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ASYMPTOTIC BEHAVIOR OF BAYES TESTS AND BAYES RISK

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In this paper the asymptotic behaviors of the Bayes test and Bayes risk are studied in both the one-sided and two-sided testing problems, where the independent observations are taken from one member of a one-parameter exponential family. Precise asymptotic expressions are found which show the Bayes procedure to be relatively insensitive to the prior distribution as the sample size increases to infinity.

1. Introduction and summary. The main shortcoming of the Bayes principle is its dependence on the prior distribution. In order to apply this principle it is necessary to assume that the parameter is a random variable with known probability distribution (prior distribution). Usually experimental data is insufficient to accurately establish the nature of this distribution. Fortunately, however, there are many situations in which the Bayes procedure becomes relatively insensitive to the prior distribution as the sample size increases.

In 1952 Chernoff [1] studied the asymptotic behavior of Bayes risk in the hypothesis testing problem involving a simple hypothesis versus a simple alternative. He proved that the natural logarithm of the Bayes risk behaves like $-nI_0$ as n tends to infinity, where I_0 is a positive constant which does not depend on the prior distribution. Efron and Truax [3] in 1968 improved upon Chernoff's result by showing that the Bayes risk behaves like $Kn^{-\frac{1}{2}}\exp(-nI_0)$ as n tends to infinity, where only the positive constant K depends on the prior distribution.

In this paper the asymptotic behaviors of the Bayes test and Bayes risk are studied in both the one-sided and two-sided testing problems, where the independent observations are taken from a one-parameter exponential family. Again it is found that the large sample behaviors do not depend very heavily on the prior distribution. Also, since two distinct probability distributions which are absolutely continuous with respect to each other can be embedded in a one-parameter exponential family (see [3], page 1414), some of the results in the following sections include the above described results as special cases.

We now introduce some notation. We represent a one-parameter exponential family by

$$P_\theta(A) = \int_A \exp(\theta x - \psi(\theta))\, d\mu(x)$$

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for Borel subsets A of the real line $\mathscr{R}$, or more simply by

$$dP_\theta(x) = \exp(\theta x - \psi(\theta))\, d\mu(x) \tag{1.1}$$

for $x \in \mathscr{R}$. Here μ is a nondegenerate probability distribution on $\mathscr{R}$, and θ takes values in

$$\Omega = \{\theta \in \mathscr{R}: \textstyle\int \exp(\theta x)\, d\mu(x) < \infty\}\,.$$

It is well known (see [9]) that Ω, called the natural parameter space of the exponential family (1.1), is an interval (finite or infinite). The function ψ is defined on Ω by

$$\psi(\theta) = \log [\textstyle\int \exp(\theta x)\, d\mu(x)]\,.$$

With the aid of Hölder's inequality together with necessary and sufficient conditions for strict inequality of Hölder's inequality (see [6]), it is easy to show that ψ is a strictly convex function. It is also known (see [9]) that ψ has finite derivatives of all orders in the interior of Ω. If X is a random variable having probability distribution P_θ, then we have

$$\psi'(\theta) = E_\theta(x) \qquad \text{and} \qquad \psi''(\theta) = \operatorname{Var}_\theta(X)\,.$$

Section 2 is devoted to the one-sided testing problem

$$H_0: \theta \leqq \theta_0 \quad \text{versus} \quad H_1: \theta > \theta_0\,,$$

where θ_0 is some fixed real number in the interior of the natural parameter space Ω. It is well known that when a sample of size n is observed and the standard zero-one loss functions are used, the Bayes test corresponding to the prior probability distribution ν has rejection region

$$\{\bar{X}_n > c_n\}\,,$$

where $\bar{X}_n$ is the sample mean and c_n is that unique real number such that

$$\int_{(-\infty,\theta_0]} \exp(n[c_n\theta - \psi(\theta)])\, d\nu(\theta) = \int_{(\theta_0,\infty)} \exp(n[c_n\theta - \psi(\theta)])\, d\nu(\theta)\,. \tag{1.2}$$

Here the prior distribution ν is defined on the Borel subsets of $\mathscr{R}$ such that $\nu(\Omega^c) = 0$, $\nu(-\infty, \theta_0] > 0$ and $\nu(\theta_0, \infty) > 0$. The Bayes risk of ν is given by

$$r_n(\nu) = \int_{(-\infty,\theta_0]} [1 - G_n(c_n; \theta)]\, d\nu(\theta) + \int_{(\theta_0,\infty)} G_n(c_n; \theta)\, d\nu(\theta)\,, \tag{1.3}$$

where $G_n(\cdot\,; \theta)$ is the distribution function of $\bar{X}_n$ under θ.

If the prior distribution ν assigns zero measure to an interval which contains θ_0, it is found that the sequence $\{c_n\}$ converges to a finite limit at least as fast as $n^{-1}\log(n)$. Also, the Bayes risk of ν converges to zero at the rate $Kn^{-\alpha}\exp(-nI_0)$, where K, α and I_0 are positive constants. On the other hand, if ν admits the possibility of parameter points arbitrarily near θ_0 on each side, then the sequence $\{c_n\}$ converges to a finite limit at least as fast as $n^{-\frac{1}{2}}$. In this case $Mn^{-\frac{1}{2}}$ is the rate at which the Bayes risk of ν converges to zero, where M is a positive constant. The constant terms K, α, I_0 and M depend on the prior distribution ν only through its behavior near its closest support to θ_0 on each side. These are the main results of Section 2.

In Section 3 a similar study is conducted of the two-sided problem

$$H_0: \theta = \theta_0 \quad \text{versus} \quad H_1: \theta \neq \theta_0 .$$

For this testing problem the prior distribution ν is defined such that $\nu(\Omega^c) = 0$, $\nu(-\infty, \theta_0) > 0$, $\nu(\{\theta_0\}) > 0$ and $\nu(\theta_0, \infty) > 0$. Let $n(\nu)$ be the smallest positive integer such that

$$\nu(\{\theta_0\}) > \int_{(-\infty,\theta_0)\cup(\theta_0,\infty)} \exp(n(\nu)\{[\psi'(\theta_0)\theta - \psi(\theta)] - [\psi'(\theta_0)\theta_0 - \psi(\theta_0)]\})\, d\nu(\theta) .$$

Such $n(\nu)$ exists because the function

$$[\psi'(\theta_0)\theta - \psi(\theta)] - [\psi'(\theta_0)\theta_0 - \psi(\theta_0)]$$

is a strictly concave function of θ which has absolute maximum equal to zero at $\theta = \theta_0$. It is known that for every $n \geqq n(\nu)$ the Bayes test corresponding to ν has acceptance region

$$\{a_n < \bar{X}_n \leqq b_n\} ,$$

where a_n and b_n are the unique real numbers such that

$$\text{(1.4)} \quad \begin{aligned} &\text{(i)} \quad a_n < \psi'(\theta_0) < b_n , \\ &\text{(ii)} \quad \nu(\{\theta_0\}) = \int_{(\infty,\theta_0)\cup(\theta_0,\infty)} \exp(n\{[a_n\theta - \psi(\theta)] - [a_n\theta_0 - \psi(\theta_0)]\})\, d\nu(\theta) , \\ &\text{(iii)} \quad \nu(\{\theta_0\}) = \int_{(\infty,\theta_0)\cup(\theta_0,\infty)} \exp(n\{[b_n\theta - \psi(\theta)] - [b_n\theta_0 - \psi(\theta_0)]\})\, d\nu(\theta) . \end{aligned}$$

Therefore, for each $n \geqq n(\nu)$ the Bayes risk of ν is given by

$$\text{(1.5)} \quad \begin{aligned} r_n(\nu) = {} & \int_{(-\infty,\theta_0)\cup(\theta_0,\infty)} [G_n(b_n; \theta) - G_n(a_n; \theta)]\, d\nu(\theta) \\ & + \nu(\{\theta_0\})[G_n(a_n; \theta_0) + (1 - G_n(b_n; \theta_0))] , \end{aligned}$$

where $G_n(\cdot\,; \theta)$ is the distribution function of $\bar{X}_n$ under θ.

The main results of Section 3 are in most cases quite similar to their one-sided counterparts found in Section 2. If there exists an interval containing θ_0 such that ν assigns zero measure to this interval with θ_0 deleted, then the two sequences $\{a_n\}$ and $\{b_n\}$ each converge to a finite limit at the rate $n^{-1}\log(n)$. In this situation the rate at which the Bayes risk of ν converges to zero is $Bn^{-\beta}\exp(-nI_1)$, where B, β and I_1 are positive constants. The final case to be considered is the situation where ν assigns positive measure to arbitrarily small intervals on each side of θ_0. In this case the two sequences $\{a_n\}$ and $\{b_n\}$ each converge to the same limit at the rate $(\log(n)/n)^{\frac{1}{2}}$; while the Bayes risk of ν converges to zero at the rate $A(\log(n)/n)^{\frac{1}{2}}$, where A is a positive constant. As in the one-sided problem, the behavior of ν near its closest support to θ_0 completely determines the constant factors B, β, I_1 and A.

