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MULTIVARIATE DISTRIBUTIONS WITH EXPONENTIAL MINIMUMS[1]

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The multivariate distribution of a set of random variables has exponential minimums if the minimum over each subset of the variables has an exponential distribution. Such distributions are shown equivalent to the more strongly structured multivariate exponential distributions described by Marshall and Olkin in 1967 in the sense that a multivariate exponential distribution can be found that gives the same marginal distribution for each minimum. The basic application of the result is that in computing the reliability of a coherent system a joint distribution for the component life lengths with exponential minimums can be replaced by a multivariate exponential distribution. It follows that the life length of the system has an increasing hazard rate average distribution. Other applications include characterizations of multivariate exponential distributions and the derivation of a positive dependence condition for multivariate distributions with exponential minimums.

1. Introduction. We will say that nonnegative random variables $T_1, \cdots, T_n$ have a joint distribution with *exponential minimums* if

$$P[\min_{i\in I} T_i > t] = e^{-\mu_I t}\,, \qquad t \geqq 0\,, \tag{1.1}$$

for some $\mu_I > 0$ and for all nonempty sets $I \subset \{1, \cdots, n\}$. In reliability theory nonnegative random variables can be used to represent the life lengths of the components in a system. Noting that the life length of a series system is the minimum of its component life lengths, the definition abstracts a familiar property of components whose life lengths have independent, exponential distributions—that the life length of each series system that can be formed from the components has an exponential distribution.

For our purposes the most important example of a distribution with exponential minimums is the multivariate exponential distribution introduced by Marshall and Olkin (1967a). Random variables $U_1, \cdots, U_n$ having this distribution can be generated by letting

$$U_i = \min\{S_J : i \in J\}\,, \qquad i = 1, \cdots, n\,, \tag{1.2}$$

where the sets J are the elements of a class $\mathscr{J}$ of nonempty subsets of $\{1, \cdots, n\}$ having the property that for each i, $i \in J$ for some $J \in \mathscr{J}$, and the random vari-

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ables S_J, $J \in \mathscr{J}$, are independent and exponentially distributed. In this paper we will say that $U_1, \cdots, U_n$ have a *multivariate exponential* distribution if they have the same joint distribution as a set of random variables generated by (1.2), because such distributions play a central role in our results. If a set of component life lengths $U_1, \cdots, U_n$ has a multivariate exponential distribution, then the random variables S_J, $J \in \mathscr{J}$, ordinarily represent independent, exponential times to occurrence for various causes of component failure. There can be a cause of failure for each component separately, for each pair of components simultaneously, and generally for each subset of components simultaneously. Also for each component there must be a cause of failure for some set of components to which the component belongs.

We will say that the joint distributions of $T_1, \cdots, T_n$ and of $U_1, \cdots, U_n$ are *marginally equivalent in minimums* if

$$(1.3) \qquad P[\min_{i \in I} T_i > t] = P[\min_{i \in I} U_i > t]\,, \qquad t \geqq 0\,,$$

for each nonempty set $I \subset \{1, \cdots, n\}$. Two joint distributions for component life lengths that are marginally equivalent in minimums are indistinguishable insofar as the distribution of the life length of any series system that can be formed from the components is concerned.

The principal result of this paper (Theorem 4.1) is that any $T_1, \cdots, T_n$ having a joint distribution with exponential minimums can be marginally equated in minimums to some $U_1, \cdots, U_n$ having a multivariate exponential distribution. The result extends (Corollary 4.3) to show, in effect, that even for the purpose of computing the reliability of a redundant system, component life distributions with exponential minimums can be replaced by multivariate exponential life distributions.

In Section 2 we make some additional remarks about distributions with exponential minimums and multivariate exponential distributions, and their places in a hierarchy of multivariate distributions with similar properties. The principal result follows from two lemmas concerning the relative reliabilities of all the systems, series and redundant, that can be formed from the same set of components. These lemmas, with some necessary preliminaries, appear in Section 3. The main result is given in Section 4, and Section 5 is devoted to some sample applications of it.

2. Joint distributions with exponential properties. We are primarily concerned in this paper with the class of joint distributions having exponential minimums and the class of multivariate exponential distributions. However, it will be helpful in interpreting and applying the results to consider a hierarchy of classes of joint distributions with similar properties.

We can consider nonnegative random variables $T_1, \cdots, T_n$ whose joint distribution satisfies one of the following conditions:

(a) $T_1, \cdots, T_n$ are independent and each T_i has an exponential distribution.
(b) $T_1, \cdots, T_n$ have a multivariate exponential distribution.

(c) $\text{Min}_{i \in I} a_i T_i$ has an exponential distribution for all $a_i > 0$, $i = 1, \cdots, n$, and all nonempty sets $I \subset \{1, \cdots, n\}$.

(d) $T_1, \cdots, T_n$ have a joint distribution with exponential minimums.

(e) Each T_i, $i = 1, \cdots, n$, has an exponential distribution.

REMARK 2.1. Each of the classes of joint distributions defined by the conditions (a), (b), (c), (d), (e) can be regarded as a legitimate class of multivariate distributions in the sense that each class has the properties:

(P_1) If the joint distribution of $T_1, \cdots, T_n$ is in the class, then the joint distribution of any subset of $T_1, \cdots, T_n$ is in the class.

(P_2) If the joint distribution of $T_1, \cdots, T_n$ is in the class, the joint distribution of $U_1, \cdots, U_m$ is in the class, and $(T_1, \cdots, T_n)$ and $(U_1, \cdots, U_m)$ are independent, then the joint distribution of $T_1, \cdots, T_n; U_1, \cdots, U_m$ is in the class.
Each class of joint distributions defined by (a), (b), (c), (d), (e) can be regarded as an extension of the class of univariate exponential distributions in the sense that each class has the property.

(P_3) If the joint distribution of $T_1, \cdots, T_n$ is in the class, then each T_i, $i = 1, \cdots, n$, has an exponential distribution.
In addition the classes of joint distributions defined by (b), (c), (d) have the important property.

