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INCREASING THE CONFIDENCE IN STUDENT'S t INTERVAL

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The usual confidence interval, based on Student's t distribution, has conditional confidence that is larger than the nominal confidence level. Although this fact is known, along with the fact that increased conditional confidence can be used to improve a confidence assertion, the confidence assertion of Student's t interval has never been critically examined. We do so here, and construct a confidence estimator that allows uniformly higher confidence in the interval and is closer (than $1 - \alpha$) to the indicator of coverage.

1. Introduction and summary. The usual confidence interval for a normal mean, when the population variance is unknown, is based on Student's t distribution. More precisely, let $x_1, \dots, x_n$ be the realized value of $X_1, \dots, X_n$, iid random variables from a normal distribution with unknown mean μ and unknown variance σ^2. The set

$$C(\bar{x}, s) = \left\{\mu : \left|\frac{\bar{x} - \mu}{s}\right| \le k\right\} \tag{1.1}$$

is a $1 - \alpha$ confidence interval for μ, where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2,$$

and k is a constant (based on the t distribution with $n - 1$ degrees of freedom) that gives $C(\bar{x}, s)$ coverage probability $1 - \alpha$.

The interval (1.1) has constant coverage probability that is equal to the confidence coefficient (defined as the infimum of the coverage probabilities). This makes the value $1 - \alpha$ particularly meaningful as a report of confidence that $C(\bar{x}, s)$ covers μ. The value $1 - \alpha$, however, is independent of the data. If we observe $\bar{X} = \bar{x}$ and $S^2 = s^2$, might this new knowledge alter our assessment of confidence in $C(\bar{x}, s)$? This question originated with Fisher (1956), who was concerned with the behavior of confidence intervals on *recognizable* subsets (subsets of the sample space). A confidence report of $1 - \alpha$ may not be appropriate if there are recognizable subsets on which confidence can be altered. The confidence coefficient may also not be appropriate if the coverage probability is not constant since the coverage probability is under-reported.

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[These points are also illustrated by Berger and Wolpert (1989) and Robert and Casella (1990).]

Many researchers have been concerned with these problems, starting with Fisher (1956, 1959). Work by Basu (1964, 1981), Buehler (1959), Robinson (1976, 1977, 1979a, b) also addresses these points. [A review of the development of conditional confidence ideas is given by Casella (1990).] A possible solution to these conditional problems was given by Kiefer (1977), who advocated using estimated (data-dependent) confidence statements.

We consider a *confidence procedure* to be a pair $\langle C(\mathbf{x}), \gamma(\mathbf{x})\rangle$, where $\gamma(\mathbf{x})$ is the reported confidence in the set $C(\mathbf{x})$. We think of $C(\mathbf{x})$ being constructed in a predetermined way and our goal is to determine a reasonable $\gamma(\mathbf{x})$. We are concerned whether, for a given $\mathbf{X} = \mathbf{x}$, the true parameter is covered by $C(\mathbf{x})$. We can treat the choice of a confidence report as an estimation problem. In other words, $\gamma(\mathbf{x})$ is treated as an estimate of the indicator function of coverage, that is, an estimate of

$$I_{C(\mathbf{x})}(\theta) = \begin{cases} 1, & \text{if } \theta \in C(\mathbf{x}), \\ 0, & \text{if } \theta \notin C(\mathbf{x}). \end{cases} \tag{1.2}$$

This approach was discussed by Berger (1985a, b, 1988) and taken in Lu and Berger (1989), George and Casella (1990) and Robert and Casella (1990). The last three papers are concerned with the usual or recentered [Hwang and Casella (1982)] confidence sets for a multivariate normal mean.

For a given $C(\mathbf{x})$ the confidence statements will be compared according to the squared error loss

$$L(\gamma, \theta, \mathbf{x}) = \left[\gamma(\mathbf{x}) - I_{C(\mathbf{x})}(\theta)\right]^2. \tag{1.3}$$

If we are concerned with the performance of $\gamma(\mathbf{x})$ from a frequentist view, we must evaluate the performance of $\gamma(\mathbf{x})$ by considering the risk function $R(\gamma, \theta) = EL(\gamma, \theta, \mathbf{X})$. A reported confidence $\gamma_1(\mathbf{x})$ will be inadmissible if there is a $\gamma_2(\mathbf{x})$ such that $R(\gamma_1, \theta) \geq R(\gamma_2, \theta)$ for all θ, with strict inequality for some θ.

