^{-H_1}L_1^{-1/2}(n)$, $b_n = n^{-H_2}L_2^{-1/2}(n)$ *and* K_H *is the constant in the spectral representation of fractional Brownian motion* (3.10).

PROOF. Since a_nU_n and b_nV_n are both tight [Taqqu (1975)], it suffices to prove convergence of finite-dimensional distributions (fdd). The proof for a fixed value of t is given here, since the proof for fdd is similar.

Let

$$Z_n = a_n b_n \sum_{k=1}^{n-1} U_n\left(\frac{k}{n}\right) Y_{k+1} \quad \text{and} \quad Z = \left(K_{H_1}K_{H_2}\right)^{-1} \int_0^1 B_{H_1}\, dB_{H_2}.$$

To obtain convergence for a fixed value of t, it suffices to show that, for any real constants p_1, p_2 and p_3,

$$p_1 a_n U_n(t) + p_2 b_n V_n(t) + p_3 Z_n \to p_1 K_{H_1}^{-1} B_{H_1}(t) + p_2 K_{H_2}^{-1} B_{H_2}^{-1} B_{H_2}(t) + p_3 Z$$

in distribution. Since convergence in distribution follows from convergence in mean square, it suffices to show that

$$E|(p_1 a_n U_n(t) + p_2 b_n V_n(t) + p_3 Z_n) - \left(p_1 K_H^{-1} B_{H_1}(t) + p_2 K_{H_2}^{-1} B_{H_2}(t) + p_3 Z\right)|^2,$$

which is bounded by

$$C\left\{E|a_n U_n(t) - K_{H_1}^{-1} B_{H_1}(t)|^2 + E|b_n V_n(t) - K_{H_2}^{-1} B_{H_2}(t)|^2 + E|Z_n - Z|^2\right\},$$

for some constant C, converges to 0 as $n \to \infty$.

As in the proof of Theorem 3.1, slowly varying functions are assumed to be constant. With a change of variables, $a_nU_n(t)$ can be written

$$\begin{aligned} a_n U_n(t) &= n^{-H_1} \sum_{k=1}^{[nt]} \int_{-\pi}^{\pi} \exp(ik\lambda)|\lambda|^{1/2-H_1} W(d\lambda) \\ &= \int_{-n\pi}^{n\pi} \frac{\exp[i\lambda([nt]+1)/n] - \exp(i\lambda/n)}{n(\exp(i\lambda/n) - 1)} |\lambda|^{1/2-H_1} W(d\lambda). \end{aligned}$$

Furthermore,

$$K_{H_1}^{-1} B_{H_1}(t) = \int_{\mathscr{R}} \frac{\exp(it\lambda) - 1}{i\lambda} |\lambda|^{1/2-H_1} W(d\lambda).$$

Let

$$D_n(\lambda) = 1_{[-n\pi, n\pi]}(\lambda) \frac{\exp[i\lambda([nt]+1)/n] - \exp(i\lambda/n)}{n(\exp(i\lambda/n) - 1)}$$

and

$$D(\lambda) = \frac{\exp(it\lambda) - 1}{i\lambda}.$$

Then

$$(3.12)\quad E|a_n U_n(t) - K_{H_1}^{-1} B_{H_1}(t)|^2 = \int_{\mathscr{R}} |D_n(\lambda) - D(\lambda)|^2 |\lambda|^{1-2H_1}\, d\lambda.$$

Clearly, $D_n(\lambda) \to D(\lambda)$ pointwise. Hence, to show that the right-hand side of (3.12) converges to 0 as $n \to \infty$, it suffices to find an integrable function which dominates $|D_n(\lambda)|^2 |\lambda|^{1-2H_1}$ for every n.

