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A NOTE ON THE RUN LENGTH TO FALSE ALARM OF A CHANGE-POINT DETECTION POLICY[1]

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A new proof is given to a known result on the average run length to false alarm of the Shiryayev–Roberts change-point detection policy when the observations are nonlattice. Via the approach of this new proof, the average run length to false alarm can be calculated in the lattice case and for the mixed-type Shiryayev–Roberts scheme.

1. Introduction and notation. The Shiryayev–Roberts change-point detection policy is an efficient process monitoring scheme. Many works have been written during the last years about the optimality properties of this policy and on how the scheme can be implemented in different settings. In this work we consider the model of independent observations from a one-parameter exponential family.

Specifically, let $f_0(x)$ be a density of some probability measure on a sample space, and consider the family of probability measures with densities

$$f_y(x) = \exp[yx - \psi(y)] f_0(x), \qquad y \in \Omega,$$

where Ω is an interval on which $\psi(\cdot)$ is finite. Without loss of generality it can be assumed that $\psi'(0) = 0$.

Let $X_1, X_2, \ldots$ be a sequence of random variables. The log-likelihood ratio of an observation X_i is

$$Z_i^y = yX_i - \psi(y).$$

Its P_1^y-mean is

$$I(y) = y\psi'(y) - \psi(y).$$

The Shiryayev–Roberts statistics are defined to be

$$R(n, y) = \sum_{k=1}^{n} \exp\left(\sum_{i=k}^{n} Z_i^y\right), \qquad n = 1, 2, \ldots.$$

Notice that the sequence of statistics satisfies the recursive relation

$$R(n, y) = [R(n-1, y) + 1] \exp(Z_n^y), \qquad R(0, y) = 1.$$

Given a probability measure F on Ω, the mixed-type statistics are

$$R(n, F) = \int \sum_{k=1}^{n} \exp\left(\sum_{i=k}^{n} Z_i^y\right) dF(y), \qquad n = 1, 2, \ldots.$$

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The appropriate stopping rules are

$$N(A, y) = \inf\{n\colon R(n, y) \geq A\},$$
$$N(A, F) = \inf\{n\colon R(n, F) \geq A\},$$

respectively.

These statistics should represent, in a sense, the likelihood of a change occurring in the past, given the information available at present. The Shiryayev–Roberts monitoring policy sets an alarm the first time that these statistics are larger than some critical value A. The model used to design the monitoring scheme assumes that the X_i's are independent. The marginal density of the random variables before the change is $f_0(x)$. After the change the marginal density is $f_y(\cdot)$. The simple Shiryayev–Roberts scheme fixes some reference value y, while the mixed-type sets a prior distribution F on the set of possible parameters after the change. The symbol P_k^y will be used to represent the first distribution model with X_k being the first observation with marginal distribution $f_y(\cdot)$, and P_∞ will represent the model of no change, that is, the observations are i.i.d. with density $f_0(\cdot)$.

The objective is to set an alarm as soon as possible after the disruption in the process occurred, subject to a constraint on the rate of false alarm—the average run length (ARL) of the scheme under the regime P_∞. Therefore, one must know the relation between the critical value A and the ARL to false alarm. Asymptotic approximations to the average run length were developed in Pollak (1987). He used the fact that $R(n, y) - n$ is a P_∞-martingale for all $y \in \Omega$ to derive the relations

$$E_\infty N(A, y) = E_\infty R(N(A, y), y) \quad \text{and} \quad E_\infty N(A, F) = E_\infty R(N(A, F), F).$$

From these relations it follows that the ARL to false alarm is at least A. Pollak was able to get first-order approximations by investigating the asymptotic properties of the overshoot. This he did for the simple scheme, assuming that the log-likelihood ratio is nonlattice, and for the mixed-type scheme, assuming that the log-likelihood ratio is strongly nonlattice.

