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NATURAL REAL EXPONENTIAL FAMILIES WITH CUBIC VARIANCE FUNCTIONS[1]

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Pursuing the classification initiated by Morris (1982), we describe all the natural exponential families on the real line such that the variance is a polynomial function of the mean with degree less than or equal to 3. We get twelve different types; the first six appear in the fundamental paper by Morris (1982); most of the other six appear as distributions of first passage times in the literature, the inverse Gaussian type being the most famous example. An explanation of this occurrence of stopping times is provided by the introduction of the notion of reciprocity between two measures or between two natural exponential families, and by classical fluctuation theory.

1. Introduction. If μ is a positive measure on the real line $\mathbb{R}$ such that $\mu(K)$ is finite for all bounded intervals K, we denote

$$L_\mu(\theta) = \int_{-\infty}^{+\infty} \exp(\theta x)\mu(dx) \leq +\infty$$

its Laplace transform, with θ in $\mathbb{R}$,

$$D(\mu) = \{\theta;\ L_\mu(\theta) < \infty\} \tag{1.1}$$

its existence domain (which is convex, by the Hölder inequality), and $\theta(\mu)$ the interior of $D(\mu)$. $\mathscr{M}$ will denote the set of such measures μ such that

1. μ is not concentrated in one point.
2. $\theta(\mu)$ is not empty.

If μ is in $\mathscr{M}$, we also denote

$$k_\mu(\theta) = \log L_\mu(\theta), \qquad \theta \in \theta(\mu), \tag{1.2}$$

called the *cumulant function* of μ (although μ is not necessarily a probability).

Recall that k_μ is a strictly convex function on $\theta(\mu)$, from the Hölder inequality, and that k_μ is real analytic on $\theta(\mu)$, from known properties of Laplace transforms.

To each μ in $\mathscr{M}$ and θ in $\theta(\mu)$, we associate the following probability distribution on $\mathbb{R}$:

$$P(\theta, \mu)(dx) = \exp\big(\theta x - k_\mu(\theta)\big)\mu(dx). \tag{1.3}$$

The set:

$$F = F(\mu) = \{P(\theta, \mu);\ \theta \in \theta(\mu)\}$$

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will be called the natural exponential family (NEF) generated by μ. We also say that μ *is a basis* of F. Note that a basis of F is by no means unique. If μ and μ' are in $\mathscr{M}$, then it is easy to check that $F(\mu) = F(\mu')$ if and only if there exists (a,b) in $\mathbb{R}^2$ such that $\mu'(dx) = \exp(ax + b)\mu(dx)$. Therefore, if μ is in $\mathscr{M}$ and $F = F(\mu)$,

$$\mathscr{B}_F = \{\mu' \in \mathscr{M};\ F(\mu') = F\} = \{\exp(ax + b)\mu(dx);\ (a, b) \in \mathbb{R}^2\}$$

is the set of bases of F. Obviously, from (1.3) $F \subset \mathscr{B}_F$.

Let us also insist on the fact that $F(\mu)$ is a *set*, which is the image of $\theta(\mu)$ in the set of probability measures on $\mathbb{R}$ by the map $\theta \mapsto P(\theta, \mu)$, and not the map itself.

Now it is easy to check, from (1.2) and (1.3), that if μ is in $\mathscr{M}$:

$$k'_\mu(\theta) = \int_{-\infty}^{+\infty} xP(\theta, \mu)(dx) \quad \text{if } \theta \text{ is in } \theta(\mu). \tag{1.4}$$

Denote by M_F the image of $\theta(\mu)$ in $\mathbb{R}$ by $\theta \mapsto k'_\mu(\theta)$ if $F = F(\mu)$; from (1.4), M_F will be called the *mean domain* of F. Note that M_F depends only on F, not on a particular μ in $\mathscr{B}_F$. Since k_μ is strictly convex on $\theta(\mu)$, $\theta \mapsto k'_\mu(\theta)$ is a bijection between $\theta(\mu)$ and M_F and we shall denote by ψ_μ: $M_F \to \theta(\mu)$ its inverse function, that is,

(1.5) if m is in M_F, $\psi_\mu(m)$ is the unique element of $\theta(\mu)$ such that $k_\mu(\psi_\mu(m)) = m$.

Therefore $m \mapsto P(\psi_\mu(m), \mu)$, where $P(\theta, \mu)$ is defined by (1.3), is a bijection between M_F and $F = F(\mu)$, and provides a new parametrization of F, sometimes called "parametrization by the mean." It is fairly easy to check that if F is a NEF and if μ and μ' are in $\mathscr{B}_F$, then $P(\psi_\mu(m), \mu) = P(\psi_{\mu'}(m), \mu')$. We shall denote

$$P(m, F) = P(\psi_\mu(m), \mu) \quad \text{if } \mu \text{ is in } \mathscr{B}_F \text{ and } m \text{ is in } M_F, \tag{1.6}$$

and we have $F = \{P(m, F);\ m \in M_F\}$.

We now come to our main subject: Taking F as an NEF, we denote

$$V_F(m) = \int_{-\infty}^{+\infty} (x - m)^2 P(m, F)(dx) \quad \text{if } m \in M_F. \tag{1.7}$$

The map $m \mapsto V_F(m)$ defined on M_F is called the *variance function* of F.

There are two main reasons to be interested in the variance function V_F of a NEF: First, it characterizes F [i.e., $M_F = M_{F_1}$, $V_F = V_{F_1}$ implies $F = F_1$; see Proposition 2.2(ii), or Morris (1982)]. Second, V_F is simple, much simpler than the generating measures of F, at least for a lot of usual distributions. This second fact was brought to light by Morris (1982) who classified in six types the NEF with the following property: V_F is the restriction to M_F of a polynomial function of m with degree less than or equal to 2. He obtained the normal, Poisson, binomial, negative binomial, gamma and a sixth family that we may call "hyperbolic cosine."

However, Morris does not cover the whole set of familiar distributions with simple variances. As an example, consider the stable distribution with parameter $1/2$ and $p > 0$ defined by

$$\mu(dx) = \frac{p}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{p^2}{2x}\right) 1_{]0,+\infty[}(x)\,dx. \tag{1.8}$$

A classical computation shows that $\theta(\mu) =]-\infty, 0[$, $k_\mu(\theta) = -p\sqrt{-2\theta}$, from which we deduce, if $F = F(\mu)$: $M_F =]0, +\infty[$, $\psi_\mu(m) = -p^2/2m^2$,

$$P(m,F)(dx) = \frac{p}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{p^2}{2x}\left(\frac{x}{m} - 1\right)^2\right) 1_{]0,+\infty[}(x)\,dx. \tag{1.9}$$

Therefore from (1.9) we see that $P(m, F)$ is a so-called inverse Gaussian distribution, considered in numerous papers, like Tweedie (1957), Folks and Chikkara (1978) and Chikkara and Guttman (1982). As is well known, the variance of $P(m, F)$ defined by (1.9) is $V_F(m) = m^3/p^2$, which is a cubic polynomial. Professor Seshadri called our attention to Morris (1982) and asked us whether there exist other NEFs with cubic variances or not. The aim of this paper is to give a detailed answer to this question. We are going to classify the NEFs such that V_F is the restriction to M_F of a polynomial with degree equal to 3 in six types: Two of them have densities, the inverse Gaussian type and Ressel type, the four others are concentrated on the set $\mathbb{N}$ of nonnegative integers: Abel type, Takács type and two arcsine (strict and extended) types. Among the six types, only the last two do not seem to have been considered in the literature and still lack a probabilistic or statistical interpretation. The remaining four, where they have been mentioned before, appear almost invariably as distribution of first passage times. We shall also provide an explanation of this fact by the technique of reciprocity (see Section 5). This idea can be traced back to a one-page paper by Tweedie (1945). Let us mention that in the three notes Letac and Mora (1986), Mora (1986) and Letac (1986) announcing the results of the present paper, five types, instead of six, were mentioned, the two arcsine types being amalgamated into one. We found it convenient (see Section 4) to split the former arcsine type in two, in order to get simpler statements.

To wind up this introduction, let us mention that the family of variance functions which are polynomials of degree less than or equal to 3 is not the only set of variance functions interesting to classify. In this respect, let us mention the "Tweedie scale," with variance $V_F(m) = Am^\alpha$ with $A > 0$, $\alpha \in \mathbb{R} \setminus]0,1[$ and $M_F =]0, +\infty[$ (except for $\alpha = 0$, where $M_F = \mathbb{R}$), which appears in Tweedie (1984) and has been frequently rediscovered [Bar-Lev and Enis (1987), Letac (1987) and Jorgensen (1987)]. One also has to mention the family $V = P + Q\sqrt{R}$, where P, Q and R are polynomials with degree $P \leq 3$, degree $Q \leq 2$ and degree $R \leq 2$, which extends to our cubic family, containing numerous useful distributions and which has still to be classified. Surprisingly enough, it does not seem to be a reasonable task to try to classify variances which are polynomials with, say, degree 4: Although a lot of them exist (see Corollary 3.3) only one is known, corresponding to $V(m) = Am^4$ in the Tweedie scale. A reason for the restriction to polynomials of degree less than or equal to 3 probably lies in the notion of reciprocity (see the remark before Section 6). We shall also be silent about possible extensions to higher dimensions with polynomial covariances [i.e., $V(m) = \Sigma m^{\alpha} V_{\alpha}$, where $m^{\alpha} = m_1^{\alpha_1} \dots m_d^{\alpha_d}$] with V_{α} are $(d.d)$ matrices; out of trivial cases (i.e., affine transformations of products of polynomial families in $\mathbb{R}$) the Wishart families are an example; but geometry and group theory are probably better tools to study multivariate NEFs than the variance function.

2. Steepness and operations on NEF. In this section we gather some facts about NEFs, mainly without proofs. We begin with the steepness property of a NEF, which is an obscure point in Morris (1982).

Let μ be in $\mathscr{M}$; the support $S(\mu)$ of μ is the smallest closed set of $\mathbb{R}$ such that $\mu(\mathbb{R} \setminus S(\mu)) = 0$; the interior of the smallest closed interval of $\mathbb{R}$ containing $S(\mu)$ is denoted by $I(\mu)$. Clearly, from (1.4), $M_{F(\mu)} \subset I(\mu)$. In most of the cases, $M_{F(\mu)} = I(\mu)$, but not always [in the second paragraph after his (2.9), Morris (1982) is wrong]. A simple example showing that $M_{F(\mu)}$ can be smaller than $I(\mu)$ can be obtained from Efron (1978): Taking

$$\mu(dx) = \frac{\exp(-|x|)\, dx}{1 + x^4},$$

one gets $\theta(\mu) =]-1, +1[$; since $k_{\mu}'(\theta)$ is odd and increasing on $]-1, +1[$, its limit when $\theta \uparrow 1$ is

$$c = \int_{-\infty}^{+\infty} x e^x \mu(dx) \Big/ \int_{-\infty}^{+\infty} e^x \mu(dx).$$

Therefore $]-c, c[= M_{F(\mu)} \neq \mathbb{R} = I(\mu)$.

The following theorem is a specialization to $\mathbb{R}$ of results of Barndorff-Nielsen (1978), Theorems 5.27 and 8.2.

THEOREM 2.1. *Let μ be in $\mathscr{M}$, let F be the NEF generated by μ and let $\theta(\mu) =]a, b[$, with $-\infty \leq a < b \leq +\infty$.*

Then $M_F = I(\mu)$ if and only if the following two conditions hold:

(i) *Either $a \notin D(\mu)$ or $\lim_{\theta \downarrow a} k_{\mu}'(\theta) = -\infty$ and $a \in D(\mu)$.*

(ii) *Either $b \notin D(\mu)$ or $\lim_{\theta \uparrow b} k_{\mu}'(\theta) = +\infty$ and $b \in D(\mu)$.*

If $M_F = I(\mu)$, F will be said to be steep; if $D(\mu) =]a, b[$, F will be said to be regular. Regularity implies steepness. The converse is false: Take μ as in (1.8), $D(\mu) = [0, +\infty[$, $M_F = I(\mu) =]0, \infty[$; $F(\mu)$ is steep and not regular.

At this point, we should emphasize that we have parted from a traditional terminology as described in Barndorff-Nielsen (1978) by defining the NEF as we do after (1.3), and not by $\{P(\theta, \mu);\ \theta \in D(\mu)\}$, where $D(\mu)$ is given in (1.1). Our definition has many advantages, especially in giving simpler statements.

The second result is a synthesis of simple properties of variance functions.

PROPOSITION 2.2. *Let μ be in $\mathscr{M}$, F the NEF generated by μ and V_F its variance function* [*see* (1.7)]. *Then*

(i) $V_F(m) > 0$ *for all m in M_F.*
(ii) $V_F(m) = k''_\mu(\psi_\mu(m)) = (\psi'_\mu(m))^{-1}$ *for all m in M_F* [*see* (1.5)].
(iii) $V_F(m)$ *is real analytic on M_F.*
(iv) *Let F_1 be another NEF such that $M_F \cap M_{F_1}$ contains a nonempty open interval O and such that $V_F(m) = V_{F_1}(m)$ for m in O. Then $F = F_1$.*

In particular, knowledge of the variance function gives knowledge of the NEF.

