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ON THE ESTIMATION OF THE EXTREME-VALUE INDEX AND LARGE QUANTILE ESTIMATION

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This paper consists of two parts. An easy proof is given for the weak consistency of Pickands' estimate for the main parameter of an extreme-value distribution. Moreover, further natural conditions are given for strong consistency and for asymptotic normality of the estimate. Next a large quantile of a distribution is estimated by a combination of extreme or intermediate order statistics. This leads to an asymptotic confidence interval.

1. Introduction.

1.1. *Estimating the extreme-value index of a probability distribution.* Suppose one is given a sequence $X_1, X_2, \dots$ of i.i.d. observations from some distribution function F. Suppose for some constants $a_n > 0$ and b_n and some $\gamma \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left\{\frac{\max(X_1, X_2, \dots, X_n) - b_n}{a_n} \le x\right\} = G_\gamma(x) \tag{1.1}$$

for all x where $G_\gamma(x)$ is one of the extreme-value distributions

$$G_\gamma(x) = \exp - (1 + \gamma x)^{-1/\gamma}. \tag{1.2}$$

Here γ is a real parameter [interpret $(1 + \gamma x)^{-1/\gamma}$ as e^{-x} for $\gamma = 0$] and x such that $1 + \gamma x > 0$. The question is how to estimate γ from a finite sample $X_1, X_2, \dots, X_n$.

A traditional method uses "yearly maxima," i.e., breaks the sample into blocks of equal size and uses maximum likelihood estimation under the assumption that the maximum in each block follows *exactly* distribution G_γ. Consistency has been proved here under certain conditions [Cohen (1988)]. By using this method some information from the sample seems to be lost.

A less traditional method consists of restricting attention to those observations from $X_1, X_2, \dots, X_n$ that exceed a certain level $M(n)$ and using the method of maximum likelihood under the assumption that these observations follow *exactly* one of the asymptotic residual lifetime distributions. Asymptotic results for this procedure have been obtained by Smith (1987).

An attractive alternative estimate has been proposed by Pickands (1975): Let $m = m(n)$ be a sequence of integers tending to infinity and let $m(n)/n \to 0$

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$(n \to \infty)$. The estimate is

$$(1.3) \qquad \hat{\gamma}_n := (\log 2)^{-1} \cdot \log \frac{X^{(n)}_{(n-m+1)} - X^{(n)}_{(n-2m+1)}}{X^{(n)}_{(n-2m+1)} - X^{(n)}_{(n-4m+1)}},$$

where $X^{(n)}_{(1)} \le X^{(n)}_{(2)} \le \cdots \le X^{(n)}_{(n)}$ are the ascending order statistics of $X_1, X_2, \dots, X_n$. Pickands proved that this estimate is weakly consistent. We shall give a short proof of this result and show that if the sequence $m(n)$ increases suitably rapidly, then there is strong consistency. Also we give quite natural and general conditions under which the estimate is asymptotically normal (Section 2). The analytical work involved in the translation of the conditions for the inverse of F into conditions for F is given in the Appendix, which should be useful in other contexts as well. In the Appendix we assume that the reader is familiar with the theory of Π-variation and Γ-variation [see, e.g., Geluk and de Haan (1987)].

Knowing the asymptotic distribution of $\hat{\gamma}_n$ is particularly important: Since there is a discontinuity in the shape of the distribution G_γ at $\gamma = 0$, one often wants to test hypotheses of the type $\gamma = 0$, $\gamma \ge 0$ or $\gamma \le 0$.

1.2. *Large quantile estimate under extreme-value conditions.* After the 1953 flood the Dutch government set the following standard for the sea dikes in the Netherlands: The probability that at any time in a given year the sea level exceeds the level of the dikes is 1 : 10,000. The question of how to give an estimate for such a level from past observations involves estimation of large quantiles of an unknown distribution function.