Our approach is closely related to the theory of conditional inference as formalized by Robinson (1979a, b). Suppose that the set $C(\mathbf{x})$ satisfies $P_\theta(\theta \in C(\mathbf{X})) = 1 - \alpha$ for all θ, and a set M and $\varepsilon > 0$ exist for which

$$P_\theta(\theta \in C(\mathbf{X}) | \mathbf{X} \in M) \geq 1 - \alpha + \varepsilon \quad \text{for all } \theta. \tag{1.4}$$

Define $\gamma(\mathbf{x}) = 1 - \alpha + \varepsilon I_M(\mathbf{x})$. Then it is immediate that the procedure $\langle C(\mathbf{x}), \gamma(\mathbf{x})\rangle$ improves on $\langle C(\mathbf{x}), 1 - \alpha\rangle$ in the sense that $R(1 - \alpha, \theta) \geq R(\gamma(\mathbf{x}), \theta)$ for all θ, with strict inequality for some θ. This is the construction we use to obtain improved confidence in Student's t, starting from a result by Brown (1967).

Brown (1967), building on the work of Buehler and Feddersen (1963), proved that there exists $\varepsilon > 0$ and a constant c such that

$$P_{\mu, \sigma^2}\left[\mu \in C(\bar{X}, S) \middle| |\bar{X}/S| \leq c\right] \geq 1 - \alpha + \varepsilon \qquad \forall\, \mu, \sigma^2. \tag{1.5}$$

It is interesting to note that Robinson (1976) showed there do not exist conditioning sets for which the inequality in (1.5) can be reversed. The existence of a set satisfying the inequality in (1.5) immediately implies that the confidence report $1 - \alpha$ can be improved upon, using a data-dependent confidence report $\gamma(\bar{x}/s)$, where $\gamma(\bar{x}/s) \geq 1 - \alpha$.

The applicability of Brown's result is limited by the fact that ε is only shown to exist, and thus an improved confidence report cannot be constructed. A main goal of this paper is to find a computable value ε_0 (along with a constant c) that satisfies (1.5). Once that is done, an improved confidence statement for the interval (1.1) can be constructed.

Section 2 contains the key result of the paper, the lemma giving an explicit value of ε that satisfies (1.5). The proof of the lemma is extremely lengthy and occupies the remainder of Section 2. (A detailed reading of the proof is not necessary to follow the main ideas of the paper.) In Section 3 we show how to construct improved confidence statements using the lemma of Section 2. We also apply a Brewster–Zidek-type construction [Brewster and Zidek (1974)] to exhibit smoother confidence estimators. The size of the gains is also investigated numerically. Section 4 contains a concluding discussion.

2. The construction lemma. In this section we state and prove the lemma needed to construct an improved estimator of confidence for the interval (1.1), giving an explicit lower bound on the conditional confidence of $C(\bar{x}, s)$.

LEMMA. *Suppose $n > 2$. Define the constant c^* by*

$$c^* = \max\left\{k, \frac{k}{\sqrt{1 + k^2} - 1}, c_0\right\} \tag{2.1}$$

with k given in (1.1) *and c_0 satisfying*

$$c_0 = \begin{cases} k + \sqrt{k^2 + 1}, & \text{if } 3^{n-2}\dfrac{\alpha}{1 - \alpha} < 1, \\ \cot \omega_0, & \text{otherwise}, \end{cases}$$

where ω_0 is the solution to

$$P\left(F_{1,n-1} < \frac{n - 1}{\cot^2 \omega_0}\right) = (1 - \alpha)\left[1 + \left(\frac{\sin \omega_0}{\sin(3\omega_0)}\right)^{n-2}\right] \tag{2.2}$$

and $F_{1,n-1}$ is an F random variable with 1 *and* $n - 1$ *degrees of freedom. Then for all $c > c^*$,*

$$P_{\mu,\sigma^2}\left(\frac{|\bar{X} - \mu|}{S} \leq k \,\middle|\, \frac{|\bar{X}|}{S} \leq c\right) \geq P_{0,\sigma^2}\left(\frac{|\bar{X}|}{S} \leq k \,\middle|\, \frac{|\bar{X}|}{S} \leq c\right). \tag{2.3}$$

TABLE 1
Values of c^*
[*the Student's t cutoff point is given in parentheses*]

Degrees of freedom	$1 - \alpha = 0.9$	$1 - \alpha = 0.95$
2	4.359 (2.920)	6.245 (4.303)
4	2.309 (2.132)	1.953 (2.776)
6	2.870 (1.943)	2.416 (2.447)
8	3.341 (1.860)	2.809 (2.306)
10	3.756 (1.812)	3.156 (2.228)
20	5.372 (1.725)	4.510 (2.086)
30	6.606 (1.697)	5.544 (2.042)
40	7.643 (1.684)	6.415 (2.021)
50	8.556 (1.676)	7.180 (2.009)

REMARK. For $\mu = 0$, the conditional probability on the right-hand side of (2.3) does not depend on σ^2, so the bound is independent of all parameters. For illustration we present Table 1, a short table of values of c^*.