PROOF. Since it is very much in the spirit of Morris (1982), we omit it. □

The next proposition sometimes gives a way to get back a μ in $\mathscr{B}_F$ when V_F is known.

PROPOSITION 2.3. *Let F be a NEF with variance function V_F on M_F. Let ψ and ψ_1 be two primitives on M_F of $m \mapsto 1/V_F(m)$ and $m \mapsto m/V_F(m)$, respectively. Then there exists a μ in $\mathscr{B}_F$ such that*

$$\exp \psi_1(m) = \int_{-\infty}^{+\infty} \exp(x\psi(m))\mu(dx) \quad \text{for } m \text{ in } M_F. \tag{2.1}$$

Furthermore, $k_\mu(\psi(m)) = \psi_1(m)$ and

$$P(m, F)(dx) = \exp(x\psi(m) - \psi_1(m))\mu(dx) \quad \text{for } m \text{ in } M_F.$$

[*See* (1.6).]

PROOF. Easy. □

This proposition has been basic in our hunt for cubic families, as described in Theorem 6.2. We shall use it in the proof of Jorgensen's theorem (Theorem 3.2).

We will now describe the influence of an affine transformation on $\mathbb{R}$. We adopt the following notation: If $(\Omega, \mathscr{A}, \mu)$ is a measure space and if φ is a measurable map from $(\Omega, \mathscr{A})$ to another measurable space $(\Omega_1, \mathscr{A}_1)$ we denote by $\mu_1 = \varphi_* \mu$ the image measure of μ by φ in $(\Omega_1, \mathscr{A}_1)$, defined by $\mu_1(A_1) = \mu(\varphi^{-1}(A_1))$ for A_1 in $\mathscr{A}_1$.

PROPOSITION 2.4. *Let $\varphi(x) = ax + b$ with $a \neq 0$ and b real, and let F be some NEF, generated by μ. Denote $\mu_1 = \varphi_* \mu$. Then*

(i) *μ_1 is in $\mathscr{M}$, and $\theta(\mu_1) = \theta(\mu)$.*
(ii) $k_{\mu_1}(\theta) = b + k_\mu(a\theta)$ *for θ in $\theta(\mu_1)$.*
(iii) *Let $F_1 = F(\mu_1)$. Then $F_1 = \varphi_* F$ and for all μ' in $\mathscr{B}_F$, one has $\varphi_* \mu'$ in $\mathscr{B}_{F_1}$.*
(iv) $M_{F_1} = \varphi(M_F)$ *and* $\varphi_* P(m, F) = P(\varphi(m), F_1)$.
(v) $V_{F_1}(m) = a^2 V_F((m - b)/a)$ *for m in M_{F_1}.*

PROOF. Easy. □

Our next concern about NEFs is division. First, one has to observe that if μ and ν are in $\mathscr{M}$ and are such that $\theta(\mu) \cap \theta(\nu)$ is not empty, there exists a $\mu * \nu$ in $\mathscr{M}$ called the *convolution* of μ and ν such that

$$(2.2) \qquad k_{\mu * \nu}(\theta) = k_\mu(\theta) + k_\nu(\theta) \quad \text{for all } \theta \text{ in } \theta(\mu * \nu) = \theta(\mu) \cap \theta(\nu).$$

Note that this is true even if μ and ν are unbounded. Now, if μ_1 is in $\mathscr{M}$, we consider [with Jorgensen (1987)] the set $\Lambda(\mu_1)$ of positive numbers p such that there exists a μ_p in $\mathscr{M}$ with $\theta(\mu_p) = \theta(\mu_1)$ and

$$(2.3) \qquad k_{\mu_p}(\theta) = pk_{\mu_1}(\theta) \quad \text{for all } \theta \text{ in } \theta(\mu_1).$$

Clearly, from (2.2), if p and p' are in $\Lambda(\mu_1)$ then $p + p'$ is in $\Lambda(\mu_1)$ and

$$(2.4) \qquad \mu_{p+p'} = \mu_p * \mu_{p'}.$$

We call μ_p the "pth *power of* μ_1." Since 1 is in $\Lambda(\mu_1)$, (2.4) shows that $\mathbb{N}^+ \subset \Lambda(\mu_1)$. If $\Lambda(\mu_1) =]0, +\infty[$, μ_1 will be said to be *infinitely divisible*.

The next proposition gives the link between the μ_p and the NEF $F(\mu_p)$.

PROPOSITION 2.5. *Let μ_1 and μ_1' in M and their pth powers μ_p and μ_p' with p in $\Lambda(\mu_1)$ and $\Lambda(\mu_1')$, respectively. Assume that $F(\mu_1) = F(\mu_1')$. Then*

(i) $\Lambda(\mu_1) = \Lambda(\mu_1')$ *(denoted by* Λ*) and* $F(\mu_p) = F(\mu_p')$ *(denoted by* F_p*) for p in* Λ.

(ii) *For θ in $\theta(\mu_1)$, the pth power of $P(\theta, \mu_1)$ [see (1.3)] is $P(\theta, \mu_p)$.*

(iii) *For p in Λ, one has $M_{F_p} = pM_{F_1}$.*

(iv) *For p in Λ and m in M_{F_p},*

$$V_{F_p}(m) = pV_{F_1}\left(\frac{m}{p}\right).$$

PROOF. Easy. □

From this proposition, we can talk about $\Lambda(F)$ and about infinite divisibility for a NEF F.

We now complete these generalities by stating without proof a specialization to $\mathbb{R}$ of a theorem of the second author, about limit variance functions. It just makes the first six lines of Section 10 of Morris (1982) more precise. This theorem will not be used in the sequel, except for one example (see Section 5.)

THEOREM 2.6 [Mora (1987)]. *Let $(F_n)_{n=1}^\infty$ be a sequence of NEFs with mean domains M_n and variance functions V_n. Assume that there exists a nonempty open interval J contained in $\bigcap_{n=1}^\infty M_n$ and a strictly positive function V on J with $\lim_{n\to\infty} V_n(m) = V(m)$ uniformly on all compact subintervals of J. Then*

(i) *There exists a natural exponential family F such that $M_F \supset J$ and such that V_F restricted to J is equal to V.*

(ii) *For all* m *in* J, $\lim_{n\to\infty} P(m, F_n) = P(m, F)$ *in the weak convergence sense.*

3. Properties of variance functions. In this section we prove a simple and basic theorem about the variance functions. We complete it by a theorem due to Jorgensen, which is not essential to the classification of cubic variances, but will throw some light on the results of Sections 4 and 5.

THEOREM 3.1. *Let* F *be a NEF, with* $M_F =]a, b[$ *with variance function* V. *Assume that there exists* $]a_1, b_1[$ *containing* $]a, b[$ *and a function* $V_1\colon]a_1, b_1[\to]0, +\infty[$ *such that*

(i) V_1 *is real analytic.*
(ii) V_1 *restricted to* $]a, b[$ *is equal to* V.

Then $]a_1, b_1[=]a, b[$.

For instance, one can check using this result that there is no NEF with a variance function defined by $M_F =]a, b[=]1, +\infty[$ and $V_F(m) = m$. Applying Theorem 3.1 to $]a_1, b_1[=]0, +\infty[$, one gets the contradiction. We have found that Theorem 3.1 is necessary in order to prove the statements at the end of Section 4, in Morris (1982).

PROOF OF THEOREM 3.1. Let μ in M be such that $F = F(\mu)$. Consider a primitive ψ of $1/V$ on $]a, b[$, and a primitive ψ_1 of $1/V_1$ on $]a_1, b_1[$ such that ψ_1 restricted to $]a, b[$ is ψ; ψ and ψ_1 are strictly increasing and we denote by $]A, B[$ and $]A_1, B_1[$ their images, and by k' and k_1' their reciprocal functions. Let k be a primitive of k' on $]A, B[$, and k_1 a primitive of k_1' on $]A_1, B_1[$ such that k_1 restricted to $]A, B[$ is k.

Since V_1 is analytic on $]a_1, b_1[$, ψ_1 is analytic on $]a_1, b_1[$. Since V_1 is nonzero on $]a_1, b_1[$, k_1' is analytic on $]A_1, B_1[$. Hence

(i) $k_1(\theta) = \log \int_{-\infty}^{+\infty} \exp(\theta x)\mu(dx)$ *if* $a < \theta < B$.
(ii) k_1 *is analytic on* $]A_1, B_1[$.
(iii) $\int_{-\infty}^{+\infty} \exp(\theta x)\mu(dx) = +\infty$ if $\theta \notin [A, B]$.

These three properties will show that $A = A_1$ and $B = B_1$. It is enough to show that $A = A_1$. Without loss of generality assume that $A_1 < A = 0$.

Property (ii) implies that there exists $R > 0$ and a sequence $(c_n)_{n=0}^{\infty}$ such that

$$\exp k_1(\theta) = \sum_{n=0}^{\infty} c_n \frac{\theta^n}{n!} \quad \text{if } -R < \theta < R.$$

We show by induction that $c_n = \int_{-\infty}^{+\infty} x^n \mu(dx)$. For $n = 0$ and from (i), $c_0 = \lim_{\theta \downarrow 0} \exp k_1(\theta) = \int_{-\infty}^{+\infty} \mu(dx)$ is clear by monotone convergence. Assume that

$c_k = \int_{-\infty}^{+\infty} x^k \mu(dx)$ for $k \leq n$; then from (i):

$$c_{n+1} = \lim_{\theta \downarrow 0} \frac{1}{\theta^{n+1}} \sum_{k=n+1}^{\infty} \frac{\theta^k}{k!} c_k = \lim_{\theta \downarrow 0} \frac{1}{\theta^{n+1}} \left(\exp(k_1(\theta)) - \sum_{k=0}^{n} c_k \frac{x^k}{k!} \right).$$

Hence, using the induction hypothesis,

$$c_{n+1} = \lim_{\theta \downarrow 0} \int_{-\infty}^{+\infty} \left(\frac{1}{\theta^{n+1}} \sum_{k=n+1}^{\infty} \frac{\theta^k x^k}{k!} \right) \mu(dx),$$

and, by monotone convergence, the induction hypothesis is extended to $n + 1$.

Thus

$$\sum_{n=0}^{\infty} c_n \frac{\theta^n}{n!} = \int_{-\infty}^{+\infty} \exp(\theta x) \mu(dx) \quad \text{if } -R < \theta < R,$$

and this gives the contradiction with the fact [implied by (iii)] that

$$\int_{-\infty}^{+\infty} \exp(\theta x) \mu(dx) = +\infty \quad \text{if } -R < \theta < 0. \qquad \square$$

We now describe an unpublished result of Jorgensen, mentioned by Bar-Lev in the discussion of Jorgensen (1987). Our thanks goes to Professor Jorgensen, who has shown us a preprint of Professor Bar-Lev's contribution to the above-mentioned discussion, and who has authorized us to reproduce his statement here.

THEOREM 3.2 (Jorgensen, unpublished). *Let* $]a, b[\subset]0, +\infty[$ *and* V: $]a, b[\to]0, +\infty[$ *of class* C^∞. *Consider the sequence* $(L_n)_{n=0}^{\infty}$ *of functions on* $]a, b[$ *defined by*

$$L_0(m) = 1, \qquad L_{n+1}(m) = V(m) L_n'(m) + m L_n(m). \tag{3.1}$$

Then there exists a NEF F concentrated on $[0, +\infty[$ *such that* $M_F =]a, b'[$, *with* $a < b \leq b' \leq +\infty$ *and such that* $V_F(m) = V(m)$ *when* $a < m < b$, *if and only if the two following conditions hold*:

(i) *For* m_0 *in* $]a, b[$, $\int_a^{m_0} (1/V(m))\, dm = +\infty$.
(ii) $L_n(m) \geq 0$ *for all integers* $n \geq 0$ *and* $a < m < b$.