(P_4) If the joint distribution of $T_1, \cdots, T_n$ is in the class and $U_j = \min_{i \in I_j} T_i$, $j = 1, \cdots, m$, where $I_1, \cdots, I_m$ are nonempty subsets of $\{1, \cdots, n\}$, then the joint distribution of $U_1, \cdots, U_m$ is in the class.
Classes of joint distributions that have property P_4 are of special interest in reliability theory, since if the life lengths of a set of components have a joint distribution in the class,. then the life lengths of any set of series systems built from the components have a joint distribution in the class. □

It is immediate that each implication in the chain

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \Rightarrow \text{(d)} \Rightarrow \text{(e)} \tag{2.1}$$

is valid, with the possible exception of (b) ⇒ (c), which is readily checked. It is possible to find examples of bivariate distributions that satisfy conditions (b) but not (a), (c) but not (b), (d) but not (c), and (e) but not (d). Thus the classes of joint distributions defined by the conditions are distinct. In these examples it is convenient to describe the distribution of nonnegative random variables T_1, T_2 by the *survival function*

$$\bar{F}(t_1, t_2) = P[T_1 > t_1, T_2 > t_2], \qquad t_1 \geqq 0,\ t_2 \geqq 0. \tag{2.2}$$

EXAMPLE 2.2. Let $\bar{F}(t_1, t_2) = \exp[-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_{12} \max(t_1, t_2)]$, $\lambda_1 + \lambda_{12} > 0$, $\lambda_2 + \lambda_{12} > 0$, $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, $\lambda_{12} \geqq 0$. This is the bivariate version of the multivariate exponential distribution. The distribution occurs if $T_1 = \min(S_1, S_{12})$ and $T_2 = \min(S_2, S_{12})$ where S_1, S_2, and S_{12} are independent with the exponential distributions $P[S_1 > t] = e^{-\lambda_1 t}$, $P[S_2 > t] = e^{-\lambda_2 t}$, $P[S_{12} > t] = e^{-\lambda_{12} t}$, $t \geqq 0$. If any of the parameters λ_1, λ_2, or λ_{12} is zero, then the corresponding random variable

S_1, S_2, or S_{12} can be omitted from the constructive representation or regarded as degenerate at infinity. □

If $\lambda_{12} > 0$ in Example 2.2, then the distribution satisfies (b) but not (a).

EXAMPLE 2.3. Let $\bar{F}(t_1, t_2) = \exp[-\xi_1 t_1 - \xi_2 t_2 - \max(\xi_3 t_1, \xi_4 t_2)]$, $\xi_1 + \xi_3 > 0$, $\xi_2 + \xi_4 > 0$, $\xi_i \geqq 0$, $i = 1, \cdots, 4$. This bivariate distribution was considered by Marshall and Olkin (1967 a) and by Fréchet (1951) in the special case $\xi_1 = 0$, $\xi_2 = 0$. The distribution occurs if $T_1 = a_1 U_1$, $a_1 > 0$, and $T_2 = a_2 U_2$, $a_2 > 0$, where U_1, U_2 have the bivariate exponential distribution described in Example 2.2. Then $\xi_1 = \lambda_1/a_1$, $\xi_2 = \lambda_2/a_2$, $\xi_3 = \lambda_{12}/a_1$, $\xi_4 = \lambda_{12}/a_2$. □

If $\xi_3 > 0$, $\xi_4 > 0$, and $\xi_3 \neq \xi_4$ in Example 2.3, then the distribution satisfies (c) but not (b).

EXAMPLE 2.4. Let $\bar{F}(t_1, t_2) = p \exp[-\xi_1 t_1 - \xi_2 t_2 - \max(\xi_3 t_1, \xi_4 t_2)] + (1-p) \times \exp[-\eta_1 t_1 - \eta_2 t_2 - \max(\eta_3 t_1, \eta_4 t_2)]$, $\xi_1 + \xi_3 > 0$, $\xi_2 + \xi_4 > 0$, $\xi_i \geqq 0$, $i = 1, \cdots, 4$, $\eta_1 + \eta_3 > 0$, $\eta_2 + \eta_4 > 0$, $\eta_i \geqq 0$, $i = 1, \cdots, 4$, $0 < p < 1$. The distribution occurs as a mixture, $T_1 = U_1$, $T_2 = U_2$ with probability p, $T_1 = V_1$, $T_2 = V_2$ with probability $1 - p$, of two bivariate distributions, for U_1, U_2 and V_1, V_2, of the type described in Example 2.3. □

If $\xi_1 + \xi_3 = \eta_1 + \eta_3$ and $\xi_2 + \xi_4 = \eta_2 + \eta_4$ in Example 2.4, the distribution satisfies (e). If in addition $\xi_1 + \xi_2 + \max(\xi_3, \xi_4) = \eta_1 + \eta_2 + \max(\eta_3, \eta_4)$, the distribution satisfies (d).

Let $\xi_1 = \xi_2 = \eta_3 = \eta_4 = c > 0$ and $\xi_3 = \xi_4 = \eta_1 = \eta_2 = 0$. Then the distribution satisfies (e) but not (d).

Let $\xi_1 = \xi_2 = \xi_3 = \xi_4 = c > 0$ and $\eta_1 = \eta_3 = c$, $\eta_2 = 0$, $\eta_4 = 2c$. Then the distribution satisfies (d). If T_1, T_2 have this distribution, then the distribution of $T_1/2$, T_2 is of the same form but with $\xi_1 = \xi_3 = 2c$, $\xi_2 = \xi_4 = c$ and $\eta_1 = \eta_3 = 2c$, $\eta_2 = 0$, $\eta_4 = 2c$. The distribution of $T_1/2$, T_2 does not satisfy (d). Thus the distribution of T_1, T_2 satisfies (d) but not (c).

In the bivariate case, condition (b) says that the survival function must have the form of Example 2.2. On the other hand, distributions with the weaker property (c) certainly need not have the simple form of Example 2.3, and Example 2.4 exhibits a very special distribution with exponential minimums.

In the literature, a number of bivariate distributions have been discussed which have exponential marginals but fail to have exponential minimums. See, e.g., Gumbel (1960) or Marshall and Olkin (1967 b).

Using the above examples, we can give bivariate illustrations of the fact that any $T_1, \cdots, T_n$ having a joint distribution with exponential minimums can be marginally equated in minimums to some $U_1, \cdots, U_n$ having a multivariate exponential distribution. Specifically, with $\lambda_1 = \xi_1 + \xi_3 - \min(\xi_3, \xi_4)$, $\lambda_2 = \xi_2 + \xi_4 - \min(\xi_3, \xi_4)$, and $\lambda_{12} = \min(\xi_3, \xi_4)$, one can easily check that the bivariate exponential distribution (Example 2.2) is marginally equivalent in minimums to the distribution of Example 2.3. It is also marginally equivalent in minimums to the distribution of Example 2.4 under the constraints required for that distribution to satisfy (d).