PROOF. Define the two sets

$$K = \{(\bar{x}, s): |\bar{x} - \mu|/s \leq k\}, \qquad C = \{(\bar{x}, s): |\bar{x}|/s \leq c\}. \tag{2.4}$$

Following Brown (1967), we will constantly refer to Figure 1 (which is a reproduction of his Figure 1). The area K is contained in $A_1A_0A_2$ and C is contained in B_1OB_2. Since K and C depend only on the ratios $\bar{x}/s$, μ/s and μ/σ, all the probabilities we consider are only functions of μ/σ. Thus,

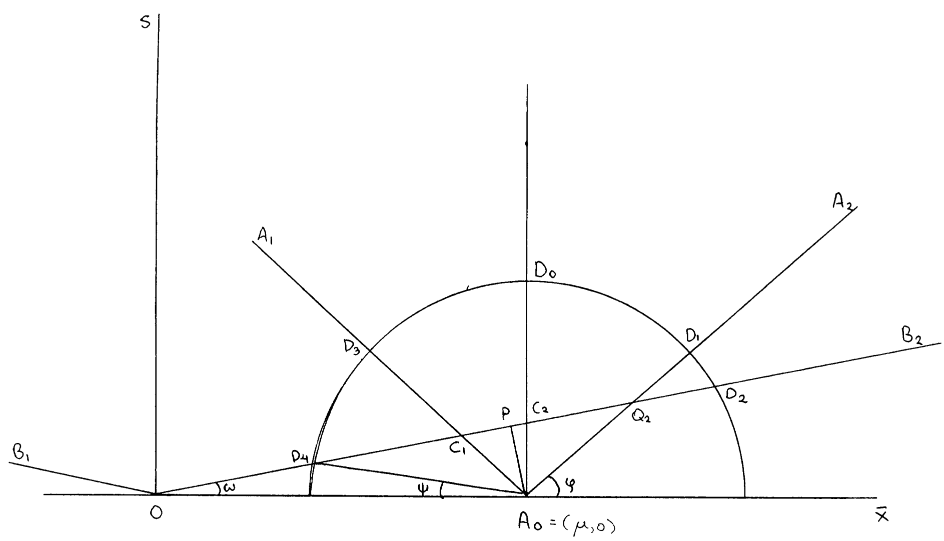

FIG. 1. *Regions used for calculation of conditional coverage probabilities.*

without loss of generality, we can assume $\sigma = 1$, and because of symmetry, we can take $\mu \geq 0$.

Let $\varphi = \cot^{-1}(k)$ and $\omega = \cot^{-1}(c)$. Since $c > k$, the lines A_0A_2 and OB_2 intersect at Q_2, while OB_1 and A_0A_1 do not intersect. Note that $c > k$ is equivalent to $\varphi > \omega$. Also, we have taken $A_0P \perp OB_2$ and $c > k/(\sqrt{1+k^2} - 1)$. The latter inequality, by the monotonicity of the cotangent and the identity $\cot(\pi/2 - 2\omega) = \tan(2\omega) = 2\tan\omega/(1 - (\tan\omega)^2)$, implies $\varphi < \pi/2 - 2\omega$, or $A_0\hat{C}_1P < A_0\hat{C}_2P$. (The "$\hat{\ }$" denotes angle.)

Consider the system of polar coordinates with A_0 as its center. Let $r^2 = (\bar{x} - \mu)^2 + s^2$ and $\theta = \arctan[s/(\bar{x} - \mu)]$. The values r and θ can be considered values of random variables, R and Θ, with probability density

$$f(r,\theta) \propto r^{n-1}e^{-nr^2/2}(\sin\theta)^{n-2}, \qquad r \geq 0, \qquad 0 \leq \theta \leq \pi.$$

It can be easily seen that R and Θ are independent. Define

$$p_1(r) = P\left\{K \cap C \cap \left\{\theta : \theta \leq \frac{\pi}{2}\right\} \middle| R = r\right\},$$

$$p_2(r) = P\left\{C \cap \left\{\theta : \theta \leq \frac{\pi}{2}\right\} \middle| R = r\right\},$$

$$p_3(r) = P\left\{K \cap C \cap \left\{\theta : \theta > \frac{\pi}{2}\right\} \middle| R = r\right\},$$

$$p_4(r) = P\left\{C \cap \left\{\theta : \theta > \frac{\pi}{2}\right\} \middle| R = r\right\},$$

$$\rho(r) = P_\mu\left\{\frac{|\bar{X} - \mu|}{S} < k \middle| \frac{|\bar{X}|}{S} < c,\ R = r\right\} = \frac{p_1(r) + p_3(r)}{p_2(r) + p_4(r)} \quad \text{for } r > A_0P.$$