Observe that

$$|D_n(\lambda)| \le 4\min(|\lambda|^{-1}, t)$$

and

$$\int_{|\lambda|>1} |\lambda|^{(1-2H_1)-1}\, d\lambda < \infty,$$

$$\int_{|\lambda|<1} t|\lambda|^{1-2H_1}\, d\lambda < \infty.$$

Hence,

$$E|a_n U_n(t) - K_{H_1}^{-1} B_{H_1}(t)|^2 \to 0.$$

Similarly,

$$E|b_n V_n(t) - K_{H_2}^{-1} B_{H_2}(t)|^2 \to 0.$$

To complete the proof, it suffices to show that $E|Z_n - Z|^2 \to 0$. We have

$$\begin{aligned} Z_n &= n^{-(H_1+H_2)} \sum_{k=1}^{n-1} \left(\sum_{j=1}^{k} X_j \right) Y_{k+1} \\ &= n^{-(H_1+H_2)} \sum_{k=1}^{n-1} \sum_{j=1}^{k} \int_{-\pi}^{\pi} \exp(ij\mu) |\mu|^{1/2-H_1} W(d\mu) \\ &\quad \times \int_{-\pi}^{\pi} \exp[i(k+1)\lambda] |\lambda|^{1/2-H_2} W(d\lambda) \\ (3.13)\quad &= \int''_{[-n\pi, n\pi]^2} \sum_{k=1}^{n-1} \frac{1}{n} \frac{\exp[i(k+1)\mu/n] - \exp(i\mu/n)}{n(\exp(i\mu/n) - 1)} \\ &\quad \times \exp\left(\frac{i(k+1)\lambda}{n} \right) |\mu|^{1/2-H_1} |\lambda|^{1/2-H_2} W(d\mu) W(d\lambda) \\ &\quad + \int_{-n\pi}^{n\pi} \sum_{k=1}^{n-1} \frac{1}{n} \frac{\exp[i(k+1)\lambda/n] - \exp(i\lambda/n)}{n(\exp(i\lambda/n) - 1)} \\ &\quad \times \exp\left(\frac{-i(k+1)\lambda}{n} \right) |\lambda|^{1-(H_1+H_2)}\, d\lambda. \end{aligned}$$

Convergence of Z_n to Z in mean square can be demonstrated by showing that the random term and the nonrandom term on the right-hand side of

(3.13) converge in mean square to the random and nonrandom terms of Z. Equation (3.11) displays the two limiting terms, multiplied by $K_{H_1}K_{H_2}$. Focusing on the random term, let

$$B_n(\mu, \lambda) = 1_{[-n\pi, n\pi]^2}(\mu, \lambda) \times \sum_{k=1}^{n-1} \frac{1}{n} \frac{\exp[i(k+1)\mu/n] - \exp(i\mu/n)}{n(\exp(i\mu/n) - 1)} \exp\left(\frac{i(k+1)\lambda}{n}\right)$$

and

$$B(\mu, \lambda) = \int_0^1 \exp(it\lambda) \frac{\exp(it\mu) - 1}{i\mu}\, dt.$$

To get convergence in mean square of the random terms, it suffices to show that

$$\int_{\mathscr{R}^2} |B_n(\mu, \lambda) - B(\mu, \lambda)|^2 |\mu|^{1-2H_1} |\lambda|^{1-2H_2}\, d\mu\, d\lambda \to 0 \tag{3.14}$$

as $n \to \infty$. Observe that

$$\begin{aligned} B_n(\mu, \lambda) &= 1_{[-n\pi, n\pi]^2}(\mu, \lambda) \frac{\exp[i(\mu+\lambda)/n]}{n(\exp(i\mu/n) - 1)} \\ &\quad \times \sum_{k=0}^{n-1} \frac{1}{n} \left(\exp\left[i\left(\frac{k}{n}\right)(\mu+\lambda)\right] - \exp\left[i\left(\frac{k}{n}\right)\lambda\right]\right) \\ &\to \frac{1}{i\mu} \int_0^1 (\exp[it(\mu+\lambda)] - \exp(it\lambda))\, dt \\ &= B(\mu, \lambda). \end{aligned} \tag{3.15}$$

Let $0 < \epsilon < 1$, and set

$$\begin{aligned} E_{n,1} &= \{(\mu, \lambda) : |\mu| < 1, |\lambda| < 1\}, \\ E_{n,2} &= \left\{(\mu, \lambda) : 1 \le |\mu| \le n\pi, 0 \le |\lambda| \le n\pi, 0 \le |\mu + \lambda| \le \frac{3n\pi}{2}\right\}, \\ E_{n,3} &= \{(\mu, \lambda) : |\mu| < 1, 1 \le |\lambda| \le n\pi, |\mu + \lambda| > \epsilon\}, \\ E_{n,4} &= \{(\mu, \lambda) : |\mu| < 1, 1 \le |\lambda| \le n\pi, |\mu + \lambda| \le \epsilon\}, \\ E_n &= \bigcup_{i=1}^{4} E_{n,i}. \end{aligned}$$

Then

$$E_n^c = [-n\pi, n\pi]^2 \setminus E_n$$
$$= \left\{(\mu, \lambda): 1 \le |\mu| \le n\pi, 0 \le |\lambda| \le n\pi, \frac{3n\pi}{2} < |\mu + \lambda| \le 2n\pi\right\}.$$

To show (3.14), it suffices to find functions g_i, $i = 1,2,3,4$, such that

$$\int_{\mathscr{R}^2} g_i(\mu, \lambda)|\mu|^{1-2H_1}|\lambda|^{1-2H_2}\, d\mu\, d\lambda < \infty$$

and

$$1_{E_{n,i}}(\mu, \lambda)|B_n(\mu, \lambda)|^2 < cg_i(\mu, \lambda),$$

for $i = 1,2,3,4$ and some constant c, and also to show that

$$\int_{E_n^c} |B_n(\mu, \lambda)|^2|\mu|^{1-2H_1}|\lambda|^{1-2H_2}\, d\mu\, d\lambda \to 0$$

as $n \to \infty$.