In this paper we propose a different approach to getting the first-order approximations. Instead of investigating the distribution of the stopped process, we approximate directly the distribution of the stopping time itself. Via this approach we are able to give not only what we believe to be a simpler proof of Pollak's result, but also an approximation of the run length in the simple lattice case and we extend his result in the mixed-type case by dropping the assumption of the log-likelihood ratios being (strongly) nonlattice. Moreover, it can be shown that the asymptotic distribution of the run length is exponential in all the cases we discuss.

The backbone of our approached is summarized in Lemma 1 in the next section. In that lemma the distribution of the Shiryayev–Roberts stopping rule is compared to the distribution of the power-1 SPRT stopping time. Recall that the simple power-1 SPRT stopping time is

$$M(A, y) = \inf\left\{n\colon \exp\left(\sum_{i=1}^{n} Z_i^y\right) \geq A\right\}.$$

Denote by H^y the asymptotic distribution, under the regime P_1^y, of the overshoot $\sum_{i=1}^{M(A,y)} Z_i^y - \log A$, and let

$$\gamma(y) = \int_0^\infty e^{-x}\, dH^y(x)$$

be the Laplace transform of H^y at 1. The (approximate) significance level of the test is $\gamma(y)/A$. In the mixed-type case the stopping rule is

$$M(A,F) = \inf\left\{n: \int \exp\left(\sum_{i=1}^{n} Z_i^y\right) dF(y) \geq A\right\},$$

and the approximate significance level is $\gamma(F)/A$, where

$$\gamma(F) = \int \gamma(y)\, dF(y).$$

[See Woodroofe (1982) and Pollak (1986) for details.]

In the next section we demonstrate the implementation of the basic approach in the simple nonlattice scheme. In Section 3 we discuss the adjustments needed to handle the simple lattice case and the mixed-type scheme.

2. The nonlattice case. In this section we consider the simplest setting. Calculating the average run length in the more complex situations requires only minor modifications of the arguments used in the proof of the result for this case.

THEOREM 1 [Pollak (1987)]. *If $y \in \Omega$, $I(y) < \infty$ and the P_1^y-distribution of Z_i^y is nonlattice, then*

$$E_\infty N(A,y) = \frac{A}{\gamma(y)}(1 + o(1)),$$

where $o(1) \to 0$ as $A \to \infty$.

Before proving the theorem, let us prove a lemma that relates the distribution of the Shiryayev–Roberts stopping time to the distribution of the power-1 SPRT.

LEMMA 1. *Let $m = m(A)$ be a sequence that satisfies*

$$\frac{A}{m} \to \infty \quad \textit{and} \quad \frac{\log A}{m} \to 0,$$

as $A \to \infty$. Under the assumptions of Theorem 1, it is true that

$$\frac{P_\infty(N(A,y) \leq m)}{P_\infty(M(A/m,y) \leq m)} \to_{A\to\infty} 1. \tag{1}$$

PROOF. From the basic definitions it follows that for any stopping time

$$P_\infty(N \le m) = \sum_{j=1}^{m} \int_{\{N=j\}} dP_\infty$$

$$(2) \qquad = \sum_{j=1}^{m} \sum_{k=1}^{j} \int_{\{N=j\}} \frac{dP_k^y}{R(j,y)}$$

$$= \sum_{k=1}^{m} \int_{\{k \le N \le m\}} \frac{dP_k^y}{R(N,y)}.$$

Let $a = \log A$ and $r(n,y) = \log R(n,y)$. Now

$$(3) \qquad \int_{\{k \le N(A,y) \le m\}} \frac{dP_k^y}{R(N(A,y),y)} = \frac{\int_{\{k \le N(A,y) \le m\}} \exp[-(r(N(A,y),y) - a)]\, dP_k^y}{A}.$$

The denominator in (1) is $[\gamma(y)m/A](1 + o(1))$, since m grows faster than $\log A$. It will be enough, therefore, to show that, for most of the k's, the value of the integral in the right-hand side of (3) is approximately $\gamma(y)$.