We consider the following idealized model: n i.i.d. observations $X_1, X_2, \dots, X_n$ are available from an unknown distribution function F. In a future year k i.i.d. observations $Y_1, Y_2, \dots, Y_k$ will be taken from F. We want to find a level x_{k, p_0} (where p_0 is a given number much less than 1) such that $P\{\max(Y_1, \dots, Y_k) \le x_{k, p_0}\} = 1 - p_0$, i.e.,

$$(1.4) \qquad F^k(x_{k, p_0}) = 1 - p_0.$$

Define the function U by

$$(1.5) \qquad U(x) := \left(\frac{1}{1 - F(x)} \right)^{\leftarrow}$$

(the arrow means inverse function), $p := 1 - (1 - p_0)^{1/k}$ and $x_p := x_{k, p_0}$. Note that (loosely speaking)

$$x_p = U\left(\frac{1}{p}\right).$$

We want to estimate x_p on the basis of the order statistics $X^{(n)}_{(1)} \le X^{(n)}_{(2)} \le \cdots \le X^{(n)}_{(n)}$ of the observations $X_1, X_2, \dots, X_n$. Let F_n be the empirical distribution function and

$$(1.6) \qquad U_n := \left(\frac{1}{1 - F_n} \right)^{\leftarrow}.$$

Note that $F_n(X_{(m)}^{(n)}) = m/n$, $m = 1, 2, \dots, n$, so that

$$X_{(n-m+1)}^{(n)} = U_n\left(\frac{n}{m}\right) \quad \text{for } m = 1, 2 \dots, n. \tag{1.7}$$

In case $p < 1/n$ nothing can be done without imposing extra conditions on F. We choose for an asymptotic theory and for imposing the extra condition that F is in the domain of attraction of some extreme value distribution.

We shall give asymptotic results for $n \to \infty$ and $p = p_n \to 0$ (the latter assumption is reasonable at least for the specific problem mentioned above, since there $p < 1/n$). For the unknown function U we write the identity, $m = 1, 2, \dots,$

$$x_p = U\left(\frac{1}{p}\right) = \frac{U(1/p) - U(n/m)}{U(n/m) - U(n/(2m))}\left\{U\left(\frac{n}{m}\right) - U\left(\frac{n}{2m}\right)\right\} + U\left(\frac{n}{m}\right). \tag{1.8}$$

Upon replacing U by its empirical counterpart U_n in appropriate places and using Lemma 2.2 we arrive at the following proposed estimate for x_p:

$$\hat{x}_{p,n} := \frac{(m/(p_n n))^{\hat{\gamma}_n} - 1}{1 - 2^{-\hat{\gamma}_n}}\left\{X_{(n-m+1)}^{(n)} - X_{(n-2m+1)}^{(n)}\right\} + X_{(n-m+1)}^{(n)}, \tag{1.9}$$

where $\hat{\gamma}_n$ is given by (1.3). Intuitively this means that in the absence of more observations (that would have allowed us to simply use the inverse empirical distribution function), one uses observed spacing to make up (modulo a multiplicative constant) for the missing spacings, like a surgeon who uses a piece of skin from elsewhere to cover a wound.

Note that we do not use the largest observation explicitly. One can argue that this makes sense because the largest observation may add too much uncertainty (larger variance if applicable).

In order to deal with the asymptotics we need to require that $m/(pn)$ has a positive limit ($n \to \infty$). We shall consider two cases:

$m \to \infty$, $m(n)/n \to 0$, hence $p_n \sim c \cdot m(n)/n \to 0$ and $n \cdot p_n \sim c \cdot m(n) \to \infty$, and m fixed, hence $p_n \sim c/n$, $0 < c < 1$.

In either case we give an asymptotic confidence interval for x_p (Section 3). Since in the first case $n \cdot p_n \to \infty$, an estimate of a simpler form than (1.9) can be used: extrapolation outside the sample is not necessary.

Somewhat related papers are Weissman (1978) and Boos (1984).

Using the same methods we find a confidence interval for the endpoint of the distribution in case $\gamma < 0$.

1.3. *Simulation results and an application.* In Section 4 we present some simulation results and an application of our results to the high tide water levels at the Dutch island Terschelling.

2. Consistency and asymptotic normality. We shall need the following simple result.

LEMMA 2.1. *If* $F(x) = 1 - e^{-x}$ *(standard exponential distribution)*, $m \le n$ *and* $m = m(n) \to \infty$ $(n \to \infty)$, *then*

$$\sqrt{2m}\left(X^{(n)}_{(n-m+1)} - X^{(n)}_{(n-2m+1)} - \log 2\right)$$

has asymptotically a standard normal distribution.