All proofs are deferred to Section 4. Since the techniques used are fairly standard, many details of analysis have been curtailed, hopefully without detriment to readability. Also, proofs of several theorems pertaining to the two-sided case have been omitted because the lines of reasoning involved are very similar to those of their one-sided counterparts. For detailed proofs of all results presented in this paper see [7].

2. The one-sided problem. Let $X_1, X_2, \cdots, X_n$ be a random sample from one member of the one-parameter exponential family (1.1), and consider the one-sided testing problem $H_0: \theta \leqq \theta_0$ versus $H_1: \theta > \theta_0$. As was pointed out in the introduction, the Bayes test with respect to the prior distribution ν is to reject if $\bar{X}_n > c_n$ where c_n is the unique solution to equation (1.2).

The support of the prior distribution ν will influence the asymptotic behavior of the Bayes risk, and we want to treat two cases separately. The first case is when some nondegenerate interval containing θ_0 receives ν-measure zero. The second case we treat is when the prior distribution has a positive density on some nondegenerate open interval containing θ_0. Accordingly, we will say ν satisfies *condition* A_1 if there are real numbers θ_1 and θ_2 in the support of ν and interior to Ω such that $\theta_1 \leqq \theta_0 < \theta_2$ and $\nu(\theta_1, \theta_2) = 0$. If such numbers exist they must be unique. (Note: If condition A_1 is satisfied we are essentially testing $\theta \leqq \theta_1$ versus $\theta \geqq \theta_2$.) We say ν satisfies *condition* A_2 in case there exists $\varepsilon > 0$ such that ν has a density ρ with respect to Lebesgue measure on $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ where $\rho(\theta_0+)$ and $\rho(\theta_0-)$ both exist and are positive.

First, let us examine the behavior of the critical constant c_n under each of these two conditions.

THEOREM 2.1. *If ν satisfies condition A_1, then*

$$\lim_n c_n = \psi'(\theta_3) \tag{2.1}$$

where θ_3 is that unique real number in (θ_1, θ_2) such that

$$\psi'(\theta_3) = [\psi(\theta_2) - \psi(\theta_1)]/(\theta_2 - \theta_1) . \tag{2.2}$$

By imposing some very mild additional conditions on ν one can say something about the rate of convergence of c_n to its limit. These conditions essentially say that ν cannot assign too little measure to neighborhoods of θ_1 and θ_2.

THEOREM 2.2. *If ν satisfies condition A_1, and if there exist $\varepsilon > 0$ and $0 < \alpha < 1$ such that*

$$\begin{aligned} &\text{(i)} \quad \nu(\theta_1 - y, \theta_1] \geqq \exp(-y^{-\alpha}) \qquad \text{for all} \quad y \in (0, \varepsilon) \\ &\text{(ii)} \quad \nu[\theta_2, \theta_2 + y) \geqq \exp(-y^{-\alpha}) \qquad \text{for all} \quad y \in (0, \varepsilon) \end{aligned} \tag{2.3}$$

then

$$\lim_n n^{\frac{1}{2}}(c_n - \psi'(\theta_3)) = 0 . \tag{2.4}$$

It should be remarked that (2.4) need not hold if (2.3) fails. On the other hand, (2.3) fails only for somewhat pathological prior distributions.

THEOREM 2.3. *If ν satisfies condition A_2, then*

$$\lim_n n^{\frac{1}{2}}(c_n - \psi'(\theta_0)) = d \tag{2.5}$$

where d is that unique real number such that

$$\Phi(d/[\psi''(\theta_0)]^{\frac{1}{2}})/\Phi(-d/[\psi''(\theta_0)]^{\frac{1}{2}}) = \rho(\theta_0-)/\rho(\theta_0+) . \tag{2.6}$$

Here Φ denotes the cdf for the standard normal distribution.

It is worth noting that in both cases, the limiting behavior of c_n depends on ν only through its properties on the support of ν nearest to θ_0. That is, under condition A_1, the limiting behavior of c_n depends on ν only through the behavior of ν near θ_1 and θ_2, and for condition A_2 only through the behavior near θ_0.

The next stage in our study is to examine the behavior of the Bayes risk and observe how it depends on the choice of prior. Recall that the Kullback-Liebler information number is defined on $\Omega \times \Omega$ by

$$I(\theta', \theta) = E_{\theta'}(\log [dP_{\theta'}/dP_\theta(X)]) ,$$

and note that if $\theta' \in \operatorname{int} \Omega$ and $\theta \in \Omega$ then

$$I(\theta', \theta) = \psi'(\theta')(\theta' - \theta) - (\psi(\theta') - \psi(\theta)) .$$

THEOREM 2.4. *If ν satisfies condition A_1, μ is nonlattice and* $\lim_n n^{\frac{1}{2}}(c_n - \psi'(\theta_3)) = 0$, *then the Bayes risk of ν has the following property*:

$$\lim_n n^{-1} \log r_n(\nu) = -I_0 \tag{2.7}$$

where

$$I_0 = I(\theta_3, \theta_1) = I(\theta_3, \theta_2) . \tag{2.8}$$

Thus, the risk decreases to zero at an exponential rate. Equation (2.7) is a generalization of Chernoff's result which was briefly discussed in Section 1.

By imposing some further restrictions on the behavior of ν at θ_1 and θ_2, we will obtain a much more precise result for the Bayes risk. The next theorem points out quite clearly how the behavior of ν near these two points determines the asymptotic behavior of the risk.

THEOREM 2.5. *Suppose*

(i) *ν satisfies condition A_1,*
(ii) *$\nu(\theta_1 - x, \theta_1] \sim \beta_1 x^{\alpha_1}$ as $x \to 0+$ where $\beta_1 > 0$, $\alpha_1 \geqq 0$,*
(iii) *$\nu[\theta_2, \theta_2 + x) \sim \beta_2 x^{\alpha_2}$ as $x \to 0+$ where $\beta_2 > 0$, $\alpha_2 \geqq 0$.*

Then

$$c_n - \psi'(\theta_3) = n^{-1}(\theta_2 - \theta_1)^{-1}[(\alpha_2 - \alpha_1) \log n + K_1 + o_n(1)] . \tag{2.9}$$

Also, if μ is nonlattice the Bayes risk of ν is

$$r_n(\nu) = K n^{-\{\frac{1}{2} + [\alpha_1(\theta_2 - \theta_3) + \alpha_2(\theta_3 - \theta_1)]/(\theta_2 - \theta_1)\}} \exp(-nI_0)(1 + o_n(1)) \tag{2.10}$$

where I_0 is as in (2.8). *K_1 and K are constants depending only on θ_1, θ_2, α_1, α_2, β_1, β_2 and their exact values can be found in* (4.6) *and* (4.7).

If μ is nonlattice the critical region $\{\bar{X}_n > c_n^*\}$ is asymptotically Bayes (i.e. the ratio of its Bayes risk to that of the Bayes test approaches one), where

$$c_n^* = \psi'(\theta_3) + n^{-1}(\theta_2 - \theta_1)^{-1}[(\alpha_2 - \alpha_1) \log n + K_1] .$$

This fact, which follows easily from the proof of Theorem 2.5, is significant because c_n^* is generally much easier to compute than c_n.

Equation (2.10) generalizes the result of Efron and Truax which was discussed in Section 1. In that case ν assigned positive mass to θ_1 and θ_2 so that $\alpha_1 = \alpha_2 = 0$. Note also that if ν and ν^* are two prior distributions both satisfying the hypotheses of Theorem 2.5 such that $\theta_1 = \theta_1^*$, $\theta_2 = \theta_2^*$, $\alpha_1 = \alpha_1^*$, $\alpha_2 = \alpha_2^*$, then

$$\lim_n r_n(\nu)/r_n(\nu^*) = K/K^* ,$$

and $K = K^*$ if $\beta_1 = \beta_1^*$, $\beta_2 = \beta_2^*$. Thus, $r_n(\nu) \sim r_n(\nu^*)$ provided ν and ν^* "behave similarly in the neighborhood of θ_1 and θ_2."