Let

$$g_1(\mu, \lambda) = 1_{[-1,1]^2}(\mu, \lambda),$$
$$g_2(\mu, \lambda) = 1_{\{|\mu|>1\}}\{|\mu|^{-2}(f(\mu + \lambda) + f(\lambda))\},$$

where $f(x) = \min\{|x|^{-2}, 1\}$; let

$$g_3(\mu, \lambda) = 1_{\{|\mu|<1, |\lambda| \ge 1, |\mu+\lambda|>\epsilon\}}|\mu + \lambda|^{-2},$$
$$g_4(\mu, \lambda) = 1_{\{|\mu|<1, |\lambda| \ge 1, |\mu+\lambda| \le \epsilon\}};$$

and observe that

$$\int_{\mathscr{R}^2} g_i(\mu, \lambda)|\mu|^{1-2H_1}|\lambda|^{1-2H_2}\, d\mu\, d\lambda < \infty, \qquad i = 1,2,3,4.$$

Since

$$|B_n(\mu, \lambda)| = \left|\sum_{k=1}^{n-1} \frac{1}{n} \exp\left(\frac{i(k+1)\lambda}{n}\right) \frac{1}{n} \sum_{j=1}^{k} \exp\left(\frac{ij\mu}{n}\right)\right| < 1,$$

it follows that

$$1_{E_{n,1}}(\mu, \lambda)|B_n(\mu, \lambda)|^2 < g_1(\mu, \lambda).$$

Observe that

$$|\exp(ix) - 1| \le 2^{1-\eta}|x|^{\eta}, \tag{3.16}$$

for $0 \le \eta \le 1$ and $x \in \mathscr{R}$, and

$$|\exp(ix) - 1| = 2\left|\sin\left(\frac{x}{2}\right)\right| > 2c_\tau\left|\frac{x}{2}\right| = c_\tau|x|, \tag{3.17}$$

for some $c_\tau > 0$ when $|x/2| < \tau < \pi$. If $\tau = \pi/2$, c_τ can be chosen equal to $\frac{1}{2}$.

By the first equality in equation (3.15) and inequalities (3.16) and (3.17),

$$
\begin{aligned}
1_{E_{n,2}}|B_n|^2 &= \left| n\left(\exp\left(\frac{i\mu}{n}\right) - 1\right)\right|^{-2} \\
&\quad \times \left| \frac{\exp[i(\mu+\lambda)] - 1}{n(\exp[i(\mu+\lambda)/n] - 1)} \right. \\
&\qquad \left. - \frac{\exp(i\lambda) - 1}{n(\exp(i\lambda/n) - 1)} \right|^2 \\
&\le c g_2(\mu, \lambda),
\end{aligned}
$$

where c is a constant.

Note that

$$
\begin{aligned}
|B_n(\mu,\lambda)| &= \left| \frac{1}{n(e^{i\mu/n} - 1)} \right. \\
&\quad \left. \times \left[\frac{e^{i(\mu+\lambda)} - 1}{n(e^{i(\mu+\lambda)/n} - 1)} - \frac{e^{i\lambda} - 1}{n(e^{i\lambda/n} - 1)} \right] \right| \\
&= \left| \frac{1}{n(e^{i\mu/n} - 1)} \right. \\
&\quad \left. \times \left[\frac{(e^{i(\mu+\lambda)} - 1)(n(e^{i\lambda/n} - 1)) - (e^{i\lambda} - 1)(n(e^{i(\mu+\lambda)/n} - 1))}{n(e^{i(\mu+\lambda)/n} - 1)n(e^{i\lambda/n} - 1)} \right] \right|.
\end{aligned}
$$

The numerator is

$$
\begin{aligned}
&\left| \exp(i\lambda)\left(n\left(\exp\left(\frac{i\lambda}{n}\right) - 1\right)\right)\exp(i\mu) - n\left(\exp\left(\frac{i\lambda}{n}\right) - 1\right) \right. \\
&\quad - (\exp(i\lambda) - 1)\left(n\left(\exp\left(\frac{i\lambda}{n}\right) - 1\right) \right. \\
&\qquad \left. \left. + n\left(\exp\left(\frac{i(\mu+\lambda)}{n}\right) - 1\right) - n\left(\exp\left(\frac{i\lambda}{n}\right) - 1\right)\right)\right| \\
&= \left| \exp(i\lambda)\left(n\left(\exp\left(\frac{i\lambda}{n}\right) - 1\right)\right)(\exp(i\mu) - 1) \right. \\
&\quad \left. - (\exp(i\lambda) - 1)\left(\exp\left(\frac{i\lambda}{n}\right)n\left(\exp\left(\frac{i\mu}{n}\right) - 1\right)\right)\right|.
\end{aligned}
$$

Hence

$$|B_n(\mu,\lambda)|^2 \le 2\left|\frac{\exp(i\mu)-1}{n(\exp(i\mu/n)-1)n(\exp[i(\mu+\lambda)/n]-1)}\right|^2 + 2\left|\frac{\exp(i\lambda)-1}{n(\exp(i\lambda/n)-1)n(\exp[i(\mu+\lambda)/n]-1)}\right|^2.$$

Therefore, by the inequality (3.16),

$$\begin{aligned} 1_{E_{n,3}}|B_n|^2 &\le c 1_{\{|\mu|<1,|\lambda|\ge 1,|\mu+\lambda|>\epsilon\}}\left\{\left|\frac{\exp(i\mu)-1}{\mu(\mu+\lambda)}\right|^2 + \left|\frac{\exp(i\lambda)-1}{\lambda(\mu+\lambda)}\right|^2\right\} \\ &\le 2cg_3(\mu,\lambda), \end{aligned}$$

for some constant c.