PROOF. We use the representation for exponential order statistics usually referred to as Rényi's representation: For each n there exist i.i.d. random variables $Z_1, Z_2, \dots$ with standard exponential distribution such that $\{X^{(n)}_{(n-m+1)} - X^{(n)}_{(n-m)}\}_{m=1}^{n} =_d \{Z_m/m\}_{m=1}^{n}$, where $X^{(n)}_{(0)} = 0$. This gives $X^{(n)}_{(n-m+1)} - X^{(n)}_{(n-2m+1)} =_d \sum_{i=m}^{2m-1} Z_i/i$. The rest of the proof is easy [use, e.g., Gnedenko and Kolmogorov (1954), Chapter 5]. □

COROLLARY 2.1. $X^{(n)}_{(n-m+1)} - X^{(n)}_{(n-2m+1)} \to \log 2$ *in probability* $(n \to \infty)$.

Further we list a well-known result [see, e.g., de Haan (1984)].

LEMMA 2.2. *Define* $U := (1/(1-F))^{\leftarrow}$ *(the inverse function). Relation* (1.1) *holds if and only if for* $x, y > 0$, $y \neq 1$,

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{U(ty) - U(t)} = \frac{x^\gamma - 1}{y^\gamma - 1} \quad \textit{locally uniformly} \left(:= \frac{\log x}{\log y} \textit{ for } \gamma = 0 \right).$$

THEOREM 2.1 (Weak consistency). *If* (1.1) *holds*, $m(n) \to \infty$ *and* $m(n)/n \to 0$ $(n \to \infty)$, *then* $\hat{\gamma}_n \to \gamma$ *in probability* $(n \to \infty)$.

PROOF. Let $A_1, A_2, \dots$ be i.i.d. exponential random variables and let $\{A^{(n)}_{(m)}\}$ be the ascending order statistics of $A_1, A_2, \dots, A_n$. Then $\{X^{(n)}_{(n-m+1)}\}_{m=1}^{n} =_d \{U(e^{A^{(n)}_{(n-m+1)}})\}_{m=1}^{n}$. Note that $m(n)/n \to 0$ implies $e^{A^{(n)}_{(n-m+1)}} \to \infty$ a.s. $(n \to \infty)$. Now

$$\frac{U\left(e^{A^{(n)}_{(n-m+1)}}\right) - U\left(e^{A^{(n)}_{(n-2m+1)}}\right)}{U\left(e^{A^{(n)}_{(n-2m+1)}}\right) - U\left(e^{A^{(n)}_{(n-4m+1)}}\right)}$$

$$= \frac{U\left(e^{A^{(n)}_{(n-2m+1)}} \cdot e^{A^{(n)}_{(n-m+1)} - A^{(n)}_{(n-2m+1)}}\right) - U\left(e^{A^{(n)}_{(n-2m+1)}}\right)}{U\left(e^{A^{(n)}_{(n-2m+1)}}\right) - U\left(e^{A^{(n)}_{(n-2m+1)}} \cdot e^{A^{(n)}_{(n-4m+1)} - A^{(n)}_{(n-2m+1)}}\right)}$$

$$\to \frac{2^\gamma - 1}{1 - 2^{-\gamma}} = 2^\gamma$$

in probability by Corollary 2.1 and Lemma 2.2. The result follows. □

THEOREM 2.2 (Strong consistency). *If* (1.1) *holds*, $m(n)/n \to 0$ *and* $m(n)/\log\log n \to \infty$ $(n \to \infty)$. *Then*

$$\hat{\gamma}_n \to \gamma \quad a.s., \qquad n \to \infty.$$

PROOF. The conditions on the sequence $m(n)$ imply $A^{(n)}_{(n-m+1)} + \log([m(n)]/n) \to 0$ a.s. [Wellner (1978), Corollary 4]. Hence $A^{(n)}_{(n-m+1)} - A^{(n)}_{(n-2m+1)} \to \log 2$ $(n \to \infty)$ a.s. The rest of the proof is as before. We thank R. Helmers for making us aware of the Wellner reference. □

THEOREM 2.3 (Asymptotic normality). *Suppose U has a positive derivative and suppose there exists a positive function a such that for $x > 0$ and $\gamma \in \mathbb{R}$ (with either choice of sign),*

$$\lim_{t\to\infty} \frac{(tx)^{1-\gamma}U'(tx) - t^{1-\gamma}U'(t)}{a(t)} = \pm\log x \tag{2.1}$$

[Π-variation, notation $\pm t^{1-\gamma}U'(t) \in \Pi(a)$]. Then

$$\sqrt{m}\,(\hat{\gamma}_n - \gamma)$$

has asymptotically a normal distribution with mean zero and variance $\gamma^2(2^{2\gamma+1}+1)/\{2(2^\gamma - 1)\log 2\}^2$ for sequences $m = m(n) \to \infty$ satisfying $m(n) = o(n/g^{\leftarrow}(n))$, where $g(t) := t^{3-2\gamma}\{U'(t)/a(t)\}^2$ $(n \to \infty)$.