Some examples of prior distributions satisfying the conditions of Theorem 2.5 would be worthwhile at this point. If ν is a prior distribution satisfying condition A_1 and if θ_1 and θ_2 are atoms of ν, then (ii) and (iii) hold with $\alpha_1 = \alpha_2 = 0$ and β_1, β_2 are the masses assigned by ν to θ_1, θ_2. If A_1 is satisfied and ν has a density ρ with respect to Lebesgue measure such that $\rho(\theta_1-)$, $\rho(\theta_2+)$ both exist and are positive, then $\nu(\theta_1 - x, \theta_1] \sim \rho(\theta_1-)x$ and $\nu[\theta_2, \theta_2 + x) \sim \rho(\theta_2+)x$ as $x \to 0+$.

When condition A_2 is fulfilled the behavior of the Bayes risk is radically different and is given in the following:

THEOREM 2.6. *If the prior distribution ν satisfies condition A_2 and μ is nonlattice, then the Bayes risk of ν is*

$$r_n(\nu) = Mn^{-\frac{1}{2}}(1 + o_n(1)) \tag{2.11}$$

where $M = ([\rho(\theta_0-) + \rho(\theta_0+)]/[2\pi\psi''(\theta_0)]^{\frac{1}{2}}) \exp(-d^2/2\psi''(\theta_0))$ *and d is given by* (2.6).

Under the hypotheses of Theorem 2.6 the critical value $c_n' = \psi'(\theta_0) + dn^{-\frac{1}{2}}$, obtained from expression (2.5), gives a test which is asymptotically Bayes. Note that if $\rho(\theta_0-) = \rho(\theta_0+)$, the most common case, this test has the extremely simple form of rejecting whenever the sample mean is larger than the population mean under θ_0.

3. The two-sided problem. The problem of testing the hypothesis $H_0: \theta = \theta_0$ against the two-sided alternative $H_1: \theta \neq \theta_0$ can be treated in a manner similar to the one-sided problem, but, as we shall see, the behavior of the Bayes risk is somewhat different. Again we will distinguish two cases. We will say the prior distribution ν satisfies *condition A_3* if $\nu(\{\theta_0\}) > 0$ and there is an open interval I containing θ_0 such that ν assigns probability zero to $I - \{\theta_0\}$. If this is the case there exist unique θ_1 and θ_2 in the support of ν such that $I = (\theta_1, \theta_2)$ satisfies the condition (i.e. $(\theta_1 \theta_2)$ is the largest such interval). Observe that in this situation we are testing $\theta = \theta_0$ versus $\theta \leqq \theta_1$ or $\geqq \theta_2$. The second case we consider is when $\nu(\{\theta_0\}) > 0$ and there exists $\varepsilon_1 > 0$ such that ν has a probability density ρ with respect to Lebesgue measure on $(\theta_0 - \varepsilon_1, \theta_0) \cup (\theta_0, \theta_0 + \varepsilon_1)$ where $\rho(\theta_0-)$, $\rho(\theta_0+)$ both exist and are positive. Under these circumstances we will say that ν satisfies *condition A_4*.

As was mentioned in the first section, every Bayes test has an acceptance region of the form $\{a_n < \bar{X}_n \leqq b_n\}$ where a_n and b_n are solutions to (1.4). In order to describe the asymptotic behavior of the critical values a_n, b_n let us define θ_3 and θ_4 as the unique real numbers such that

$$\text{(i)} \quad \psi'(\theta_3) = [\psi(\theta_0) - \psi(\theta_1)]/(\theta_0 - \theta_1),$$
$$\text{(ii)} \quad \psi'(\theta_4) = [\psi(\theta_2) - \psi(\theta_0)]/(\theta_2 - \theta_0).$$

THEOREM 3.1. *If ν satisfies condition A_3, then*

$$\lim_n a_n = \psi'(\theta_3) \qquad \textit{and} \qquad \lim_n b_n = \psi'(\theta_4).$$

THEOREM 3.2. *If ν satisfies condition A_4, then*

$$a_n - \psi'(\theta_0) = -[\psi''(\theta_0)n^{-1}\log n]^{\frac{1}{2}}(1 + o_n(1)),$$
$$b_n - \psi'(\theta_0) = [\psi''(\theta_0)n^{-1}\log n]^{\frac{1}{2}}(1 + o_n(1)).$$

THEOREM 3.3. *If ν satisfies condition A_3, μ is nonlattice and if*

$$\lim_n n^{\frac{1}{2}}(a_n - \psi'(\theta_3)) = 0 = \lim_n n^{\frac{1}{2}}(b_n - \psi'(\theta_4)),$$

then the Bayes risk of ν has the property

$$\lim_n n^{-1}\log r_n(\nu) = -I_1$$

where

$$I_1 = \min\{I(\theta_3, \theta_1), I(\theta_4, \theta_2)\} = \min\{I(\theta_3, \theta_0), I(\theta_4, \theta_0)\}.$$

Because of the hypotheses of this theorem we will need conditions on ν which will insure that

$$\lim_n n^{\frac{1}{2}}(a_n - \psi'(\theta_3)) = 0 = \lim_n n^{\frac{1}{2}}(b_n - \psi'(\theta_4)).$$

As in the previous section, the requirement is essentially that ν cannot assign too little measure to neighborhoods of θ_1 and θ_2.

THEOREM 3.4. *Suppose ν satisfies condition A_3. If there exist $\varepsilon > 0$ and $0 < \alpha < 1$ such that*

$$\text{(i)} \quad \nu(\theta_1 - y, \theta_1] \geqq \exp(-y^{-\alpha}) \qquad \textit{for all} \quad y \in (0, \varepsilon),$$
$$\text{(ii)} \quad \nu[\theta_2; \theta_2 + y) \geqq \exp(-y^{-\alpha}) \qquad \textit{for all} \quad y \in (0, \varepsilon),$$

then

$$\lim_n n^{\frac{1}{2}}(a_n - \psi'(\theta_3)) = 0 = \lim_n n^{\frac{1}{2}}(b_n - \psi'(\theta_4)).$$

Under certain rather mild restrictions on the prior distribution ν, which are satisfied for all common choices of ν satisfying condition A_3, we obtain the next theorem which gives precise asymptotic expressions for the critical constants a_n, b_n of the Bayes test and for the Bayes risk. Fortunately, the dependence of these expressions on the prior distribution is not too severe.

THEOREM 3.5. *Suppose ν satisfies condition A_3 and has the following two properties:*

$$\text{(i)} \quad \nu(\theta_1 - x, \theta_1] \sim \beta_1 x^{\alpha_1} \quad \textit{as} \quad x \to 0+ \quad \textit{where} \quad \beta_1 > 0,\ \alpha_1 \geqq 0,$$
$$\text{(ii)} \quad \nu[\theta_2, \theta_2 + x) \sim \beta_2 x^{\alpha_2} \quad \textit{as} \quad x \to 0+ \quad \textit{where} \quad \beta_2 > 0,\ \alpha_2 \geqq 0.$$

Then

$$a_n - \psi'(\theta_3) = n^{-1}(\theta_1 - \theta_0)^{-1}[\alpha_1 \log n + K_2 + o_n(1)]\,, \qquad \textit{and}$$
$$b_n - \psi'(\theta_4) = n^{-1}(\theta_2 - \theta_0)^{-1}[\alpha_2 \log n + K_3 + o_n(1)]\,.$$

Also, if μ *is nonlattice the Bayes risk of* ν *is*

$$r_n(\nu) = \{An^{-[\frac{1}{2}+\alpha_1(\theta_0-\theta_3)/(\theta_0-\theta_1)]}\exp[-nI(\theta_3, \theta_0)] + Bn^{-[\frac{1}{2}+\alpha_2(\theta_4-\theta_0)/(\theta_2-\theta_0)]}\exp[-nI(\theta_4, \theta_0)]\}(1 + o_n(1))\,,$$

where K_2, K_3, A *and* B *are constants depending only on* θ_0, θ_1, θ_2, α_1, α_2, β_1, β_2 *and* $\nu(\{\theta_0\})$. *Their exact values can be found in* (4.10), (4.11), (4.12), (4.13).

The final theorem of this section describes the behavior of the Bayes risk when the prior distribution satisfies condition A_4.

THEOREM 3.6. *If* ν *satisfies condition* A_4 *and* μ *is nonlattice, then*

(3.1) $$r_n(\nu) = C(n^{-1} \log n)^{\frac{1}{2}}(1 + o_n(1))\,,$$

where

$$C = ([\rho(\theta_0-) + \rho(\theta_0+)]/[\psi''(\theta_0)]^{\frac{1}{2}})\,.$$

Note the rate in which the Bayes risk converges to zero in this theorem is not only much slower than when the prior distribution satisfies condition A_3, but is also slightly slower than the rate found in Theorem 2.6 which is the one-sided analogue of this theorem. It is also interesting to observe that neither the rate of convergence $(n^{-1} \log n)^{\frac{1}{2}}$ nor the constant C depends on the amount of mass that ν assigns to θ_0. Further, the large sample insensitivity of the Bayes procedure to the prior distribution is again apparent in this theorem.