In order to evaluate this integral, let us investigate the properties of the process $r(k-1,y), r(k,y), \dots, r(k-1+n,y), \dots,$ under the distribution P_k^y, given the value of $r(k-1,y)$. Easy calculations show that

$$(4) \qquad r(k-1+n,y) - \sum_{i=k}^{k-1+n} Z_i^y = \log\left(R(k-1,y) + 1 + \sum_{j=k+1}^{k-1+n} \exp\left\{ -\sum_{i=k}^{j-1} Z_i^y \right\} \right).$$

Under the regime P_k^y, the right-hand side of equation (4) is bounded by the random variable $\log(R(k-1,y) + 1 + W_1(k,y))$, where

$$W_1(k,y) = \sum_{j=k+1}^{\infty} \exp\left\{ -\sum_{i=k}^{j-1} Z_i^y \right\}.$$

This random variable is finite with probability 1 [Pollak (1985), Lemma 3]. As a result we get that the difference in (4) satisfies the first condition of Theorem A.7 in Siegmund (1986), provided that $R(k-1,y) = o(A)$.

From the relation

$$|\log(r + 1 + w_1) - \log(r + 1 + w_2)| \le |\log(1 + w_1) - \log(1 + w_2)|,$$

$r > 0$, $w_1 > 0$, $w_2 > 0$ and from the a.s.-convergence of the sequence

$$W(k,n,y) = \sum_{j=k+1}^{k-1+n} \exp\left\{ -\sum_{i=k}^{j-1} Z_i^y \right\}$$

to $W_1(k,y)$ we get that the second condition of Theorem A.7 [Siegmund (1986)] holds as well, uniformly in k and in the value of $R(k-1,y)$.

Given $\varepsilon > 0$, consider the k's that are smaller than $(1-\varepsilon)m$. Let $m' = m'(A)$ be a sequence such that $m' = o(A)$ and $m = o(m')$. Define

$$B(k, m') = \left\{ \sup_{1 \le j \le k-1} R(j, y) < m' \right\} \subset \{k \le N(A, y)\}.$$

By Doob's inequality it follows that the P_k^y-probability of the event $B(k, m')$ is at least $1 - m/m'$. From nonlinear renewal theory we get that the overshoot $r(N(A, y), y) - a$, given the value of $R(k-1, y)$, has the same asymptotic distribution as the overshoot of the random walk. By integrating on the event $B(k, m')$, we find that the value of the integral, on the right-hand side of (3), is in the interval $(1 \pm \varepsilon)\gamma(y)$, when a is big, uniformly in k. This is true since $(1-\varepsilon)m/a$ goes to infinity. Hence

$$\int_{\{k \le N(A, y) \le m\}} \exp[-(r(N(A, y), y) - a)]\, dP_k^y \tag{5}$$

$$\ge (1-\varepsilon)\left(1 - \frac{m}{m'}\right)\gamma(y)$$

$$\le \frac{m}{m'} + (1+\varepsilon)\gamma(y). \tag{6}$$

Plugging (6), (5) and (3) into (2) gives

$$P_\infty(N(A, y) \le m) \ge \left[(1-\varepsilon)^2\left(1 - \frac{m}{m'}\right)\gamma(y)\right]\frac{m}{A}$$

$$\le \left[\varepsilon + (1-\varepsilon)\frac{m}{m'} + (1+\varepsilon)^2\gamma(y)\right]\frac{m}{A},$$

and the proof is complete, since m/m' goes to zero and ε is arbitrary. □

PROOF OF THEOREM 1. Fix $0 < \varepsilon < 1$ (small) and $0 < b < \infty$ (large). Choose the sequence $m = m(A)$ so that m is an integer. Given an integer j, define

$$\bar{S}_{j,m} = \{jm \le N(A, y)\}, \qquad S_{j,m} = \bar{S}_{j,m} \setminus \bar{S}_{j+1,m}.$$