The following corollary is essentially due to Bar-Lev in the discussion of Jorgensen (1987), although his statement is slightly more informal:

COROLLARY 3.3. *Let* $V(m) = \sum_{n=1}^{\infty} a_n m^n$ *be the sum of a nonzero entire series with nonnegative coefficients, positive radius of convergence R and* $V(0) = 0$. *Then there exists a NEF F such that* $M_F =]0, R[$ *and* $V_F(m) = V(m)$ *if* $0 < m < R$. *Furthermore, F is infinitely divisible.*

PROOF. Theorem 3.2 is applied to $]a, b[=]0, R[$. The C^∞ condition is fulfilled since V is analytic. Condition (i) is fulfilled since $1/V$ has a pole at 0. Condition (ii) is fulfilled since the L_n are analytic functions on $]-R, R[$ with nonnegative coefficients. $M_F =]0, b'[$ implies $b' = R$, since $b' > R$ would imply analyticity of V on R and would contradict the definition of R. To check that F is infinitely divisible, observe that for any $p > 0$, the function V_p defined by $V_p(m) = pV(m/p)$ satisfies the same hypothesis as V in Corollary 3.3, and use Proposition 2.5. □

PROOF OF THEOREM 3.2. $\boxed{\Rightarrow}$ Consider two primitives ψ and ψ_1 of the functions $m \mapsto 1/V(m)$ and $m \mapsto m/V(m)$ on $]a, b[$. From Proposition 2.3, there exists μ in $\mathscr{B}_F$ such that

$$\exp \psi_1(m) = \int_0^\infty \exp(x\psi(m))\mu(dx) \quad \text{if } a < m < b.$$

Since μ is concentrated on $[0, +\infty[$, $\theta(\mu)$ contains some half-line $]-\infty, t[$. Furthermore, $M_F =]a, b'[$, from the hypothesis. Therefore $\lim_{m \downarrow a} \psi_\mu = -\infty$ [see (1.5)], with ψ_μ restricted to $]a, b[$ equal to ψ. Hence $\psi(]a, b[) =]-\infty, T[$ for some $T \le +\infty$. Without loss of generality, by changing μ in $\exp(Tx)\mu(dx)$, one may assume $T \ge 0$ (this only involves a change of primitive ψ). Hence

$$\exp \psi_1\big(k_\mu'(\theta)\big) = \int_0^\infty \exp(\theta x)\mu(dx) \quad \text{if } \theta < T. \tag{3.2}$$

It is now easy to show from (3.2) by induction on n that

$$\left(\frac{d}{d\theta}\right)^n \exp \psi_1\big(k_\mu'(\theta)\big) = \exp\big(\psi_1\big(k_\mu'(\theta)\big)L_n\big(k_\mu'(\theta)\big) \quad \text{if } \theta < T, \tag{3.3}$$

where L_n is defined by (3.1). Since k_μ' when restricted to $]-\infty, T[$ is a bijection onto $]a, b[$, property (ii) is shown by (3.3) since from (3.2) the first member of (3.3) is greater than or equal to 0. Property (i) is a reformulation of $\lim_{m \downarrow a} \psi(m) = -\infty$.

$\boxed{\Leftarrow}$ We define $\psi(m) = \int_{m_0}^m dx/V(x)$ and $\psi_1(m) = \int_{m_0}^m x\,dx/V(x)$, for fixed m_0 in $]a, b[$ and m in $]a, b[$. From (i) there exists $T \le +\infty$ such that $\psi(]a, b[) =]-\infty, T[$. Note that $T > 0$. Consider the inverse function k' of ψ. Hence if $f(\theta) = \exp(\psi_1 \circ k')(\theta)$, f is in the C^∞ class on $]-\infty, T[$, and its derivatives satisfy

$$f^{(n)}(\theta) = f(\theta)L_n(k'(\theta)), \qquad \forall\, \theta < T.$$

Hence $f^{(n)}(\theta) \ge 0$ on $]-\infty, T[$, $V_n f$ is absolutely monotone and [see Feller (1966), page 416] there exists a positive measure on $[0, +\infty[$ such that

$$f(\theta) = \int_0^\infty \exp(\theta x)\mu(dx).$$

From here it is easy to conclude that $V(m) = V_{F(\mu)}(m)$ if $a < m < b$. Clearly, if $M_{F(\mu)} =]a', b'[$, with $a' \le a < b \le b'$, $a' < a$ is impossible since (i) implies that $\liminf_{m \to a} V(m)_{F(\mu)} = 0$. □

4. NEF concentrated on $\mathbb{N}$. The case where μ, in $\mathscr{M}$, is concentrated on the set $\mathbb{N}$ of nonnegative integers is quite important for our classification problem. After recalling the Lagrange formula, we adapt Proposition 2.2 to the case where μ is concentrated on $\mathbb{N}$, when the information on μ is the generating function:

$$(4.1)\qquad f_\mu(z) = \sum_{n=0}^{\infty} \mu_n z^n, \quad \text{where } \mu(dx) = \sum_{n=0}^{\infty} \mu_n \delta_n(dx),$$

rather than the familiar cumulant function k_μ of (1.2) (recall that δ_n is the Dirac mass on n). After three fundamental examples, we build a technique (Theorem 4.5.) for constructing new variance functions on $\mathbb{N}$ from old ones, and from the fundamental examples, we get three cubic types concentrated on $\mathbb{N}$.

THEOREM 4.1 (Lagrange's formula). *Let g be analytic in a disc $D(0, r)$, $r > 0$, with $g(0) \neq 0$. Then there exists an $R > 0$ and an analytic function h on $D(0, R)$ such that*

$$(4.2)\qquad h(w) - wg(h(w)) = 0 \quad \text{for } w \text{ in } D(0, R).$$

Furthermore, if F is analytic on $D(0, r)$, then for all w in $D(0, R)$ one has

$$(4.3)\qquad F(h(w)) = F(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\left(\frac{d}{dz} \right)^{n-1} (F'(z)(g(z))^n \right]_{z=0}.$$

PROOF. See, for example, Dieudonné (1971) [where $g(0) \neq 0$ is inadvertently omitted]. □

COROLLARY 4.2. *With the same hypothesis and notation, for $p \neq 0$,*

$$(g(h(w)))^p = \sum_{n=0}^{\infty} w^n \mu_n,$$

with

$$(4.4)\qquad \mu_n = \frac{p}{p+n} \frac{1}{n!} \left[\left(\frac{d}{dz} \right)^n (g(z))^{n+p} \right]_{z=0} \quad \text{if } n \neq -p,$$

$$(4.5)\qquad \mu_n = \frac{p}{(n-1)!} \left[\left(\frac{d}{dz} \right)^{n-1} \left(\frac{g'(z)}{g(z)} \right) \right]_{z=0} \quad \text{if } n = -p.$$

PROOF. Apply (4.3) to $F(z) = (g(z))^p$, with an r small enough such that F is analytic in $D(0, r)$. If $n \neq -p$, observe that

$$\left(\frac{d}{dz} \right)^{n-1} \left[(g(z))^n \frac{d}{dz} (g(z))^p \right] = \frac{p}{p+n} \left(\frac{d}{dz} \right)^n (g(z))^{n+p},$$

to get (4.4). Equality (4.5) is straightforward from (4.3). □

Consider now a positive measure μ on $\mathbb{N}$ defined by

$$\mu(dx) = \sum_{n=0}^{\infty} \mu_n \delta_n(dx) \tag{4.6}$$

and its generating function f_μ as in (4.1); denote by $R(\mu)$ the radius of convergence of the entire series (4.1). Clearly, μ is in $\mathcal{M}$ if and only if $R(\mu) > 0$; in this case:

$$\theta(\mu) =]-\infty, R(\mu)[, \qquad k_\mu(\theta) = \log f_\mu(e^\theta), \tag{4.7}$$

$$P(\log z, \mu)(dx) = \sum_{n=0}^{\infty} \frac{\mu_n z^n}{f_\mu(z)} \delta_n(dx) \quad \text{if } 0 < z < R(\mu) \tag{4.8}$$

[see (1.3)]. Note that the family $F(\mu)$ is not necessarily steep: For a counterexample, take $\mu_n = (n+1)^{-3}$. Then $I(\mu) =]0, +\infty[$, $R(\mu) = 1$, $M_{F(\mu)} =]0, b[$, with

$$b = \sum_{n=0}^{\infty} n(n+1)^{-3} \Big/ \sum_{n=0}^{\infty} (n+1)^{-3} = \frac{\pi^2}{6\zeta(3)} - 1.$$

The next proposition enables us to compute $V_{F(\mu)}$ from f_μ, under the mild condition (4.9).

PROPOSITION 4.3. *Consider* μ *defined by* (4.6) *with* $R = R(\mu) > 0$. *Assume that*

$$\mu_0 > 0 \quad \text{and} \quad \mu_1 > 0, \tag{4.9}$$

and consider the NEF $F = F(\mu)$. *Then* $M_F =]0, b[$ *with* $0 < b \leq +\infty$.

Furthermore, there exists an open subset U *of the complex disc* $D(0, R)$ *containing the real segment* $[0, R[$ *and on which* $f_\mu'(z) \neq 0$. *Defining*

$$g_\mu(z) = \frac{f_\mu(z)}{f_\mu'(z)} \quad \text{for } z \text{ in } U,$$

there exists an open subset O *of the complex plane containing the real segment* $[0, b[$ *and an analytic function*

$$h_\mu : O \to U$$

such that

$$h_\mu(m) - m g_\mu(h_\mu(m)) = 0, \qquad \forall\, m \in O. \tag{4.10}$$

With this notation, one has

$$P(m, F)(dx) = \sum_{n=0}^{\infty} \frac{(h_\mu(m))^n \mu_n}{f_\mu(h_\mu(m))} \delta_n(dx), \tag{4.11}$$

$$V_F(m) = \frac{h_\mu(m)}{h_\mu'(m)} \quad \text{for } 0 < m < b. \tag{4.12}$$

PROOF. Define $U = \{z \in D(0, R);\ f_\mu'(z) \neq 0\}$. Condition (4.9) implies 0 is in U and $g_\mu(0) = \mu_0/\mu_1 \neq 0$. From (4.7),

$$k_\mu'(\theta) = \frac{e^\theta}{g_\mu(e^\theta)} = m, \tag{4.13}$$

for all θ in $\theta(\mu)$. Hence $M_F =]0, b[$, with $b = \lim_{z \uparrow R}(z/g_\mu(z))$.

Consider now $\psi_\mu(m)$, defined by (1.5), for $0 < m < b$, and $h_\mu(m) = \exp \psi_\mu(m)$. Then from (4.13) h_μ satisfies (4.10) for $0 < m < b$, and is a real analytic function on $]0, b[$. Now, since $g_\mu(0) \neq 0$, Theorem 4.1 can be applied and there exists $r > 0$ and an analytic function h_μ on $D(0, r)$ such that (4.10) is also fulfilled on $D(0, r)$. Since $D(0, r) \cap]0, b[$ is not empty, the two h_μ coincide on it and there exists an open subset O of the complex plane containing $D(0, r) \cup]0, b[$ and an analytic function h_μ: $O \to U$ such that (4.10) is true on O.

Formulas (4.11) and (4.12) are easy consequences of (1.6), of $V_F(\psi_\mu'(m))^{-1}$ (Proposition 2.2) and of 4.10. □

We illustrate the previous proposition by three fundamental examples. We keep the notation of Proposition 4.3.

EXAMPLE A (The Poisson type).

$$\mu(dx) = \sum_{n=0}^{\infty} \frac{1}{n!}\delta_n(dx),$$

$$R(\mu) = +\infty, \qquad f_\mu(z) = \exp z, \qquad g_\mu(z) = 1, \qquad M_F =]0, +\infty[,$$

$$U = O = \mathbb{C}, \qquad h_\mu(m) = m, \qquad V_F(m) = m.$$

EXAMPLE B (The negative binomial type). Let $a > 0$ be a fixed number.

$$\mu(dx) = \sum_{n=0}^{\infty} a(a+1)\dots(a+n-1)\frac{\delta_n(dx)}{n!},$$

$$R(\mu) = 1, \qquad f_\mu(z) = (1-z)^{-1}, \qquad U = D(0,1), \qquad M_F =]0, +\infty[,$$

$$g_\mu(z) = \frac{1}{a}(1-z), \qquad O = \left\{m \in \mathbb{C};\ Re m > -\frac{a}{2}\right\},$$

$$h_\mu(m) = \frac{m}{m+a}, \qquad V_F(m) = m\left(1 + \frac{m}{a}\right).$$

EXAMPLE C (The strict arcsine type). Let $a > 0$ be fixed. μ is defined by

$$f_\mu(z) = \exp(a \arcsin z) \quad \text{with } R(\mu) = 1.$$

It is an advanced calculus exercise to check that

$$f_\mu(z) = \sum_{n=0}^{\infty} \frac{p_n(a)}{n!} z^n,$$

where the $p_n(a)$ are polynomials with respect to a defined by

$$p_{2n}(a) = \prod_{k=0}^{n-1} (a^2 + 4k^2),$$

(4.14)

$$p_{2n+1}(a) = a \prod_{k=0}^{n-1} \left(a^2 + (2k+1)^2\right).$$

To check it, one has to use the differential equation

$$f_\mu'(z) = \frac{a}{(1 - z^2)^{1/2}} f_\mu(z).$$

Clearly, from (4.14) $\mu_n = p_n(a)/n!$ will define a positive measure on $\mathbb{N}$.

Now $U = D(0,1)$, $g_\mu(z) = (1/a)(1 - z^2)^{1/2}$, $M_F =]0, +\infty[$,

$$O = \{m \in \mathbb{C};\ |m|^2 \leq |m^2 + a^2|\}$$

and $h_\mu(m) = m/\sqrt{m^2 + a^2}$. This gives the following variance function:

$$V_F(m) = m\left(1 + \frac{m^2}{a^2}\right) \quad \text{for } 0 < m. \tag{4.15}$$

[Note that (4.15) gives a cubic variance, and (4.15) falls in the scope of Corollary 3.3.]