We remark that with sums playing the role of minimums, there are conditions analagous to (a) through (e) which are appropriate for normal distributions. In the normal case, (b) is replaced by the condition that the random variables have a multivariate normal distribution and (c) is replaced by the condition that all linear combinations of the random variables are normally distributed. Since these conditions coincide, the analogy between the exponential and the normal cases is only partial. Of course, the essential ingredient in both cases is a function σ of n variables (the minimum or the sum) such that if $X_1, \cdots, X_n$ are independent random variables with distribution in some class, then $\sigma(X_1, \cdots, X_n)$ is a random variable with distribution in the same class.

3. Measures on coherent structure functions. A function $\phi(\mathbf{x}) = \phi(x_1, \cdots, x_n)$, where $x_i = 0$ or 1, $i = 1, \cdots, n$, and $\phi(\mathbf{x}) = 0$ or 1, is a *coherent structure function of order n* if ϕ is non-decreasing in each of its arguments and $\phi(\mathbf{0}) = \phi(0, \cdots, 0) = 0$, $\phi(\mathbf{1}) = \phi(1, \cdots, 1) = 1$. Coherent structure functions are used in reliability theory to relate the performance of two-state systems (system functioning if $\phi = 1$, failed if $\phi = 0$) to their, also two-state, components (ith component functioning if $x_i = 1$, failed if $x_i = 0$). The definition permits a coherent structure function to be constant in some but not all of its arguments. Thus the class of coherent structure functions of order n describes all the coherent systems that can be formed using at least one and no more than n components.

To introduce random performance for the components of a system, the deterministic indicator variables $x_1, \cdots, x_n$ could be replaced by binary random variables $X_1, \cdots, X_n$. In this case the reliability of a system is the probability that it functions, i.e.

$$P[\phi(\mathbf{X}) = 1] = \textstyle\sum_{\mathbf{x}} \phi(\mathbf{x}) P[\mathbf{X} = \mathbf{x}] \,. \tag{3.1}$$

From (3.1) it is easy to see that if ϕ_1, ϕ_2 are structure functions of order n, then

$$\phi_1 \leqq \phi_2 \quad \text{implies} \quad P[\phi_1(\mathbf{X}) = 1] \leqq P[\phi_2(\mathbf{X}) = 1] \tag{3.2}$$

and if $x \vee y = 1 - (1 - x)(1 - y) = x + y - xy$ for $x, y = 0$ or 1, then

$$\begin{aligned} P[\{\phi_1(\mathbf{X}) \vee \phi_2(\mathbf{X})\} = 1] &= P[\phi_1(\mathbf{X}) = 1] \\ &\quad + P[\phi_2(\mathbf{X}) = 1] - P[\phi_1(\mathbf{X})\phi_2(\mathbf{X}) = 1] \,. \end{aligned} \tag{3.3}$$

Equations (3.2) and (3.3) express familiar relationships between the reliabilities of two systems that are formed from the same components. The following lemma is in part a converse to the preceding remarks.

LEMMA 3.1. *Let $m(\phi)$ be a nonnegative function defined for all coherent structure functions of order n and such that:*

(a) $\phi_1 \leqq \phi_2$ *implies* $m(\phi_1) \leqq m(\phi_2)$.

(b) $m(\phi_1 \vee \phi_2) = m(\phi_1) + m(\phi_2) - m(\phi_1 \phi_2)$.

Then there exists a nonnegative function $w(\mathbf{x})$ *such that*

$$\text{(c)} \quad m(\phi) = \textstyle\sum_{\mathbf{x}} w(\mathbf{x})\phi(\mathbf{x})$$

for all coherent structure functions of order n. Note: Since $\phi(\mathbf{0}) = 0$ for all coherent structure functions, $w(\mathbf{0})$ can be defined arbitrarily.

PROOF. An elementary, but involved, proof that (c) holds can be made by an induction on the *length* l of ϕ as l decreases from n to 1, where $l = \min_{\{\mathbf{x}:\phi(\mathbf{x})=1\}} \sum_{i=1}^{n} x_i$. The details would be a departure from our main purpose, and are omitted. □

The application of Lemma 3.1 that concerns us occurs in a context that requires some additional definitions and observations.

A *path set* of a coherent structure function is a set of indices $P \subset \{1, \cdots, n\}$ such that $\prod_{i \in P} x_i = 1$ implies $\phi(\mathbf{x}) = 1$. A path set is *minimal* if it properly contains no other path set. A minimal path set corresponds to an irreducible collection of components which by all functioning insure that the system functions. A coherent structure function can be defined by listing its minimal path sets $P_1, \cdots, P_p$ and then represented by

$$\phi(\mathbf{x}) = \textstyle\prod_{i \in P_1} x_i \vee \cdots \vee \prod_{i \in P_p} x_i \,. \tag{3.4}$$

The representation (3.4) corresponds to forming a series system from the components in each minimal path set and than placing the series systems in parallel.

The *coherent life function* $\tau(\mathbf{t}) = \tau(t_1, \cdots, t_n)$, $t_i \geqq 0$, $i = 1, \cdots, n$, corresponding to a coherent structure function ϕ can be defined through a representation analagous to (3.4), i.e.,

$$\tau(\mathbf{t}) = \max_{j=1,\cdots,p} \min_{i \in P_j} t_i \,, \tag{3.5}$$

where $P_1, \cdots, P_p$ are the minimal path sets of ϕ. The life function of a coherent system gives the life length of the system as a function of the life lengths of its components. The properties of coherent life functions are discussed in Esary and Marshall (1970). It is easy to see from (3.4) and (3.5) that if τ_1, τ_2 correspond to ϕ_1, ϕ_2, then:

$$\phi_1 \leqq \phi_2 \Leftrightarrow \tau_1 \leqq \tau_2 \,, \tag{3.6}$$

$$\begin{aligned} &\min(\tau_1, \tau_2) \quad \text{corresponds to} \quad \phi_1\phi_2 \,, \\ &\max(\tau_1, \tau_2) \quad \text{corresponds to} \quad \phi_1 \vee \phi_2 \,. \end{aligned} \tag{3.7}$$

The *dual* of a coherent structure function ϕ is the coherent structure function ϕ^D defined by $\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x})$. For example, the dual of a series system is a parallel system built from the same components and conversely. It is easy to see from the definition of ϕ^D that:

$$\phi_1 \leqq \phi_2 \Leftrightarrow \phi_1^D \geqq \phi_2^D \,, \tag{3.8}$$

$$(\phi_1 \vee \phi_2)^D = \phi_1^D \phi_2^D \,. \tag{3.9}$$

The *dual* τ^D of the coherent life function τ that corresponds to ϕ is the coherent life function that corresponds to ϕ^D.