Finally,

$$1_{E_{n,4}}|B_n|^2 \le g_4(\mu,\lambda).$$

To show that the integral on E_n^c goes to zero, observe that, by (3.16) and (3.17),

$$\begin{aligned} &\left|\sum_{k=1}^{n-1}\frac{\exp[i(k+1)\mu]-\exp(i\mu)}{\exp(i\mu)-1}\exp[i(k+1)\lambda]\right| \\ &\qquad \le \sum_{k=1}^{n-1}\left|\frac{\exp(ik\mu)-1}{\exp(i\mu)-1}\right| \le 2^{2-\eta}|\mu|^{\eta-1}\sum_{k=1}^{n-1}|k|^{\eta}, \end{aligned}$$

where η is a number between 0 and 1, to be specified later. Hence

$$\begin{aligned} &\int_{E_n^c}|B_n(\mu,\lambda)|^2|\mu|^{1-2H_1}|\lambda|^{1-2H_2}\,d\mu\,d\lambda \\ &\quad = n^{-2(H_1+H_2)}\int_{E_n^c/n}\left|\sum_{k=1}^{n-1}\frac{\exp[i(k+1)\mu]-\exp(i\mu)}{\exp(i\mu)-1}\exp[i(k+1)\lambda]\right|^2 \\ &\qquad\qquad \times|\mu|^{1-2H_1}|\lambda|^{1-2H_2}\,d\mu\,d\lambda \\ &\quad \le n^{-2(H_1+H_2)+2(\eta+1)}c\int_{\pi/2<|\mu|,\,|\lambda|<\pi}|\mu|^{2(\eta-1)}|\mu|^{1-2H_1}|\lambda|^{1-2H_2}\,d\mu\,d\lambda, \end{aligned}$$

for some constant c. The region of integration is a result of the inequality $3\pi/2<|\mu+\lambda|$. Since μ and λ are bounded away from 0, the integral is finite, and $n^{-2(H_1+H_2)+2(\eta+1)}\to 0$ when $0<\eta<H_1+H_2-1$. This completes the proof of the convergence of the random term on the right-hand side of (3.13). Convergence of the nonrandom term can be shown using similar arguments. □

Using similar techniques, we get the following corollary.

COROLLARY 3.1. *We have*

$$\left(a_nU_n, b_nV_n, a_nb_nn^{-j}\sum_{k=1}^{n-1}\left\{\sum_{l_j=1}^{k}\sum_{l_{j-1}=1}^{l_j}\cdots\sum_{l_1=1}^{l_2}U_{l_2}\left(\frac{l_1}{l_2}\right)\right\}Y_{k+1}\right)$$

$$\to_{\mathscr{L}}\left(K_{H_1}^{-1}B_{H_1}, K_{H_2}^{-1}B_{H_2}, \left(K_{H_1}K_{H_2}\right)^{-1}\right.$$

$$\left.\times\int_0^1\left\{\int_0^t\int_0^{t_j}\cdots\int_0^{t_2}B_{H_1}(t_1)\,dt_1\cdots dt_j\right\}dB_{H_2}(t)\right),$$

where

$$\int_0^1\left\{\int_0^t\int_0^{t_j}\cdots\int_0^{t_2}B_{H_1}(t_1)\,dt_1\cdots dt_j\right\}dB_{H_2}(t)$$

is defined as

$$K_{H_1}K_{H_2}\int_{\mathscr{R}^2}''\int_0^1\exp(it\lambda)\left\{\int_0^t\int_0^{t_j}\cdots\int_0^{t_2}\frac{\exp(it_1\mu)-1}{i\mu}\,dt_1\cdots dt_j\right\}dt$$

$$\times|\mu|^{1/2-H_1}|\lambda|^{1/2-H_2}W(d\mu)W(d\lambda)$$

$$+K_{H_1}K_{H_2}\int_{\mathscr{R}}\int_0^1\exp(-it\mu)\left\{\int_0^t\int_0^{t_j}\cdots\int_0^{t_2}\frac{\exp(it_1\mu)-1}{i\mu}\,dt_1\cdots dt_j\right\}dt$$

$$\times|\mu|^{1-(H_1+H_2)}\,d\mu.$$

4. Roots equal to 1 and −1. Let $\{e_t\}$ be a long-memory process, as defined by (3.2), with $\frac{1}{2}<H<1$, and consider the component u_t such that