It is worth mentioning here a criterion for the following problem: Given the variance function V of some NEF F on $\mathbb{R}$, how does one decide whether F is concentrated on $\mathbb{N}$ [with condition (4.9)] or not? Here is a precise answer.

PROPOSITION 4.4. *Let F be a NEF on $\mathbb{R}$ with variance function V defined on M_F. Then F is concentrated on $\mathbb{N}$ such that* (4.9) *holds if and only if*

(i) $M_F =]0, b[$ *for some* $0 < b \leq +\infty$.

(ii) *There exists an open subset O of the complex plane containing* $[0, b[$, *and an analytic function ψ_1' on O such that $\psi_1'(m) = m/V(m)$ if $0 < m < b$ and such that $\psi_1'(0) = 1$.*

In this case, if ψ_1 is a primitive of ψ_1' in O, if G is an analytic function in O such that

$$\frac{G'(m)}{G(m)} = \frac{1}{m}(1 - \psi_1'(m)) \quad \text{and} \quad G(0) = 1$$

and if

$$(4.16)\qquad \begin{cases} \mu_0 = \exp\psi_1(0), \\ \mu_n = \dfrac{1}{n!}\left[\left(\dfrac{d}{dm}\right)^{n-1}\left(\exp(\psi_1(m))\psi_1'(m)(G(m))\right)^n\right]_{m=0}, \end{cases}$$

then $\mu(dx) = \sum_{n=0}^{\infty}\mu_n\delta_n(dx)$ *generates* F.

PROOF. $\boxed{\Rightarrow}$ Using Proposition 4.3, (i) is obvious. Taking μ in $\mathscr{B}_F$, ψ_1' can be defined by $\psi_1'(m) = mh_\mu'(m)/h_\mu(m)$. To check that $\psi_1'(0) = 1$, recall that, from the Lagrange formula (4.3) applied to $F(z) = z$ and $g = g_\mu$, one gets

$$h_\mu'(0) = g_\mu(0) = \frac{\mu_0}{\mu_1}.$$

We now prove both $\boxed{\Leftarrow}$ and the last part of the proposition. Since $\psi_1'(0) = 1$, there exists an analytic function G in O satisfying (4.10). Since

$$\frac{G'(m)}{G(m)} = \frac{1}{m} - \frac{1}{V(m)},$$

there exists a primitive ψ of $1/V$ on $]0, b[$ such that

$$(4.17)\qquad G(m) = m\exp(-\psi(m)) \quad \text{for } 0 < m < b.$$

Note that $\lim_{m\downarrow 0}\psi(m) = -\infty$, from (ii). Denote $R = \lim_{m\uparrow b}\log\psi(m) \le +\infty$.

Since G is analytic in O, one applies Theorem 4.1 to $g = G$. Therefore there exists $r > 0$ and h analytic in $D(0, r)$ valued in O such that

$$(4.18)\qquad h(w) - wG(h(w)) = 0 \quad \text{if } w \in D(0, r).$$

Similar considerations to the proof of Proposition 4.3 allow us to claim the existence of an open set U of $\mathbb{C}$ containing $D(0, r)$ such that there exists an analytic function $h\colon U \to O$ which still satisfies (4.18) for w in U.

Introduce a primitive ψ_1 of ψ_1' in O and apply the Lagrange formula (4.3) to $F(m) = \exp\psi_1(m)$. One gets

$$(4.19)\qquad \exp\psi_1(h(w)) = \sum_{n=0}^{\infty}\mu_n w^n \quad \text{for } w \in U,$$

where μ_n is defined by (4.16). Using (4.17) and (4.18), the formula (4.19) becomes

$$(4.20)\qquad \exp\psi_1(m) = \sum_{n=0}^{\infty}\mu_n\exp(n\psi(m)) \quad \text{for } m \text{ in } O.$$

Now we use Proposition 2.3 to claim that there exists μ in $\mathscr{M}$ such that (2.1) holds. Using the uniqueness of Laplace transforms and comparing with (4.20), one gets

$$\mu = \sum_{n=0}^{\infty}\mu_n\delta_n.$$

The last thing to verify is condition (4.6). Actually,

$$\mu_0 = \exp \psi_1(0) > 0 \quad \text{and} \quad \mu_1 = 1,$$

since $G(0) = 1$. □

Actually, Proposition 4.4 is an interesting theoretical result to identify a variance function on integers, and we shall use it in Proposition 6.1. However, we have found the explicit formula (4.16) rather deceptive when one has to apply it to concrete examples in order to compute μ_n. For instance, we know from Corollary 3.3 and Proposition 4.4 that there exists a NEF F on $\mathbb{N}$ such that

$$M_F =]0, +\infty[\quad \text{and} \quad V_F(m) = m\left(1 + \frac{m}{p_1}\right)\cdots\left(1 + \frac{m}{p_k}\right),$$

with $p_1, \dots, p_k > 0$. But (4.16) is not really helpful in computing μ_n; through Corollary 3.3, μ_n is positive and (4.16) provides only an inequality for which it is generally difficult to get a direct proof. Similar considerations hold with

$$V_F(m) = \frac{m}{(1 - m/p_1)\cdots(1 - m/p_k)} \quad \text{on } M_F =]0, +\infty[.$$

We shall now explain a basic trick to obtain new variance functions on $\mathbb{N}$ from the old ones.

THEOREM 4.5. *Let $p > 0$, $g(z) = \sum_{n=0}^{\infty} g_n z^n$ with radius of convergence $R(g) > 0$. Assume*

(i) $g_n \geq 0$ *for all n in $\mathbb{N}$, $g_0 > 0$ and $g_1 > 0$.*
(ii) $\mu_n = p/(p+n) \cdot 1/n![(d/dz)^n(g(z))^{n+p}]_{z=0} \geq 0$ *for all n in $\mathbb{N}$.*

Let F and F_1 be the NEF generated by $\mu = \sum \mu_n \delta_n$ and $\nu = \sum g_n \delta_n$, respectively. Then, if

$$M_{F_1} =]0, b_1[\quad \text{and} \quad M_F =]0, b[,$$

one has $b = +\infty$ if $b_1 \geq 1$ and $b = b_1 p/(1 - b_1)$ if $0 < b_1 < 1$. Furthermore,

$$V_F(m) = \frac{(m+p)^3}{p^2} V_{F_1}\left(\frac{m}{m+p}\right) \quad \text{if } 0 < m < b. \tag{4.21}$$

PROOF. From Corollary 4.2, $R(\mu) > 0$, μ is in $\mathscr{M}$ and F exists. Now μ fulfills condition (4.9), since

$$\mu_0 = g_0^p > 0 \quad \text{and} \quad \mu_1 = p g_0^p g_1 > 0.$$

Hence from Proposition 4.3 $M_F =]0, b[$ for some b in $]0, +\infty]$.

Let ψ and ψ_1 be primitives of $m \mapsto 1/V_F(m)$ and $m \mapsto m/V_F(m)$ on $]0, b[$ such that

$$\exp \psi_1(m) = \sum_{n=0}^{\infty} \mu_n \exp(n\psi(m)) \quad \text{for } 0 < m < b$$

(see Proposition 2.3). Hence if $w = \exp\psi(m)$ is in $]0, R(\mu)[$ and if h is defined by (4.2) from g in $D(0, R(\mu))$, one gets from Corollary 4.2 and formula (4.2):

$$g(h(w)) = \frac{h(w)}{w} = \exp\frac{\psi_1(m)}{p}, \tag{4.22}$$

$$h(w) = \exp\left(\psi(m) + \frac{\psi_1(m)}{p}\right) \quad \text{for } 0 < w < R(\mu). \tag{4.23}$$

The aim of the following lengthy and obscure calculations is to eliminate ψ_1 between (4.22) and (4.23), in order to have an explicit expression of ψ and get $V_F(m) = (\psi'(m))^{-1}$. Formulas (4.22) and (4.23) together give

$$\exp\left(\frac{1}{p}\psi_1(m)\right) = g\left(\exp\left(\psi(m) + \frac{\psi_1(m)}{p}\right)\right) \quad \text{for } 0 < m < b. \tag{4.24}$$

We take the derivative in (4.24) in m, cancel the $V_F(m)$ and get

$$\frac{m}{m+p}\exp(-\psi(m)) = g'\left(\exp\left(\psi(m) + \frac{\psi_1(m)}{p}\right)\right) \quad \text{for } 0 < m < b. \tag{4.25}$$

Since g has positive coefficients, g is increasing on $]0, R(g)[$ and its inverse g^{-1} is defined on

$$I =]g(0), g(R(g))[. \tag{4.26}$$

Hence from (4.24):

$$\exp\left(\psi(m) + \frac{\psi_1(m)}{p}\right) = g^{-1}\left(\exp\frac{\psi_1(m)}{p}\right) \quad \text{for } 0 < m < b. \tag{4.27}$$

Carrying (4.27) to (4.25),

$$\frac{m}{m+p}\exp(-\psi(m)) = (g'\circ g^{-1})\left(\exp\frac{\psi_1(m)}{p}\right) \quad \text{for } 0 < m < b. \tag{4.28}$$

Multiply (4.28) by $\exp(-\psi_1(m)/p)$ and use (4.27) again:

$$\begin{aligned} &\frac{m}{m+p}\,\frac{1}{g^{-1}(\exp\psi_1(m)/p)} \\ &\quad = \exp\left(-\frac{\psi_1(m)}{p}\right)(g'\circ g^{-1})\left(\exp\frac{\psi_1(m)}{p}\right). \end{aligned} \tag{4.29}$$

Now we introduce $\varphi\colon I \to \mathbb{R}$, where I is (4.26):

$$\varphi(z) = (g'\circ g^{-1})(z)\frac{g^{-1}(z)}{z}. \tag{4.30}$$

(4.29) can be written in a simpler way:

$$\frac{m}{m+p} = \varphi\left(\exp\frac{\psi_1(m)}{p}\right) \quad \text{for } 0 < m < b. \tag{4.31}$$

Now we observe that $\varphi(I) = M_{F_1} =]0, b_1[$ (recall that F_1 is the NEF generated by $\nu = \sum_{n=0}^{\infty} g_n \delta_n$): This comes from the easy formula (4.13). Furthermore, φ is a bijection between I and M_{F_1}, as the composition of two injective functions $w \mapsto g'(w)w/g(w)$ and $z \mapsto g^{-1}(z)$. Denote by G: $]0, b_1[\to I$ the reciprocal function of φ. Then we have

$$\frac{G'(m)}{G(m)} = \frac{m}{V_{F_1}(m)} \quad \text{for } m \text{ in } M_{F_1}. \tag{4.32}$$

To check (4.32), use $(\varphi \circ g)(e^\theta) = m$ if $e^\theta < R(g)$, get $\theta = \log((g^{-1} \circ G)(m))$ and take the derivative to obtain

$$\frac{1}{V_{F_1}(m)} = \frac{1}{(g^{-1} \circ G)(m)} \frac{1}{g'(g^{-1} \circ G)(m)} G'(m). \tag{4.33}$$

Since the denominator of the second member of (4.33) is

$$\varphi(G(m))G(m) = mG(m),$$

(4.32) is proved.

Coming back to (4.31), we compose with G to get

$$\exp \frac{\psi_1(m)}{p} = G\left(\frac{m}{m+p}\right)$$

or

$$\psi_1(m) = pG\left(\frac{m}{m+p}\right), \qquad 0 < m < b. \tag{4.34}$$

Taking derivatives in m of (4.34) gives

$$\frac{m}{V_F(m)} = \frac{p^2}{(m+p)^2} \frac{G'(m/(m+p))}{G(m/(m+p))}, \qquad 0 < m < b,$$

and finally from (4.32), (4.21) is proven. □

We now give three basic applications of the previous theorem, corresponding to our Examples A, B and C.

Example D (The Abel type). We apply Theorem 4.5 to $g(z) = \exp z$. Hence, from Example A, F_1 belongs to the Poisson type, $V_{F_1}(m) = m$ and $b_1 = +\infty$. The computation of μ_n gives

$$\mu_n = p\frac{(n+p)^{n-1}}{n!} \quad \text{for } n \text{ in } \mathbb{N}. \tag{4.35}$$

Since $\mu_n > 0$, Theorem 4.5 is applicable for any $p > 0$, and the corresponding exponential family F has a variance function described by

$$M_F =]0, +\infty[, \qquad V_F(m) = m\left(1 + \frac{m}{p}\right)^2 \tag{4.36}$$

from (4.32). We have chosen to call this set of families the Abel type instead of "generalized Poisson" as in Consul and Jain (1973). This alludes to a famous formula due to Abel; see, for instance, Comtet (1974), Theorem C; page 130, which should be applied to $f(t) = \exp tz$, $t = p$, $u = -1$, to get the generating function $f_\mu(z)$ of (4.25). Recall that from Theorem 4.5 and Corollary 4.2 the generating function f_μ of μ is

$$f_\mu(z) = \exp(ph(z)), \quad \text{where } h(z)\exp(-h(z)) = z.$$

Note that this family has already appeared in the literature. Notable references are Pyke (1958), Theorem 2, Note 3, and Consul and Jain (1973) [suitably corrected by Nelson (1975)]. Abel's type also appears in Consul and Shenton (1972), entry 6 and (up to an affinity) entry 5, Table 6.1. We understand that Consul is currently preparing a book on the subject.

