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UNBIASED TESTING IN EXPONENTIAL FAMILY REGRESSION

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Let $(X_{ij}, \mathbf{z}_i), i = 1, 2, \ldots, k, j = 1, 2, \ldots, n_i$, be independent observations such that $\mathbf{z}_i$ is a fixed $r \times 1$ vector [r can be 0 (no $\mathbf{z}$'s observed) or $1, 2, \ldots, k-1$], and X_{ij} is distributed according to a one-parameter exponential family which is log concave with natural parameter θ_i. We test the hypothesis that $\theta = \mathbf{Z}\beta$, where $\theta = (\theta_1, \ldots, \theta_k)', \mathbf{Z}$ is the matrix whose ith row is $\mathbf{z}_i'$ and $\beta = (\beta_1, \ldots, \beta_r)'$ is a vector of parameters. We focus on $r = 2$ and $\mathbf{z}_i' = (1, z_i), i = 1, 2, \ldots, k, z_i < z_{i+1}$. The null hypothesis on hand is thus of the form $\theta_i = \beta_1 + \beta_2 z_i$. In such a case the model under the null hypothesis becomes logistic regression in the binomial case, Poisson regression in the Poisson case and linear regression in the normal case. We consider mostly the one-sided alternative that the second-order differences of the natural parameters are nonnegative. Such testing problems test goodness of fit vs. alternatives in which the natural parameters behave in a convex way. We find classes of tests that are unbiased and that lie in a complete class. We also note that every admissible test of constant size is unbiased. In some discrete situations we find the minimal complete class of unbiased admissible tests. Generalizations and examples are given.

1. Introduction and summary. Consider the model where $(X_{ij}, \mathbf{z}_i), i = 1, 2, \ldots, k, j = 1, 2, \ldots, n_i$, are observed. Here X_{ij} has a one-parameter exponential family density with natural parameter θ_i. That is, for each $i = 1, 2, \ldots, k$ and any $j = 1, 2, \ldots, n_i$, the density of X_{ij} is

$$f_{X_{ij}}(x_{ij}, \theta_i) = e^{-M(\theta_i) + x_{ij}\theta_i} h_i(x_{ij}). \tag{1.1}$$

Further assume $h_i(x_{ij})$ is log concave for i; that is, the distributions are PF_2. The dominating measure for each x_{ij} is Lebesgue measure on $(-\infty, \infty)$ for the continuous case and counting measure on $\{0, \pm 1, \pm 2, \ldots\}$ for the case where X_{ij} is integer valued. The vector $\mathbf{z}_i$ is an $r \times 1$ vector of fixed constants such that the matrix $\mathbf{Z}$ of order $k \times r$ is of rank $r < k$ and consists of the k rows $\mathbf{z}_i'$. Note that $r = 0, 1, \ldots, k-1$. When $r = 0$ no $\mathbf{z}_i$'s are observed. When $r > 0$ the $\mathbf{z}_i$ represent the independent variables in a potential regression model. Let

$$S_i = \sum_{j=1}^{n_i} X_{ij}, \qquad X_i = S_i / n_i, \qquad \mathbf{S}' = (S_1, \ldots, S_k),$$
$$\mathbf{X}' = (X_1, \ldots, X_k), \qquad \boldsymbol{\theta}' = (\theta_1, \ldots, \theta_k).$$

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Sufficient statistics are $\mathbf{S}$ or $\mathbf{X}$. For ease of exposition only, we take $n_i = n = 1$. (See Section 5 for generalizations.)

Consider testing H: $\boldsymbol{\theta} = \mathbf{Z}\boldsymbol{\beta}$, where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_r)'$ is an $r \times 1$ vector of parameters. We focus on $r = 2$ and $\mathbf{z}_i' = (1, z_i), i = 1, 2, \dots, k, z_i < z_{i+1}$. Problems with this null hypothesis are called goodness of fit or lack of fit. In the case $r = 2$, the null hypothesis specifies that the natural parameters lie on a straight line. McCullagh (1986) studies lack of fit for discrete data. Tsiatis (1980) and others study logistic regression. Draper and Smith (1966) study normal regression. Zelterman (1988) studies lack of fit (goodness of fit) for specialized alternatives while Dean and Lawless (1989) and many others study Poisson regression.

Consider next the one-sided alternative $K/H = K - H$, where K specifies that the natural parameters behave in a convex manner. That is, the testing problem is

$$H\colon \theta_i = \beta_1 + \beta_2 z_i \text{ vs. } K/H, \tag{1.2}$$

where K: $(\theta_{i+2} - \theta_{i+1})/(z_{i+2} - z_{i+1}) \geq (\theta_{i+1} - \theta_i)/(z_{i+1} - z_i)$. Another way to describe the alternative is that second-order generalized differences in the natural parameters θ_i are nonnegative. Embedded in both null and alternative hypotheses are some popular change point problems. See, for example, Miao (1988), Kim and Siegmund (1989), Loader (1992) and others.

For testing (1.2) we find a class of unbiased tests that also lie in a complete class. In the integer-valued case the class includes and often coincides with the class of all admissible, unbiased tests. In the continuous case the class contains all the admissible unbiased tests. From a practical point of view, any test used in practice should be in the complete class and should be unbiased.

To describe the class of tests with the desirable properties, we start by noting that (1.2) is of the form H: $\mathbf{A}_1\boldsymbol{\theta} = 0$ and K: $\mathbf{A}_1\boldsymbol{\theta} \geq 0$, where $\mathbf{A}_1$ represents the first $(k-2)$ rows of the $k \times k$ upper triangular matrix

$$\mathbf{A} = \begin{pmatrix} r_1 & -(r_1 + r_2) & r_2 & 0 & & 0 \\ 0 & r_2 & -(r_2 + r_3) & r_3 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & & 0 & r_{k-1} & -(r_{k-1} + r_k) \\ 0 & \cdots & & & 0 & r_k \end{pmatrix}, \tag{1.3}$$

where $r_i = 1/\Delta_i$, $i = 1, 2, \dots, k-1$, $\Delta_i = z_{i+1} - z_i$, $r_k > 0$. The particular choice of $r_k > 0$ is irrelevant. For concreteness, we define $r_k = 1$ (corresponding to $z_{k+1} = z_k + \Delta_k = z_k + 1$).