In this paper the relevant way of introducing random performance for the components in a system is to represent their life lengths by nonnegative random variables $T_1, \cdots, T_n$. In this case the reliability function or *survival probability* of a system is

$$\bar{F}_\tau(t) = P[\tau(\mathbf{T}) > t], \qquad t \geqq 0. \tag{3.10}$$

If $\bar{F}_\tau$ is absolutely continuous on $[0, \varepsilon)$ for some $\varepsilon > 0$, then $\bar{F}_\tau(0) = 1$, and an initial hazard rate, i.e.

$$\alpha(\tau) = \frac{d}{dt}\{-\log \bar{F}_\tau(t)\}\Big|_{t=0} = -\frac{d\bar{F}_\tau(t)}{dt}\Big|_{t=0}, \tag{3.11}$$

exists for $\bar{F}_\tau$. The derivatives in (3.11) should be interpreted as derivatives from the right.

REMARK 3.2. It is shown in Esary and Marshall (1970), Application 5.3, that *if each T_i, $i = 1, \cdots, n$, is an absolutely continuous random variable, then $\tau(\mathbf{T})$ is an absolutely continuous random variable for each coherent life function τ of order n.* The same argument can be used to show that if each T_i, $i = 1, \cdots, n$, is absolutely continuous on $[0, \varepsilon)$, then each $\tau(\mathbf{T})$ is absolutely continuous on $[0, \varepsilon)$. □

LEMMA 3.3. *Let $T_1, \cdots, T_n$ be nonnegative random variables with a joint distribution such that an initial hazard rate $\alpha(\tau)$ exists for $\bar{F}_\tau(t) = P[\tau(\mathbf{T}) > t]$ for each coherent life function τ of order n. For each coherent structure function ϕ of order n let $m(\phi) = \alpha(\tau^D)$ where τ^D corresponds to ϕ^D, the dual of ϕ. Then m is nonnegative and satisfies conditions* (a) *and* (b) *of Lemma* 3.1.

PROOF. It is immediate from (3.11) that m is nonnegative, since all survival probabilities $\bar{F}(t)$ are non-increasing in t.

To show that m satisfies (a) observe that $\tau_1 \leqq \tau_2$ implies that $\bar{F}_{\tau_1}(t) \leqq \bar{F}_{\tau_2}(t)$, $t \geqq 0$. Since $\bar{F}_{\tau_1}(0) = \bar{F}_{\tau_2}(0) = 1$, it follows from (3.11) that $\tau_1 \leqq \tau_2$ implies that $\alpha(\tau_1) \geqq \alpha(\tau_2)$. Let τ_1, τ_2 correspond to ϕ_1, ϕ_2. Then from (3.6) and (3.8)

$$\phi_1 \leqq \phi_2 \Rightarrow \phi_1^D \geqq \phi_2^D \Rightarrow \tau_1^D \geqq \tau_2^D \Rightarrow \alpha(\tau_1^D) \leqq \alpha(\tau_2^D) \Rightarrow m(\phi_1) \leqq m(\phi_2).$$

To show that m satisfies (b) observe that $\bar{F}_{\max(\tau_1,\tau_2)}(t) = \bar{F}_{\tau_1}(t) + \bar{F}_{\tau_2}(t) - \bar{F}_{\min(\tau_1,\tau_2)}(t)$, $t \geqq 0$. It follows from (3.11) that $\alpha\{\max(\tau_1, \tau_2)\} = \alpha(\tau_1) + \alpha(\tau_2) - \alpha\{\min(\tau_1, \tau_2)\}$. Let τ_1, τ_2 correspond to ϕ_1, ϕ_2. Then from (3.7) and (3.9) $\min(\tau_1^D, \tau_2^D)$ corresponds to $\phi_1^D\phi_2^D = (\phi_1 \vee \phi_2)^D$, and $\max(\tau_1^D, \tau_2^D)$ corresponds to $\phi_1^D \vee \phi_2^D = (\phi_1\phi_2)^D$. Thus

$$\begin{aligned} m(\phi_1 \vee \phi_2) &= \alpha\{\min(\tau_1^D, \tau_2^D)\} \\ &= \alpha(\tau_1^D) + \alpha(\tau_2^D) - \alpha\{\max(\tau_1^D, \tau_2^D)\} \\ &= m(\phi_1) + m(\phi_2) - m(\phi_1\phi_2). \end{aligned}$$ □

4. Marginal equivalences. If $U_1, \cdots, U_n$ have a multivariate exponential distribution, i.e., can be assumed to be generated by (1.2) from a family of independent random variables S_J, $J \in \mathscr{J}$, with the exponential distributions $P[S_J > t] = e^{-\lambda_J t}$, $t \geqq 0$, $\lambda_J > 0$, then it is immediate that

$$P[\min_{i \in I} U_i > t] = e^{-\eta_I t}, \qquad t \geqq 0, \tag{4.1}$$

with

$$\eta_I = \sum_{J \in \mathscr{J}(I)} \lambda_J, \tag{4.2}$$

where $\mathscr{J}(I) = \{J \in \mathscr{J}: I \cap J \text{ is not empty}\}$.

Now suppose $T_1, \cdots, T_n$ have a joint distribution with exponential minimums. Then from (1.1) and (1.3), the construction of $U_1, \cdots, U_n$ which are multivariate exponential and marginally equivalent in minimums to $T_1, \cdots, T_n$ begins with the solution of the system of equations

$$\mu_I = \sum_{\{J:\, I \cap J \text{ is not empty}\}} \lambda_J, \tag{4.3}$$

where I and J range over the class of nonempty subsets of $\{1, \cdots, n\}$. The solution is for the λ_J's in terms of the given μ_I's. If $\lambda_J \geqq 0$ for each $J \subset \{1, \cdots, n\}$, then independent, exponential random variables S_J can be generated, as above, for those J such that $\lambda_J > 0$. Then $U_1, \cdots, U_n$ generated in accordance with (1.2) will have a multivariate exponential distribution and be marginally equivalent to $T_1, \cdots, T_n$. As indicated, in this construction $\mathscr{J}$ is the class of nonempty subsets of $\{1, \cdots, n\}$ for which $\lambda_J > 0$.

The system of equations (4.3) has a pattern which facilitates solution, e.g. in the bivariate case

$$\mu_1 = \lambda_1 + \lambda_{12}$$
$$\mu_2 = \lambda_2 + \lambda_{12}$$
$$\mu_{12} = \lambda_1 + \lambda_2 + \lambda_{12},$$

where the subscripts should be interpreted as lists of elements in the sets I and J of (4.3). The question is whether the λ's obtained will always be nonnegative. The following theorem answers this question.