Finally, let us observe that, similarly as in the one-sided problem, we are able to find asymptotically Bayes tests which are easy to construct (see Theorems 3.2 and 3.5).

4. Proofs of theorems stated in Sections 2 and 3. This section is devoted to the proofs of main results.

PROOF OF THEOREM 2.1. One can easily show the sequence $\{c_n\}$ is bounded, where c_n is the solution of (1.2). If b is any limit point of the sequence, then there is a subsequence $\{c_{n_k}\}$ such that $c_{n_k} \to b$. Taking n_kth roots of both sides of (1.2) and letting $k \to \infty$, it is routine to show that the l.h.s. converges to $\exp(b\theta_1 - \psi(\theta_1))$ and the r.h.s. converges to $\exp(b\theta_2 - \psi(\theta_2))$. Thus, any limit point $b = [\psi(\theta_2) - \psi(\theta_1)]/(\theta_2 - \theta_1) = \psi'(\theta_3)$, which proves Theorem 2.1.

PROOF OF THEOREM 2.2. Define $g_n(\theta) = c_n\theta - \psi(\theta)$. From (1.2)

$$\exp(ng_n(\theta_1)) \int_{(-\infty,\theta_1]} \exp(n[g_n(\theta) - g_n(\theta_1)])\, d\nu(\theta) = \exp(ng_n(\theta_2)) \int_{[\theta_2,\infty)} \exp(n[g_n(\theta) - g_n(\theta_2)])\, d\nu(\theta)\,,$$

so that

(4.1) $$c_n - \psi'(\theta_3) = [n(\theta_2 - \theta_1)]^{-1}\{\log \int_{(-\infty,\theta_1]} \exp(n[g_n(\theta) - g_n(\theta_1)])\, d\nu(\theta) - \log \int_{[\theta_2\infty)} \exp(n[g_n(\theta) - g_n(\theta_2)])\, d\nu(\theta)\}\,.$$

Thus, it will be sufficient to prove

$$\lim_n n^{-\frac{1}{2}} \log \int_{(-\infty,\theta_1]} \exp(n[g_n(\theta) - g_n(\theta_1)])\, d\nu(\theta) = 0 , \tag{4.2}$$

and

$$\lim_n n^{-\frac{1}{2}} \log \int_{[\theta_2,\infty)} \exp(n[g_n(\theta) - g_n(\theta_2)])\, d\nu(\theta) = 0 . \tag{4.3}$$

First we will prove the following:

LEMMA. *Define $H(t)$ to be the* df *of a nonnegative random variable such that for some positive constants M, c, ε and some $0 < \alpha < 1$ we have*

$$H(t) \geqq M \exp(-ct^{-\alpha}) \qquad \text{for all} \quad t \in (0, \varepsilon) .$$

Then

$$n^{-\frac{1}{2}} \log \int_{(0,\infty)} e^{-nt}\, dH(t) \to 0 .$$

PROOF. Applying integration by parts

$$\int_{(0,\infty)} e^{-nt}\, dH(t) = n \int_0^\infty H(t)e^{-nt}\, dt \geqq n \int_0^\varepsilon H(t)e^{-nt}\, dt$$
$$\geqq nM \int_0^\varepsilon \exp[-(ct^{-\alpha} + nt)]\, dt .$$

The function $ct^{-\alpha} + nt$ has a minimum when $t = (c\alpha/n)^{1/(\alpha+1)}$ and is decreasing on $(0, [c\alpha/n]^{1/(\alpha+1)})$. Thus, for all sufficiently large n

$$\int_0^\varepsilon \exp[-(ct^{-\alpha} + nt)]\, dt \geqq \int_{\frac{1}{2}(c\alpha/n)^{1/(\alpha+1)}}^{(c\alpha/n)^{1/(\alpha+1)}} \exp[-(ct^{-\alpha} + nt)]\, dt$$
$$\geqq \tfrac{1}{2}(c\alpha/n)^{1/(\alpha+1)} \exp[-\beta n^{\alpha/(\alpha+1)}] ,$$

where $\beta = (2^\alpha + \alpha/2)c^{1/(\alpha+1)}\alpha^{-\alpha/(\alpha+1)}$. Therefore,

$$0 \geqq \liminf n^{-\frac{1}{2}} \log \int_{(0,\infty)} e^{-nt}\, dH(t)$$
$$\geqq \liminf n^{-\frac{1}{2}}\{\log [(nM/2)(c\alpha/n)^{1/(\alpha+1)}] - \beta n^{\alpha/(\alpha+1)}\} = 0 ,$$

since $0 < \alpha < 1$.

Now we return to the proof of Theorem 2.2. For all sufficiently large n the strictly concave function g_n is increasing on $(-\infty, \theta_1] \cap \Omega$ and decreasing on $[\theta_2, \infty) \cap \Omega$, so we can write

$$\int_{(-\infty,\theta_1]} \exp(n[g_n(\theta) - g_n(\theta_1)])\, d\nu(\theta) = \int_{(0,\infty)} e^{-nt}\, dG_n(t) ,$$

where $G_n(t) = \nu\{\theta : \theta \leqq \theta_1 \text{ and } g_n(\theta_1) - g_n(\theta) < t\}$. There exist positive real numbers b and δ such that

$$g_n(\theta_1) - g_n(\theta) = c_n(\theta_1 - \theta) - (\psi(\theta_1) - \psi(\theta)) < -b(\theta - \theta_1)$$

for all $\theta \in [\theta_1 - \delta, \theta_1)$, provided n is sufficiently large. Then

$$G(t) = \nu(\theta_1 - t/b, \theta_1] \leqq G_n(t) \qquad \text{for all} \quad 0 \leqq t \leqq b\delta .$$

Therefore,

$$\int_{(0,b\delta)} e^{-nt}\, dG_n(t) \geqq \int_{(0,b\delta)} e^{-nt}\, dG(t) \qquad \text{for all sufficiently large} \quad n.$$

Equation (4.2) follows immediately from the above lemma by letting $H(t) = G(t)/\nu(-\infty, \theta_1]$. The proof of (4.3) is similar.

PROOF OF THEOREM 2.3. For convenience of notation let us set $\alpha_n = c_n - \psi'(\theta_0)$ and rewrite equation (1.2), which is the defining relationship for c_n, as

$$(4.4)\quad \int_{(-\infty,\theta_0]} \exp(n[\alpha_n(\theta-\theta_0) - I(\theta_0,\theta)])\, d\nu(\theta) = \int_{(\theta_0,\infty)} \exp(n[\alpha_n(\theta-\theta_0) - I(\theta_0,\theta)])\, d\nu(\theta)\,.$$

It is easily seen that $\lim_n \alpha_n = 0$, for otherwise (4.4) would fail for infinitely many n.

For a fixed $\delta > 0$ we will first obtain upper and lower bounds for $\int_{(\theta_0-\delta,\theta_0]} \exp(n[\alpha_n(\theta-\theta_0) - I(\theta_0,\theta)])\, d\nu(\theta)$. Define

$$H_1(\delta) = \inf\{[\psi''(\theta)]^{\frac{1}{2}} : \theta_0 - \delta \leqq \theta \leqq \theta_0\}$$

and

$$H_2(\delta) = \sup\{[\psi''(\theta)]^{\frac{1}{2}} : \theta_0 - \delta \leqq \theta \leqq \theta_0\}\,.$$

By Taylor's theorem $I(\theta_0,\theta) = \frac{1}{2}\psi''(\zeta(\theta))(\theta-\theta_0)^2$ for some $\zeta(\theta)$ between $\theta_0 - \delta$ and θ_0 so that

$$\tfrac{1}{2}H_1^2(\delta)(\theta-\theta_0)^2 \leqq I(\theta_0,\theta) \leqq \tfrac{1}{2}H_2^2(\delta)(\theta-\theta_0)^2 \qquad \text{if}\quad \theta_0 - \delta \leqq \theta \leqq \theta_0\,.$$

If δ is sufficiently small ν has density ρ in $(\theta_0 - \delta, \theta_0]$ and we have the upper bound

$$\int_{(\theta_0-\delta,\theta_0]} \exp(n[\alpha_n(\theta-\theta_0) - I(\theta_0,\theta)])\rho(\theta)\, d\theta \leqq \int_{(\theta_0-\delta,\theta_0]} \exp(n[\alpha_n(\theta-\theta_0) - \tfrac{1}{2}H_1^2(\theta-\theta_0)^2])\rho(\theta)\, d\theta\,,$$

where the r.h.s. is easily evaluated as

$$S_{n1} = \rho(\theta_1)\exp(n\alpha_n^2/2H_1^2)(2\pi/nH_1^2)^{\frac{1}{2}}[\Phi(-n^{\frac{1}{2}}\alpha_n/H_1) - \Phi(-n^{\frac{1}{2}}H_1\delta - n^{\frac{1}{2}}\alpha_n/H_1)]\,,$$

θ_1 lies between $\theta_0 - \delta$ and θ_0 and Φ is the df of the standard normal distribution. Similarly, a lower bound for the integral is S_{n2} which is the same as S_{n1} except that H_1 is replaced by H_2 and θ_1 is replaced by θ_2, a number between $\theta_0 - \delta$ and θ_0.