The inverse matrix $\mathbf{B} = \mathbf{A}^{-1}$ is

$$\mathbf{B} = \begin{pmatrix} \Delta_1 & (\Delta_1 + \Delta_2) & \cdots & & (\Delta_1 + \Delta_2 + \cdots + \Delta_k) \\ 0 & \Delta_2 & & & (\Delta_2 + \cdots + \Delta_k) \\ 0 & 0 & \Delta_3 & & (\Delta_3 + \cdots + \Delta_k) \\ \vdots & & & \ddots & \\ 0 & & & 0 & \Delta_k \end{pmatrix}. \tag{1.4}$$

Now let $\mathbf{Y} = \mathbf{B}'\mathbf{X}$ and let $\mathbf{Y} = \binom{\mathbf{Y}^{(1)}}{\mathbf{Y}^{(2)}}$, where $\mathbf{Y}^{(1)}$ is of order $(k-2) \times 1$. Thus the jth component of $\mathbf{Y}^{(1)}$ is

$$(\mathbf{Y}^{(1)})_j = \left(\sum_{i=1}^{i} \Delta_i\right) X_i + \left(\sum_{i=2}^{j} \Delta_i\right) X_2 + \cdots + \Delta_j X_j = \sum_{i=1}^{j} (z_{j+1} - z_i) X_i, \tag{1.5}$$

$j = 1, 2, \ldots, k-2$. Further, $\mathbf{Y}^{(2)}$ has components

$$\left[z_k \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} z_i X_i\right] \quad \text{and} \quad \left[(z_k + 1) \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} z_i X_i\right]. \tag{1.6}$$

Hence working conditionally on $\mathbf{Y}^{(2)} = \mathbf{y}^{(2)}$ simply means working conditionally on $\sum_{i=1}^{k} X_i$ and $\sum_{i=1}^{k} z_i X_i$ being constant.

DEFINITION. Let $\mathcal{C}$ denote the class of test functions $\varphi(\mathbf{y}^{(1)}, \mathbf{y}^{(2)})$ of H vs. K/H such that:

(i) $\varphi(\mathbf{y})$ is constant size α,
(ii) $\varphi(\mathbf{y})$ is nondecreasing in $\mathbf{y}^{(1)}$ [for fixed $\mathbf{y}^{(2)}$],
(iii) $\varphi(\mathbf{y})$ has convex acceptance sections for almost all fixed $\mathbf{y}^{(2)}$. In the integer-valued case, randomization can only occur at extreme points of these convex sections.

A MAIN RESULT. Each test φ in $\mathcal{C}$ is unbiased. All admissible constant-size tests are unbiased and are in $\mathcal{C}$. Moreover, $\mathcal{C}$ is contained in a nontrivial complete class.

In Cohen, Kemperman and Sackrowitz (1993), the problem of testing H: $\mathbf{A}_1\boldsymbol{\theta} = \mathbf{0}$ vs. K/H, where K: $\mathbf{A}_1\boldsymbol{\theta} \geq \mathbf{0}$, is studied for a class of matrices $\mathbf{A}_1$ (each associated with a different testing problem). In that reference, it is assumed that the $\mathbf{X}_{ij} \sim N(\theta_i, \sigma^2), \sigma^2$ unknown. See Section 5 for a comparison of the results in this paper with results in the referenced paper.

To prove unbiasedness results, we introduce a notion of positive dependence called weakly conditionally increasing in sequence (WCIS). This is done in Section 2 where the connection between WCIS, CIS and the notion of association is made.

In Section 3 we return to the regression problem starting with a complete class containing the class $\mathcal{C}$. In Section 4 we prove the results on unbiased testing and indicate how WCIS is used. The verification of the WCIS property is given in Section 4 which also contains some new results on total positivity and further results on stochastic ordering. Section 5 contains some related results, extensions and examples.

2. WCIS.

We need some definitions.

DEFINITION 2.1. The random vector $\mathbf{V}$ is stochastically greater than or equal to a random vector $\mathbf{U}$ ($\mathbf{U} \leq_{\text{st}} \mathbf{V}$) if

$$Eh(\mathbf{U}) \leq Eh(\mathbf{V}), \tag{2.1}$$

for all nondecreasing h for which the expectations in (2.1) exist.

DEFINITION 2.2. The random variables in $\mathbf{U} = (U_1, \dots, U_k)'$ are said to be conditionally increasing in sequence (CIS) if, for $j = 1, \dots, k-1$,

$$[U_j \mid U_{j+1} = u_{j+1}, \dots, U_k = u_k] \leq_{\text{st}} [U_j \mid U = u^*_{j+1}, \dots, U_k = u^*_k], \tag{2.2}$$

for $u_{j+1} \leq u^*_{j+1}, \dots, u_k \leq u^*_k$ [see Tong (1990), page 92]. (We reordered the variables in Tong because it is more natural for our application.) In Cohen and Sackrowitz (1992), Remark 2.8, it is shown that (2.2) is equivalent to

$$\begin{aligned}&[U_1, \dots, U_j \mid U_{j+1} = u_{j+1}, \dots, U_k = u_k]\\ &\qquad \leq_{\text{st}} [U_1, \dots, U_j \mid U_{j+1} = u^*_{j+1}, \dots, U_k = u^*_k].\end{aligned} \tag{2.3}$$

DEFINITION 2.3. The random variables in $\mathbf{U}$ are said to be weak conditionally increasing in sequence (WCIS) if, for $j = 1, \dots, k-1$,

$$\begin{aligned}&[U_1, \dots, U_j \mid U_{j+1} = u_{j+1}, \dots, U_k = u_k]\\ &\qquad \leq_{\text{st}} [U_1, \dots, U_j \mid U_{j+1} = u^*_{j+1}, \dots, U_k = u_k],\end{aligned} \tag{2.4}$$

for $u_{j+1} \leq u^*_{j+1}$. In other words, WCIS means that, for each $j = 1, 2, \dots, k$, the random vector $[U_1, \dots, U_j]$ given $U_{j+1}, \dots, U_k$ need only be stochastically nondecreasing in $U_{j+1} = u_{j+1}$.

DEFINITION 2.4. The random variables in $\mathbf{U}$ are said to be (positively) associated (A) if

$$Eh_1(\mathbf{U})h_2(\mathbf{U}) \geq Eh_1(\mathbf{U})Eh_2(\mathbf{U}), \tag{2.5}$$

holds for all nondecreasing h_1, h_2 for which expectations in (2.5) exist. [Esary, Proschan and Walkup (1967) introduced the notion of association.]