EXAMPLE E (The Takács type). We apply Theorem 4.5 to $g(z) = (1-z)^{-a}$, with $a > 0$. Hence, from Example B, F_1 belongs to the negative binomial type, $V_{F_1}(m) = m(1 + m/a)$ and $b_1 = +\infty$. The computation of μ_n gives

$$\mu_n = \frac{p}{p+n}\frac{1}{n!}a(n+p)[a(n+p)+1] \times [a(n+p)+2]\dots[a(n+p)+n-1]. \tag{4.37}$$

Since $\mu_n > 0$, Theorem 4.5 is applicable for any $p > 0$, and the corresponding exponential family F has a variance function described by

$$M_F =]0, +\infty[, \qquad V_F(m) = m\left(1 + \frac{m}{p}\right)\left(1 + \frac{a+1}{a}\cdot\frac{m}{p}\right). \tag{4.38}$$

We have chosen, somewhat arbitrarily, to give to this set of families the name of an illustrious probabilist among other people who have been considering various subsets of our Takács type; some authors call this set "generalized negative binomial distributions" [Nelson (1975)].

Actually, the members of the family F generated by (4.37) appear in the literature for various values of the parameters a and p. We find that Charalambides (1986) provides a good bibliography on the subject, including Takács (1962) and Jain and Consul (1971). Up to affinities, Takács type appears in entries 1, 2, 9 and 12 of Table 6.1 of Consul and Shenton (1972). The most famous particular case probably corresponds to $a = p = 1$. In this case (4.37) becomes the Catalan number

$$\mu_n = \frac{(2n)!}{n!(n+1)!}.$$

Therefore

$$f_\mu(z) = \sum_{n=0}^{\infty}\frac{(2n)!}{n!(n+1)!}z^n = \frac{1-\sqrt{1-4z}}{2z}.$$

The distributions generated by this μ or its images by affinities appear in fluctuation theory [see, for instance, Feller (1966), page 396, (79)]. For this reason, Example E was previously called the "fluctuation type" [Mora (1986)].

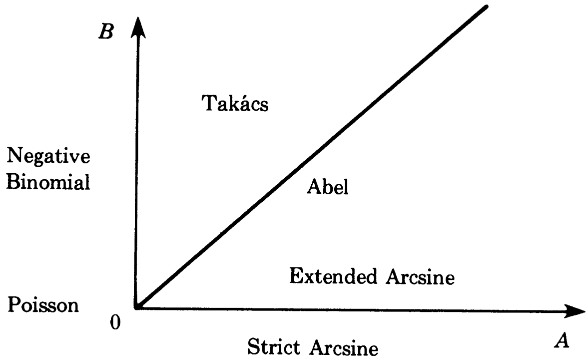

FIG. 1.

TABLE 1

$0 = A = B$	Poisson
$0 = A < B$	Negative binomial
$0 = B < A$	Strict arcsine
$0 < A = B$	Abel
$0 < A < B$	Takács
$0 < B < A$	Extended arcsine

EXAMPLE F (The extended arcsine type). We apply Theorem 4.5 to $g(z) = \exp(a \arcsin z)$ with $a > 0$. Hence, from Example C, F_1 belongs to the strict arcsine type, $V_{F_1}(m) = m(1 + m^2/a^2)$ and $b_1 = +\infty$. The computation of μ_n gives

$$\mu_n = \frac{p}{p+n}\frac{1}{n!}p_n(a(n+p)), \tag{4.39}$$

where the polynomials $(p_n)_{n=0}^{\infty}$ are defined by (4.14).

Again μ_n is positive, Theorem 4.5 is applicable for any $p > 0$ and the corresponding exponential family F has a variance function described by

$$M_F =]0, +\infty[, \qquad V_F(m) = m\left(1 + \frac{2}{a}\frac{m}{p} + \frac{1+a^2}{a^2}\left(\frac{m}{p}\right)^2\right). \tag{4.40}$$

Examples C and G do not seem to appear anywhere in the literature.

We have summarized Examples A, B, C, D, E and F in Figure 1. These examples are all concentrated on $\mathbb{N}$ with a cubic variance of the following form:

$$M_F =]0, +\infty[, \qquad V_F(m) = m(1 + 2Bm + A^2m^2),$$

where $B \geq 0$ and $A \geq 0$. Therefore we have Table 1.

Of course, there is a seventh type of cubic on $\mathbb{N}$ which is simply the binomial type: For each integer $N > 0$, one has the family F described by

$$(4.41) \qquad M_F =]0, N[, \qquad V_F = m\left(1 - \frac{m}{N}\right),$$

corresponding to $A = 0$ and $B = -1/N$. Application of Theorem 4.5 to this would give again a Takács family. We shall see (Theorem 6.2) that these seven types are the only cubic ones concentrated on $\mathbb{N}$ (up to an affinity).

5. Ressel families: The reciprocity of two exponential families.

A. Reciprocity. Up to now, the variance functions which are polynomials in m with degree equal to 3 that we have considered are, for $M_F =]0, +\infty[$, $V_F(m) = m^3/p^2$ (inverse Gaussian) and $V_F(m) = m(1 + 2Bm + A^2m^2)$ with $B \geq 0$ and $A > 0$.

All of them can be proved to be variance functions by means of Corollary 3.3, but Theorem 4.5 was devised in order to describe explicitly the distributions corresponding to $m(1 + 2Bm + A^2m^2)$.

Now we have to consider the same question for the following variance:

$$(5.1) \qquad M_F =]0, +\infty[, \qquad V_F(m) = \frac{m^2}{p}\left(1 + \frac{m}{p}\right), \qquad p > 0.$$

Again, from Corollary 3.3, we can claim that (5.1) is the variance function of a steep natural exponential family F. We decided to call it the *Ressel family of power* p. Actually, Ressel (private communication) has shown us a proof of the fact that (5.1) defines a variance function; this proof is described in Mora (1986). It turns out that this proof, when generalized, yields nothing but Corollary 3.3.

One should also mention that another proof of the same result can be obtained from Theorem 2.5 applied to the sequence of variance functions:

$$M_{F_n} =]0, +\infty[, \qquad V_{F_n}(m) = \frac{m}{p}\left(\frac{1}{n} + m\right)\left(1 + \frac{m}{p}\right).$$

[Note that F_n is the image by $x \mapsto x/pn$ of a Takács family with variance $m(1 + m/p)(1 + m/np^2)$: Use Proposition 2.4(v).]

One can use the approach of Theorem 2.5 to obtain, by a tedious limiting process, an explicit distribution generating the Ressel family of power p [see Letac (1986)]. But the concept of reciprocity that we are going to introduce, together with known results about Lévy processes, will enable us to get the Ressel distributions (i.e., generating Ressel families) in a natural and straightforward way (see Proposition 5.5).

DEFINITION 5.1. If μ is in $\mathscr{M}$, we denote

$$\tilde{\theta}(\mu) = \{\theta;\ \theta \in \theta(\mu) \text{ and } k'_\mu(\theta) > 0\}.$$

A pair (μ, μ_1) of elements in $\mathcal{M}$ will be said to be *reciprocal* if

(i) $\tilde{\theta}(\mu)$ and $\tilde{\theta}(\mu_1)$ are nonempty.

(ii) The image of $\tilde{\theta}(\mu)$ by $\theta \mapsto -k_\mu(\theta)$ is $\tilde{\theta}(\mu_1)$, and the image of $\tilde{\theta}(\mu_1)$ by $\theta \mapsto -k_{\mu_1}(\theta)$ is $\tilde{\theta}(\mu)$.

(iii) $-k_{\mu_1}(-k_\mu(\theta)) = \theta$ for all θ in $\tilde{\theta}(\mu)$.

The most famous example of a reciprocal pair is probably

$$(5.2)\qquad \begin{cases} \mu(dx) = \exp\left(-\dfrac{x^2}{2p}\right)\dfrac{dx}{\sqrt{2\pi p}}, \\ \mu_1(dx) = \dfrac{1}{2\pi p}\dfrac{1}{x^{3/2}}\exp\left(-\dfrac{1}{2\sqrt{px}}\right)1_{]0,+\infty[}(x)\,dx, \end{cases} \qquad p > 0,$$

for which $\theta(\mu) = \mathbb{R}$, $k_\mu(\theta) = p\theta^2/2$, $\tilde{\theta}(\mu) =]0, +\infty[$ and $\theta(\mu_1) = \tilde{\theta}(\mu_1) =]-\infty, 0[$, $k_\mu'(\theta) = -\sqrt{-2\theta/p}$. We shall later see other examples.

We indicate some simple facts about reciprocity in the next proposition.

PROPOSITION 5.1. *Let (μ, μ_1) be a reciprocal pair. Then*

(i) *The two subsets of $\mathbb{R}^2$, equipped with its Euclidean structure*:

$$\{(x, y);\ x \in \tilde{\theta}(\mu),\ y = k_\mu(x)\} \quad \text{and} \quad \{(x, y);\ x \in \tilde{\theta}(\mu_1),\ y = k_{\mu_1}(x)\}$$

are symmetrical with respect to the line $y + x = 0$.

(ii) *If (a, b) is in $\mathbb{R}^2$,*

$$\mu'(dx) = \exp(ax + b)\mu(dx) \quad \text{and} \quad \mu_1'(dx) = \exp(bx + a)\mu_1(dx)$$

defined a reciprocal pair (μ', μ_1'). Furthermore, if $F = F(\mu)$ and $F_1 = F(\mu_1)$, the map $\mu' \mapsto \mu_1'$ is a bijection between $\mathcal{B}_F$ and $\mathcal{B}_{F_1}$.

(iii) *One has equivalence between the three facts*: (a) *μ and μ_1 are probabilities*; (b) *$\tilde{\theta}(\mu)$ and $\tilde{\theta}(\mu_1)$ contain 0 in their closure*; (c) *$\tilde{\theta}(\mu)$ or $\tilde{\theta}(\mu_1)$ contain 0 in their closure.*

PROOF. (i) and (ii) are obvious, as for (iii) (b) ⇒ (c) and (b) ⇒ (a); now (c) ⇒ (b) comes from (i). Let us prove that (a) ⇒ (c).

From (a), 0 belongs to the closure of $\theta(\mu)$ and $\theta(\mu_1)$. If (c) is false, there exists an interval $]\alpha, \beta[= \{\theta;\ k_\mu'(\theta) < 0\}$ with $-\infty \le \alpha \le 0 < \beta < +\infty$. Now, since μ is a probability, $\lim_{\theta \downarrow 0} k_\mu(\theta) < 0$. Hence $k_\mu(\beta) = 0$ and $k_\mu'(\beta) = 0$. A parallel reasoning with μ_1 shows the existence of $\beta_1 > 0$ such that $k_\mu(\beta_1) < 0$ and $k_\mu'(\beta) = 0$, with $\tilde{\theta}(\mu_1) \subset]\beta_1, +\infty[$. Now use (i): $\tilde{\theta}(\mu_1)$ must be contained in $]-\infty, k_\mu(\beta)[\subset]-\infty, 0[$, a contradiction. □

A consequence of Proposition 5.1 for NEFs is the following: Assume that (μ, μ_1) is a reciprocal pair and denote $F = F(\mu)$ and $F_1 = F(\mu_1)$. Since each element ν of $\mathscr{B}_F$ has a reciprocal ν_1 in $\mathscr{B}_{F_1}$, and since $\nu \mapsto \nu_1$ is a bijection between $\mathscr{B}_F$ and $\mathscr{B}_{F_1}$, this leads us to the following definition.

DEFINITION 5.2. Let F and F_1 be two NEFs on $\mathbb{R}$. Then (F, F_1) is a *reciprocal pair* if there exists a reciprocal pair (μ, μ_1) in $\mathscr{B}_F \times \mathscr{B}_{F_1}$.

We shall characterize reciprocal pairs (F, F_1) by the variance functions by means of the following theorem.