We will omit the dependence on r for notational convenience, whenever no confusion arises. Obviously,

$$P_\mu\left\{\frac{|\bar{X} - \mu|}{S} < k \middle| \frac{|\bar{X}|}{S} < c\right\} = E_\mu(\rho).$$

If $\mu = 0$, $\rho(r)$ is independent of r, $1 - \alpha < \rho(r) < 1$. Define $W(\omega) = \int_\omega^{\pi/2}(\sin\theta)^{n-2}\,d\theta$, then $\rho(r)$ can be written as

$$\rho(r) = \frac{W(\varphi)}{W(\omega)} \equiv 1 - \alpha + \varepsilon_0, \tag{2.5}$$

which defines ε_0. Observe that for every μ,

$$\lim_{r\to\infty} \rho(r) = 1 - \alpha + \varepsilon_0. \tag{2.6}$$

To show that this limit is a lower bound, we must consider a number of cases. We will consider two cases depending on the range of c and for each of these cases there will be four subcases, depending on the range of r.

Case I: $c > k + \sqrt{1 + k^2}$ or, equivalently, $\varphi > 2\omega$ or $A_0Q_2 < A_0O$.

Case Ia: $r > A_0O$. Let $\psi = D_4\hat{A}_0O$ and $\xi = D_2\hat{A}_0x$ (x is the positive x-axis on the graph). Note that D_4 lies on the line OB_1. Let D'_0 and D'_4 be the reflections of D_0 and D_4 with respect to the x axis, respectively. Clearly $D'_4\hat{A}_0O = \psi$. By simple geometry we know that the sum of the radian measure of the arcs $D'_4D'_0$ and D_2D_0 equals twice the measure of $O\hat{C}_2A_0$; hence $\psi + \xi = 2\omega$.

Taking the derivative of $p_2 + p_4$ with respect to ψ, we have

$$(2.7)\qquad \begin{aligned}\frac{d(p_2 + p_4)}{d\psi} &\propto \frac{d}{d\psi}[W(\psi) + W(2\omega - \psi)]\\ &= -(\sin\psi)^{n-2} + [\sin(2\omega - \psi)]^{n-2},\end{aligned}$$

which is positive since $\xi = 2\omega - \psi > \psi$ and $n > 2$. Since ψ is increasing in r, $d(p_2 + p_4)/dr > 0$. On the other hand, for $r > A_0Q_2 > A_0C_1$, p_1 and p_3 are constant; hence ρ is decreasing in r. Using (2.6), we conclude that $\rho > 1 - \alpha + \varepsilon_0$.

Case Ib: $A_0O > r > A_0Q_2$. Similar to the previous case, p_1 and p_3 are constant, whereas now both p_2 and p_4 are increasing in r. Therefore ρ decreases to $\rho(A_0O)$, which is greater than $1 - \alpha + \varepsilon_0$.

Case Ic: $A_0Q_2 > r > A_0C_1$. Let $\psi = D_4\hat{A}_0O$ as above and observe that ψ is decreasing in r. If $\psi > \omega$, then $\rho > 1 - \alpha + \varepsilon_0$ and $r = A_0Q_2$ corresponds to $\psi = \varphi - 2\omega$. Hence we are interested in ψ such that $\varphi - 2\omega < \psi < \omega$. (If $\varphi > 3\omega$, then $\rho > 1 - \alpha + \varepsilon_0$ trivially.) Since p_4 is decreasing in ψ,

$$(2.8)\qquad \begin{aligned}\rho &= \frac{W(2\omega) + W(\varphi)}{W(2\omega + \psi) + W(\psi)}\\ &\geq \frac{W(2\omega + \psi) + W(\varphi)}{W(2\omega + \psi) + W(\varphi - 2\omega)} \geq \frac{W(3\omega) + W(\varphi)}{W(3\omega) + W(\varphi - 2\omega)}.\end{aligned}$$

Comparing the lower bound in (2.8) to (2.5), to show $\rho \geq 1 - \alpha + \varepsilon_0$, it suffices to show

$$(2.9)\qquad [W(3\omega) + W(\varphi)]W(\omega) - [W(3\omega) + W(\varphi - 2\omega)]W(\varphi) \geq 0$$

for $2\omega < \varphi < 3\omega$ and $0 < \omega < \pi/8$. This last restriction is a consequence of the fact that $2\omega < \varphi < \pi/2 - 2\omega$.