THEOREM 2.5. *The following implications are true*:

$$CIS \overset{(a)}{\Rightarrow} WCIS \overset{(b)}{\Rightarrow} A. \tag{2.6}$$

Furthermore, all implications are strict for $k \geq 3$.

PROOF. The implication (a) follows from (2.3) and (2.4). To prove (b), we need to show that (2.4) implies (2.5). For notational convenience take $h_1 = Q$, $h_2 = W$ and let $Q = Q_0$, $W = W_0$ be given nondecreasing functions on $\mathbb{R}^k$. For $j = 1, 2, \dots, k-1$ define

$$Q_j(u_{j+1}, \dots, u_k) = E[Q_0(U_1, \dots, U_k) \mid U_{j+1} = u_{j+1}, \dots, U_k = u_k] \tag{2.7}$$

and $Q_k = EQ_0(U_1, \dots, U_k)$. Similarly, define $W_j(u_{j+1}, \dots, u_k)$ and W_k. Since $Q_0(u_1, \dots, u_j, u_{j+1}, \dots, u_k)$ is nondecreasing as a function of u_{j+1} and also as a function of $u_1, \ldots, u_j$, the stated condition of the lemma implies that $Q_j(u_{j+1}, \ldots, u_k)$ is nondecreasing as a function of u_{j+1} (keeping $u_{j+2}, \ldots, u_k$ fixed). Similarly, $W_j(u_{j+1}, \ldots, u_k)$ is nondecreasing in u_{j+1}. By virtue of the one-dimensional correlation inequality, if follows, for $j = 0, 1, \ldots, k-1$, that

$$
\begin{aligned}
&E\big[Q_j(U_{j+1}, \ldots, U_k)W_j(U_{j+1}, \ldots, U_k) \mid U_{j+2}, \ldots, U_k\big] \\
&\quad \geq E\big[Q_j(U_{j+1}, \ldots, U_k) \mid U_{j+2}, \ldots, U_k\big] \\
&\qquad \times E\big[W_j(U_{j+1}, \ldots, U_k) \mid U_{j+2}, \ldots, U_k\big] \\
&\quad = Q_{j+1}(U_{j+2}, \ldots, U_k)W_{j+1}(U_{j+2}, \ldots, U_k).
\end{aligned} \tag{2.8}
$$

Now (2.8) implies

$$
\begin{aligned}
&EQ_j(U_{j+1}, \ldots, U_k)W_j(U_{j+1}, \ldots, U_k) \\
&\qquad \geq EQ_{j+1}(U_{j+2}, \ldots, U_k)W_{j+1}(U_{j+2}, \ldots, U_k).
\end{aligned} \tag{2.9}
$$

This implies (2.5) since the right-hand side of (2.9) with $j = k - 1$ reduces to $E[Q_0(U_1, \ldots, U_k)]E[W_0(U_1, \ldots, U_k)]$. □

A sufficient condition for random vectors $\mathbf{U}$ and $\mathbf{V}$ to be stochastically ordered is as follows: let $f_{\mathbf{U}}(\cdot)$ and $f_{\mathbf{V}}(\cdot)$ denote the densities of $\mathbf{U}$ and $\mathbf{V}$, respectively. Consider $r(\mathbf{u}) = f_{\mathbf{V}}(\mathbf{u})/f_{\mathbf{U}}(\mathbf{u})$. Conditions implying $\mathbf{U} \leq_{\mathrm{st}} \mathbf{V}$ ($\mathbf{U}$ is stochastically less than or equal to $\mathbf{V}$) are the pair (2.10) and (2.11) where

$$r(\mathbf{u}) \text{ is nondecreasing in } \mathbf{u} \tag{2.10}$$

[nondecreasing means that $r(\mathbf{u})$ is nondecreasing in u_i for $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k$ fixed, for $i = 1, 2, \ldots, k$],

$$\mathbf{U} \text{ is A.} \tag{2.11}$$

This is essentially proved in Perlman and Olkin (1980). In light of this and Theorem 2.5 we have the following result.

COROLLARY 2.6. *Suppose* $\mathbf{U}$ *is WCIS and* (2.10) *holds. Then* $\mathbf{U} \leq_{\mathrm{st}} \mathbf{V}$.

3. Complete class. We now return to the statistical model of Section 1. Recall $\mathbf{Y} = \mathbf{B}'\mathbf{X}$ and let $\nu = A\boldsymbol{\theta}$. Since the X_i follow an exponential family and are independent, it follows from (1.1) that the joint distribution of $\mathbf{Y}$ can be written as

$$
\begin{aligned}
f_{\mathbf{Y}}(\mathbf{y}, \nu) &= \exp\big[-M^*(\nu) + \mathbf{y}'\nu\big]h^*(\mathbf{y}) \\
&= \exp\big[-M^*(\nu) + \mathbf{y}^{(1)'}\nu^{(1)} + \mathbf{y}^{(2)'}\nu^{(2)}\big]h^*(\mathbf{y}).
\end{aligned} \tag{3.1}
$$

The hypotheses are H: $\nu^{(1)} = 0$ vs. K/H, where K: $\nu^{(1)} \geq 0$. Hence, from (3.1), it follows that $Y^{(2)}$ is a complete sufficient statistic under H. Also, from Lehmann (1986), page 144, it follows that all exact-size α tests must have Neyman structure; that is, all exact tests must be such that, conditional on $\mathbf{y}^{(2)}$, they have size α. From Lehmann (1986), page 58, we note that the conditional density of $\mathbf{Y}^{(1)} \mid \mathbf{Y}^{(2)} = \mathbf{y}^{(2)}$ can be written as

$$f_{\nu^{(1)}}(\mathbf{y}^{(1)} \mid \mathbf{y}^{(2)}) = C_{\mathbf{y}^{(2)}}(\nu^{(1)}) \exp[\mathbf{y}^{(1)\prime}\nu^{(1)}] h_{\mathbf{y}^{(2)}}(\mathbf{y}^{(1)}). \tag{3.2}$$

Conditional on $\mathbf{y}^{(2)}$, then the problem is to test the simple hypothesis H: $\nu^{(1)} = 0$ vs. K/H, where K: $\nu^{(1)} \geq 0$.