THEOREM 5.2. *Let F and F_1 be two NEFs on $\mathbb{R}$, and denote $\tilde{M}_F = M_F \cap]0, +\infty[$ and $\tilde{M}_{F_1} = M_{F_1} \cap]0, +\infty[$. Then*

(A) *(F, F_1) is a reciprocal pair if and only if the three following conditions hold*:

(i) *$\tilde{M}_F$ and $\tilde{M}_{F_1}$ are nonempty.*
(ii) *$m \mapsto 1/m$ is a bijective mapping from $\tilde{M}_F$ onto $\tilde{M}_{F_1}$.*
(iii) *$V_F(m) = m^3 V_{F_1}(1/m)$ for all m in $\tilde{M}_F$.*

(B) *Furthermore, if (F, F_1) is a reciprocal pair of NEFs, then (μ, μ_1) in $F \times F_1$ is a reciprocal pair if and only if there exists m in $\tilde{M}_F$ such that*

$$\mu = P(m, F) \quad \text{and} \quad \mu_1 = P\left(\frac{1}{m}, F_1\right).$$

PROOF. (A) $\boxed{\Rightarrow}$ Let (μ, μ_1) in $\mathscr{B}_F \times \mathscr{B}_{F_1}$ be a reciprocal pair of M.

(i) $\tilde{\theta}(\mu)$ is not empty; hence its image $\tilde{M}_F$ by $\theta \mapsto k'_\mu(\theta)$ is not empty. The same is true for F_1.

(ii) Let m in $\tilde{M}_F$, and θ in $\tilde{\theta}(\mu)$ be such that $m = k'_\mu(\theta)$. From Definition 5.1(iii):

$$-k'_\mu(\theta) = \left(-k'_{\mu_1}(-k_\mu(\theta))\right)^{-1}.$$

Hence

$$k'_{\mu_1}\left(-k_\mu(\theta)\right) = \frac{1}{m}$$

and there exists $y = -k_\mu(\theta)$ in $\tilde{\theta}(\mu_1)$ such that $k'_{\mu_1}(y) = 1/m$, and $1/m$ is in $\tilde{M}_{F_1}$. Interchanging F and F_1 completes the proof of (ii).

(iii) Differentiating with respect to θ in $\tilde{\theta}(\mu)$ both sides of

$$k'_\mu(\theta) k'_{\mu_1}\left(-k_\mu(\theta)\right) = 1,$$

we get

$$k''_\mu(\theta) k'_{\mu_1}\left(-k_\mu(\theta)\right) = \left(k'_\mu(\theta)\right)^2 k''_{\mu_1}\left(-k_\mu(\theta)\right). \tag{5.3}$$

If m is in $\tilde{M}_F$ and $\theta = \psi_\mu(m)$, then $k''_\mu(\theta) = V_F(m)$ [see Proposition 1.2(ii)]; if

$y = \psi_{\mu_1}(1/m)$, then $k''_{\mu_1}(y) = V_{F_1}(1/m)$. Using (5.1) and these remarks, (5.3) becomes

$$V_F(m)\frac{1}{m} = m^2 V_{F_1}\left(\frac{1}{m}\right) \quad \text{for } m \text{ in } \tilde{M}_F.$$

(A) $\boxed{\Leftarrow}$ The converse is easily obtained: One jumps from (iii) to equality (5.3). Integrating (5.3) yields $k'_\mu(\theta)k'_{\mu_1}(-k_\mu(\theta)) = c$ for some positive c and one can get the existence of (a, b) in $\mathbb{R}^2$ such that

$$\mu'(dx) = \exp(ax + b)\mu(dx) \quad \text{and} \quad \mu_1'(dx) = \exp(bx + a)\mu_1(dx)$$

are reciprocal.

(B) $\boxed{\Leftarrow}$ Let $\mu = P(m, F)$ with m in $\tilde{M}_F$. Proposition 5.1 implies the existence of μ_1 in $\mathscr{B}_{F_1}$ such that (μ, μ_1) is a reciprocal pair. Since $m > 0$, the open interval $\tilde{\theta}(\mu)$ contains 0. From Proposition 5.1(iii) the reciprocal μ_1 of μ is a probability and $k'_{\mu_1}(0) = 1/m$. Therefore $\mu_1 = P(1/m, F_1)$ is in F_1.

(B) $\boxed{\Rightarrow}$ Let (μ, μ_1) be in $F \times F_1$ and be reciprocal. Proposition 5.1(iii) implies 0 in the closure of $\tilde{\theta}(\mu)$. Since μ is in F, 0 is in $\theta(\mu)$. Therefore if 0 is not in $\tilde{\theta}(\mu)$, $k'_\mu(0) = 0$, and so 0 is not in $\theta(\mu_1)$, a contradiction with the fact that μ_1 is in F_1. Hence 0 is in $\tilde{\theta}(\mu)$, and there exists m in $\tilde{M}_F$ such that $\mu = P(m, F)$. A parallel reasoning shows the existence of m in $\tilde{M}_F$ such that $\mu_1 = P(m_1, F_1)$, and $mm_1 = 1$ is easily verified. □

The two most famous examples of reciprocal pairs (F, F_1) of NEFs are probably the following:

$$(5.4) \quad \begin{cases} \text{Normal} & M_F = \mathbb{R}, & V_F(m) = p \\ \text{Inverse Gaussian} & M_{F_1} =]0, +\infty[, & V_F(m) = pm^3, \end{cases} \qquad (p > 0 \text{ fixed}),$$

$$(5.5) \quad \begin{cases} \text{Exponential} & M_F =]0, +\infty[, & V_F(m) = m^2, \\ \text{Poisson} & M_{F_1} =]0, +\infty[, & V_F(m) = m. \end{cases}$$

One can point out that the reciprocal F_1 of F does not necessarily exist. Furthermore, F_1 can be drastically modified, or can disappear, if we make an affinity on F. For instance, making the change $x \mapsto 1 - x$ on the Poisson family gives

$$(5.6) \qquad M_F =]-\infty, 1[, \qquad V_F(m) = 1 - m,$$

and its reciprocal F, will be defined, by applying Theorem 5.2, by

$$(5.7) \qquad M_{F_1} =]1, +\infty[, \qquad V_{F_1}(m) = m^2(m - 1).$$

Shifting F_1 by $x \mapsto x - 1$ gives a family F_2:

$$(5.8) \qquad M_{F_2} =]0, +\infty[, \qquad V_{F_2}(m) = (m + 1)^2 m,$$

which is an Abel family concentrated on integers, and quite different from a translated exponential.

To provide a simple example of a nonexisting reciprocal, one can consider F generated by $\mu = \delta_1 + \delta_2$. Hence

$$M_F =]1, 2[, \qquad V_F(m) = (m - 1)(2 - m)$$

and we shall see, as a consequence of our classification Theorem 6.2, that

$$M_{F_1} =]1/2, 1[, \qquad V_{F_1}(m) = m(1 - m)(2m - 1)$$

cannot define a variance function.

Of course, the members of the Tweedie scale, as described in Jorgensen (1987) or Bar-Lev and Enis (1987) by

(5.9) $\quad M_F =]0, +\infty[, \qquad V_F(m) = Am^\beta \quad \text{with } \beta \notin [0, 1[\text{ and } A > 0,$

provide other examples of reciprocal pairs. Note that if $2 < \beta < 3$, F has no reciprocal, and that $\beta = 3/2$ gives a self-reciprocal family.

It is worth mentioning that one can have reciprocity, and even self-reciprocity, for families such that $M_F = M_{F_1} = \mathbb{R}$. A natural example can be obtained by considering a Brownian notion $B(t)$ in the plane with positive drift in the direction of the y axis, starting from point $(0, -1)$. It is not hard to verify that if μ is the distribution of the hitting point of the x axis, with suitable units one has $k_\mu(\theta) = 1 - \sqrt{1 - \theta^2}$, $\theta(\mu) =]-1, +1[$. Clearly, $F = F(\mu)$ is self-reciprocal and spreads on all of $\mathbb{R}$.

B. The use of Lévy processes. Although a general probabilistic interpretation of the reciprocity of a pair (μ, μ_1) is still lacking, certain cases can be explained by means of classical fluctuation theory. For information on fluctuation theory, we refer to the beautiful survey paper by Bingham (1975).

Recall that a Lévy process $(X(t))_{t \geq 0}$ on $\mathbb{R}$ is a process with independent stationary increments such that $X(0) = 0$ (this implies that the distribution has an infinitely divisible law) and with continuous trajectories on the right and the existence of left limits. X will be said to be *spectrally negative* if its Lévy measure does not charge $]0, +\infty[$. For us, the main feature of spectrally negative Lévy processes is the following:

(5.10) If $x > 0$, denote $h(X)(x) = \inf\{t > 0;\ X(t) \geq x\}$ with $h(X)(x) = +\infty$ if this set is empty. Then $X(h(X)(x)) = x$ if $h(X)(x) < \infty$.

In other words one cannot go above the level x without touching x before: This comes from the fact that X has nó positive jumps. Some examples of this situation are

(5.11) X = Brownian motion possibly with drift δt.

(5.12) X is an extreme stable process, i.e., $E(\exp \theta X(t)) = \exp(at\theta^\alpha)$, with $a > 0$ and $1 < \alpha < 2$ [see, e.g., Bingham (1975)].

(5.13) $X(t) = t - U(T)$, where $U(t)$ is a subordinator, i.e., an increasing Lévy process.

The connection with reciprocity comes with the following result which is nothing but a restatement in the terminology of this section of Proposition 2, page 721, of Bingham (1975).

THEOREM 5.3. *Let X be a spectrally negative Lévy process and $h(X)$ defined by* (5.10). *Denote by μ the distribution of $X(1)$ and by ν_x the distribution of $h(X)(x)$ restricted to $[0, +\infty[$ (i.e., $\nu_x([0, +\infty[) = P[\exists\ t;\ X(t) \geq x]$. Then (μ, ν_1) is a reciprocal pair. More generally, for all $x > 0$, the distribution of $X(1)/x$ and ν_x give a reciprocal pair.*

REMARKS. The proof of this is not difficult and makes standard use of martingales. Jumping from (μ, ν_1) to $(\text{Law}\, X(1)/x, \nu_x)$ is just a matter of rescaling. Last, $(\nu_x)_{x>0}$ is a convolution semigroup (i.e., $\nu_{x+x'} = \nu_x * \nu_{x'}$). Therefore $F(\nu_1)$ is always infinitely divisible.

Before applying Theorem 5.3 to Ressel families, let us survey some other applications. Taking X like in (5.11) will provide

$$\mu(dx) = \exp\left(-\frac{(x-\delta)^2}{2p}\right)\frac{dx}{\sqrt{2\pi p}}$$

and ν_1 will have the cumulant transform

$$k_{\nu_1}(\theta) = \delta - \sqrt{\delta^2 - 2p\theta} \quad \text{with } \theta(\mu) = \left]-\infty, \frac{\delta}{2p}\right[.$$

Note that ν_1 is the familiar inverse Gaussian law if $\delta > 0$. If $\delta < 0$, ν_1 will be an inverse Gaussian law multiplied by $\exp 2\delta$. If $\delta = 0$, we get (5.2). These facts make (5.4) precise.

Applying Theorem 5.3 to (5.12) will explain reciprocity in the Tweedie scale (5.9). Actually, an easy computation shows that if

$$k_\mu(\theta) = A\theta^\alpha, \quad \text{with } \theta(\mu) =]0, +\infty[,\ A > 0 \text{ and } 1 < \alpha < 2,$$

then, denoting $F = F(\mu)$, $\beta = (2-\alpha)/(1-\alpha)$ and $B = (\alpha - 1)(A\alpha)^{1/(\alpha-1)}$, one has $M_F =]0, +\infty[$ and $V_F(m) = Bm^\beta$. (Recall that since μ is spread on $\mathbb{R}$, F is not steep.) The reciprocal family F_1 will be described by

$$M_{F_1} =]0, +\infty[\quad \text{and} \quad V_F = Bm^{3-\beta},$$

corresponding to the family generated by a stable subordinator with parameter $\alpha_1 - 1/\alpha$, with $\beta_1 = 3 - \beta = (2-\alpha_1)/(1-\alpha_1)$ [see Tweedie (1984), Jorgensen (1987) and Bar-Lev and Enis (1987)].

We now apply Theorem 5.3 to (5.13) when $U(t) = N(t)$ is a standard Poisson process with intensity ν [i.e., $E(N(t)) = \nu t$]. Then the NEF F generated by $X(1) = 1 - N(1)$ is defined by (5.6), the distribution of $h(X)(1)$ belongs to the family F_1 described by (5.7) and from (5.8) $h(X)(1) - 1$ has a distribution which belongs to the Abel family [see Pyke (1958), Theorem 2, note 3].

For the purpose of this article, our main interest in Theorem 5.3 is its application to (5.13) when $U(t)$ is a gamma process, that is, the distribution of

$U(t)$ is

$$u^{t-1}e^{-u}1_{]0,+\infty[}(u)\frac{du}{\Gamma(t)}. \tag{5.14}$$

Therefore if F is the NEF generated by $(1/p)X(1) = (1 - U(1))1/p$ with $p > 0$ its variance function is

$$M_F = \left]-\infty, p^{-1}\right[, \qquad V_F(m) = \frac{1}{p^2}(1 - pm)^2 \tag{5.15}$$

[here we apply Proposition 2.4(v)].