Also, for δ sufficiently small

$$T_{n2} \leqq \int_{(\theta_0,\theta_0+\delta)} \exp(n[\alpha_n(\theta-\theta_0) - I(\theta_0,\theta)])\, d\nu(\theta) \leqq T_{n1}\,,$$

where

$$T_{ni} = \rho(\bar\theta_i)\exp(n\alpha_n^2/2H_i^2)(2\pi/nH_i^2)^{\frac{1}{2}}[\Phi(n^{\frac{1}{2}}H_i\delta - n^{\frac{1}{2}}\alpha_n/H_i) - \Phi(-n^{\frac{1}{2}}\alpha_n/H_i)]\,,$$

$\bar\theta_i$ lies between θ_0 and $\theta_0 + \delta$, $i = 1, 2$.

Now since $\lim_n \alpha_n = 0$ and $\alpha_n(\theta-\theta_0) - I(\theta_0,\theta)$ has, for all sufficiently large n, its maximum value on $(-\infty, \theta_0 - \delta]$ at $\theta_0 - \delta$ it follows that

$$\int_{(-\infty,\theta_0-\delta]} \exp(n[\alpha_n(\theta-\theta_0) - I(\theta_0,\theta)])\, d\nu(\theta) \leqq \exp[-(n/2)I(\theta_0,\theta_0-\delta)]\,.$$

From this,

$$\int_{(-\infty,\theta_0]} \exp(n[\alpha_n(\theta-\theta_0) - I(\theta_0,\theta)]) \sim \int_{(\theta_0-\delta,\theta_0]} \exp(n[\alpha_n(\theta-\theta_0) - I(\theta_0,\theta)])\, d\nu(\theta)\,,$$

and an analogous argument gives the asymptotic equivalence of the integral over (θ_0, ∞) and $(\theta_0, \theta_0 + \delta)$.

Next, we prove that $\{n^{\frac{1}{2}}\alpha_n\}$ is a bounded sequence. Suppose it tends to $-\infty$ through some subsequence. Then examine T_{n1}/S_{n2} along this subsequence. Using the well-known relation

$$1 - \Phi(x) \sim \exp(-\tfrac{1}{2}x^2)/[x(2\pi)^{\frac{1}{2}}] \qquad \text{as } x \to \infty ,$$

we find,

$$T_{n1} \sim \rho(\bar{\theta}_1)/n|\alpha_n|$$

and

$$S_{n2} \sim \rho(\underline{\theta}_2)\exp(n\alpha_n^2/2H_2^2)(2\pi/nH_2^2)^{\frac{1}{2}}$$

so that

$$T_{n1}/S_{n2} \sim [\rho(\bar{\theta}_1)/\rho(\underline{\theta}_2)]\exp(-n\alpha_n^2/2H_2^2)(H_2^2/2\pi n\alpha_n^2)^{\frac{1}{2}}$$

which has limit zero contradicting $1 \leqq \liminf T_{n1}/S_{n2}$.

In the same way, if $n^{\frac{1}{2}}\alpha_n \to +\infty$ through some subsequence we can contradict $\limsup T_{n2}/S_{n1} \leqq 1$.

Finally, let d be any limit point of $n^{\frac{1}{2}}\alpha_n$. Letting n tend to infinity through a subsequence such that $n^{\frac{1}{2}}\alpha_n \to d$ we have

$$T_{n1}/S_{n2} \to (\rho(\bar{\theta}_1)/\rho(\underline{\theta}_2))(H_2/H_1)\exp[\tfrac{1}{2}d^2(H_1^{-1} - H_2^{-1})]\Phi(d/H_1)/\Phi(-d/H_2)$$

and

$$T_{n2}/S_{n1} \to (\rho(\bar{\theta}_2)/\rho(\underline{\theta}_1))(H_1/H_2)\exp[\tfrac{1}{2}d^2(H_2^{-1} - H_1^{-1})]\Phi(d/H_2)/\Phi(-d/H_1) .$$

Since δ was arbitrary and $H_i^2(\delta) \to \psi''(\theta_0)$ as $\delta \to 0$, it follows that

$$\Phi(d/[\psi''(\theta_0)]^{\frac{1}{2}})/\Phi(-d/[\psi''(\theta_0)]^{\frac{1}{2}}) = \rho(\theta_0-)/\rho(\theta_0+) ,$$

and there is only one such limit point, proving Theorem 2.3.

PROOF OF THEOREM 2.4. Because of (1.3) we need asymptotic expressions for $1 - G_n(c_n; \theta)$ and $G_n(c_n; \theta)$. For $\theta \in (-\infty, \theta_1] \cap \Omega$ one obtains after some calculations

$$\begin{aligned} &1 - G_n(c_n; \theta) \\ &\quad = \exp(n[g_n(\theta) - g_n(\theta_3)]) \textstyle\int_0^1 \{F_n(k_n - (\log t)/[\sigma_3 n^{\frac{1}{2}}(\theta_3 - \theta)]) - F_n(k_n)\}\, dt , \end{aligned}$$

where $g_n(\theta) = c_n\theta - \psi(\theta)$, $\sigma_3 = [\psi''(\theta_3)]^{\frac{1}{2}}$, $k_n = n^{\frac{1}{2}}(c_n - \psi'(\theta_3))/\sigma_3$ and F_n is the df of the normalized sum $n^{\frac{1}{2}}(\bar{X}_n - \psi'(\theta_3))/\sigma_3$ under θ_3. Next we apply the following theorem due to Esséen.

THEOREM (Esséen). *If the independent random variables $Z_1, Z_2, \cdots, Z_n$ are identically distributed and have finite third moments, then*

$$H_n(x) - \Phi(x) = (2\pi n)^{-\frac{1}{2}}\exp(-x^2/2)Q_1(x) + o(n^{-\frac{1}{2}})$$

uniformly in x. Here H_n is the df *of the normalized sum, Φ is the standard normal distribution function and Q_1 is a polynomial of degree* 2 (see [5], page 210 for exact form of Q_1).

Using this theorem we obtain for $\theta \in (-\infty, \theta_1] \cap \Omega$

$$1 - G_n(c_n; \theta) = \frac{\exp(n[g_n(\theta) - g_n(\theta_3)])}{(2\pi n)^{\frac{1}{2}}(\theta_3 - \theta)\sigma_3} (1 + [\theta_3 - \theta]o_n(1)),$$

where $o_n(1)$ goes to zero uniformly for $\theta \in (-\infty, \theta_1] \cap \Omega$.

Similarly, for $\theta \in [\theta_2, \infty) \cap \Omega$

$$G_n(c_n; \theta) = \frac{\exp(n[g_n(\theta) - g_n(\theta_3)])}{(2\pi n)^{\frac{1}{2}}(\theta - \theta_3)\sigma_3} (1 + [\theta - \theta_3]o_n(1)),$$

where $o_n(1)$ converges to zero uniformly for $\theta \in [\theta_2, \infty) \cap \Omega$.

A routine calculation will show the Bayes risk of ν is

$$r_n(\nu) = \frac{1 + o_n(1)}{\sigma_3(2\pi n)^{\frac{1}{2}}} \int_{(-\infty,\theta_1]\cup[\theta_2,\infty)} \frac{\exp(n[g_n(\theta) - g_n(\theta_3)])}{|\theta_3 - \theta|} d\nu(\theta)$$

$$(4.5) \qquad = \frac{(1 + o_n(1))}{\sigma_3(2\pi n)^{\frac{1}{2}}} \exp(n[g_n(\theta_1) - g_n(\theta_3)]) \times \int_{(-\infty,\theta_1]\cup[\theta_2,\infty)} \frac{\exp(n[g_n(\theta) - g_n(\theta_1)])}{|\theta - \theta_3|} d\nu(\theta).$$

Since the nth root of the integral converges to 1 and

$$g_n(\theta_1) - g_n(\theta_3) \to -I(\theta_3, \theta_1),$$

formula (2.7) is valid.