THEOREM 4.1. *Let $T_1, \cdots, T_n$ be random variables whose joint distribution has exponential minimums. Then there exist random variables $U_1, \cdots, U_n$ with a multivariate exponential distribution such that the joint distribution of $T_1, \cdots, T_n$ and $U_1, \cdots, U_n$ are marginally equivalent in minimums.*

PROOF. Since the distribution of each T_i, $i = 1, \cdots, n$, is exponential and thus absolutely continuous it follows from Remark 3.2 that the distribution of $\tau(\mathbf{T})$ is absolutely continuous for each coherent life function τ of order n. Thus from (3.11) an initial hazard rate $\alpha(\tau)$ exists for each coherent life function τ of order n. Then it follows from Lemma 3.3 that $m(\phi) = \alpha(\tau^D)$ satisfies the hypotheses of Lemma 3.1. Then from Lemma 3.1

$$\alpha(\tau) = m(\phi^D) = \sum_{\mathbf{x}} w(\mathbf{x})\phi^D(\mathbf{x}), \tag{i}$$

where w is nonnegative, for all coherent life functions τ of order n, where ϕ^D is the dual of the coherent structure function ϕ corresponding to τ. According to Lemma 3.1, $w(\mathbf{0})$ may be chosen arbitrarily. It is convenient here to choose $w(\mathbf{0}) = 0$.

For each nonempty $I \subset \{1, \cdots, n\}$ let $\tau_I(\mathbf{t}) = \min_{i \in I} t_i$. Since $T_1, \cdots, T_n$ have a joint distribution with exponential minimums, then $P[\tau_I(\mathbf{T}) > t] = e^{-\mu_I t}, t \geqq 0$, $\mu_I > 0$, from (1.1). From (3.11), $\alpha(\tau_I) = \mu_I$. The coherent structure function corresponding to τ_I is $\phi_I(\mathbf{x}) = \prod_{i \in I} x_i$. By its definition $\phi_I{}^D(\mathbf{x}) = 1$ if and only if $x_i = 1$ for some $i \in I$. Thus from (i)

$$\text{(ii)} \qquad \mu_I = \sum_{\mathbf{x} \in A(I)} w(\mathbf{x})$$

where $A(I) = \{\mathbf{x} : x_i = 1 \text{ for some } i \in I\}$.

For each $J \subset \{1, \cdots, n\}$ let $\lambda_J = w(\mathbf{x}^J) \geqq 0$ where $x_i{}^J = 1$ if $i \in J$, $x_i{}^J = 0$ if $i \notin J$. Then (ii) can be written as

$$\text{(iii)} \qquad \mu_I = \sum_{J \in \mathscr{J}(I)} \lambda_J ,$$

where $\mathscr{J}(I) = \{J \in \mathscr{J} : I \cap J \text{ is not empty}\}$ for $\mathscr{J} = \{J : \lambda_J > 0\}$.

For the empty set E, $\lambda_E = w(\mathbf{0}) = 0$, by our choice of $w(\mathbf{0})$. Thus *$\mathscr{J}$ is a class of nonempty sets.* Since $\mu_I > 0$ for all nonempty I, and in particular $\mu_i = \mu_{\{i\}} > 0$ when $I = \{i\}$, then from (ii) there exists a J containing i such that $\lambda_J > 0$. Thus *for each $i = 1, \cdots, n$ there exists a $J \in \mathscr{J}$ such that $i \in J$.* Thus $\mathscr{J}$, as defined above, has the properties required of a class $\mathscr{J}$ to be used in generating a multivariate exponential distribution by means of (1.2).

Now corresponding to the sets $J \in \mathscr{J}$ construct independent random variables S_J with exponential distributions $P[S_J > t] = e^{-\lambda_J t}$, $t \geqq 0$. Let $U_i = \min\{S_J : i \in J\}$, $i = 1, \cdots, n$. Then $U_1, \cdots, U_n$ have a multivariate exponential distribution from (1.2). From (4.2) and (iii), $\eta_I = \mu_I$ for all nonempty I. It follows from (4.1) and (1.1) that $T_1, \cdots, T_n$ and $U_1, \cdots, U_n$ are marginally equivalent in minimums. □

We will say that the joint distributions of $T_1, \cdots, T_n$ and $U_1, \cdots, U_n$ are *marginally equivalent in coherent life functions* if

$$\text{(4.4)} \qquad P[\tau(\mathbf{T}) > t] = P[\tau(\mathbf{U}) > t] , \qquad t \geqq 0$$

for each coherent life function τ of order n. The reliability interpretation of marginal equivalence in coherent life functions is the same as the reliability interpretation of marginal equivalence in minimums, except that general, redundant systems now take the place of series systems.

THEOREM 4.2. *The joint distribution of the random variables $T_1, \cdots, T_n$ is marginally equivalent in minimums to the joint distribution of the random variables $U_1, \cdots, U_n$ if and only if the joint distribution of $T_1, \cdots, T_n$ is equivalent in coherent life functions to the joint distribution of $U_1, \cdots, U_n$.*

PROOF. (if) Marginal equivalence in coherent life functions implies marginal equivalence in minimums because the class of coherent life functions of order n includes the functions $\tau_I(\mathbf{t}) = \min_{i \in I} t_i$ for all nonempty sets $I \subset \{1, \cdots, n\}$.

(only if) Consider some coherent life function τ of order n. Let $P_1, \cdots, P_p$ be the minimal path sets of the structure function ϕ corresponding to τ. Let $\tau_j(\mathbf{t}) = \min_{i \in P_j} t_i$, $j = 1, \cdots, p$. Then from (3.5) by the standard inclusion and exclusion argument

$$\begin{aligned} P[\tau(\mathbf{T}) > t] = \textstyle\sum_{j=1}^{p} P[\tau_j(\mathbf{T}) > t] \\ - \textstyle\sum_{j,k=1;j\neq k}^{p} P[\min\{\tau_j(\mathbf{T}), \tau_k(\mathbf{T})\} > t] \\ + \\ \vdots \\ (\pm)\, P[\min\{\tau_1(\mathbf{T}), \cdots, \tau_p(\mathbf{T})\} > t]\,, \qquad t \geqq 0\,. \end{aligned}$$