REMARK. Note that (2.1) implies $t^{\gamma-1}a(t)/U'(t) \to 0$, hence $g(t)/t \to \infty$ $(t \to \infty)$.

Before we prove this theorem we first formulate the conditions on U in terms of the distribution function F and its density.

THEOREM 2.4. *Suppose U has a positive derivative U'. Equivalent are (with either choice of sign):*

(a) $\pm t^{1-\gamma}U'(t) \in \Pi(a)$.

(b) *For $\gamma > 0$: $\pm t^{1+1/\gamma}F'(t) \in \Pi(b)$. For $\gamma < 0$: $U(\infty) := \lim_{t\to\infty} U(t) < \infty$ and $\mp t^{-1-1/\gamma}F'(U(\infty) - t^{-1}) \in \Pi(b)$.*

For $\gamma = 0$: Let $f_0 = (1-F)/F'$ and $x^* := \sup\{x \mid F(x) < 1\}$. There exists a positive function α with $\alpha(t) \to 0$ $(t \uparrow x^*)$ such that for $x > 0$ locally uniformly,

$$\lim_{t\uparrow x^*}\left[\left(\frac{1 - F(t + xf_0(t))}{1 - F(t)} - e^{-x}\right)\Big/\alpha(t)\right] = \pm\frac{x^2}{2}e^{-x}.$$

REMARK. In case $\gamma = 0$ the following condition is sufficient for (b): Suppose F is three times differentiable, $\pm f_0' > 0$, $\lim_{t\to\infty} f_0''(t)f_0(t)/f_0'(t) = 0$ and $\lim_{t\to\infty} f_0'(t) = 0$. Then (b) holds with $\gamma = 0$, the plus sign at the right-hand side and $\alpha = f_0'$ (Theorem A.8).

REMARK. If F satisfies Theorem 2.4(b), then U satisfies (2.1) with

$$a(t) = \begin{cases} \gamma^3 t^{1-\gamma}\{U(t)\}^{1-1/\gamma}b(U(t)), & \gamma > 0, \\ \alpha(U(t))f_0(U(t)) = tU'(t)\alpha(U(t)), & \gamma = 0, \\ -\gamma^3 t^{1-\gamma}\{U(\infty) - U(t)\}^{1-1/\gamma}b(1/\{U(\infty) - U(t)\}), & \gamma < 0. \end{cases}$$

The proof of this theorem will be given in the Appendix (Theorems A.1, A.3, A.8 and A.10). The normal distribution satisfies the conditions of Theorem 2.4 and we then have asymptotic normality of $\hat{\gamma}_n$ for sequences $m(n) \to \infty$ satisfying $m(n) = o(\log^2 n)$. See the end of this section. For distributions like the Cauchy distribution we have the following theorem.

THEOREM 2.5. *Suppose that one of the following conditions holds*:

(a) *For some* $\gamma > 0$, $\rho > 0$ *and* $c > 0$ *the function* $t^{1+1/\gamma}F'(t) - c$ *is of constant sign and*

$$\lim_{t\to\infty} \frac{(xt)^{1+1/\gamma}F'(tx) - c}{t^{1+1/\gamma}F'(t) - c} = x^{-\rho}$$

[*regular variation with exponent* $-\rho$, *notation* $\pm\{t^{1+1/\gamma}F'(t) - c\} \in RV_{-\rho}$].

(b) *For some* $\gamma < 0$, $\rho > 0$ *and* $c > 0$ *the function*

$$\pm\left\{t^{-1-1/\gamma}F'\left(U(\infty) - t^{-1}\right) - c\right\} \in RV_{-\rho}.$$

Then

$$\sqrt{m}\,\{\hat{\gamma}_n - \gamma\}$$

has asymptotically a normal distribution with mean zero and variance $\gamma^2(2^{2\gamma+1} + 1)/\{2(2^\gamma - 1)\log 2\}^2$ *for sequences* $m = m(n) \to \infty$ *satisfying* $m(n) = o(n/g^{\leftarrow}(n))$ $(n \to \infty)$, *where* $g^{\leftarrow}$ *is the inverse function of* $g(t) := t^{3-2\gamma}\{U'(t)/(t^{1-\gamma}U'(t) - c^\gamma|\gamma|^{1+\gamma})\}^2$.