THEOREM 3.1. *The class of tests $\varphi(\mathbf{y})$ where $\varphi(\mathbf{y})$ is nondecreasing in $\mathbf{y}^{(1)}$ for fixed $\mathbf{y}^{(2)}$ and has convex acceptance sections is a complete class in the continuous case. In the integer-valued case the same conclusion holds for $\varphi(\mathbf{y})$ provided we require in addition that randomization is only permitted at the extreme points of these convex acceptance sections.*

PROOF. See Eaton (1970). For the integer-valued case, see also Matthes and Truax (1967) and Cohen, Gatsonis and Marden (1983). □

REMARK. In many integer-valued cases, including binomial and Poisson distributions, the class is minimal complete. See again the last two references above.

4. Unbiased tests. In this section we prove the main application, namely that tests lying in the class $\mathcal{C}$ are unbiased for testing H vs. K/H. To prove unbiasedness, our plan is to prove conditional unbiasedness, where the conditioned variables are $y^{(2)}$. Conditional unbiasedness implies unbiasedness. To achieve conditional unbiasedness, we will demonstrate that the distribution of $\mathbf{Y}^{(1)} \mid \mathbf{Y}^{(2)} = y^{(2)}$ under the alternative $K/H, K$: $\nu^{(1)} \geq \mathbf{0}$, is stochastically larger than the distribution of $\mathbf{Y}^{(1)} \mid \mathbf{Y}^{(2)} = y^{(2)}$ under H: $\nu^{(1)} = \mathbf{0}$. To prove such a stochastic ordering, we will invoke Corollary 2.6. Hence we will prove the following result.

THEOREM 4.1. *Let $\varphi(\mathbf{y}) = \varphi(\mathbf{y}^{(1)}, \mathbf{y}^{(2)})$ be a size α test. Then, if $\varphi(\mathbf{y})$ is nondecreasing in $\mathbf{y}^{(1)}$ for fixed $\mathbf{y}^{(2)}$, $\varphi(\mathbf{y})$ is unbiased.*

PROOF. By the remarks preceding the statement of the theorem, the result is proved if

$$r(\mathbf{y}^{(1)}) = f_{\nu^{(1)}}(\mathbf{y}^{(1)} \mid \mathbf{y}^{(2)}) / f_{\mathbf{0}}(\mathbf{y}^{(1)} \mid \mathbf{y}^{(2)}) \tag{4.1}$$

is nondecreasing in $\mathbf{y}^{(1)}$, and

$$\mathbf{Y}^{(1)} \mid \mathbf{Y}^{(2)} \text{ is WCIS under } H. \tag{4.2}$$

That condition (4.1) holds follows immediately from (3.2). The remainder of this section is devoted to proving (4.2). The proof culminates with Theorem 4.8 below. □

Our remaining task is to show (4.2) which can be written as

$$
\begin{aligned}
&[(Y_1,\dots,Y_m)\mid Y_{m+1}=y_{m+1},\dots,Y_{k-2}=y_{k-2}]\\
&\quad\le_{\mathrm{st}}[(Y_1,\dots,Y_m)\mid Y_{m+1}=y^*_{m+1},\dots,Y_{k-2}=y_{k-2}]
\end{aligned}
\tag{4.3}
$$

under H assuming $Y^{(2)'}=(Y_{k-1},Y_k)$ are fixed, where $y_{m+1}\le y^*_{m+1}$, $m=1,\dots,k-3$. (Since Y_{k-1},Y_k are fixed throughout, from this point we will not include them in the argument.) We note from (3.2) that the distribution of $\mathbf{Y}^{(1)}\mid\mathbf{Y}^{(2)}$ is parameter free under H. Hence in establishing our goal it suffices to assume that the random variables $X_1,\dots,X_k$ under H are independent with PF_2 densities $h_i(x)$, $i=1,2,\dots,k$. Recall $\mathbf{Y}=\mathbf{B}'\mathbf{X}$ or $\mathbf{X}=\mathbf{A}'\mathbf{Y}$, where $\mathbf{B}=\mathbf{A}^{-1}$, $\mathbf{A}$ and $\mathbf{B}$ are given in (1.3) and (1.4).

We need some new results in total positivity and stochastic ordering. Let $g(x,u)$ be a joint density of (x,u).

LEMMA 4.2. *Suppose $g(x,u)$ is TP_2 and, moreover, log concave in (x,u). Then, for $a>0$, $f(x,u)=g(x,u-ax)$ is log concave and TP_2 in (x,u).*

PROOF. The log concavity of f in (x,u) is immediate since the transformation $(x,u-ax)$ is linear in (x,u). Furthermore, log concavity implies that both f and g are PF_2 in each variable. Let $x\le x^*$, $u\le u^*$. Then

$$
\begin{aligned}
&f(x,u)f(x^*,u^*)-f(x^*,u)f(x,u^*)\\
&\quad=g(x,u-ax)g(x^*,u^*-ax^*)-g(x^*,u-ax^*)g(x,u^*-ax)\ge 0,
\end{aligned}
\tag{4.4}
$$

by Lemma 5.2 of Karlin (1968), page 126. □

REMARK. It can be shown that the result of Lemma 4.2 holds also for $f(x,u)=g(ax-bu,cx+du)$ for each choice of nonnegative constants a,b,c,d. Even more generally, the TP_2 property holds for $f(x,u)=g(\alpha(x)-\beta(u),-\gamma(x)+\delta(u))$ where $\alpha,\beta,\gamma,\delta$ are nondecreasing functions.

We next state two well-known results.

LEMMA 4.3. *If $g(x_1,\dots,x_k)$ is log concave in $(x_1,x_2,\dots,x_k)$, then for any $m=1,2,\dots,k-1$, the marginal $f(x_1,\dots,x_m)=\int_{R^{k-m}}g(x_1,\dots,x_k)\prod_{i=m+1}^k dx_i$ is log concave in $(x_1,\dots,x_m)$.*

PROOF. See Dharmadhikari and Joag-Dev (1988), page 61.

LEMMA 4.4. *If $\omega_1(x,y)$ is TP_2 in (x,y) and $\omega_2(y,z)$ is TP_2 in (y,z), then $\omega_3(x,z)=\int\omega_1(x,y)\omega_2(y,z)\,dy$ is TP_2 in (y,z).*

PROOF. See Karlin (1968), page 123.

LEMMA 4.5. *For $m=1,2,\dots,k-2$, the conditional distribution of $[(Y_1,Y_2,\dots,Y_m)\mid(Y_{m+1},Y_{m+2},\dots,Y_{k-2})]$ is the same as the conditional distribution of*

$[(Y_1, Y_2, \dots, Y_m) \mid (Y_{m+1}, Y_{m+2})]$. *As a consequence, the conditional distribution of* $[(Y_1, Y_m) | (Y_{m+1}, Y_{m+2}, \dots, Y_{k-2})]$ *is the same as the conditional distribution of* $[(Y_1, Y_m) \mid (Y_{m+1}, Y_{m+2})]$.