Consider two auxiliary processes and stopping times:

$$R''(n, y) = \begin{cases} \sum_{k=1}^{n} \exp\left(\sum_{i=k}^{n} Z_i^y\right), & n \le jm - 1, \\ \sum_{k=1}^{jm-1} \exp\left(\sum_{i=k}^{n} Z_i^y\right), & n \ge jm, \end{cases}$$

$$R'(n, y) = \begin{cases} 0, & n \le jm - 1, \\ \sum_{k=jm}^{n} \exp\left(\sum_{i=k}^{n} Z_i^y\right), & n \ge jm, \end{cases}$$

and

$$N''(A, y) = \inf\{n \ge jm \colon R''(n, y) \ge A\},$$
$$N'(A, y) = \inf\{n \colon R'(n, y) \ge A\}.$$

Notice that $R(n, y) = R''(n, y) + R'(n, y)$.

Let $\eta = \varepsilon/2$. It is easy to check that

$$P_\infty(S_{j,m}) \geq P_\infty\big(\overline{S}_{j,m};\, N'(A,y) < (j+1)m\big) \tag{7}$$

and

$$\begin{aligned} P_\infty(S_{j,m}) \leq{}& P_\infty\big(\overline{S}_{j,m};\, N'((1-\eta)A,y) < (j+1)m\big) \\ &+ P_\infty\big(\overline{S}_{j,m};\, N''(\eta A,y) < (j+1)m\big). \end{aligned} \tag{8}$$

Consider the last term in (8). Using the same techniques as in the proof of Lemma 1 we get

$$\begin{aligned} P_\infty\big(\overline{S}_{j,m};\, N''(\eta A,y) < (j+1)m\big) &\leq \frac{\sum_{k=1}^{jm-1} P_k^y\big(\overline{S}_{j,m}\big)}{\eta A} \\ &\leq \frac{\sum_{k=1}^{jm-1} P_1^y(N(A,y) \geq mj-k+1)}{\eta A} \\ &\leq \frac{E_1^y(M(A,y))}{\eta A}. \end{aligned} \tag{9}$$

Hence, the last term in (8) is of order $(\log A)/A$.

Let us use induction to show that the probability of the event $\overline{S}_{j,m}$ is bounded from below for $1 \leq j \leq bA/m$ (and for large A). It is enough to prove that

$$P_\infty\big(\overline{S}_{j,m}\big) \geq \left(1 - \frac{2\gamma m}{A}\right)^j, \qquad 1 \leq j \leq bA/m. \tag{10}$$

The claim in (10) for $j = 1$ follows from Lemma 1. Assume that the claim holds for $j \leq t$ ($< bA/m$). For all these j's it follows that the right-hand side of (10) is bounded away from zero by a constant that does not depend on A or on t. (The constant does depend on b.) The stopping time $N'(A, y)$ is independent of the event $\overline{S}_{j,m}$. From the induction assumption, equation (8) and Lemma 1 we get, for large enough A, that

$$\begin{aligned} P_\infty\big(\overline{S}_{t+1,m}\big) &= \prod_{j=0}^{t}\Big[1 - P_\infty\big(S_{j,m} \mid \overline{S}_{j,m}\big)\Big] \\ &\geq \prod_{j=0}^{t}\left[1 - \frac{m}{A}\left\{\gamma\frac{1+\varepsilon}{1-\eta} + C\frac{\log A}{m}\right\}\right], \end{aligned}$$

where $C < \infty$ does not depend on t or on A, and the proof of (10) is complete.

From (7)–(10) and Lemma 1 it follows that, when A is big,

$$(1-\varepsilon)\gamma(y)\frac{m}{A} \leq P_\infty\big(S_{j,m} \mid \overline{S}_{j,m}\big) \leq (1+\varepsilon)\gamma(y)\frac{m}{A}, \tag{11}$$

for all $0 \leq j \leq bA/m$.