PROOF OF THEOREM 2.3. Assume for the moment that $+t^{1-\gamma}U'(t) \in \Pi(a)$. Then $a \in RV_0$ and $\lim_{t\to\infty} a(t)/\{t^{1-\gamma}U'(t)\} = 0$ [see, e.g., Geluk and de Haan (1987)]. This also implies $F \in D(G_\gamma)$. Write $V(t) := U(e^t)$. We have

$$\frac{V'(t) - e^{-\gamma x}V'(t + x)}{\beta(t)} \to -x \quad \text{locally uniformly} \tag{2.2}$$

for some positive function β satisfying $\beta(t + x) \sim e^{\gamma x}\beta(t)$ locally uniformly and $\beta(t)/V'(t) \to 0$ $(t \to \infty)$. Now

$$\begin{aligned} &V(t + x) - V(t) - e^{\gamma x}V(t) + e^{\gamma x}V(t - x) \\ &= \int_0^x \{V'(t + s) - e^{\gamma x}V'(t + s - x)\}\, ds \\ &= \beta(t)\int_0^x \frac{V'(t + s) - e^{\gamma x}V'(t + s - x)}{\beta(t + s)} \cdot \frac{\beta(t + s)}{\beta(t)}\, ds, \end{aligned}$$

hence locally uniformly

$$\lim_{t\to\infty} \frac{V(t + x) - V(t) - e^{\gamma x}V(t) + e^{\gamma x}V(t - x)}{\beta(t)} = x \cdot \frac{e^{\gamma x} - 1}{\gamma}. \tag{2.3}$$

We write as in the proof of Theorem 2.1,

$$\frac{X^{(n)}_{(n-m+1)} - X^{(n)}_{(n-2m+1)}}{X^{(n)}_{(n-2m+1)} - X^{(n)}_{(n-4m+1)}} - 2^{\gamma}$$

$$\begin{aligned}(2.4)\quad &= \frac{V\Big(\big\{A^{(n)}_{(n-m+1)} - A^{(n)}_{(n-2m+1)}\big\} + A^{(n)}_{(n-2m+1)}\Big) - V\Big(A^{(n)}_{(n-2m+1)}\Big)}{V\Big(A^{(n)}_{(n-2m+1)}\Big) - V\Big(\big\{A^{(n)}_{(n-4m+1)} - A^{(n)}_{(n-2m+1)}\big\} + A^{(n)}_{(n-2m+1)}\Big)} - 2^{\gamma}\\ &= \frac{V\Big(\big\{A^{(n)}_{(n-m+1)} - A^{(n)}_{(n-2m+1)}\big\} + A^{(n)}_{(n-2m+1)}\Big) - V\Big(A^{(n)}_{(n-2m+1)}\Big)}{V\Big(A^{(n)}_{(n-2m+1)}\Big) - V\Big(A^{(n)}_{(n-4m+1)}\Big)}\\ &\quad - 2^{\gamma}\frac{V\Big(A^{(n)}_{(n-2m+1)}\Big) - V\Big(\big\{A^{(n)}_{(n-4m+1)} - A^{(n)}_{(n-2m+1)}\big\} + A^{(n)}_{(n-2m+1)}\Big)}{V\Big(A^{(n)}_{(n-2m+1)}\Big) - V\Big(A^{(n)}_{(n-4m+1)}\Big)}.\end{aligned}$$

In view of the result of Lemma 2.1 we introduce

$$Q_n := \sqrt{2m}\left(A^{(n)}_{(n-m+1)} - A^{(n)}_{(n-2m+1)} - \log 2\right),$$
$$R_n := \sqrt{4m}\left(A^{(n)}_{(n-2m+1)} - A^{(n)}_{(n-4m+1)} - \log 2\right).$$

Note that Q_n and R_n are independent and asymptotically standard normal.