PROOF. Since $\mathbf{X} = \mathbf{A}'\mathbf{Y}$, from (1.3) we have, for $j = 1, 2, \dots, (k-2)$,

$$X_j = r_j Y_j - (r_{j-1} + r_j) Y_{j-1} + r_{j-2} Y_{j-2}, \tag{4.5}$$

and from (1.5)

$$Y_j = \left(\sum_{i=1}^{j} \Delta_i \right) X_1 + \left(\sum_{i=2}^{j} \Delta_i \right) X_2 + \cdots + \Delta_j X_j. \tag{4.6}$$

Now (4.5) implies that the event $Y_{m+1} = y_{m+1}, Y_{m+2} = y_{m+2}, \dots, Y_{k-2} = y_{k-2}$ is equivalent to the event $Y_{m+1} = y_{m+1}, Y_{m+2} = y_{m+2}, X_{m+3} = x_{m+3}, \dots, X_{k-2} = x_{k-2}$. However, from (4.6), $Y_1, Y_2, \dots, Y_m$ are independent of $X_{m+3}, \dots, X_{k-2}$ since the $Y_1, \dots, Y_m$ depend only on $(X_1, \dots, X_m)$. □

LEMMA 4.6. *The conditional density of* $[Y_1, Y_m \mid Y_{m+1}, Y_{m+2}]$ *is* TP_2 *(and log concave).*

The proof is given in the Appendix.

An immediate consequence of Lemma 4.6 is the following result.

COROLLARY 4.7. *For* $m = 2, \dots, k-2$, *the conditional distributions of* $Y_1 \mid Y_m, Y_{m+1}, Y_{m+2}$ *and* $Y_m \mid Y_1, Y_{m+1}, Y_{m+2}$ *are stochastically nondecreasing in* Y_m *and* Y_1, *respectively.*

THEOREM 4.8. *For* $i = 2, \dots, k-2$, *let* $W_i(y_1, y_2, \dots, y_i)$ *be a nondecreasing function of* $(y_1, \dots, y_i)$. *Then, for fixed* y_{i+1}, y_{i+2}, $Y_{i+1} = y_{i+1}$, $Y_{i+2} = y_{i+2}$.

$$E\big[W_i(Y_1, \dots, Y_i) \mid (Y_i, Y_{i+1}, Y_{i+2})\big] \text{ is nondecreasing in } Y_i. \tag{4.7}$$

$$E\big[W_i(Y_1, \dots, Y_i) \mid (Y_1, Y_{i+1}, Y_{i+2})\big] \text{ is nondecreasing in } Y_1. \tag{4.8}$$

Note (4.7) is equivalent to the desired result (4.3).

PROOF. For $i = 2$, (4.7) and (4.8) easily follow from Corollary 4.7. That is, let $y_2 \le y_2^*$. Then

$$\begin{aligned} E\big[W_2(Y_1, Y_2) \mid (y_2, y_3, y_4)\big] &= E\big[W_2(Y_1, y_2) \mid (y_2, y_3, y_4)\big] \\ &\le E\big[W_2(Y_1, y_2) \mid (y_2^*, y_3, y_4)\big] \\ &\le E\big[W_2(Y_1, y_2^*) \mid (y_2^*, y_3, y_4)\big] \\ &= E\big[W_2(Y_1, Y_2) \mid (y_2^*, y_3, y_4)\big]. \end{aligned}$$

Now assume (4.7) and (4.8) to be true for $i = 2, \dots, m$. To establish (4.7) and (4.8) for $i = m + 1$, begin by defining

$$\begin{aligned} &W^*(Y_1, Y_{m+1}, Y_{m+2}, Y_{m+3}) \\ &\quad = E\big[W_{m+1}(Y_1, \dots, Y_{m+1}) \,|\, Y_1, Y_{m+1}, Y_{m+2}, Y_{m+3}\big]. \end{aligned} \tag{4.9}$$

Note from the proof given in Lemma 4.5 that the conditional distribution of $(Y_2, \dots, Y_m)$, given $(Y_1, Y_{m+1}, Y_{m+2}, Y_{m+3})$, does not depend on Y_{m+3}. Thus W^* is nondecreasing in Y_1 for fixed Y_{m+1}, Y_{m+2} as (4.8) holds for $i = m$.

Next, let $\widehat{X}_l = X_{l+1}$, $l = 1, 2, \dots, k-3$, and define

$$\widehat{Y}_j = \left(\sum_{i=2}^{j+1} \Delta_i\right)\widehat{X}_1 + \left(\sum_{i=3}^{j+1} \Delta_i\right)\widehat{X}_2 + \cdots + \Delta_{(j+1)}\widehat{X}_j,$$

so that from (4.6) we have

$$Y_j = \left(\sum_{i=1}^{j} \Delta_i\right)X_1 + \widehat{Y}_{j-1}, \qquad j = 2, \dots, k-2.$$

The variables $\widehat{X}_j$ are independent of X_1 and are related to $\widehat{Y}_j$ as X_j's were related to Y_j; that is, $\widehat{Y} = \widehat{A}'\widehat{X}$, where the matrix $\widehat{A}$ has the identical structure as A only the order of the square matrix is one less than that of A. Therefore

$$\begin{aligned} &E\big[W_{m+1}(Y_1, \dots, Y_{m+1}) \,|\, Y_1 = y_1, Y_{m+1} = y_{m+1}, Y_{m+2} = y_{m+2}, Y_{m+3} = y_{m+3}\big] \\ &\quad = E\left[W_{m+1}\left(y_1, [(\Delta_1 + \Delta_2)/\Delta_1]y_1 + \widehat{Y}_1, \dots, \left[\left(\sum_{i=1}^{m+1} \Delta_i\right)\Big/\Delta_1\right]y_1 + \widehat{Y}_m\right) \,\middle|\right. \\ &\qquad \widehat{Y}_m = y_{m+1} - \left[\sum_{i=1}^{m+1} \Delta_i/\Delta_1\right]y_1, \widehat{Y}_{m+1} = y_{m+2} - \left[\sum_{i=1}^{m+2} \Delta_i/\Delta_1\right]y_1, \\ &\qquad \left.\widehat{Y}_{m+2} = y_{m+3} - \left[\sum_{i=1}^{m+3} \Delta_i/\Delta_1\right]y_1\right]. \end{aligned} \tag{4.10}$$

Now (4.10) is nondecreasing in y_{m+1} by virtue of (4.7) for $i = m$. Hence $W^*(Y_1, Y_{m+1}, Y_{m+2}, Y_{m+3})$ is nondecreasing in Y_1 and Y_{m+1}.