Proof of Theorem 2.5. To prove (2.9) we need asymptotic expressions for the integrals in (4.1). For all sufficiently large n we can write

$$\int_{(-\infty,\theta_1]} \exp(n[g_n(\theta) - g_n(\theta_1)]) d\nu(\theta) = n \int_0^\infty \nu\{\theta : \theta \leqq \theta_1 \text{ and } y > g_n(\theta_1) - g_n(\theta)\} \exp(-ny)\, dy$$

and

$$\int_{[\theta_2,\infty)} \exp(n[g_n(\theta) - g_n(\theta_2)]) d\nu(\theta) = n \int_0^\infty \nu\{\theta : \theta \geqq \theta_2 \text{ and } y > g_n(\theta_2) - g_n(\theta)\} \exp(-ny)\, dy.$$

Thus, the following well-known lemma will be useful.

Lemma. *Suppose f is a nonnegative bounded measurable function defined on* $(0, \infty)$ *such that $f(y) \sim \beta y^\alpha$ as $y \to 0+$, where $\beta > 0$ and $\alpha \geqq 0$. Then*

$$n \int_0^\infty f(y) \exp(-ny)\, dy \sim \beta n^{-\alpha}\Gamma(\alpha + 1) \qquad \textit{as} \quad n \to \infty.$$

After some calculations using this lemma, one obtains

$$\log \int_{(-\infty,\theta_1]} \exp(n[g_n(\theta) - g_n(\theta_1)]) d\nu(\theta) = \log\left\{\frac{\beta_1\Gamma(\alpha_1 + 1)}{[\psi'(\theta_3) - \psi'(\theta_1)]^{\alpha_1}}\right\} - \alpha_1 \log n + o_n(1),$$

and

$$\log \int_{[\theta_2,\infty)} \exp(n[g_n(\theta) - g_n(\theta_1)])\, d\nu(\theta)$$
$$= \log\left\{\frac{\beta_2\Gamma(\alpha_2+1)}{[\psi'(\theta_2) - \psi'(\theta_3)]^{\alpha_2}}\right\} - \alpha_2 \log n + o_n(1)\,.$$

Hence, (2.9) is established with

$$K_1 = \log \frac{\beta_1\Gamma(\alpha_1+1)[\psi'(\theta_2) - \psi'(\theta_3)]^{\alpha_2}}{\beta_2\Gamma(\alpha_2+1)[\psi'(\theta_3) - \psi'(\theta_1)]^{\alpha_1}}\,. \tag{4.6}$$

To establish (2.10) observe that from (4.5) we have

$$r_n(\nu) = (1 + o_n(1))(\sigma_3^2 2\pi n)^{-\frac{1}{2}}$$
$$\times \left\{\frac{\exp(n[g_n(\theta_1) - g_n(\theta_3)])}{\theta_3 - \theta_1} \int_{(-\infty,\theta_1]} \frac{\exp(n[g_n(\theta) - g_n(\theta_1)])}{(\theta_3 - \theta)/(\theta_3 - \theta_1)}\, d\nu(\theta)\right.$$
$$\left. + \frac{\exp(n[g_n(\theta_2) - g_n(\theta_3)])}{\theta_2 - \theta_3} \int_{[\theta_2,\infty)} \frac{\exp(n[g_n(\theta) - g_n(\theta_2)])}{(\theta - \theta_3)/(\theta_2 - \theta_3)}\, d\nu(\theta)\right\}.$$

By proceeding similarly as above it is easy to show

$$\int_{(-\infty,\theta_1]} \frac{\exp(n[g_n(\theta) - g_n(\theta_1)])}{(\theta_3 - \theta)/(\theta_3 - \theta_1)}\, d\nu(\theta)$$
$$= \beta_1\Gamma(\alpha_1+1)n^{-\alpha_1}[\psi'(\theta_3) - \psi'(\theta_1)]^{-\alpha_1}(1 + o_n(1))\,,$$

and

$$\int_{[\theta_2,\infty)} \frac{\exp(n[g_n(\theta) - g_n(\theta_2)])}{(\theta - \theta_3)/(\theta_2 - \theta_3)}\, d\nu(\theta)$$
$$= \beta_2\Gamma(\alpha_2+1)n^{-\alpha_2}[\psi'(\theta_2) - \psi'(\theta_3)]^{-\alpha_2}(1 + o_n(1))\,.$$

Since

$$\exp(n[g_n(\theta_1) - g_n(\theta_3)]) = \exp\{-nI_0 + n(c_n - \psi'(\theta_3))(\theta_1 - \theta_3)\}$$

and

$$\exp(n[g_n(\theta_2) - g_n(\theta_3)]) = \exp\{-nI_0 + n(c_n - \psi'(\theta_3))(\theta_2 - \theta_3)\}\,,$$

it is routine to combine the above formulas obtaining equation (2.10) with

$$\begin{aligned} K = {} & \beta_1\Gamma(\alpha_1+1)[2\pi\psi''(\theta_3)]^{-\frac{1}{2}}(\theta_3 - \theta_1)^{-1}[\psi'(\theta_3) - \psi'(\theta_1)]^{-\alpha_1} \\ & \times \left[\frac{\beta_2\Gamma(\alpha_2+1)[\psi'(\theta_3) - \psi'(\theta_1)]^{\alpha_1}}{\beta_1\Gamma(\alpha_1+1)[\psi'(\theta_2) - \psi'(\theta_3)]^{\alpha_2}}\right]^{(\theta_3-\theta_1)/(\theta_2-\theta_1)} \\ & + \beta_2\Gamma(\alpha_2+1)[2\pi\psi''(\theta_3)]^{-\frac{1}{2}}(\theta_2 - \theta_3)^{-1}[\psi'(\theta_2) - \psi'(\theta_3)]^{-\alpha_2} \\ & \times \left[\frac{\beta_1\Gamma(\alpha_2+1)[\psi'(\theta_2) - \psi'(\theta_3)]^{\alpha_2}}{\beta_2\Gamma(\alpha_2+1)[\psi'(\theta_3) - \psi'(\theta_1)]^{\alpha_1}}\right]^{(\theta_2-\theta_3)/(\theta_2-\theta_1)}. \end{aligned} \tag{4.7}$$

PROOF OF THEOREM 2.6. Similarly as in the proof of Theorem 2.4 write

$$1 - G_n(c_n;\theta)$$
$$= \exp(n[g_n(\theta) - g_n(\theta_0)]) \int_0^1 \{F_n(k_n - (\log t)/[\sigma_0 n^{\frac{1}{2}}(\theta_0 - \theta)]) - F_n(k_n)\}\, dt$$

for $\theta \in (-\infty, \theta_0) \cap \Omega$, and

$$G_n(c_n;\theta) = \exp(n[g_n(\theta) - g_n(\theta_0)]) \int_0^1 \{F_n(k_n) - F_n(k_n + (\log t)/[\sigma_0 n^{\frac{1}{2}}(\theta - \theta_0)])\}\, dt$$

for $\theta \in (\theta_0, \infty) \cap \Omega$, where $\sigma_0 = [\psi''(\theta_0)]^{\frac{1}{2}}$, $k_n = n^{\frac{1}{2}}(c_n - \psi'(\theta_0))/\sigma_0$ and F_n is the df of the normalized sum $n^{\frac{1}{2}}(\bar{X}_n - \psi'(\theta_0))/\sigma_0$ under θ_0. Since the integrals in the above expressions are bounded above by 1, it is easy to see that

$$n^{\frac{1}{2}} \int_{(-\infty,\theta_0-\delta]} [1 - G_n(c_n; \theta)]\, d\nu(\theta) \to 0 \qquad \text{as } n \to \infty$$

and

$$n^{\frac{1}{2}} \int_{[\theta_0+\delta,\infty)} G_n(c_n; \theta)\, d\nu(\theta) \to 0 \qquad \text{as } n \to \infty$$

for every $\delta > 0$. Thus, as expected, the asymptotic behavior is determined by integration over a neighborhood of θ_0.