$$(1-B)^a u_t = e_t. \tag{4.1}$$

It follows from (3.1.3) of CW that there exists an $a\times a$ matrix M such that $M\mathbf{u}_t = U_t$, where $U_t=(u_t(1),\dots,u_t(a))'$ with $u_t(j)=(1-B)^{a-j}u_t$, $j=1,\dots,a$. Clearly, $u_t(1)=\sum_{k=1}^t e_k$, $(1-B)u_t(j+1)=u_t(j)$ and $u_t(j+1)=\sum_{k=1}^t u_k(j)$, $j=0,\dots,a-1$. Let $B_H(t)$ denote a fractional Brownian motion with parameter H, $\frac{1}{2}<H<1$. Define the following recursions:

$$F_0(t)=C_1B_H(t),\qquad F_j(t)=\int_0^t F_{j-1}(s)\,ds; \tag{4.2}$$

$$F=(\sigma_{jl}),\qquad \sigma_{jl}=\int_0^1 F_{j-1}(t)F_{l-1}(t)\,dt,\qquad j,l=1,\dots,a; \tag{4.3}$$

$$\xi=\left(C_1\int_0^1F_0(t)\,dB_H(t),\dots,C_1\int_0^1F_{a-1}(t)\,dB_H(t)\right)'; \tag{4.4}$$

and

$$J_n=N_n^{-1}M,\qquad N_n=\operatorname{diag}(n,\dots,n^a), \tag{4.5}$$

where the integrals in (4.4) are defined in Corollary 3.1 and $C_1 = K_H^{-1}$. With this notation, we are now ready to prove the following result.

THEOREM 4.1. *We have*

$$n^{-2H+1}L^{-1}(n) J_n \sum \mathbf{u}_{t-1}\mathbf{u}'_{t-1} J'_n \to_{\mathscr{L}} F, \tag{4.6}$$

$$(J'_n)^{-1}\left(\sum \mathbf{u}_{t-1}\mathbf{u}'_{t-1}\right)^{-1} \sum \mathbf{u}_{t-1} e_t \to_{\mathscr{L}} F^{-1}\boldsymbol{\xi}. \tag{4.7}$$

PROOF. Since $u_t(1) = \sum_{k=1}^t e_k$, it follows from Theorem 3.3 that

$$n^{-H}L^{-1/2}(n) u_{[nt]}(1) \to_{\mathscr{L}} C_1 B_H(t),$$

in $D[0, 1]$. Accordingly, we have, for $j = 1, \dots, a$,

$$\left(n^{j-1+H}L^{1/2}(n)\right)^{-1} u_{[nt]}(j) \to_{\mathscr{L}} F_{j-1}(t),$$

and

$$\left(n^{j+l+2H-1}L(n)\right)^{-1} \sum u_t(j) u_t(l) \to_{\mathscr{L}} \sigma_{jl}.$$

Since all quantities in the left-hand side of (4.6) and (4.7) are functionals of $n^{-H}L^{-1/2}(n)u_{[nt]}(1)$, the continuous mapping theorem together with Theorem 3.3 and Corollary 3.1 imply (4.6) and (4.7). □

Note that the proper normalization for (4.6) involves H. This is due to the presence of the fractional Brownian motion $B_H(t)$. When $H = \frac{1}{2}$, $n^{1-2H} = 1$ and Theorem 4.1 reduces to Theorem 3.1.2 of CW.

The case for roots equal to -1 can be dealt with in a similar manner. Specifically, following (3.2.2) of CW, if

$$(1 + B)^b y_t = e_t, \tag{4.8}$$

then there exists a $b \times b$ matrix $\tilde{M}$ such that $\tilde{M}\mathbf{v}_t = V_t$, where $V_t = (v_t(1), \dots, v_t(b))'$, $V_t(j) = (1 + B)^{b-j} v_t$, $j = 1, \dots, b$. Again $v_t(1) = \sum_{k=1}^t (-1)^{t-k} e_k$ and $(-1)^t v_t(j+1) = \sum_{k=1}^t (-1)^k v_k(j)$, $j = 0, \dots, b-1$. Let $\tilde{F} = (\tilde{\sigma}_{jl})$,

$$K_n = \tilde{N}_n^{-1}\tilde{M}, \qquad \tilde{N}_n = \operatorname{diag}(n, \dots, n^b) \tag{4.9}$$

and

$$\boldsymbol{\eta} = -\left(C_2 \int_0^1 \tilde{F}_0(t)\, dB(t), \dots, C_2 \int_0^1 \tilde{F}_{b-1}(t)\, dB(t)\right)', \tag{4.10}$$

where B is a Brownian motion and $\tilde{F}_j$ and $\tilde{\sigma}_{jl}$ are the same as F_j and σ_{jl}, with $B_H(t)$ replaced by B and C_1 replaced by C_2, for $j, l = 1, \dots, b$. We have the following result.