Finally,

$$\begin{aligned} &E\big[W_{m+1}(Y_1, \dots, Y_{m+1}) \,|\, Y_{m+1}, Y_{m+2}, Y_{m+3}\big] \\ &\quad = E\big[W^*(Y_1, Y_{m+1}, Y_{m+2}, Y_{m+3}) \,|\, Y_{m+1}, Y_{m+2}, Y_{m+3}\big] \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &E\big[W_{m+1}(Y_1, \dots, Y_{m+1}) \,|\, Y_1, Y_{m+2}, Y_{m+3}\big] \\ &\quad = E\big[W^*(Y_1, Y_{m+1}, Y_{m+2}, Y_{m+3}) \,|\, Y_1, Y_{m+2}, Y_{m+3}\big]. \end{aligned} \tag{4.12}$$

The nondecreasing property of (4.11) and (4.12) in Y_{m+1} and Y_1, respectively, follows from Corollary 4.7. □

THEOREM 4.9. *All admissible tests of constant size are unbiased.*

PROOF. All admissible tests must be nondecreasing in $y^{(1)}$ for fixed $y^{(2)}$ and have conditionally convex acceptance sections by Theorem 3.1. All constant-size tests have constant conditional size by completeness. All tests with conditionally constant size which are conditionally nondecreasing are unbiased by Theorem 4.1. Thus the theorem is proved. □

REMARK. We used the notion of WCIS in proving Theorem 4.1. The relevant variables do not have the CIS property.

5. Generalizations and remarks for statistical problem.

5.1. *Unequal sample sizes.* The results of Sections 3 and 4 hold with only minor modifications if the sample sizes at each z_i are n_i. We would then consider the vector $\mathbf{S}' = (S_1, \dots, S_k)$ whose natural parameter would still be $\theta' = (\theta_1, \dots, \theta_k)$. Hence no changes would be required in defining the matrix $\mathbf{A}$. Note that the development in Section 4 only used the fact that under H, X_i were independent with PF_2 densities $h_i(x)$, $i = 1, 2, \dots, k$, which is also true of the S_i.

5.2. *Regression through the origin vs. increasing ratios or star shaped with respect to the origin.* The model to be tested is $\theta_i = \beta z_i$. For this problem $r = 1$ (see Section 1) and the $\mathbf{A}$ matrix is

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{z_1} & \frac{1}{z_2} & 0 & \cdots & & 0 \\ 0 & -\frac{1}{z_2} & \frac{1}{z_3} & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & & \cdots & 0 & \frac{-1}{z_{k-1}} & \frac{1}{z_k} \\ 0 & \cdots & & & 0 & -\frac{1}{z_k} \end{pmatrix}, \tag{5.1}$$

$0 < z_i < z_{i+1}$. Thus $\mathbf{A}_1$ is now the top $(k-1) \times k$ submatrix of $\mathbf{A}$. Under the alternative, (θ_i / z_i) is increasing. The analogue of Theorem 4.1 is proved by noting that the distribution of $\mathbf{Y}^{(1)}$, of order $(k-1) \times 1$, given $\mathbf{Y}^{(2)}$ is MTP_2. This is established as in Cohen, Perlman and Sackrowitz (1990).

A remark is in order about the practicality of the formulation. Testing the null hypothesis that the natural parameters lie on any line through the origin vs. the alternative that the ratios (θ_i / z_i) are increasing is somewhat realistic, although perhaps not frequently of interest. However, testing the null hypothesis that the natural parameters lie on a line through the origin with a nonnegative slope vs. the alternative that the ratios increase in such a way that they lie above a line is contained in the null hypothesis that the parameters lie on any line through the origin, and the alternative is contained in the alternative that the ratios increase. This implies that any test which is unbiased for the larger null and larger alternative is also unbiased for the smaller null and smaller alternative.

5.3. *Linear regression with zero slope vs. convex increasing.* For this problem the $\mathbf{A}$ matrix is as in (1.3) except that the last row is deleted and a first row of the form $(-r_1, r_1, \dots, 0)$ is added. The relevant $\mathbf{Y} = (\mathbf{Y}^{(1)'}, \mathbf{Y}^{(2)})$ is obtained by finding $\mathbf{B} = \mathbf{A}^{-1}$, and so

$$\mathbf{B} = \begin{pmatrix} -\left(\sum_{i=1}^{k} \Delta_i\right) & -\left(\sum_{i=2}^{k} \Delta_i\right) & \cdots & & -\Delta_k \\ -\left(\sum_{i=2}^{k} \Delta_i\right) & -\left(\sum_{i=2}^{k} \Delta_i\right) & -\left(\sum_{i=3}^{k} \Delta_i\right) & \cdots & -\Delta_k \\ \vdots & & & & \vdots \\ -(\Delta_k + \Delta_{k-1}) & & & & \\ -\Delta_k & -\Delta_k & & & -\Delta_k \end{pmatrix}. \tag{5.2}$$

It follows from (5.2) that

$$Y_j = \sum_{i=j}^{k} \Delta_i T_i, \tag{5.3}$$

where $T_i = \sum_{l=1}^{i} X_l$; that is, T_i are the partial sums of X_l. Clearly, $Y^{(2)} = Y_k = -T_k$.

Now any test which is nondecreasing in $y^{(1)}$ for fixed $y^{(2)}$ is unbiased. To see this, first note that this problem is contained in the problem of testing homogeneity vs. the "simple order" $\theta_1 \le \cdots \le \theta_k$. Any test which is unbiased for the larger problem is certainly unbiased for the smaller problem. Size α tests which are nonincreasing in the partial sums are unbiased for the larger problem by virtue of Cohen, Perlman and Sackrowitz (1990). Finally, note that tests which are nondecreasing in $y^{(1)}$ are nonincreasing in the partial sums. To see this, write $\varphi(y_1, \dots, y_k) = \varphi^*(t_1, \dots, t_k)$. Assume φ is nondecreasing in $y^{(1)}$ and then we must show φ^* is nonincreasing in $(t_1, \dots, t_{k-1})$ for t_k fixed. First, let $t_2, \dots, t_k$ be fixed and let t_1 decrease. Note from (5.3), since all $\Delta_i > 0$, y_1 increases and no other y_i is effected. Hence φ^* is nonincreasing in t_1. Similarly treat the other t_i, $i = 2, \dots, k-1$.