Straightforward calculations show that

$$(12)\qquad E_\infty N(A,y) \ge \frac{A}{\gamma(y)}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)(1-2b\gamma(y)\exp[-\gamma(y)b]).$$

Using the fact that $E_\infty N(A,y) \ge E_\infty(N(A,y)-k \mid N(A,y)\ge k)$, for all k, and the bound

$$P_\infty\left(\overline{S}_{\lfloor bA/m\rfloor, m}\right) \le \frac{\exp[-\gamma(y)(1-\varepsilon)b]}{1-\gamma(y)m/A},$$

we get

$$(13)\qquad \begin{aligned} &\left(1-\frac{\exp[-\gamma(y)(1-\varepsilon)b]}{1-\gamma(y)m/A}\right)E_\infty N(A,y) \\ &\quad \le \frac{A}{\gamma(y)}\left(\frac{1+\varepsilon}{(1-\varepsilon)^2}+\frac{\gamma(y)b\exp[-\gamma(y)(1-\varepsilon)b]}{1-\gamma(y)m/A}\right). \end{aligned}$$

Finally, combine (13) and (12), and send $\varepsilon \to 0$ and $b \to \infty$. □

3. The lattice case and the mixture-type case. In this section we extend the method of the proof of Theorem 1 to the case when the P_1^y-distribution of Z_i^y is on a lattice, and we handle the calculation of $E_\infty N(A,F)$.

Let us start with the lattice case. Assume that the distribution of Z_i^y, under the regime P_1^y, is arithmetic with span d. Let H^y be the asymptotic distribution of

$$\sum_{i=1}^{M(A,y)} Z_i^y - a,$$

as $a \to \infty$, a/d an integer. Let G^y be the distribution of $\log(W_0(y)+1+W_1(y))$, where $W_1(y)=W_1(1,y)$ and $W_0(y)$ is independent of $W_1(y)$ with the same distribution as the P_∞-distribution of

$$\sum_{j=1}^{\infty} \exp\left\{\sum_{i=1}^{j} Z_i^y\right\}.$$

Determine a distribution K^y by the condition

$$K^y(jd+B) = H^y((j+1)d)\sum_{i=0}^{\infty} G^y(id+B)$$

for $j \ge 0$ and Borel sets $B \subset [0,d)$. Define

$$\gamma'(y) = \int e^{-x}K^y(dx).$$

THEOREM 2. *If $y \in \Omega$, $I(y) < \infty$ and the P_1^y-distribution of Z_i^y is arithmetic with span d, then*

$$E_\infty N(A,y) = \frac{A}{\gamma'(y)}(1+o(1)),$$

where $o(1) \to 0$ as $A \to \infty$.

PROOF. Consider (4) as a process in n. Under the regime P_k^y it converges to the random variable

$$\log(R(k-1,y)+1+W_1(k,y)). \tag{14}$$

Let us show that the distribution of this random variable is continuous.

It is enough to prove that the distribution of $W_1(y)$ is continuous. Assume that it is not. Let Δ be the support of Z_i^y, and let

$$w=\arg\max P_1^y(W_1(y)=x).$$

From the relation

$$P_1^y(W_1(y)=w)=\sum_{z\in\Delta}P_1^y(W_1(y)=e^z w-1)P_1^y(z)$$

we get that $P_1^y(W_1(y)=w)=P_1^y(W_1(y)=e^z w-1)$, and hence, in particular, that

$$P_1^y(W_1(y)=w)=P_1^y\big(W_1(y)=e^{nz}w-e^{(n-1)z}-\cdots-e^z-1\big),$$

for all $n\geq 1$ and $z\in\Delta$. The cardinality of this set is not finite, therefore, $P_1^y(W_1(y)=w)=0$, and the distribution of $W_1(y)$ is indeed continuous.