THEOREM 4.2. *We have*

$$L^{-1}(n) \tilde{K}_n \sum \mathbf{v}_{t-1}\mathbf{v}'_{t-1} K'_n \to_{\mathscr{L}} \tilde{F}, \tag{4.11}$$

$$(K'_n)^{-1}\left(\sum \mathbf{v}_{t-1}\mathbf{v}'_{t-1}\right)^{-1}\left(\sum \mathbf{v}_{t-1} e_t\right) \to_{\mathscr{L}} \tilde{F}^{-1}\boldsymbol{\eta}. \tag{4.12}$$

PROOF. The proof follows an argument similar to the proof of Theorem 4.1. The difference is that now the building block, $\Sigma(-1)^k e_k$, when properly normalized, converges to a Brownian motion, by Theorem 3.2. Details can be found in Theorem 3.2.1 of CW. □

5. Complex roots. In this section, we consider the case of a conjugate pair of complex roots in (2.1), that is,

$$(1 - 2\cos\theta B + B^2)^d x_t = e_t. \tag{5.1}$$

Define the component vector $\mathbf{x}_t = (x_t, \dots, x_{t-2d+1})'$, $y_t(j) = (\gamma(B))^{d-j} x_t$, $j = 0, \dots, d$, $\gamma(B) = (1 - 2\cos\theta B + B^2)$ and $Y_t = (y_t(1), y_{t-1}(1), \dots, y_t(d), y_{t-1}(d))'$. As in CW, there exists a $2d \times 2d$ matrix C such that $C\mathbf{x}_t = Y_t$. Clearly, for $j = 0, \dots, d-1$,

$$\gamma(B) y_t(j+1) = y_t(j), \tag{5.2}$$

and

$$y_t(j+1) = \csc\theta \sum_{k=1}^{t} \sin(t-k+1)\theta\, y_k(j). \tag{5.3}$$

For $j = 0, \dots, d$, define the basic building blocks

$$S_t(j) = \sum_{k=1}^{t} \cos k\theta\, y_k(j), \qquad T_t(j) = \sum_{k=1}^{t} \sin k\theta\, y_k(j),$$

$$S_t = S_t(0), \qquad T_t = T_t(0).$$

Observe that quantities such as $\Sigma_{t=1}^n y_t(k) y_t(j)$, $\Sigma_{t=1}^n y_{t-1}(k) y_t(j)$, $\Sigma_{t=1}^n y_t(j) e_{t+1}$ and $\Sigma_{t=1}^n y_{t-1}(j) e_{t+1}$ can be expressed as recursive sums of $S_t(j-1)$ and $T_t(k-1)$ as given in Lemmas 3.3.1–3.3.6 of CW. All the arguments given in Section 3.3 of CW can be carried over to this analysis. Specifically, let

$$\begin{aligned}
f_0(t) &= B_1(t), \qquad g_0 = B_2(t),\\
f_j(t) &= \tfrac{1}{2}\csc\theta\left(\sin\theta\int_0^t f_{j-1}(s)\,ds - \cos\theta\int_0^t g_{j-1}(s)\,ds\right),\\
g_j(t) &= \tfrac{1}{2}\csc\theta\left(\cos\theta\int_0^t f_{j-1}(s)\,ds + \sin\theta\int_0^t g_{j-1}(s)\,ds\right),\\
\zeta_{2j-1} &= \tfrac{1}{2}\csc\theta\left(\int_0^1 f_{j-1}(s)\,dB_2(s) - \int_0^1 g_{j-1}(s)\,dB_1(s)\right),\\
\zeta_{2j} &= \tfrac{1}{2}\csc\theta\left\{\cos\theta\left(\int_0^1 f_{j-1}(s)\,dB_2(s) - \int_0^1 g_{j-1}(s)\,dB_1(s)\right),\right.\\
&\qquad \left. -\sin\theta\left(\int_0^1 f_{j-1}(s)\,dB_1(s) + \int_0^1 g_{j-1}(s)\,dB_2(s)\right)\right\},\\
\sigma_{2k-1,2j-1} &= \sigma_{2k,2j}\\
&= \tfrac{1}{4}\csc^2\theta\left(\int_0^1 f_{k-1}(s) f_{j-1}(s)\,ds + \int_0^1 g_{k-1}(s) g_{j-1}(s)\,ds\right),
\end{aligned} \tag{5.4}$$

$$\sigma_{2k-1,2j} = \sigma_{2j,2k-1}$$
$$= \tfrac{1}{4}\csc^2\theta\left\{\cos\theta\left(\int_0^1 f_{k-1}(s)f_{j-1}(s)\,ds + \int_0^1 g_{k-1}(s)g_{j-1}(s)\,ds\right)\right.$$
$$\left. -\sin\theta\left(\int_0^1 f_{j-1}(s)g_{k-1}(s)\,ds - \int_0^1 g_{j-1}(s)f_{k-1}(s)\,ds\right)\right\},$$
$$\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_{2d})' \quad \text{and} \quad D = (\sigma_{ij}), \quad \text{a } 2d \times 2d \text{ matrix},$$

where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions with variance $C_1^2(\theta)t$ and $C_2^2(\theta)t$, respectively. By virtue of the arguments in Lemmas 3.3.1–3.3.6 in CW, we have the following joint convergence results.