We start by evaluating the denominator of (2.4) asymptotically. Note that $t^{1-\gamma}U'(t) \in \Pi$ implies $V'(t+x) \sim e^{\gamma x}V'(t)$ locally uniformly ($t \to \infty$). Hence

$$\begin{aligned}V\Big(A^{(n)}_{(n-2m+1)}\Big) - V\Big(A^{(n)}_{(n-4m+1)}\Big) &= V'\Big(A^{(n)}_{(n-2m+1)}\Big)\\ &\quad\times \int_{-\log 2 - R_n/\sqrt{4m}}^{0} \frac{V'\Big(A^{(n)}_{(n-2m+1)} + s\Big)}{V'\Big(A^{(n)}_{(n-2m+1)}\Big)}\,ds\\ &\sim V'\Big(A^{(n)}_{(n-2m+1)}\Big)\cdot \gamma^{-1}(1-2^{-\gamma})\end{aligned}$$

in probability ($n \to \infty$), with the usual convention $\log 2 =: (1-2^{-\gamma})/\gamma$ when $\gamma = 0$.

For the numerator of (2.4) we proceed as follows:

$$\begin{aligned}&\sqrt{m}\,\frac{V\Big(A^{(n)}_{(n-2m+1)} + \log 2 + Q_n/\sqrt{2m}\Big) - V\Big(A^{(n)}_{(n-2m+1)}\Big)}{V'\Big(A^{(n)}_{(n-2m+1)}\Big)}\\ &\quad -\sqrt{m}\,2^{\gamma}\frac{V\Big(A^{(n)}_{(n-2m+1)}\Big) - V\Big(A^{(n)}_{(n-2m+1)} - \log 2 - R_n/\sqrt{4m}\Big)}{V'\Big(A^{(n)}_{(n-2m+1)}\Big)}\\ &= \sqrt{m}\int_0^{Q_n/\sqrt{2m}} \frac{V'\Big(A^{(n)}_{(n-2m+1)} + \log 2 + s\Big)}{V'\Big(A^{(n)}_{(n-2m+1)}\Big)}\,ds\\ &\quad -\sqrt{m}\,2^{\gamma}\int_{-R_n/\sqrt{4m}}^{0} \frac{V'\Big(A^{(n)}_{(n-2m+1)} - \log 2 + s\Big)}{V'\Big(A^{(n)}_{(n-2m+1)}\Big)}\,ds\\ &\quad +\sqrt{m}\Big[V\Big(A^{(n)}_{(n-2m+1)} + \log 2\Big) - V\Big(A^{(n)}_{(n-2m+1)}\Big)\\ &\qquad -2^{\gamma}V\Big(A^{(n)}_{(n-2m+1)}\Big) + 2^{\gamma}V\Big(A^{(n)}_{(n-2m+1)} - \log 2\Big)\Big]\Big/\Big[V'\Big(A^{(n)}_{(n-2m+1)}\Big)\Big].\end{aligned}$$

Now $V'(t+x) \sim e^{\gamma x} V'(t)$ locally uniformly ($t \to \infty$), hence the sum of the first two terms converges in distribution to $2^{\gamma - 1/2} Q - 2^{-1} R$ where Q and R are independent and standard normal. Our aim is to make the last term negligible by choosing the sequence $m(n)$ appropriately.

Using (2.3) we get that the last term converges to $(\sqrt{2}\gamma)^{-1}(2^\gamma - 1)\log 2$ for any sequence $m(n)$ with

$$\sqrt{m} \sim \frac{V'\left(A^{(n)}_{(n-2m+1)}\right)}{\sqrt{2}\,\beta\left(A^{(n)}_{(n-2m+1)}\right)}, \qquad n \to \infty. \tag{2.5}$$

We now investigate what sequences $m(n)$ satisfy (2.5).

Note that [see, e.g., Smirnov (1949)]

$$A^{(n)}_{(n-2m+1)} + \log \frac{2m(n)}{n} \to 0 \quad \text{in probability, } n \to \infty,$$

so that (2.5) reads

$$\begin{aligned}\sqrt{2m} &\sim \left[V'\left(-\log \frac{2m(n)}{n}\right)\right] \Big/ \left[\beta\left(-\log \frac{2m(n)}{n}\right)\right] \\ &= \left[\left(\frac{n}{2m}\right)^{1-\gamma} U'\left(\frac{n}{2m}\right)\right] \Big/ \left[a\left(\frac{n}{2m}\right)\right],\end{aligned}$$

where a is the auxiliary function for $t^{1-\gamma}U'(t) \in \Pi$ or

$$n \sim \left(\frac{n}{2m}\right)^{3-2\gamma} \left\{U'\left(\frac{n}{2m}\right) \Big/ a\left(\frac{n}{2m}\right)\right\}^2 =: g\left(\frac{n}{2m}\right),$$

with $g \in RV_1$. The function g has an asymptotic inverse $g^{\leftarrow} \in RV_1$. So (2.5) is equivalent to

$$m(n) \sim \frac{n}{2g^{\leftarrow}(n)}, \qquad n \to \infty \tag{2.6}$$

and the latter sequence is RV_0. Thus the sequences $m(n)$ for which the condition holds tend to infinity rather slowly.