Differentiating the l.h.s. of (2.9) with respect to φ, we find that this derivative is positive if $W(\varphi - 2\omega) > W(\omega)$, which is true. Thus (2.9) holds, for $0 < \omega < \pi/8$, if the inequality holds for $\varphi = 2\omega$, the smallest value of φ. Substituting this value in (2.9) and rearranging terms gives the condition

$$(2.10)\qquad \frac{W(3\omega)}{W(3\omega) + W(0) - W(\omega)} \geq \frac{W(2\omega)}{W(\omega)}.$$

Inverting numerators and denominators in (2.10), after some algebra, we obtain the equivalent expression

$$1 + \frac{\int_{2\omega}^{3\omega}(\sin\theta)^{n-2}\,d\theta}{\int_{3\omega}^{\pi/2}(\sin\theta)^{n-2}\,d\theta} \le \frac{\int_{\omega}^{2\omega}(\sin\theta)^{n-2}\,d\theta}{\int_{0}^{\omega}(\sin\theta)^{n-2}\,d\theta}. \tag{2.11}$$

The l.h.s. of (2.11) is less than 2 since the interval of integration of the denominator is wider and lies on the right of the interval of integration of the numerator, since $\omega < \pi/8$ and $(\sin\theta)^{n-2}$ is increasing. Hence a sufficient condition for (2.11) to hold is

$$\int_0^{2\omega}(\sin\theta)^{n-2}\,d\theta - 3\int_0^{\omega}(\sin\theta)^{n-2}\,d\theta \ge 0 \tag{2.12}$$

for $\omega > 0$. Since (2.12) is true for $\omega = 0$, it suffices to show that the l.h.s. of (2.12) is nondecreasing in ω. Differentiating with respect to ω, we need to establish $2(\sin 2\omega)^{n-2} - 3(\sin\omega)^{n-2} \ge 0$ or, equivalently, $(2\cos\omega)^{n-2} \ge 3/2$. This last inequality is true for $0 < \omega < \pi/8$, $n > 2$ since $\cos\omega \ge \cos\pi/8 = 0.9239$, thus establishing (2.10).

Case Id: $A_0C_1 > r > A_0P$. Clearly $\rho = 1 > 1 - \alpha + \varepsilon_0$ for this range of r.

Case II: $c < k + \sqrt{k^2+1}$ or, equivalently, $\varphi < 2\omega$ or $A_0O < A_0Q_2$. We partition the space of possible values of r in a similar way.

Case IIa: $r > A_0Q_2$. Here ρ decreases to $1 - \alpha + \varepsilon_0$, as before.

Case IIb: $A_0Q_2 > r > A_0O$. The functions p_1 and p_2 are equal and increasing in r, whereas p_4 is decreasing and p_3 is constant. Hence

$$\frac{dp_1}{dr} = \frac{dp_2}{dr} > \frac{d(p_2+p_4)}{dr}. \tag{2.13}$$

The derivative of ρ has the same sign as

$$\frac{dp_2}{dr}(p_2+p_4) - \frac{d(p_2+p_4)}{dr}(p_1+p_3), \tag{2.14}$$

which is positive because of (2.13) and $p_2 + p_4 > p_1 + p_3$. Hence ρ is increasing in r and it is greater than $1 - \alpha + \varepsilon_0$ if and only if $\rho(A_0O) > 1 - \alpha + \varepsilon_0$.

Case IIc: $A_0O > r > A_0C_1$. Define $\psi = D_4\hat{A}_0O$. If $\psi \ge \omega$ we have $\rho > 1 - \alpha + \varepsilon_0$, so $\rho > 1 - \alpha + \varepsilon_0$ if and only if

$$W(2\omega+\varphi)\left[W(\varphi) - (1-\alpha+\varepsilon_0)W(\psi)\right] \ge 0 \tag{2.15}$$

for all ψ such that $0 < \psi < \omega$.

Since (2.15) is true for $\psi = \omega$, it is sufficient (but not necessary) that the l.h.s. be nonincreasing in ψ. Differentiating with respect to ψ, the derivative is nonpositive if and only if

$$\frac{1}{1-\alpha+\varepsilon_0} - 1 \ge \left(\frac{\sin\psi}{\sin(2\omega+\psi)}\right)^{n-2}. \tag{2.16}$$

Since the r.h.s. of (2.16) is increasing in ψ, using the definition of $1 - \alpha + \varepsilon_0$

[from (2.5)], an equivalent condition is

$$\frac{W(\omega)}{W(\varphi)} - 1 \geq \left(\frac{\sin \omega}{\sin 3\omega}\right)^{n-2} \tag{2.17}$$

for $0 < \omega < \varphi$. Now observe that the l.h.s., as a function of ω, decreases to 0, whereas the r.h.s. is increasing. Hence if

$$\frac{W(0)}{W(\varphi)} - 1 > \left(\frac{1}{3}\right)^{n-2}, \tag{2.18}$$

inequality (2.17) holds for $\omega < \omega_0$, where ω_0 is the value of ω that makes (2.17) an equality.