An application of Esséen's theorem yields

$$\begin{aligned}&\int_0^1 \{F_n(k_n - (\log t)/[\sigma_0 n^{\frac{1}{2}}(\theta_0 - \theta)]) - F_n(k_n)\}\, dt \\ &\qquad = \int_0^1 \{\Phi(k_n - (\log t)/[\sigma_0 n^{\frac{1}{2}}(\theta_0 - \theta)]) - \Phi(k_n)\}\, dt + n^{-\frac{1}{2}}K_n(\theta)\end{aligned}$$

for $\theta \in (-\infty, \theta_0) \cap \Omega$, where

$$\begin{aligned}K_n(\theta) = \int_0^1 \{&(2\pi)^{-\frac{1}{2}}Q_1(k_n - (\log t)/[\sigma_0 n^{\frac{1}{2}}(\theta_0 - \theta)]) \\ &\times \exp(-\tfrac{1}{2}\{k_n - (\log t)/[\sigma_0 n^{\frac{1}{2}}(\theta_0 - \theta)]\}^2) \\ &- (2\pi)^{-\frac{1}{2}}Q_1(k_n)\exp(-\tfrac{1}{2}k_n^2)\}\, dt + o_n(1)\,.\end{aligned}$$

The term $o_n(1)$ converges to zero uniformly in θ. Since $k_n \to d/\sigma_0$ where d is given by (2.7), routine analysis establishes for $\theta \in (\theta_0 - \delta, \theta_0)$

$$\begin{aligned}&\int_0^1 \{F_n(k_n - (\log t)/[\sigma_0 n^{\frac{1}{2}}(\theta_0 - \theta)]) - F_n(k_n)\}\, dt \\ &\qquad = (1 + o_n(1)) \int_0^1 \{\Phi(k_n - (\log t)/[\sigma_0 n^{\frac{1}{2}}(\theta_0 - \theta)]) - \Phi(k_n)\}\, dt\,,\end{aligned}$$

where $o_n(1)$ converges to zero uniformly on $(\theta_0 - \delta, \theta_0)$.

Similarly, for $\theta \in (\theta_0, \theta_0 + \delta)$

$$\begin{aligned}&\int_0^1 \{F_n(k_n) - F_n(k_n + (\log t)/[\sigma_0(\theta - \theta_0)n^{\frac{1}{2}}])\}\, dt \\ &\qquad = (1 + o_n(1)) \int_0^1 \{\Phi(k_n) - \Phi(k_n + (\log t)/[\sigma_0(\theta - \theta_0)n^{\frac{1}{2}}])\}\, dt\,,\end{aligned}$$

where $o_n(1)$ converges to zero uniformly on $(\theta_0, \theta_0 + \delta)$.

Now we are ready to prove

$$\begin{aligned}\text{(4.8)}\qquad &\lim_n n^{\frac{1}{2}} \int_{(-\infty,\theta_0]} (1 - G_n(c_n; \theta)]\, d\nu(\theta) \\ &\qquad = \frac{\rho(\theta_0-)}{\sigma_0(2\pi)^{\frac{1}{2}}} \{\exp(-d^2/2\sigma_0^2) - (2\pi)^{\frac{1}{2}}(d/\sigma_0)\Phi(-d/\sigma_0)\}\end{aligned}$$

and

$$\begin{aligned}\text{(4.9)}\qquad &\lim_n n^{\frac{1}{2}} \int_{(\theta_0,\infty)} G_n(c_n; \theta)\, d\nu(\theta) \\ &\qquad = \frac{\rho(\theta_0+)}{\sigma_0(2\pi)^{\frac{1}{2}}} \{\exp(-d^2/2\sigma_0^2) + (2\pi)^{\frac{1}{2}}(d/\sigma_0)\Phi(d/\sigma_0)\}\,.\end{aligned}$$

Since $I(\theta_0, \theta) = \frac{1}{2}\psi''(\zeta(\theta))(\theta - \theta_0)^2$ for some $\zeta(\theta)$ between θ and θ_0, it follows that for given $0 < a < 1$ we have

$$\rho(\theta) < \rho(\theta_0-)/a \qquad \text{and} \qquad I(\theta_0, \theta) > \tfrac{1}{2}a\sigma_0^2(\theta_0 - \theta)^2$$

for all θ sufficiently near θ_0 on the left. Hence,

$$\begin{aligned}
&\limsup_n n^{\frac{1}{2}} \textstyle\int_{(-\infty,\theta_0]} [1 - G_n(c_n; \theta)]\, d\nu(\theta) \\
&\quad = \limsup_n n^{\frac{1}{2}} \textstyle\int_{\theta_0-\delta}^{\theta_0} [1 - G_n(c_n; \theta)]\rho(\theta)\, d\theta \\
&\quad = \limsup_n n^{\frac{1}{2}}(1 + o_n(1)) \textstyle\int_{\theta_0-\delta}^{\theta_0} \exp(n[g_n(\theta) - g_n(\theta_0)]) \\
&\qquad \times \textstyle\int_0^1 \{\Phi(k_n - [\log t]/[\sigma_0 n^{\frac{1}{2}}(\theta_0 - \theta)]) - \Phi(k_n)\}\, dt\, \rho(\theta)\, d\theta \\
&\quad = \limsup_n n\sigma_0(1 + o_n(1)) \textstyle\int_{\theta_0-\delta}^{\theta_0} (\theta_0 - \theta) \exp[-nI(\theta_0, \theta)]\rho(\theta) \\
&\qquad \times \textstyle\int_{k_n}^{\infty} \{\Phi(y) - \Phi(k_n)\} \exp[-n^{\frac{1}{2}}\sigma_0(\theta_0 - \theta)y]\, dy\, d\theta \\
&\quad = \limsup_n n^{\frac{1}{2}}(1 + o_n(1)) \textstyle\int_{\theta_0-\delta}^{\theta_0} \exp[-nI(\theta_0, \theta)]\rho(\theta) \\
&\qquad \times \textstyle\int_{k_n}^{\infty} (2\pi)^{-\frac{1}{2}} \exp[-\frac{1}{2}x^2 - n^{\frac{1}{2}}\sigma_0(\theta_0 - \theta)x]\, dx\, d\theta \\
&\quad \leqq \limsup_n \rho(\theta_0-)a^{-1}(n/2\pi)^{\frac{1}{2}} \textstyle\int_{\theta_0-\delta}^{\theta_0} \int_{k_n}^{\infty} \exp[-\frac{1}{2}x^2 - xn^{\frac{1}{2}}\sigma_0(\theta_0 - \theta) \\
&\qquad - \frac{1}{2}na\sigma_0^2(\theta_0 - \theta)^2]\, dx\, d\theta \\
&\quad = \rho(\theta_0-)(a\sigma_0)^{-1}(2\pi)^{-\frac{1}{2}} \textstyle\int_{d/\sigma_0}^{\infty} \int_0^{\infty} \exp(-\frac{1}{2}x^2 - xy - \frac{1}{2}ay^2)\, dy\, dx\,.
\end{aligned}$$

Since a can be chosen arbitrarily close to 1 from the left, it follows that

$$\begin{aligned}
&\limsup_n n^{\frac{1}{2}} \textstyle\int_{(-\infty,\theta_0]} [1 - G_n(c_n; \theta)]\, d\nu(\theta) \\
&\qquad \leqq \rho(\theta_0-)\sigma_0^{-1}\{(2\pi)^{-\frac{1}{2}} \exp(-d^2/2\sigma_0^2) - (d/\sigma_0)\Phi(-d/\sigma_0)\}\,.
\end{aligned}$$

Similarly, the reverse inequality holds when taking the limit inferior of the l.h.s., which establishes (4.8). The proof of (4.9) is similar. Formula (2.11) is an immediate consequence of (1.3), (4.8) and (4.9).

Development of theory for the two-sided problem proceeds similarly as in the one-sided case, and most of the proofs involve the same basic lines of reasoning. For this reason the proofs of Theorems 3.1 through 3.5 have been omitted. The proof of Theorem 3.6, being somewhat unique, is outlined below. Detailed proofs of all results presented in this paper can be found in [7].