Since $T_1, \cdots, T_n$ and $U_1, \cdots, U_n$ are marginally equivalent in minimums, then $P[\tau_j(\mathbf{T}) > t] = P[\tau_j(\mathbf{U}) > t]$, $t \geqq 0$, $j = 1, \cdots, p$. Also, noting that $\min\{\tau_j(\mathbf{t}), \tau_k(\mathbf{t})\} = \min_{i \in P_j \cup P_k} t_i$, then $P[\min\{\tau_j(\mathbf{T}), \tau_k(\mathbf{T})\} > t] = P[\min\{\tau_j(\mathbf{U}), \tau_k(\mathbf{U})\} > t]$, $t \geqq 0$, $j, k = 1, \cdots, p$, and so on. Thus it follows from the expansion that $P[\tau(\mathbf{T}) > t] = P[\tau(\mathbf{U}) > t]$, $t \geqq 0$. □

COROLLARY 4.3. *Let $T_1, \cdots, T_n$ be random variables whose joint distribution has exponential minimums. Then there exist random variables $U_1, \cdots, U_n$ with a multivariate exponential distribution such that the joint distributions of $T_1, \cdots, T_n$ and $U_1, \cdots, U_n$ are marginally equivalent in coherent life functions.*

PROOF. This corollary is an immediate consequence of Theorems 4.1 and 4.2. □

REMARK 4.4. Coherent life functions can also be defined through the representation $\tau(\mathbf{t}) = \min_{j=1,\cdots,k} \max_{i \in K_j} t_i$, where $K_1, \cdots, K_k$ are the minimal cut sets of the structure function ϕ corresponding to τ. See Esary and Marshall (1970). We can say that $T_1, \cdots, T_n$ and $U_1, \cdots, U_n$ are *marginally equivalent in maximums* if $P[\max_{i \in I} T_i > t] = P[\max_{i \in I} U_i > t]$, $t \geqq 0$, for each nonempty $I \subset \{1, \cdots, n\}$. An argument parallel to the proof of Theorem 4.2 can then be used to show that marginal equivalence in maximums and marginal equivalence in minimums are the same. Thus marginal equivalence in minimums, maximums, and coherent life functions are all the same. □

Theorem 4.1 allows for the possibility that dependent random variables $T_1, \cdots, T_n$ are marginally equivalent in minimums to independent exponential random variables $U_1, \cdots, U_n$. This is indeed a real possibility, as is illustrated by the following example.

EXAMPLE 4.5. Let H be a bivariate distribution function with the properties (a) $H(u, 1) = u$, $0 \leqq u \leqq 1$, (b) $H(1, v) = v$, $0 \leqq v \leqq 1$, (c) $H(u, u) = u^2$, $0 \leqq u \leqq 1$. Then $\bar{F}(t_1, t_2) = H(e^{-t_1}, e^{-t_2})$ is a survival function marginally equivalent in minimums to $\bar{G}(t_1, t_2) = e^{-t_1-t_2}$, $t_1 \geqq 0$, $t_2 \geqq 0$. Properties (a), (b) and (c) are satisfied if H has a density equal to 2 in the shaded area and density zero elsewhere, as illustrated in Figure 1 or 2 below.

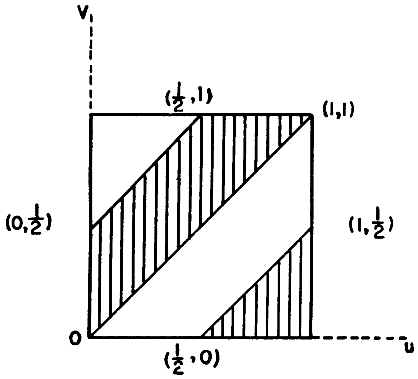

FIG. 1

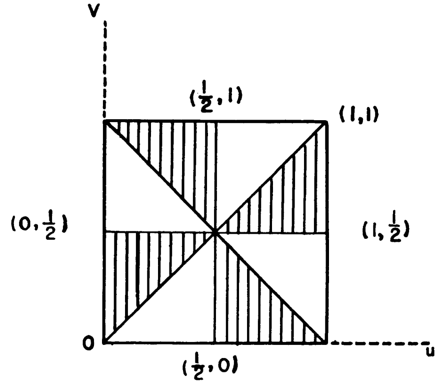

FIG. 2

One can check that Figure 1 leads to the survival function

$$\begin{aligned}\bar{F}(t_1, t_2) = {} & 2\exp[-t_2 - \max(t_1, t_2)] - \exp[-2\max(t_1, t_2)] \\ & + [\max(e^{-t_1} - \tfrac{1}{2}, 0)]^2 - [\max(e^{-t_2} - \tfrac{1}{2}, 0)]^2 \\ & + [\max(e^{-t_2} - e^{-t_1} - \tfrac{1}{2}, 0)]^2 - [\max(e^{-t_1} - e^{-t_2} - \tfrac{1}{2}, 0)]^2\end{aligned}$$

A somewhat longer expression is derivable from Figure 2. □

The two survival functions derived from Figures 1 and 2 in Example 4.3 satisfy condition (d) of Section 2 but not condition (c).

It is possible to say that $T_1, \cdots, T_n$ and $U_1, \cdots, U_n$ are *jointly equivalent in minimums* if $\min_{i \in I_j} T_i$ and $\min_{i \in I_j} U_i$, $j = 1, \cdots, m$, have the same joint distribution for nonempty $I_1, \cdots, I_m \subset \{1, \cdots, n\}$ and all m, and to similarly define joint equivalence in maximums and coherent life functions. As with marginal equivalence these three notions are the same, but they are also the same as ordinary equivalence between two distributions. On the other hand, it is possible to consider special joint equivalences between two distributions that do not reduce to ordinary equivalence. For example let T_1, T_2 have the distribution derived from Figure 1 or Figure 2 in Example 4.5. Then $\min(T_1, T_2)$ and $\max(T_1, T_2)$ have the same joint distribution as $\min(U_1, U_2)$ and $\max(U_1, U_2)$, where U_1, U_2 are independent copies of T_1, T_2, if and only if $H(u, v) + H(v, u) - H(u, u) = H(u, 1)H(1, v) + H(v, 1)H(1, u) - H(u, 1)H(1, u)$ for all $0 \leqq u \leqq v \leqq 1$, a condition which is satisfied in both figures.

5. Sample applications. The following applications are intended as examples of how the main results of the paper can be used.