Note that integrals of the form $\int(\sin\theta)^{n-2}\,d\theta$ can be expressed as beta integrals, using the transformation $u = \sin^2\theta$. Then using the relationship between the beta and F distribution, equality in (2.17) is exactly condition (2.2). Moreover, condition (2.18) reduces to the requirement $3^{n-2}\alpha/(1-\alpha) > 1$. Therefore, for $c > c_0 = \cot\omega_0$, $\rho > 1 - \alpha + \varepsilon_0$.

Case IId: $A_0C_1 > r > A_0P$. For the range of r it is clear that $\rho = 1$; hence $\rho > 1 - \alpha + \varepsilon_0$.

Since $\rho(r) \geq 1 - \alpha + \varepsilon_0$ for every μ, with equality for $\mu = 0$, the lemma is proved by taking expectations over r. □

The assumptions of the lemma are clearly sufficient but not necessary. Numerical evidence shows that the result holds for c smaller than the bounds given in the statement of the lemma. However, if $c < k$ the r.h.s. of (2.3) equals 1 and, as Brown (1967) points out, if $c < 1/k$ the conditional probability tends to 0 as μ tends to ∞.

3. Increased confidence for the t interval. The lemma of Section 2 gives an easily computable bound on the conditional coverage probability. We now use this bound to construct a post-data confidence estimate $\gamma(\bar{x}/s)$ which improves on $1 - \alpha$.

THEOREM 3.1. *Let $c > c^*$ and k be fixed constants, where c^* and k satisfy the assumptions of the lemma of Section 2. Define $\gamma_c(\bar{x}/s)$ as follows:*

$$\gamma_c(\bar{x}/s) = \begin{cases} \dfrac{1-\alpha}{P\left(|t_{n-1}| < c\sqrt{n-1}\right)}, & \text{if } \dfrac{|\bar{x}|}{s} < c, \\[2ex] 1 - \alpha, & \text{if } \dfrac{|\bar{x}|}{s} \geq c, \end{cases} \tag{3.1}$$

where t_{n-1} *denotes a t random variable with* $n-1$ *degrees of freedom. Then*

$$E_{\mu,\sigma^2}\Big[\big(\gamma_c(\bar{X}/S) - I_{C(\bar{X},S)}(\mu)\big)^2\Big] < E_{\mu,\sigma^2}\Big[\big(1-\alpha - I_{C(\bar{X},S)}(\mu)\big)^2\Big] \tag{3.2}$$

for all μ *and* σ^2, *where* $C(\bar{x}, s)$ *is the* $1-\alpha$ *Student's t interval of* (1.1).

PROOF. Since $\gamma_c(\bar{x}/s) = 1-\alpha$ for $|\bar{x}|/s \geq c$, it suffices to show

$$\begin{aligned} &E_{\mu,\sigma^2}\left[\big(\gamma_c(\bar{X}/S) - I_{C(\bar{X},S)}(\mu)\big)^2 \,\Big|\, \frac{|\bar{X}|}{S} < c\right] \\ &\quad < E_{\mu,\sigma^2}\left[\big(1-\alpha - I_{C(\bar{X},S)}(\mu)\big)^2 \,\Big|\, \frac{|\bar{X}|}{S} < c\right] \end{aligned} \tag{3.3}$$

for all μ. The last inequality can be seen to be true by observing that, for $|\bar{X}|/S < c$,

$$\gamma_c(\bar{X}/S) = P_{0,\sigma^2}\left(\frac{|\bar{X}|}{S} < k \,\Big|\, \frac{|\bar{X}|}{S} < c\right) > 1-\alpha \tag{3.4}$$

and using inequality (2.3) [as in Robinson (1979a)]. □

Since a confidence report based on a partition of possible values of $|\bar{x}|/s$ improves upon $1-\alpha$, it seems plausible that, by taking a finer partition, we could construct a report that is even better. The technique of further partitioning has been introduced by Brewster and Zidek (1974) and has been applied in similar problems in Goutis and Casella (1991), Shorrock (1990) and Kubokawa (1991). We have the following theorem.