Using the same arguments as in the proof of Lemma 1 and nonlinear renewal theory for arithmetic random variables [Woodroofe (1982), Theorem 4.3], it can be shown that

$$\begin{aligned}&\int_{\{k\leq N(A,y)\leq m\}}\exp[-(r(N(A,y),y)-a)]\,dP_k^y\\&\qquad\geq(1-\varepsilon)\left(1-\frac{m}{m'}\right)\gamma'(y),\end{aligned}\tag{15}$$

provided that k is big enough. Consider only the k's that are between εm and $(1-\varepsilon)m$, conclude a version of Lemma 1 and the theorem follows. □

THEOREM 3. *If F has a positive continuous density with respect to Lebesgue measure on Ω, then*

$$E_\infty N(A,F)=\frac{A}{\gamma(F)}(1+o(1)),$$

where $o(1)\to 0$ as $A\to\infty$.

PROOF. Use the same arguments as in the proof of relation (2) to get

$$P_\infty(N\leq m)=\sum_{k=1}^{m}\int_\Omega\int_{\{k\leq N\leq m\}}\frac{dP_k^y}{R(N,F)}\,dF(y). \tag{16}$$

Let $r(n,F)=\log R(n,F)$, and fix $\omega\in\Omega\setminus\{0\}$.

$$\begin{aligned}&\int_{\{k\leq N(A,F)\leq m\}}\frac{dP_k^\omega}{R(N(A,F),F)}\\&\quad=\frac{\int_{k\leq\{N(A,F)\leq m\}}\exp[-(r(N(A,F),F)-a)]\,dP_k^\omega}{A}.\end{aligned}$$

In order to show that the integral on the right-hand side of the above equation converges to $\gamma(\omega)$, we need to demonstrate that the process

$$
\begin{aligned}
(17)\qquad & r(k-1+n, F) - \sum_{i=k}^{k-1+n} Z_i^{\omega} \\
&= \log \int \exp\left\{ \sum_{i=k}^{k-1+n} (Z_i^y - Z_i^{\omega}) \right\} \\
&\quad \times [R(k-1, y) + 1 + W(k, n, y)]\, dF(y),
\end{aligned}
$$

given the σ-algebra generated by the first $k-1$ observations, is slowly changing under the regime P_k^{ω}. The approximation of the integral should hold uniformly in k, for most of the k's. This can be proved via the approach of the proof of Theorem 4 in Pollak (1987).

The rest of the proof of Theorem 3 is similar to the proof of Theorem 1. □

REMARKS.

(i) The difference between the approach presented in this paper, for the calculation of the average run length to false alarm of the Shiryayev–Roberts procedure, and the approach of Pollak (1987) is that he evaluated the distribution of the normalized overshoot $R(N(A, y), y)/A$, while we consider directly the limiting distribution of the normalized run length $N(A, y)/A$.

(ii) It follows immediately from the proofs of Theorems 1–3 that the normalized run lengths converge to exponential distributions under the regime P_∞. The expectation of these distributions are $1/\gamma(y)$, $1/\gamma'(y)$ and $1/\gamma(F)$, respectively.

(iii) Theorem 1 was proved in Pollak (1987). Notice, though, that the proof there is not complete. In order to prove his Lemma 1, one needs to show, in effect, that condition 1(c) of Gordon and Pollak [(1993), Theorem 1] holds. The lemma does not follow from the fact that $P_\infty\{\mathbf{Q}(j, L_j) > A/\sqrt{C}\}$ is arbitrarily small, independently of A.

(iv) Theorem 3 is not identical to Theorem 2 in Pollak (1987). He assumed that the P_1^y-distribution of X_i is strongly nonlattice for all $y \in \Omega$, and in our version this assumption is dropped. The price we pay, however, is that we assume that the mixing distribution F has a continuous positive density.

(v) Theorem 2 is, to the best of our knowledge, a new result.

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