5.1. *A characterization of multivariate exponential distributions.* In keeping with (2.2) the joint distribution of nonnegative random variables $T_1, \cdots, T_n$ can be described by the *survival function*

$$\bar{F}(\mathbf{t}) = \bar{F}(t_1, \cdots, t_n) = P[T_1 > t_1, \cdots, T_n > t_n], \quad t_1 \geqq 0, \cdots, t_n \geqq 0. \tag{5.1}$$

If $U_1, \cdots, U_n$ have a multivariate exponential distribution, i.e., can be assumed to be generated by (1.2) from a family of independent random variables S_J, $J \in \mathscr{J}$ with the exponential distributions $P[S_J > t] = e^{-\lambda_J t}$, $t \geqq 0$, $\lambda_J > 0$, then it is immediate that the survival function $\bar{G}$ of $U_1, \cdots, U_n$ can be written as

$$\bar{G}(\mathbf{t}) = \exp[-\textstyle\sum_{J \in \mathscr{J}} \lambda_J \max_{i \in J} t_i]. \tag{5.2}$$

Given a simplex $0 \leqq t_{i_1} \leqq \cdots \leqq t_{i_n}$, let $I_1 = \{i_1, \cdots, i_n\} = \{1, \cdots, n\}$, $I_2 = \{i_2, \cdots, i_n\}, \cdots, I_n = \{i_n\}$. Then it is again immediate that for any $\mathbf{t}$ in the simplex the survival function of $U_1, \cdots, U_n$ can be written in the specialized form

$$\begin{aligned} \bar{G}(\mathbf{t}) &= \exp[-t_{i_1} \textstyle\sum_{J \in \mathscr{J}(I_1)} \lambda_J] \exp[-(t_{i_2} - t_{i_1}) \sum_{J \in \mathscr{J}(I_2)} \lambda_J] \cdots \\ &\qquad \exp[-(t_{i_n} - t_{i_{n-1}}) \textstyle\sum_{J \in \mathscr{J}(I_n)} \lambda_J] \\ &= \textstyle\prod_{j=1}^{n} P[\min_{i \in I_j} U_i > t_{i_j} - t_{i_{j-1}}], \end{aligned} \tag{5.3}$$

where as before $\mathscr{J}(I) = \{J \in \mathscr{J} : I \cap J \text{ is not empty}\}$ and $t_{i_0} = 0$.

Application 5.1. Let $T_1, \cdots, T_n$ be nonnegative random variables such that:

(a) The joint distribution of $T_1, \cdots, T_n$ has exponential minimums.

(b) On each simplex $0 \leqq t_{i_1} \leqq \cdots \leqq t_{i_n}$ the survival function $\bar{F}$ of $T_1, \cdots, T_n$ satisfies

$$\bar{F}(\mathbf{t}) = \textstyle\prod_{j=1}^{n} P[\min_{i \in I_j} T_i > t_{i_j} - t_{i_{j-1}}].$$

Then $T_1, \cdots, T_n$ have a multivariate exponential distribution.

Proof. It follows from (a), by Theorem 4.1, that there exist $U_1, \cdots, U_n$ with a multivariate exponential distribution which is marginally equivalent in minimums to the joint distribution of $T_1, \cdots, T_n$. Let $\bar{G}$ be the survival function of $U_1, \cdots, U_n$. Then from (b) and (5.3), $\bar{F} = \bar{G}$ on each simplex. Thus $\bar{F} = \bar{G}$ and $T_1, \cdots, T_n$ have the same multivariate exponential distribution as $U_1, \cdots, U_n$. □

Application 5.1 can be used to show that if $T_1, \cdots, T_n$ are nonnegative random variables such that;

(c) each $n - 1$ dimensional marginal distribution of the joint distribution of $T_1, \cdots, T_n$ is a multivariate exponential distribution,

(d) the survival function $\bar{F}$ of $T_1, \cdots, T_n$ satisfies

$$\bar{F}(s_1 + t, \cdots, s_n + t) = \bar{F}(s_1, \cdots, s_n)\bar{F}(t, \cdots, t) \quad \text{for all } s_1 \geqq 0, \cdots, s_n \geqq 0 \text{ and } t \geqq 0,$$

then $T_1, \cdots, T_n$ have a multivariate exponential distribution. This confirms a result of Marshall and Olkin (1967a, Lemma 2.2 and page 39).

Application 5.1 can also be used to show that if $T_1, \cdots, T_n$ are nonnegative random variables such that;

(e) $P[T_i > s_i] < 1$ for some $s_i > 0$, $i = 1, \cdots, n$,

(f) on each simplex $0 \leqq t_{i_1} \leqq \cdots \leqq t_{i_n}$ the survival function $\bar{F}$ of $T_1, \cdots, T_n$ has the form

$$\bar{F}(\mathbf{t}) = \exp[-(\xi_1 t_{i_1} + \cdots + \xi_n t_{i_n})] ,$$

where $\xi_1, \cdots, \xi_n$ depend only on the simplex ,

then $T_1, \cdots, T_n$ have a multivariate exponential distribution.

5.2. IHRA *distributions for system lives.* A nonnegative random variable T has an *increasing hazard rate average* (IHRA) distribution if $\{-\log P[T > t]\}/t$ is non-decreasing in t. A component or system whose life length has an IHRA distribution undergoes deleterious aging or wearout in one of the possible stochastic senses that can be given to the term wearout. Exponential distributions, for which $\{-\log P[T > t]\}/t$ is constant in t, are boundary members of the class of IHRA distributions. It is shown in Birnbaum, Esary, and Marshall (1966, Theorem 4.2) that the class of IHRA distributions is the closure, under limits in distribution, of the class of distributions for $\tau(T_1, \cdots, T_n)$, $n = 1, 2, \cdots$, where τ is a coherent life function and $T_1, \cdots, T_n$ are independent, exponentially distributed random variables.

REMARK 5.2. If τ is a coherent life function of order n, and $U_1, \cdots, U_n$ have a multivariate exponential distribution, then $\tau(U_1, \cdots, U_n)$ has an IHRA distribution. The remark can be proved by assuming that $U_1, \cdots, U_n$ are generated according to (1.2) so that $U_i = \min\{S_J : i \in J\}$ where S_J, $J \in \mathscr{J}$, are independent, exponentially distributed random variables. In effect each component in the coherent system described by τ is replaced by a series system of new components whose life lengths are represented by the S_J's. It is easily seen that result is a new coherent system with independent, exponentially distributed component life lengths. Thus $\tau(U_1, \cdots, U_n)$ has an IHRA distribution by the previously mentioned result of Birnbaum, Esary, and Marshall (1966). □

APPLICATION 5.3. If τ is a coherent life function of order n, and $T_1, \cdots, T_n$ have a joint distribution with exponential minimums, then $\tau(T_1, \cdots, T_n)$ has an IHRA distribution.