THEOREM 3.2. *Let* $\mathbf{c}_2 = (c_1, c_2)$, $c_1 > c_2 > c^*$, *and k be constants, where* c^* *and k satisfy the assumptions of the lemma of Section* 2. *Define* $\gamma_{\mathbf{c}_2}(\bar{x}/s)$ *as follows*:

$$\gamma_{\mathbf{c}_2}(\bar{x}/s) = \begin{cases} \dfrac{1-\alpha}{P\big(|t_{n-1}| < c_2\sqrt{n-1}\big)}, & \textit{if } \dfrac{|\bar{x}|}{s} \leq c_2, \\[2ex] \dfrac{1-\alpha}{P\big(|t_{n-1}| < c_1\sqrt{n-1}\big)}, & \textit{if } c_2 < \dfrac{|\bar{x}|}{s} < c_1, \\[2ex] 1-\alpha, & \textit{if } c_1 \leq \dfrac{|\bar{x}|}{s}. \end{cases}$$

Then

$$E_{\mu,\sigma^2}\Big[\big(\gamma_{\mathbf{c}_2}(\bar{X}/S) - I_{C(\bar{X},S)}(\mu)\big)^2\Big] < E_{\mu,\sigma^2}\Big[\big(\gamma_{c_1}(\bar{X}/S) - I_{C(\bar{X},S)}(\mu)\big)^2\Big]$$

for all μ *and* σ^2, *where* γ_{c_1} *is given in* (3.1).

PROOF. The result follows immediately after observing that

$$P_{0,\sigma^2}\left\{\frac{|\bar{X}|}{S} < k \,\middle|\, \frac{|\bar{X}|}{S} < c\right\}$$

is decreasing in c. □

An immediate consequence of the above theorem is that $\gamma_{\mathbf{c}_2}(\bar{x}/s)$ improves upon $1 - \alpha$. We can easily generalize and take more cutoff points. We create an array $\mathbf{c}_m = (c_{m,1}, c_{m,2}, \dots, c_{m,m})$ such that $c^* < c_{m,1} < c_{m,2} < \cdots < c_{m,m} < +\infty$, $\lim_{m\to\infty} c_{m,m} = +\infty$, $\lim_{m\to\infty} c_{m,1} = c^*$, and $\lim_{m\to\infty} \max_i (c_{m,i} - c_{m,i-1}) = 0$. As $m \to +\infty$ the reported confidence will tend to $\gamma(\bar{x}/s)$ defined by

$$\gamma(\bar{x}/s) = \begin{cases} \dfrac{1-\alpha}{P\big(|t_{n-1}| < (|\bar{x}|/s)\sqrt{n-1}\big)}, & \text{if } |\bar{x}|/s > c^*, \\[2ex] \dfrac{1-\alpha}{P\big(|t_{n-1}| < c^*\sqrt{n-1}\big)}, & \text{if } |\bar{x}|/s < c^*. \end{cases} \tag{3.5}$$

Note that $P(|t_{n-1}| < (|\bar{x}|/s)\sqrt{n-1}) = 1 - p(\mathbf{x})$, where $p(\mathbf{x})$ is the p value associated with the hypothesis $\mu = 0$. By the dominated convergence theorem, $\gamma(\bar{x}/s)$ dominates $1 - \alpha$ in terms of risk.

For an observed value of $\bar{x}/s$, the confidence report attached to the t interval will be $\gamma(\bar{x}/s)$ of (3.5). To give some idea of the shape of this function, Figure 2 shows values of $\gamma(\bar{x}/s)$ for $n = 5$ and $n = 9$. It can be seen that a confidence report of almost 94% confidence is possible when using a nominal

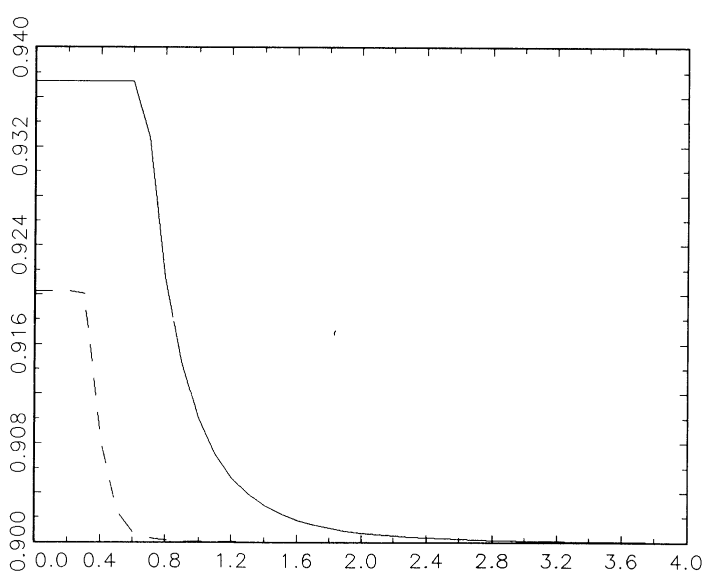

FIG. 2. *Values of $\gamma(\bar{x}/s)$ of (3.5) for $n = 5$ (solid line), $n = 9$ (dotted line) and $1 - \alpha = 0.90$.*