Using now Theorem 5.2(iii) and Theorem 5.3, if F_1 is the NEF generated by the distribution of

$$h(X)(p) = \inf\{t;\ t - U(t) = p\} \tag{5.16}$$

the variance function of F_1 will be

$$M_{F_1} =]p, +\infty[, \qquad V_{F_1}(m) = m\left(\frac{m}{p} - 1\right)^2.$$

Taking the image F_2 of F_1 by $x \mapsto x - p$, that is, the family generated by $h(X)(p) - p$, we get the Ressel family (5.1) with power p; thus Theorem 5.3 provides a third proof of the existence of this family.

It does provide more than its mere existence. While it is impossible to get a generating measure of (5.1) from the analytic expression of the variance (with the help of Proposition 2.3, for instance), the previous probabilistic interpretation of (5.1) is going to give us this generating measure in a much simpler way than in Letac (1986). For this, we use the following result, initially due to Zolotarev (1964); for a self-contained proof in English, see Borovkov (1965). The result is quoted in Bingham (1975), page 725.

THEOREM 5.4. *Let X be a spectrally negative Lévy process and $h(X)$ defined by* (5.10). *Then for all x and $t > 0$,*

$$\Pr[h(X)(x) \le t] = -x\frac{\partial}{\partial x}\int_0^t \Pr[X(u) \ge x]\frac{du}{u}.$$

Furthermore, if the Lévy measure $\pi(dx)$ is absolutely continuous and unbounded, then the distributions of $X(t)$ and $h(X)(x)$ are absolutely continuous; denoting by $p(x,t)$ and $q(x,t)$ their respective densities, one has for all x and $t > 0$,

$$tq(x,t) = xp(x,t). \tag{5.17}$$

C. Ressel families. We now apply Theorem 5.4 to $X(t) = t - U(t)$, where U is defined by (5.14). Since for $\theta < 1$

$$E(\exp(\theta U(t))) = \exp\left(t\int_0^\infty \frac{1 - e^{\theta x}}{x}e^{-x}\,dx\right)$$

[see, e.g., Feller (1966), page 427, (78)], the Lévy measure fulfills the requirement of Theorem 5.4. The density of $X(t) = t - U(t)$ is, for $x < t$, from (5.14):

$$p(x, t) = (t - x)^{t-1} e^{x-t} 1_{]-\infty, t[}(x) \frac{1}{\Gamma(t)}.$$

Hence from (5.17) the distribution of $h(X)(p)$ is

$$q(p, t)\, dt = p(t - p)^{t-1} e^{p-t} 1_{]p, +\infty[}(t) \frac{1}{t\Gamma(t)}\, dt. \tag{5.18}$$

Note that since $E(X(t)) = 0$, the process X satisfies $P(\limsup_{t\to\infty} X(t) = +\infty) = 1$ [see Bingham (1975), Theorem 3.c]. Therefore (5.18) is not a defective distribution, but a real probability measure. Shifting (5.18) by $x \mapsto x - p$, we get the main result of this section.

PROPOSITION 5.5. *If $p > 0$, the following measure*:

$$\tau_p(dt) = \frac{p t^{t+p-1} e^{-t}}{\Gamma(t + p + 1)} 1_{]0, +\infty[}(t)\, dt$$

is a probability measure which generates the Ressel NEF with power p and variance function (5.1).

Theorem 5.3 has provided a case of probabilistic interpretation of reciprocity. There is no space here to give some details on similar interpretations when the spectrally negative Lévy process X is replaced by a random walk on the set of relative integers $\mathbb{Z}$ which is right-continuous in the Spitzer (1964) sense. We shall content ourselves with the following statement, without proof.

THEOREM 5.6. *Let $U_1, \dots, U_m, \dots$ be a sequence of independent and identically distributed random variables, taking their values in the set $\mathbb{N}$ of nonnegative integers. Denote $S_0 = 0$, $S_n = n - U_1 - U_2 - \cdots - U_n$ and for k in $\mathbb{N}$: $T_k = \inf\{n;\ S_n = k\}$, with $T_k = +\infty$ if this set is empty. Then*

(i) $kP[S_n = k] = nP[T_k = n]$ *for n and $k > 0$.*

(ii) *The distributions of S_1/k and T_k (restricted to $T_k < \infty$) give a reciprocal pair.*

(iii) *Let $\{N(t);\ t \ge 0\}$ be a Poisson process independent of $(U_n)_{n\ge1}$, $X(t) = S_{N(t)}$ and $H(k) = \inf\{t;\ N(t) = T_k\}$.*

Then the distributions of $X(1)/k$ and $H(k)$ (restricted to $T_k < \infty$) give a reciprocal pair.

REMARKS. (ii) and (iii) are easy to check by using martingales. (i) is the Kemperman theorem and is a perfect analog of (5.17). See Wendel (1975) for an elegant proof of (i) by the Lagrange formula Theorem 4.1. Result (ii) for $k = 1$ partially explains Theorem 4.5 for $p = 1$ only. In the notation of Theorem 4.5, F_1 would be the family of the distribution of U_1 and F the family of the distribution of $T_1 - 1$; using Theorems (5.6)(ii) and (5.2)(iii) gives Theorem 4.5 for $p = 1$.

There is a final remark about reciprocity and cubic variances. The formula $V_{F_1}(m) = m^3 V_F(1/m)$ of Theorem 5.2 shows that the set of cubic variances is closed under reciprocity, and makes this set especially interesting to classify. Theorems 5.3 and 5.6 explain why NEFs with variance of degree equal to 3 have appeared in the literature through the distributions of first passage times.

6. The classification of cubic variance functions. In this section we are going to prove that all cubic families have been met before in this paper or among the Morris quadratic families. (Recall that quadratic means "of degree less than or equal to 2," as cubic means "of degree less than or equal to 3.") Of course, this must be understood as "up to an affinity."

First, we give a list of 12 cubic types in Table 2. There, p is a positive number and is the power parameter (see Proposition 2.5). Note that all the types except the "binomial" are infinitely divisible.

The μ column of the table shows the generating measures of the family. In the continuous cases (lines 1, 5, 6, 11 and 12) we have chosen μ as a probability (however, it is not in the family for lines 11 and 12). We have given in line 6 a closed formula instead of the first formula of page 73 of Morris (1982). The name "hyperbolic cosine" for line 6 alludes to the fact that the Fourier transform is $L_\mu(i\theta) = (\cosh\theta)^{-p}$.

PROPOSITION 6.1. *None of the following functions* V: $M \to]0, +\infty[$ *are variance functions*:

(i) *Let* a *be noninteger and positive*

$$M =]0, a[\quad \text{and} \quad V(m) = m\left(1 - \frac{m}{a}\right).$$

(ii) *Let* $a > 0$,

$$M =]0, a[\quad \text{and} \quad V(m) = m\left(1 - \frac{m}{a}\right)^2.$$

(iii) *Let* $0 < a < b$,

$$M =]0, a[\quad \text{and} \quad V(m) = m\left(1 - \frac{m}{a}\right)\left(1 - \frac{m}{b}\right).$$

(iv) *Let* $0 < b < a$,

$$M =]0, +\infty[\quad \text{and} \quad V(m) = m(1 - 2bm + a^2m^2).$$

[(iv) *should be compared to the end of Section* 3.]

PROOF. From Proposition 4.4, if any of these functions is a variance function, it is associated to an exponential family F generated by a measure $\mu = \sum_{n=0}^{\infty} \mu_n \delta_n$ concentrated on nonnegative integers such that μ_0 and μ_1 are positive, and there exist primitives ψ and ψ_1 on M of $m \mapsto 1/V(m)$ and $m \mapsto m/V(m)$ such that

$$\exp \psi_1(m) = \sum_{n=0}^{\infty} \mu_n \exp(n\psi(m)) \quad \text{for } m \text{ in } M \tag{6.1}$$

(see Proposition 2.3). We are going to compute explicitly primitives ψ and ψ_1 in each of the four cases and show that μ_n defined by (6.1) is negative for some n, getting a contradiction.

(i) $$\exp \psi_1(m) = \left(1 - \frac{m}{a}\right)^{-1}, \qquad \exp \psi(m) = \frac{m}{a}\left(1 - \frac{m}{a}\right)^{-1}.$$

Denoting $z = m/(a - m)$, (6.1) becomes

$$(1 + z)^a = \sum_{n=0}^{\infty} \mu_n z^n.$$

Since a is not an integer, there exists an integer $N \geq 1$ such that $N - 1 < a < N$; hence $\mu_{N+1} = [1/(N+1)!]a(a-1)\dots(a-N+1)(a-N) < 0$.

(ii) $$\exp \psi_1(m) = \exp\left(\frac{-am}{a-m}\right), \qquad \exp \psi(m) = \frac{m}{a-m} \exp \frac{m}{a-m}.$$

Denoting $z = m/(a - m)$, (6.1) becomes

$$\exp(-az) = \sum_{n=0}^{\infty} \mu_n (z \exp z)^n.$$

Denote now $w = ze^z$ and apply Corollary 4.2 to $g(z) = \exp(-z)$. With the notation $h(w)$ of Theorem 4.1, one gets

$$\exp(-ah(w)) = \sum_{n=0}^{\infty} \mu_n w^n$$

and formula (4.4) gives

$$\mu_n = \frac{a}{a+n} \frac{1}{n!} \left[\frac{d^n}{dz^n} \exp(-z(a+n))\right]_{z=0}$$

which is clearly negative if n is odd.

(iii) To simplify, denote $r = b/(b - a) > 1$. Then

$$\exp \psi_1(m) = \left(1 - \frac{m}{b}\right)^{ra} \left(1 - \frac{m}{a}\right)^{-ra},$$

$$\exp \psi(m) = \frac{m}{b-m}\left(1 - \frac{m}{b}\right)^{-r} \left(1 - \frac{m}{a}\right)^{r} \left(\frac{b}{a} - 1\right).$$

Denoting $z = (b/a - 1)m/b - m$ in $(0,1)$, (6.1) becomes

$$(1 - z)^{-ra} = \sum_{n=0}^{\infty} \mu_n \left(z(1-z)^{-r}\right)^n.$$

TABLE 2

	Degree	Name of the type	M_F	$V_F(m)$	$\mu(dx)$	Reference
1	0	Gaussian	$\mathbb{R}$	p	$\exp\left(-\frac{x^2}{2p}\right)\frac{dx}{\sqrt{2\pi p}}$	Morris (1982)
2	1	Poisson	$]0, +\infty[$	m	$\sum_{k=0}^{\infty}\frac{1}{k!}\delta_k(dx)$	Morris (1982)
3	2	Binomial, with integer parameter N	$]0, N[$	$m\left(1-\frac{m}{N}\right)$	$\sum_{k=0}^{\infty}\binom{N}{k}\delta_k(dx)$	Morris (1982)
4	2	Negative binomial	$]0, +\infty[$	$m\left(1+\frac{m}{p}\right)$	$\sum_{k=0}^{\infty}p(p+1)\dots(p+k-1)\frac{\delta_k(dx)}{k!}$	Morris (1982)
5	2	Gamma	$]0, +\infty[$	$\frac{m^2}{p}$	$x^{p-1}1_{]0,+\infty[}(x)e^{-x}\frac{dx}{\Gamma(p)}$	Morris (1982)
6	2	Hyperbolic cosine	$\mathbb{R}$	$p\left(1+\frac{m^2}{p^2}\right)$	$\frac{2^{p-2}}{\Gamma(p)(\Gamma(p/2))^2}\left\|\Gamma\left(\frac{p}{2}+i\frac{x}{2}\right)\right\|^2 dx$	Morris (1982)

7	3	Abel	]0, +∞[	$m\left(1+\frac{m}{p}\right)^2$	$\sum_{k=0}^{\infty} p(p+k)^{k-1}\frac{\delta_k(dx)}{k!}$	(4.35)
8	3	Takács $a>0$	]0, +∞[	$m\left(1+\frac{m}{p}\right)\left(1+\frac{a+1}{a}\frac{m}{p}\right)$	$\delta_0+\sum_{k=1}^{\infty} ap\prod_{j=1}^{k-1}(a(p+k)+j)\frac{\delta_k(dx)}{k!}$	(4.37)
9	3	Strict arcsine	]0, +∞[	$m\left(1+\frac{m^2}{p^2}\right)$	$\sum_{k=0}^{\circ} p_k(p)\frac{\delta_k(dx)}{k!}$, p_k on (4.14)	(4.14)
10	3	Large arcsine $a>0$	]0, +∞[	$m\left(1+\frac{2}{1}\frac{m}{p}+\frac{1+a^2}{a^2}\frac{m^2}{p^2}\right)$	$\sum_{k=0}^{\infty}\frac{p}{p+k}p_k(a(p+k))\frac{\delta_k(dx)}{k!}$, p_k on (4.14)	(4.39)
11	3	Ressel	]0, +∞[	$\frac{m^2}{p}\left(1+\frac{m}{p}\right)$	$\frac{px^{x+p-1}e^{-x}}{\Gamma(x+p+1)}1_{]0,+\infty[}(x)\,dx$	Proposition 5.5
12	3	Inverse Gaussian	]0, +∞[	$\frac{m^3}{p^2}$	$x^{-3/2}\exp\left(-\frac{p^2}{2x}\right)\frac{p}{\sqrt{2\pi}}1_{]0,+\infty[}(x)\,dx$	(1.9)

We now apply Corollary 4.2 to $g(z) = (1 - z)^r$, with $p = a$. Therefore from (4.4), if $n \neq a$,

$$\mu_n = \frac{-a}{n-a}\frac{1}{n!}\left[\left(\frac{d}{dz}\right)^n (1-z)^{r(n-a)}\right]_{z=0}.$$

Let us take $n \geq ra/(r-1)$. This implies $r(n-a) - k > 0$ for all integers $k = 0, 1, \dots, n-1$. Hence

$$\left[\left(\frac{d}{dz}\right)^n (1+z)^{r(n-a)}\right]_{z=0} > 0.$$

This implies $\mu_{2n} < 0$ if $2n \geq ra/(r-1)$.