5.4. *Change point problems.* Several change point problems have their null hypotheses and alternative hypotheses contained in null and alternative hypotheses, respectively, that we study. For example, one change point problem tests homogeneity H: $\theta_1 = \cdots = \theta_k$ vs. K/H, where K: $\theta_1 = \cdots = \theta_m < \theta_{m+1} = \cdots = \theta_k$ for some $m = 1, 2, \dots, k-1$. Such an alternative is contained in the alternative that natural parameters are nondecreasing and so tests which are unbiased for the larger problem are unbiased for the smaller problem.

Another change point problem model is to test that the points (z_i, θ_i) lie on a straight line vs. the alternative that instead, for some choice of $m = 2, \dots, k-1$, this is true of the (z_i, θ_i) with $i \le h$ and also for the points (z_i, θ_i), $i \ge h$, but where the two slopes can be different. See, for example, Kim and Siegmund (1989), Miao (1988) and Loader (1992). A one-sided change point problem is one where the slope of the second line is larger (smaller) than the slope of the first line. As such, the null hypothesis is equivalent to our null hypothesis and the alternative is contained in our alternative. Whereas we do not necessarily advocate generating a procedure that is designed to be good for our broader hypotheses (when the smaller hypotheses are of interest), it follows nevertheless that if the test for the smaller hypotheses does not satisfy the conditions of Theorem 3.1 it will be inadmissible. If it does satisfy the conditions of Theorem 4.1 it will be unbiased.

5.5. *Normal case.* When X_i are normally distributed with known common variance, the results of this paper apply. However because the normal distribution has special properties, more extensive results are obtainable even in the case where the common variance is unknown. The methods for the normal case are decidedly different from those used here. The results for the normal case appear in Cohen, Kemperman and Sackrowitz (1993). There, the lack of fit problem is but one example of a general case that is treated. The general case involves testing H: $\mathbf{A}_1\boldsymbol{\theta} = 0$ vs. K/H, where K: $\mathbf{A}_1\boldsymbol{\theta} \ge 0$, where the matrix $\mathbf{A}_1$ is such that $(\mathbf{A}_1\mathbf{A}_1')^{-1} = G^{-1} > 0$. For the general case, in addition to a result like Theorem 4.1, other sufficient conditions for a test to be unbiased are given. It is sometimes possible to verify one set of sufficient conditions and not another set.

5.6. *Examples.* For all the problems mentioned the conditional likelihood ratio test with constant conditional size function qualifies as an unbiased test which lies in $\mathcal{C}$. To see this, refer to (3.2). Since under H_0 the conditional distribution is parameter free, we easily recognize from (3.2) that the conditional likelihood ratio is nondecreasing in $y^{(1)}$ for fixed $y^{(2)}$. From Birnbaum (1955) it follows that the acceptance sections are convex as well.

Many other examples of unbiased tests which lie in $\mathcal{C}$ can be offered. Suppose γ is a $(k-r)$ vector whose elements are nonnegative. Let $\mathcal{K}_\alpha(\mathbf{y}^{(2)})$ be critical values such that

$$P\Big\{\gamma'\mathbf{Y}^{(1)} \ge \mathcal{K}_\alpha\big(\mathbf{y}^{(2)}\big)\Big\} = \alpha. \tag{5.4}$$

Then tests which reject when $\gamma'\mathbf{Y}^{(1)} \ge \mathcal{K}_\alpha(\mathbf{y}^{(2)})$ are unbiased and lie in $\mathcal{C}$. (In the discrete case randomization at extreme points may be required.)

Still another unbiased test in $\mathcal{C}$ is one which rejects when

$$\max(Y_1, Y_2, \dots, Y_{k-r}) \ge \mathcal{K}_\alpha\big(\mathbf{y}^{(2)}\big).$$

APPENDIX

PROOF OF LEMMA 4.6. In light of Lemma 4.5, the density of $[(Y_1, Y_m) \mid Y_{m+1}, \dots, Y_{k-2}]$ is the same as $[(Y_1, Y_m) \mid Y_{m+1}, Y_{m+2}]$. The density of this random vector is obtained from

$$\prod_{j=1}^{m+2} h_j\big(r_j y_j - (r_{j-1} + r_j) y_{j-1} + r_{j-1} y_{j-2}\big)$$

($y_j = 0$ if $j \le 0$), by integrating out the variables $y_2, y_3, \dots, y_{m-1}$. We can set aside the factor $h_1(r_1 y_1) h_{m+2}(r_{m+2} y_{m+2} - (r_{m+1} + r_{m+2}) y_{m+1} + r_{m+1} y_m)$ since this factor has no influence on the desired TP_2 property.

Now consider the function

$$\begin{aligned} f_m(y_{m-2}, y_{m-1}, y_m) = {} & h_m\big(r_m y_m - (r_{m-1} + r_m) y_{m-1} + r_{m-1} y_{m-2}\big) \\ & \times h_{m+1}\big(r_{m+1} y_{m+1} - (r_m + r_{m+1}) y_m + r_m y_{m-1}\big) \end{aligned} \tag{A.1}$$

and recursively define

$$\begin{aligned} & f_j(y_{j-2}, y_{j-1}, y_m) \\ & \quad = \int h_j\big(r_j y_j - (r_{j-1} + r_j) y_{j-1} + r_{j-1} y_{j-2}\big) f_{j+1}(y_{j-1}, y_j, y_m)\, dy_j, \end{aligned} \tag{A.2}$$

$j = m - 1, m - 2, \dots, 2$. We want to show that $f_2(y_0, y_1, y_m) = f_2(y_1, y_m)$ is a TP_2 function.