THEOREM 5.1. *The following hold*:

(i) $\sqrt{2}\,n^{-j-1/2}L^{-1/2}(n)\big(S_{[nt]}(j), T_{[nt]}(j)\big) \to_{\mathscr{L}} \big(f_j(t), g_j(t)\big)$ *in* $D[0,1] \times D[0,1]$;

(ii) $n^{-(k+j)}L^{-1}(n)\sum_{t=1}^{n} y_t(k)y_t(j) \to_{\mathscr{L}} \sigma_{2k,2j}$;

(iii) $n^{-(k+j)}L^{-1}(n)\sum_{t=1}^{n} (k)y_t(j) \to_{\mathscr{L}} \sigma_{2k-1,2j}$;

(iv) $n^{-j}L^{-1}(n)\sum_{t=1}^{n} y_t(j)e_{t+1} \to_{\mathscr{L}} \zeta_{2j}$;

(v) $n^{-j}L^{-1}(n)\sum_{t=1}^{n} y_{t-1}(j)e_{t+1} \to_{\mathscr{L}} \zeta_{2j-1}$.

PROOF. In view of Theorem 3.2 and arguments similar to Lemmas 3.3.1–3.3.6 in CW, (i)–(v) are readily established. □

THEOREM 5.2. *Let* $N_n = \operatorname{diag}(nI_2, \dots, n^dI_2)$, *where* I_2 *is the* 2×2 *identity matrix. Let* $L_n = N_n^{-1}C$, *where* C *is such that* $C\mathbf{x}_t = Y_t$. *Then*

$$L^{-1}(n)L_n\sum \mathbf{x}_{t-1}\mathbf{x}_{t-1}'L_n' \to_{\mathscr{L}} D, \tag{5.5}$$

$$(L_n')^{-1}\left(\sum \mathbf{x}_{t-1}\mathbf{x}_{t-1}'\right)^{-1}\left(\sum \mathbf{x}_{t-1}e_t\right) \to_{\mathscr{L}} D^{-1}\boldsymbol{\zeta}. \tag{5.6}$$

PROOF. This follows directly from Theorem 5.1 and the fact that $C\mathbf{x}_t = Y_t$. The rest of the proof is exactly the same as Theorem 3.3.4 of CW. □

6. Main results. We are now ready to prove our main result. Suppose $\{y_t\}$ follows

$$\alpha(B)y_t = e_t, \tag{6.1}$$

with $\alpha(B)$ satisfying (2.2) and $\{e_t\}$ satisfying (3.2.), with $\frac{1}{2} < H < 1$. For $k = 1, \dots, l$, define the corresponding $L_n(k)$, $\boldsymbol{\zeta}_k$ and D_k as in Section 5 by replacing d by d_k. Also, define F, $\boldsymbol{\xi}$, $\tilde{F}$ and $\boldsymbol{\eta}$ as in Theorems 4.1 and 4.2, respectively. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)'$ and let $\hat{\boldsymbol{\alpha}}_n = (\hat{\alpha}_1, \dots, \hat{\alpha}_p)'$ be its least squares estimate. Let

$$A_n = \operatorname{diag}\big(n^{-2H+1}L^{-1}(n)I_a, L^{-1}(n)I_{b+\Sigma d_k}, n^{-1}L^{-1}(n)I_q\big),$$

where I_m is the $m \times m$ identity matrix, $\Delta = \operatorname{diag}(F, \tilde{F}, D_1, \dots, D_l, \Sigma)$, Q is such that $Q\mathbf{y}_t = (\mathbf{u}'_t, \mathbf{v}'_t, \mathbf{x}'_t(1), \dots, \mathbf{x}'_t(l), \mathbf{z}'_t)'$ and $G_n = \operatorname{diag}(J_n, K_n, L_n(1), \dots, L_n(l), M_n)$, with J_n, K_n and $L_n(k)$ defined in Sections 4 and 5, Σ defined in Section 3, N' the limit in (3.3) and $M_n = I_q$.

THEOREM 6.1. *We have*

$$A_n G_n Q \sum \mathbf{y}_{t-1}\mathbf{y}'_{t-1} Q' G'_n \to_{\mathscr{L}} \Delta \tag{6.2}$$

and

$$\begin{aligned}&(Q'G'_n)^{-1}(\hat{\boldsymbol{\alpha}}_n - \boldsymbol{\alpha})\\ &\quad \to_{\mathscr{L}} \big((F^{-1}\boldsymbol{\xi})', (\tilde{F}^{-1}\boldsymbol{\eta})', (D_1^{-1}\boldsymbol{\zeta}_1)', \dots, (D_k^{-1}\boldsymbol{\zeta}_k)', N'\big)'.\end{aligned} \tag{6.3}$$