(iv) Without loss of generality, we write

$$V(m) = m\frac{(m-r)^2 + q^2}{r^2 + q^2},$$

with $r > 0$ and $q > 0$. Hence

$$\exp\psi_1(m) = C_1 \exp\left(\frac{r^2+q^2}{q}\arctan\frac{m-r}{q}\right),$$

$$\exp\psi(m) = Cm\left((m-r)^2 + q^2\right)^{-1/2}\exp\left(\frac{r}{a}\arctan\frac{m-r}{q}\right),$$

where C_1 and C are constants that we will choose later. Denote

$$\theta = \arctan\frac{m-r}{q} \quad \text{and} \quad \alpha = \arctan\frac{r}{q};$$

(6.1) becomes

$$C_1\exp\left(\theta\left(\frac{r^2+q^2}{q}\right)\right) = \sum_{n=0}^{\infty}\mu_n\left(C\frac{q}{\cos\alpha}\sin(\theta+\alpha)\exp\left(\frac{r}{q}\theta\right)\right)^n, \tag{6.2}$$

for $0 > -\alpha$. In (6.2) we denote $z = \sin(\theta + \alpha)$, $p = (r^2 + q^2)/r$ and we now choose the constants by $C_1 = \exp(p\alpha)$ and $C = \exp((r/q)\alpha)$. The new form of (6.2) is

$$\exp\left(p\frac{r}{q}\arcsin z\right) = \sum_{n=0}^{\infty}\mu_n\left(z\exp\left(\frac{r}{q}\arcsin z\right)\right)^n.$$

It remains to apply Corollary 4.2 to

$$g(z) = \exp\left(-\frac{r}{q}\arcsin z\right).$$

If $n \neq p$, (4.4) gives

$$\mu_n = \frac{-p}{n-p}\frac{1}{n!}\left[\left(\frac{d}{dz}\right)^n\exp\left(-(n-p)\frac{r}{q}\arcsin z\right)\right]_{z=0}.$$

Clearly, $\mu_{2n} < 0$ if $2n > p$. □

We now state and prove the main result of this paper, which is the theorem of classification of the cubic families.

THEOREM 6.2. *Let F be a natural exponential family on $\mathbb{R}$ such that its variance function V_F is the restriction to the mean domain M_F of a polynomial P with degree less than or equal to 3. Then there exists an affinity φ: $x \mapsto ax + b$, $a \neq 0$, of $\mathbb{R}$ to itself such that the image $\varphi_* F$ of F by this affinity is one of the twelve types of the array.*

REMARKS. This theorem contains the statement of Morris (1982) about the existence of exactly six quadratic types, up to affinity. However, his paper contains some obscure points due to the omission of statements corresponding to our Theorem 3.1 and Proposition 6.1(i). For this reason, we give a complete proof of Theorem 6.2.

PROOF. Denote $M_F =]\alpha, \beta[$, with $-\infty \leq \alpha < \beta \leq +\infty$. We make two observations about the polynomial P:

(a) P has no 0 in $]\alpha, \beta[$.

(b) Either $\alpha = -\infty$, or $P(\alpha) = 0$. Similarly, either $\beta = +\infty$ or $P(\beta) = 0$. This is a consequence of Theorem 3.1 applied to the analytic function P.

We now prove the theorem by discussing $\delta \circ P =$ degree of P.

1. If $\delta \circ P = 0$, (b) implies $M_F = \mathbb{R}$ and F is a normal family (line 1).

2. If $\delta \circ P = 1$, $P(x) = ax + b$ with $a \neq 0$, and M_F is a half-line with endpoint $-b/a$. The affinity $\varphi(x) = x/a + b/a^2$ changes F in $F_1 = \varphi_* F$, and Proposition 2.4(v) gives $M_{F_1} =]0, +\infty[$ and $V_{F_1}(m) = m$; thus F_1 is the Poisson family corresponding to line 2.

3. $\delta \circ P = 2$, P has two distinct real roots and $M_F =]\alpha, \beta[$ is bounded. Therefore from (b) the roots of P are α and β, and there exists $A > 0$ such that

$$P(x) = \frac{(\beta - x)(x - \alpha)}{A}.$$

The affinity $\varphi(x) = A[(x - \alpha)/(\beta - \alpha)]$ changes F in $F_1 = \varphi_* F$, and from Proposition 2.4(v):

$$M_{F_1} =]0, +\infty[\quad \text{and} \quad V_{F_1}(m) = m\left(1 - \frac{m}{A}\right).$$

Then either A is not an integer and from Proposition 6.1(i), F_1 does not exist, or A is an integer $N > 0$, and F_1 is a binomial family corresponding to line 3.

4. $\delta \circ P = 2$, p has two distinct real roots and $M_F =]\alpha, \beta[$ is not bounded. Therefore from (b) either $\beta = +\infty$, $P(\alpha) = 0$ and there exists $r < \alpha$ and $p > 0$ such that

$$P(x) = \frac{(x - \alpha)(x - r)}{p},$$

and the affinity $\varphi(x) = p[(x - \alpha)/(\alpha - r)]$ will send F to the negative binomial family F_1 of line 4, on $\alpha = -\infty$, $P(\beta) = 0$ and there exists $r > \beta$ and p such that

$$P(x) = \frac{(\beta - x)(r - x)}{p}$$

and $\varphi(x) = p/r - \beta(\beta - x)$ will send us to line 4 again.

5. $\delta \circ P = 2$, P has a double root. Then from (b): Either $\beta = +\infty$ and $P(\alpha) = 0$, and there exists $p > 0$ such that $P(x) = (1/p)(x - \alpha)^2$, then the affinity $\varphi(x) = c(x - \alpha)$, where $c > 0$ will send us to line 5, and $\varphi_* F$ is a gamma family; or $\alpha = -\infty$ and $P(\beta) = 0$, then $P(x) = (1/p)(x - \beta)^2$ for $p > 0$ and $\varphi(x) = c(x - \beta)$ with $c < 0$ will send us to line 5.

Note that here the particular value of c is irrelevant, since a gamma family is invariant under scaling: See the characterization of the gamma families after Proposition 2.4.

6. $\delta \circ P = 2$, P has no real root. Therefore from (b) $M_F = \mathbb{R}$ and there exists $p > 0$, r real and $A > 0$ such that

$$V_F(m) = \frac{p}{A^2}\left(1 + \frac{A^2(m - r)^2}{p^2}\right).$$

The affinity $\varphi(x) = A^{-1}x + r$ will send us to line 6 of the hyperbolic cosine.

In the remaining cases $\delta \circ P = 3$ and P has necessarily a real root. Without loss of generality, by means of a translation, we may assume that $P(0) = 0$. Therefore we suppose that there exists a polynomial Q with degree 2 such that $P(x) = xQ(x)$. Since we have already done a translation, we are now only allowed to do scaling $\varphi(x) = x/A$, with $A \neq 0$.

7. Q has a double root different from 0. Then if $M_F =]\alpha, \beta[$ is bounded, either $\alpha = 0$ and $Q(\beta) = 0$, or $Q(\alpha) = 0$ and $\beta = 0$. In the first case, there exists $a > 0$ such that

$$P(x) = x\frac{(\beta - x)^2}{\beta a},$$

and the scaling $\varphi(x) = ax/\beta$ shows that $F_1 \subset \varphi_* F$ would have the variance function

$$M_{F_1} =]0, a[, \qquad V(m) = m\left(1 - \frac{m}{a}\right)^2,$$

which is excluded by Proposition 6.1(ii). A similar analysis excludes $Q(\alpha) = 0$ and $\beta = 0$. Therefore, in this case, M_F is unbounded.

If $\beta = +\infty$, then from (a) either $\alpha = 0$ or $\alpha > 0$. If $\alpha = 0$, there exists $p > 0$ and $r > 0$ such that

$$P(x) = \frac{x(x + r)^2}{pr},$$

and the scaling $\varphi(x) = rx/p$ sends F onto the Abel family of line 7. If $\alpha > 0$, a translation $x \mapsto x - \alpha$ would send us to case 11 and we defer its study.

If $\alpha = -\infty$, then either $\beta = 0$ and a similar analysis will send F to line 7, or $\beta < 0$ and we defer until case 11.

8. Q has distinct real roots different from 0. Assume that $M_F =]\alpha, \beta[$ is bounded. From (b), $P(\alpha) = P(\beta) = 0$. Since P has three simple roots, up to a translation and if necessary changing x to $-x$, we can assume that $\alpha = 0$, $Q(\beta) = 0$ and that the second r of Q is larger than β. Therefore there exists $A > 0$, $a = \beta/A$ and $b = r/A$ such that

$$P(x) = Ax\left(1 - \frac{x}{aA}\right)\left(1 - \frac{x}{bA}\right).$$

The scaling $\varphi(x) = x/A$ show that $F_1 = \varphi_* F$ would have the variance function, where $0 < a < b$,

$$M_{F_1} =]0, a[, \qquad V(m) = m\left(1 - \frac{m}{a}\right)\left(1 - \frac{m}{b}\right),$$

which is excluded by Proposition 6.1(iii). Therefore M_F is unbounded.

Since P has three simple roots, up to a translation and if necessary changing x in $-x$, we can assume that $\alpha = 0$ and $\beta = +\infty$. Therefore there exist three positive numbers $p > 0$, $a > 0$ and $A > 0$ such that

$$P(x) = Ax\left(1 + \frac{x}{Ap}\right)\left(1 + \frac{a+1}{a}\frac{x}{Ap}\right).$$

The scaling $\varphi(x) = x/A$ shows that $F_1 = \varphi_* F$ is the Takács family of line 8.

9. Q has no real roots and its roots are purely imaginary. Therefore $M_F =]-\infty, 0[$ or $M_F =]0, +\infty[$. Changing x to $-x$ leads to the second case, and there exist p and $A > 0$ such that

$$P(x) = Ax\left(1 + \left(\frac{x}{Ap}\right)^2\right).$$

We get a strict arcsine family (line 9) by the scaling $\varphi(x) = x/A$.

10. Q has no real roots and its roots are not purely imaginary. Therefore $M_F =]-\infty, 0[$ or $M_F =]0, +\infty[$. Changing x to $-x$ leads to the second case. Assume that the roots of Q have a positive real part. Therefore there exist $A > 0$ and $0 < b < a$ such that

$$P(x) = Ax\left(1 - 2b\frac{x}{A} + \frac{a^2x^2}{A^2}\right).$$

Scaling $\varphi(x) = x/A$ and Proposition 6.1(iv) show that this is impossible. Hence the roots of Q have negative real parts, there exist $A > 0$, $0 < b < a$ such that

$$P(x) = Ax\left(1 + 2b\frac{x}{A} + \frac{a^2x^2}{A^2}\right)$$

and scaling $\varphi(x) = x/A$ sends F onto the large arcsine family on line 10.

11. Q has two distinct roots and one of them is 0. Again we exclude the case where $M_F =]\alpha, \beta[$ is bounded like in case 7 by Proposition 6.1(iii). Therefore

changing x to $-x$ if necessary gives $M_F =]0, +\infty[$ and there exist $p > 0$ and $A > 0$ such that

$$P(x) = \frac{x^2}{p}\left(1 + \frac{x}{Ap}\right).$$

The scaling $\varphi(x) = x/A$ yields the Ressel family (line 11).

12. Q has a double root on 0. Therefore $M_F =]-\infty, 0[$ or $M_F =]0, +\infty[$, and changing x to $-x$ leads to the second case. Therefore there exists $p > 0$ such that

$$P(x) = \frac{x^3}{p^2},$$

and this is the inverse Gaussian family of line 12. □

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Note added in proof. Professors V. Seshadri and T. Speed have recently informed the authors that formula (5.18) is formula (20) in a paper written by David G. Kendall [Some problems in the theory of dams. *J. Roy. Statist. Soc. Ser. B* **19** 207–212 (1957)]. According to this, Kendall–Reosel families could be a more appropriate name for line 11 of the classification of the cubic families. We apologize for having missed this reference.

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