PROOF. This application is immediate from Corollary 4.3 and Remark 5.2. □

5.3. *Positive dependence.* Random variables $T_1, \cdots, T_n$ are *associated* if $\mathrm{Cov}[f(\mathbf{T}), g(\mathbf{T})] \geqq 0$ for all pairs f, g of non-decreasing functions for which the covariance in question exists. It is shown in Esary, Proschan, and Walkup (1967, Theorem 5.1) that if $T_1, \cdots, T_n$ are associated, then:

(5.4) $$P[T_1 > t_1, \cdots, T_n > t_n] \geqq \prod_{i=1}^n P[T_i > t_i] , \quad -\infty \leqq t_i < +\infty, \; i = 1, \cdots, n .$$

(5.5) $$P[T_1 \leqq t_1, \cdots, T_n \leqq t_n] \geqq \prod_{i=1}^n P[T_i \leqq t_i] , \quad -\infty < t_i \leqq +\infty, \; i = 1, \cdots, n .$$

We will say that random variables that satisfy (5.4) are positively right quadrant dependent and that random variables that satisfy (5.5) are positively left quadrant dependent. In the bivariate case (5.4) and (5.5) are equivalent and define the notion of positive quadrant dependence introduced by Lehmann (1966). It is often reasonable to suppose that exposure to a common service environment will produce some kind of positive dependence, such as association or positive quadrant dependence between the life lengths of the components in a system.

REMARK 5.4. If $U_1, \cdots, U_n$ have a multivariate exponential distribution, then $U_1, \cdots, U_n$ are associated. Again assume that $U_1, \cdots, U_n$ are generated according to (1.2), so that $U_i = \min\{S_J : i \in J\}$ where the S_J, $J \in \mathscr{J}$ are independent. Then the remark follows immediately from results given in Esary, Proschan, and Walkup (1967, Theorem 2.1) that independent random variables are associated, and property P_4 of association, that non-decreasing functions of associated random variables are associated. □

APPLICATION 5.5. If $T_1, \cdots, T_n$ have a joint distribution with exponential minimums, then

(a) $$P[T_1 > t, \cdots, T_n > t] \geqq \prod_{i=1}^n P[T_i > t], \qquad t \geqq 0.$$

(b) $$P[T_1 \leqq t, \cdots, T_n \leqq t] \geqq \prod_{i=1}^n P[T_i \leqq t], \qquad t \geqq 0.$$

If $T_1, \cdots, T_n$ have a joint distribution such that $\min_{i \in I} a_i T_i$ has an exponential distribution for all $a_i > 0$, $i = 1, \cdots, n$, and all nonempty sets $I \subset \{1, \cdots, n\}$, then $T_1, \cdots, T_n$ are positively right quadrant dependent and positively left quadrant dependent.

PROOF. Suppose that the joint distribution of $T_1, \cdots, T_n$ has exponential minimums. Using Theorem 4.1, let $U_1, \cdots, U_n$ have a multivariate exponential distribution that is marginally equivalent in minimums to the joint distribution of $T_1, \cdots, T_n$. Then from Remark 5.4 and (5.4)

$$\begin{aligned} P[T_1 > t, \cdots, T_n > t] &= P[\min_{i=1,\cdots,n} T_i > t] \\ &= P[\min_{i=1,\cdots,n} U_i > t] = P[U_1 > t, \cdots, U_n > t] \\ &\geqq \prod_{i=1}^n P[U_i > t] = \prod_{i=1}^n P[T_i > t], \qquad t \geqq 0. \end{aligned}$$

Thus $T_1, \cdots, T_n$ satisfy (a). The proof that $T_1, \cdots, T_n$ satisfy (b) is similar, using that part of Remark 4.4 that says that marginal equivalence in minimums is the same as marginal equivalence in maximums.

Now suppose that $a_1 T_1, \cdots, a_n T_n$ have a joint distribution with exponential minimums for all $a_i > 0$, $i = 1, \cdots, n$. Then from (a)

$$\begin{aligned} P[T_1 > t_1, \cdots, T_n > t_n] &= P[a_1 T_1 > t, \cdots, a_n T_n > t] \\ &\geqq \prod_{i=1}^n P[a_i T_i > t] = \prod_{i=1}^n P[T_i > t_i], \end{aligned}$$

$t_i \geqq 0$, $i = 1, \cdots, n$, where $a_i t_i = t$. Thus $T_1, \cdots, T_n$ are positively right quadrant dependent. That $T_1, \cdots, T_n$ are positively left quadrant dependent follows similarly from (b). □

The bivariate distributions derived from Figures 1 or 2 in Example 4.5 have exponential minimums and so satisfy conditions (a) and (b) of Application 5.5, but neither distribution is positive quadrant dependent. From (5.4) positive quadrant dependence would require that $H(u, v) \geqq H(u, 1)H(1, v)$, $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, a condition not satisfied by either figure.

On the other hand we suspect that distributions that satisfy condition (c) of Section 2 have stronger dependence properties than positive right and left quadrant dependence.

REFERENCES

[1] BARLOW, RICHARD E. and PROSCHAN, FRANK (1965). *Mathematical Theory of Reliability.* Wiley, New York.

[2] BIRNBAUM, Z. W., ESARY, J. D. and MARSHALL, A. W. (1966). A stochastic characterization of wear-out for components and systems. *Ann. Math. Statist.* **37** 816–825.

[3] BIRNBAUM. Z. W., ESARY, J. D. and SAUNDERS, S. C. (1961). Multi-component systems and structures and their reliability. *Technometrics* **3** 55–77.

[4] ESARY, J. D. and MARSHALL, A. W. (1970). Coherent life functions. *SIAM J. Appl. Math.* **18** 810–814.

[5] ESARY, J. D., PROSCHAN, F. and WALKUP, D. W. (1967). Association of random variables with applications. *Ann. Math. Statist.* **38** 1466–1474.

[6] FRÉCHET, M. (1951). Sur les tableaux de corrélation dont les marges sont données. *Ann. Univ. Lyon Sect. A Ser.* 3 **14** 53–77.

[7] GUMBEL, E. J. (1960). Bivariate exponential distributions. *J. Amer. Statist. Assoc.* **55** 698–707.

[8] LEHMANN, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–1153.

[9] MARSHALL, A. W. and OLKIN, I. (1967a). A multivariate exponential distribution. *J. Amer. Statist. Assoc.* **62** 30–44.

[10] MARSHALL, A. W. and OLKIN, I. (1967b). A generalized bivariate exponential distribution. *J. Appl. Probability* **4** 291–302.

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