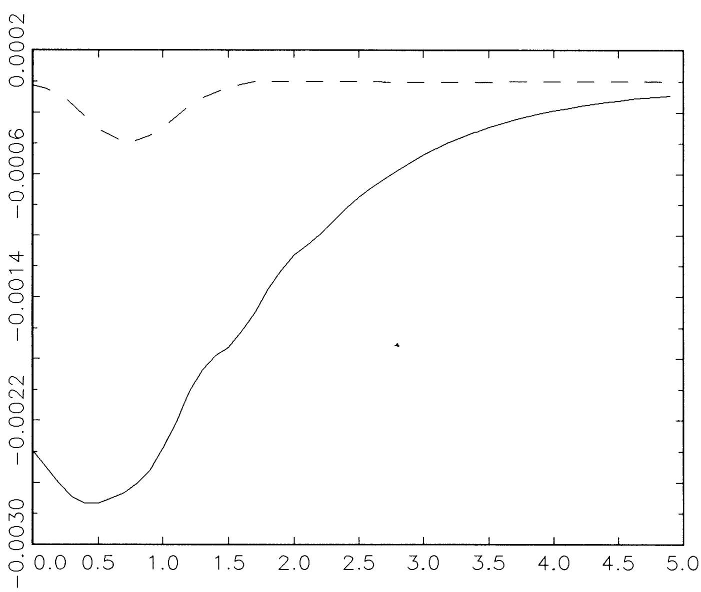

FIG. 3. *Difference in risks* $E_{\mu,\sigma^2}[\{\gamma_c(\bar{X}/S) - I_{C(\bar{X},S)}(\mu)\}^2] - E_{\mu,\sigma^2}[\{1-\alpha - I_{C(\bar{X},S)}(\mu)\}^2]$ *for* $n = 5$ *(solid line),* $n = 9$ *(dotted line) and* $1-\alpha = 0.90$.

90% interval. Figure 3 shows the difference in the risks plotted against μ/σ. Although the improvement is not large, $\gamma(\bar{x}/s)$ does dominate $1-\alpha$. Note that the size of the risk improvement is really only of secondary concern, as the experimenter may be able to assert a sizeable increase in confidence. Indeed, the experimenter does not really care about the magnitude of the risk improvement as long as the confidence is maximally increased. As the degrees of freedom become large, for the usual confidence coefficients, the constant k is rather small. Since

$$c^* \geq \frac{k}{\sqrt{1+k^2} - 1} > \frac{1}{k},$$

the bound c^* is large, so $\gamma(\bar{x}/s)$ is quite close to $1-\alpha$ and we cannot expect a large improvement.

4. Discussion. For the t interval, the existence of a subset for which the conditional confidence level can be bounded above the nominal $1-\alpha$ for all parameter values shows that an increased confidence assertion is appropriate. Such sets can be used in the construction of a confidence estimator that allows a confidence assertion uniformly higher than $1-\alpha$, and is closer to the true indicator of coverage.

Many of the arguments used here took advantage of properties of the normal and t distribution, so the extent of generalizability is not clear. It is probably the case, however, that our results can be extended to the F

distribution in the analysis of variance, allowing for increased confidence in Scheffé's simultaneous intervals [as Olshen (1973) was able to extend some of Brown's (1967) arguments to the analysis of variance]. Also, the improvements realized are a function of prior information about the location of the mean. The inequality (1.5) and the results in this paper will hold for any conditioning set of the form $\{(\bar{x}, s): |\bar{x} - \mu_0|/s \le c\}$, where μ_0 is a specified constant. The value of μ_0 should represent the experimenter's best prior estimate of the mean μ, as the greatest improvement will occur when μ_0 is close to μ.

For the case of known variance, the usual interval does not allow the existence of sets satisfying (1.4) and, moreover, the usual confidence report cannot be improved upon. Indeed, Hwang and Brown (1991) have shown that $1 - \alpha$ is an admissible confidence report in four or fewer dimensions. For five or more dimensions, however, the confidence in the usual set can be improved [Robert and Casella (1990)]. Thus a condition such as (1.4) is sufficient, but not necessary, for the existence of an improved confidence statement.

Perhaps the key idea is Fisher's, that an interval's confidence should be evaluated on recognizable subsets. Then if the confidence, conditional on such a subset, is different from the nominal value that information should be used to improve the confidence assertion. Fisher concentrated on sets defined by ancillary statistics, while here the conditioning set is the acceptance region of a hypothesis test, and is not based on an ancillary statistic. The moral may be to examine recognizable sets of interest, dictated by the problem at hand, and see if conditional confidence assertions are affected. If so, the nominal confidence coefficient, $1 - \alpha$, may not be an adequate confidence report.

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