Define $\beta_{m+1} = 1 + r_{m+1}/r_m$, and further recursively

$$\beta_j = 1 + (r_j/r_{j-1})\Big(1 - (1/\beta_{j+1})\Big), \tag{A.3}$$

$j = m, m - 1, \dots, 2$. (Thus $\beta_j > 1$; it would be natural to let $\beta_{m+2} = \infty$.) In particular,

$$\begin{aligned} \beta_m &= 1 + \big[(r_m r_{m+1})/r_{m-1}(r_m + r_{m+1})\big] \\ &= \big[(r_m + r_{m-1})(r_m + r_{m+1}) - r_m^2\big]/r_{m-1}(r_m + r_{m+1}). \end{aligned}$$

Further, let

$$\rho_{m+2} = 1 \quad \text{and} \quad \rho_j = \rho_{j+1}/\beta_j, \quad j = m + 1, \dots, 2.$$

Thus,

$$\rho_j = 1/\beta_j \beta_{j+1} \cdots \beta_m \beta_{m+1}.$$

In particular, $\rho_{m+1} = 1/\beta_{m+1} = r_m/(r_m + r_{m+1})$. We will work with the quantities

$$\begin{aligned} v_j = \beta_j y_{j-1} - y_{j-2}, \qquad & w_j = y_m - \rho_j y_{j-2}, \\ j = 2, \dots, m; v_2 = \beta_2 y_1; & \, w_2 = y_m. \end{aligned} \tag{A.4}$$

Note that since $\rho_{j+1} = \rho_j\beta_j$,

$$w_{j+1} = w_j - \rho_j v_j. \tag{A.5}$$

Now in (A.1) we make a change of variable in integration by letting

$$y_j = \xi + y_{j-1}/\beta_{j+1}.$$

Then, using (A.3) and (A.4), we see that

$$\begin{aligned} &r_j y_j - (r_{j-1} + r_j)y_{j-1} + r_{j-1}y_{j-2} \\ &\quad = r_j\xi - \left[r_{j-1} + r_j - (r_j/\beta_{j+1})\right]y_{j-1} + r_{j-1\,j-2} \\ &\quad = r_j\xi - r_{j-1}[\beta_j y_{j-1} - y_{j-2}] = r_j\xi - r_{j-1}v_j. \end{aligned}$$

Hence (A.2) leads to the recursion formula

$$\begin{aligned} f_j(y_{j-2}, y_{j-1}, y_m) = \int & h_j(r_j\xi - r_{j-1}v_j) \\ & \times f_{j+1}\Big(y_{j-1}, \xi + (y_{j-1}/\beta_{j+1}), y_m\Big)\, d\xi. \end{aligned} \tag{A.6}$$

Consider the TP_2 function

$$\begin{aligned} g_m^*(v, w) = {} & h_m\Big(r_m w - \left[(r_{m-1} + r_m)/\beta_m\right]v\Big) \\ & \times h_{m+1}\Big(r_{m+1}y_{m+1} + (r_m/\beta_m)v - (r_m + r_{m+1})w\Big). \end{aligned} \tag{A.7}$$

It follows from (A.1) and (A.6) that

$$f_m(y_{m-2}, y_{m-1}, y_m) = g_m^*(v_m, w_m). \tag{A.8}$$

Namely,

$$\begin{aligned} &r_m w_m - \left[(r_{m-1} + r_m)/\beta_m\right]v_m \\ &\quad = r_m(y_m - \rho_m y_{m-2}) - \left[(r_{m-1} + r_m)/\beta_m\right]\beta_m y_{m-1} - y_{m-2} \\ &\quad = r_m y_m - (r_{m-1} + r_m)y_{m-1} + r_{m-1}y_{m-2} \end{aligned}$$

and further,

$$\begin{aligned} &(r_m/\beta_m)v_m - (r_m + r_{m+1})w_m \\ &\quad = (r_m/\beta_m)(\beta_m y_{m-1} - y_{m-2}) - (r_m + r_{m+1})(y_m - \rho_m y_m) \\ &\quad = -(r_m + r_{m+1})y_m + r_m y_{m-1}. \end{aligned}$$

Here, we used the fact that

$$-r_m\rho_m + (r_{m-1} + r_m)/\beta_m = (-r_m\rho_{m+1} + r_{m-1} + r_m)/\beta_m = r_{m-1}$$

[see (A.3)] and

$$-(r_m\beta_m) + (r_m + r_{m+1})\rho_m = \left[-r_m + (r_m + r_{m+1})\rho_{m+1}\right]/\beta_m = 0.$$

Next, apply (A.6) with $j = m - 1$. Here, from (A.8),

$$f_m\left(y_{m-2}, \xi + (y_{m-2}/\beta_m), y_m\right) = g_m^*(\beta_m \xi, w_m).$$

Replacing y_{m-1} by $\xi + (y_{m-2}/\beta_m)$ causes $v_m = \beta_m y_{m-1} - y_{m-2}$ to be replaced by $\beta_m \xi$ (and has no effect on $w_m = y_m - \rho_m y_{m-2}$). In this way, (A.5) and (A.6) yield

$$\begin{aligned} f_{m-1}(y_{m-3}, y_{m-2}, y_m) &= g_{m-1}(v_{m-1}, w_m) \\ &= g_{m-1}(v_{m-1}, w_{m-1} - \rho_{m-1} v_{m-1}) \\ &= g_{m-1}^*(v_{m-1}, w_{m-1}). \end{aligned}$$

Here, g_{m-1} and g_{m-1}^* are defined by

$$g_{m-1}(v, w) = \int h_{m-1}(r_{m-1}\xi - r_{m-2}v) g_m^*(\beta_m \xi, w)\, d\xi$$

and

$$g_{m-1}^*(v, w) = g_{m-1}(v, w - \rho_m v).$$

Lemma 4.3 implies that $g_{m-1}(v, w)$ is log concave in (v, w). Lemma 4.4 implies that $g_{m-1}(v, w)$ is TP_2. Hence Lemma 4.2 can be applied to establish that $g_{m-1}^*(v, w)$ is TP_2. This procedure can be continued to imply by induction that

$$f_j(y_{j-2}, y_{j-1}, y_m) = g_j(v_j, w_{j+1}) = g_j^*(v_j, w_j),$$

$j = m, m-1, \dots, 2$. Hence g_j and g_j^* are recursively defined by

$$g_j(v, w) = \int h_j(r_j \xi - r_{j-1} v) g_{j+1}^*(\beta_{j+1}\xi, w)\, d\xi$$

and

$$g_j^*(v, w) = g_j(v, w - \rho_{j+1} v),$$

$j = m-1, m-2, \dots, 2$. Since g_m^* is a TP_2 function, it follows that all the functions g_j and g_j^* are TP_2. We conclude that

$$f_2(y_1, y_m) = g_2^*(v_2, w_2) = g_2^*(\beta_2 y_1, y_m)$$

is TP_2. □

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