The following formulas define the constants which appear in the statement of Theorem 3.5.

$$K_2 = \log[\nu(\{\theta_0\})(\psi'(\theta_3) - \psi'(\theta_1))^{\alpha_1}/\beta_1\Gamma(\alpha_1 + 1)] \tag{4.10}$$

$$K_3 = \log[\nu(\{\theta_0\})(\psi'(\theta_2) - \psi'(\theta_4))^{\alpha_2}/\beta_2\Gamma(\alpha_2 + 1)] \tag{4.11}$$

$$A = \frac{(2\pi)^{-\frac{1}{2}}(\theta_0 - \theta_1)\nu(\{\theta_0\})}{(\theta_0 - \theta_3)(\theta_3 - \theta_1)[\psi''(\theta_3)]^{\frac{1}{2}}}\left[\frac{\beta_1\Gamma(\alpha_1 + 1)}{\nu(\{\theta_0\})[\psi'(\theta_3) - \psi'(\theta_1)]^{\alpha_1}}\right]^{(\theta_0-\theta_3)/(\theta_0-\theta_1)} \tag{4.12}$$

$$B = \frac{(2\pi)^{-\frac{1}{2}}(\theta_2 - \theta_0)\nu(\{\theta_0\})}{(\theta_4 - \theta_0)(\theta_2 - \theta_4)[\psi''(\theta_4)]^{\frac{1}{2}}}\left[\frac{\beta_2\Gamma(\alpha_2 + 1)}{\nu(\{\theta_0\})[\psi'(\theta_2) - \psi'(\theta_4)]^{\alpha_2}}\right]^{(\theta_4-\theta_0)/(\theta_2-\theta_0)} \tag{4.13}$$

PROOF OF THEOREM 3.6. Let $d_n = n^{\frac{1}{2}}(a_n - \psi'(\theta_0))/\sigma_0$ and $k_n = n^{\frac{1}{2}}(b_n - \psi'(\theta_0))/\sigma_0$ where $\sigma_0 = [\psi''(\theta_0)]^{\frac{1}{2}}$. From Theorem 3.2

$$d_n \sim -(\log n)^{\frac{1}{2}} \quad \text{and} \quad k_n \sim (\log n)^{\frac{1}{2}}\,. \tag{4.14}$$

It will now be shown that

$$\exp(-\tfrac{1}{2}d_n^2) = [(2\pi)^{\frac{1}{2}}\rho(\theta_0-)/\sigma_0\nu(\{\theta_0\})]n^{-\frac{1}{2}}(1 + o_n(1))\,, \tag{4.15}$$

$$\exp(-\tfrac{1}{2}k_n^2) = [(2\pi)^{\frac{1}{2}}\rho(\theta_0+)/\sigma_0\nu(\{\theta_0\})]n^{-\frac{1}{2}}(1 + o_n(1))\,. \tag{4.16}$$

Equation (1.4, ii) can be written

$$\nu(\{\theta_0\}) = \int_{(-\infty,\theta_0)\cup(\theta_0,\infty)} \exp\{-nI(\theta_0, \theta) - n^{\frac{1}{2}}(\theta_0 - \theta)\sigma_0 d_n\}\, d\nu(\theta)\,,$$

from which we obtain

$$\int_{(\theta_0 - n^{-\frac{1}{2}}\log n, \theta_0)} \exp\{-nI(\theta_0, \theta) - n^{\frac{1}{2}}(\theta_0 - \theta)\sigma_0 d_n\}\, d\nu(\theta) \to \nu(\{\theta_0\})\,.$$

From Taylor's theorem

$$I(\theta_0, \theta) = \tfrac{1}{2}\sigma_0^2(\theta_0 - \theta)^2 + \tfrac{1}{6}\psi'''(\zeta(\theta))(\theta_0 - \theta)^3$$

for some $\zeta(\theta)$ between θ_0 and θ. Further, for $\theta \in (\theta_0 - n^{-\frac{1}{2}}\log n, \theta_0)$

$$\rho(\theta)\exp\{-(n/6)\psi'''(\zeta(\theta))(\theta_0 - \theta)^3\} = \rho(\theta_0-)(1 + o_n(1))$$

where $o_n(1)$ converges to zero uniformly on $(\theta_0 - n^{-\frac{1}{2}}\log n, \theta_0)$. Therefore,

$$\begin{aligned}
\lim_n n^{\frac{1}{2}} &\exp(-\tfrac{1}{2}d_n^2)\nu(\{\theta_0\}) \\
&= \lim_n n^{\frac{1}{2}}\exp(-\tfrac{1}{2}d_n^2)\rho(\theta_0-)(1 + o_n(1)) \\
&\qquad \times \int_{\theta_0 - n^{-\frac{1}{2}}\log n}^{\theta_0} \exp\{-(n\sigma_0^2/2)(\theta_0 - \theta)^2 - n^{\frac{1}{2}}(\theta_0 - \theta)\sigma_0 d_n\}\, d\theta \\
&= (2\pi)^{\frac{1}{2}}\rho(\theta_0-)/\sigma_0\,,
\end{aligned}$$

so (4.15) is valid. The proof of (4.16) is similar.

Next, from Esséen's theorem and the behavior of normal tails, it is shown that

$$\nu(\{\theta_0\})[G_n(a_n; \theta_0) + (1 - G_n(b_n; \theta_0))] = o([(\log n)/n]^{\frac{1}{2}})\,. \tag{4.17}$$

It remains to find an asymptotic formula for the integral in (1.5). Similarly as in the one-sided problem we obtain the following expressions:

$$\begin{aligned}
1 - G_n(a_n; \theta) &= \exp[-nI(\theta_0, \theta) - n^{\frac{1}{2}}(\theta_0 - \theta)\sigma_0 d_n] \\
&\qquad \times \int_0^1 \{F_n(d_n - (\log t)/[(\theta_0 - \theta)\sigma_0 n^{\frac{1}{2}}]) - F_n(d_n)\}\, dt
\end{aligned}$$

and

$$\begin{aligned}
1 - G_n(b_n; \theta) &= \exp[-nI(\theta_0, \theta) - n^{\frac{1}{2}}(\theta_0 - \theta)\sigma_0 k_n] \\
&\qquad \times \int_0^1 \{F_n(k_n - (\log t)/[(\theta_0 - \theta)\sigma_0 n^{\frac{1}{2}}]) - F_n(k_n)\}\, dt
\end{aligned}$$

for $\theta \in (-\infty, \theta_0) \cap \Omega$; also,

$$\begin{aligned}
G_n(a_n; \theta) &= \exp[-nI(\theta_0, \theta) + n^{\frac{1}{2}}(\theta - \theta_0)\sigma_0 d_n] \\
&\qquad \times \int_0^1 \{F_n(d_n) - F_n(d_n + (\log t)/[(\theta - \theta_0)\sigma_0 n^{\frac{1}{2}}])\}\, dt
\end{aligned}$$

and

$$\begin{aligned}
G_n(b_n; \theta) &= \exp[-nI(\theta_0, \theta) + n^{\frac{1}{2}}(\theta - \theta_0)\sigma_0 k_n] \\
&\qquad \times \int_0^1 \{F_n(k_n) - F_n(k_n + (\log t)/[(\theta - \theta_0)\sigma_0 n^{\frac{1}{2}}])\}\, dt
\end{aligned}$$

for $\theta \in (\theta_0, \infty) \cap \Omega$. Analysis using these expressions will yield the desired asymptotic formula. More precisely, the equation

$$\begin{aligned}
\int_{(-\infty,\theta_0)} &[1 - G_n(a_n; \theta)]\, d\nu(\theta) \\
&= (1 + o_n(1)) \int_{(-\infty,\theta_0)} \exp[-nI(\theta_0, \theta) - n^{\frac{1}{2}}(\theta_0 - \theta)\sigma_0 d_n] \\
&\qquad \times \int_0^1 \{\Phi(d_n - (\log t)/[(\theta_0 - \theta)n^{\frac{1}{2}}\sigma_0]) - \Phi(d_n)\}\, dt\, d\nu(\theta)
\end{aligned}$$

is obtained by applying Esséen's theorem and considering separately the integral over $(-\infty, \theta_0 - n^{-\frac{1}{2}} \log n]$ and $(\theta_0 - n^{-\frac{1}{2}} \log n, \theta_0)$. Further, the r.h.s. integral over $(\theta_0 - n^{-\frac{1}{2}} \log n, \theta_0)$ is found to be asymptotically equivalent to

$$[(\log n)/n]^{\frac{1}{2}}(\rho(\theta_0-)/\sigma_0)$$

while the integral over $(-\infty, \theta_0 - n^{-\frac{1}{2}} \log n]$ goes to zero faster than $1/n$, which implies

$$\int_{(-\infty,\theta_0)} [1 - G_n(a_n; \theta)]\, d\nu(\theta) = [(\log n)/n]^{\frac{1}{2}}(\rho(\theta_0-)/\sigma_0)(1 + o_n(1)) .$$

Similarly,

$$\int_{(\theta_0,\infty)} G_n(b_n; \theta)\, d\nu(\theta) = [(\log n)/n]^{\frac{1}{2}}(\rho(\theta_0+)/\sigma_0)(1 + o_n(1)) .$$

These results combine to yield

$$(4.18) \qquad \int_{(-\infty,\theta_0)\cup(\theta_0,\infty)} \{G_n(b_n; \theta) - G_n(a_n; \theta)\}\, d\nu(\theta) = [(\log n)/n]^{\frac{1}{2}}([\rho(\theta_0-) + \rho(\theta_0+)]/\sigma_0)(1 + o_n(1)) .$$

Formula (3.1) follows from (1.5), (4.17) and (4.18), proving Theorem 3.6.

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