PROOF. By virtue of Theorems 3.1, 4.1, 4.2 and 5.1, we only need to establish (2.6); the off-diagonal submatrices of $A_n G_n Q \Sigma \mathbf{y}_{t-1}\mathbf{y}'_{t-1} Q' G'_n$ converge to zero in probability. This follows directly from Theorems 2.1, 3.4.1 and 3.4.2 of CW. For example, the (1, 1) entry of $n^{-H+1/2}L^{-1}(n)J_n \Sigma \mathbf{u}_{t-1}\mathbf{v}'_{t-1}K'_n$ is

$$\begin{aligned}&n^{-H-3/2}L^{-1}(n)\sum u_{t-1}(1)v_{t-1}(1)\\ &\quad = n^{-H-3/2}L^{-1}(n)\sum \cos((t-1)\pi)u_{t-1}(1)\big((-1)^{t-1}v_{t-1}(1)\big).\end{aligned}$$

Direct calculation shows $E(u_t(1)(-1)^t v_t(1)) = o(n^{H+1/2}L(n))$. By virtue of Theorem 2.1 of CW, we conclude that $\Sigma \cos((t-1)\pi)u_{t-1}(1)((-1)^{t-1}v_{t-1}(1)) = o_p(n^{H+3/2}L(n))$. Similar arguments can be used to deal with other components in exactly the same manner. Details can be found in Theorem 3.4.1 of CW and are omitted here. □

Note that when $H = \frac{1}{2}$, Theorem 6.1 reduces to Theorem 3.5.1 of CW.

7. Concluding remarks.

1. In this paper, a unified asymptotic theory for the study of both long-memory and unit root phenomena is formulated, providing a comprehensive link between these two methods for understanding low-frequency behavior. In particular, a new weak convergence result involving fractional Brownian motion is derived.
2. The results hold not only for specific parametric models such as ARIMA(0, d, 0), but for any long-memory process $\{e_t\}$ satisfying (3.1) with $\frac{1}{2} < H < 1$.

3. An important application of our results will be to use the limiting distributions to test for the presence of unit roots in fractional ARMA models. Since this will involve tabulations of the percentiles of fractional Brownian motions, it will be interesting to explore the computational aspects of tabulating the limiting distributions through the stochastic integral formulae given in Section 3. Testing for a unit root in a fractional integrated model may, however, suffer from the misspecification of the fractional parameter d [Sowell (1990)]. It will thus be interesting to find out to what extent such a phenomenon will affect the computational results.
4. It will be interesting to extend our results to the case of near unit root as given in Chan and Wei (1987). In view of the recent paper of Jeganathan (1991), such an extension is plausible and is currently being undertaken.

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REFERENCES

BANERJEE, A., DOLADO, J. J., GALBRAITH, J. W. and HENDRY, D. F. (1993). *Co-Integration, Error Correction, and the Economic Analysis of Non-Stationary Data*. Oxford Univ. Press.

BERAN, J. (1992). Statistical methods for data with long-range dependence. *Statist. Sci.* **7** 404–427.

CHAN, N. H. and WEI, C. Z. (1987). Asymptotic inference for nearly nonstationary AR(1) processes. *Ann. Statist.* **15** 1050–1063.

CHAN, N. H. and WEI, C. Z. (1988). Limiting distributions of least squares estimates of unstable autoregressive processes. *Ann. Statist.* **16** 367–401.

DIEBOLD, F. X. (1988). Random walk versus fractional integration: power comparisons of scalar and joint tests of the variance–time function. Technical Report 41, Federal Reserve Board, Washington, DC.

DOBRUSHIN, R. L. and MAJOR, P. (1979). Non-central limit theorems for non-linear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete* **50** 27–52.

JEGANATHAN, P. (1991). On the asymptotic behavior of least squares estimators in AR time series with roots near the unit circle. *Econometric Theory* **7** 269–306.

MAJOR, P. (1981). *Multiple Wiener–Itô Integrals. Lecture Notes in Math.* **849**. Springer, New York.

PROTTER, P. (1990). *Stochastic Integration and Differential Equations*. Springer, New York.

ROBINSON, P. M. (1994). Time series with strong dependence. In *Advances in Econometrics. Sixth World Congress* (C. A. Sims, ed.) **1** 47–95. Cambridge Univ. Press.

SOWELL, F. (1990). Fractional unit root distribution. *Econometrica* **50** 495–505.

STOCK, J. H. and WATSON, M. W. (1988). Variable trends in economic time series. *Economic Perspectives* **2** 147–174.

TAQQU, M. S. (1975). Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrsch. Verw. Gebiete* **31** 287–302.

ZYGMUND, A. (1988). *Trigonometric Series*, 2nd ed. Cambridge Univ. Press.

DEPARTMENT OF MATHEMATICS
HONG KONG UNIVERSITY
OF SCIENCE AND TECHNOLOGY
CLEAR WATER BAY, KOWLOON
HONG KONG

DEPARTMENT OF STATISTICS
CARNEGIE MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213-3890