Let $m_0(n)$ be the sequence of integers defined by

$$m_0(n) := [n/2g^{\leftarrow}(n)]. \tag{2.7}$$

We claim that the statement of the theorem holds for any sequence of integers $m(n) \to \infty$ satisfying

$$m(n) = o(m_0(n)), \qquad n \to \infty. \tag{2.8}$$

To see this recall that

$$\begin{aligned}&\sqrt{2m_0}\left[V\left(\log \frac{n}{2m_0} + \log 2\right) - V\left(\log \frac{n}{2m_0}\right)\right.\\ &\quad \left. - 2^\gamma V\left(\log \frac{n}{2m_0}\right) + 2^\gamma V\left(\log \frac{n}{2m_0} - \log 2\right)\right] \Big/ \left[V'\left(\log \frac{n}{2m_0}\right)\right] \\ &\quad \to \frac{2^\gamma - 1}{\gamma} \log 2, \qquad n \to \infty.\end{aligned}$$

Since (2.8) makes $\sqrt{m}$ of smaller order than $\sqrt{m_0}$ and $\log(n/m)$ of no smaller order than $\log(n/m_0)$, we must have

$$\sqrt{m}\left[V\left(\log\frac{n}{2m}+\log 2\right)-V\left(\log\frac{n}{2m}\right)-2^{\gamma}V\left(\log\frac{n}{2m}\right)+2^{\gamma}V\left(\log\frac{n}{2m}-\log 2\right)\right]$$
$$\div\left[V'\left(\log\frac{n}{2m}\right)\right]\to 0,\qquad n\to\infty$$

and the statement of the theorem holds for the sequence $m(n)$. For later use we mention that also [start from (2.2) instead of (2.3)]

$$\lim_{n\to\infty}\sqrt{m}\left[V'\left(\log\frac{n}{m}+s\right)-e^{\gamma s}V'\left(\log\frac{n}{m}\right)\right]\Big/\left[V'\left(\log\frac{n}{m}\right)\right]=0 \tag{2.9}$$

for all s, locally uniformly.

The proof in case $-t^{1-\gamma}U'(t)\in\Pi$ is similar. □

PROOF OF THEOREM 2.5. Note that $\pm\{t^{1+1/\gamma}F'(t)-c\}\in RV_{-\rho}$ if and only if $\mp\{t^{1-\gamma}U'(t)-c^{\gamma}\gamma^{\gamma+1}\}\in RV_{-\rho\gamma}$, hence $(t\to\infty)$

$$\frac{1}{\gamma\rho}\cdot\frac{V'(t)-e^{-\gamma x}V'(t+x)}{V'(t)-e^{\gamma t}c^{\gamma}\gamma^{1+\gamma}}\to-\frac{e^{-\rho\gamma x}-1}{\rho\gamma}\quad\text{locally uniformly.} \tag{2.10}$$

The rest of the proof is similar to that of Theorem 2.3 and is omitted. □

REMARK 2.1. Note that $g(t)\in RV_{1+2\rho\gamma}$ so that $t/g^{\leftarrow}(t)\in RV_{2\rho\gamma/(1+2\rho\gamma)}$. So here the asymptotic normality holds for sequences $m(n)$ increasing more rapidly than in the situation of Theorem 2.3.

REMARK 2.2. It is possible to state the conditions of Theorems 2.3 and 2.5 in a unified way. The conditions (2.1) and (2.10) can be replaced with: $U'\in RV_{\gamma-1}$ and

$$\lim_{t\to\infty}\frac{(tx)^{1-\gamma}U'(tx)-t^{1-\gamma}U'(t)}{a(t)}=\pm\frac{x^{-\gamma\rho}-1}{-\gamma\rho}$$

for some $\rho\ge 0$ and some positive function a. Note that for $\gamma=0$ the limit does not depend on ρ; this is the reason why the case $\gamma=0$ is not present in Theorem 2.5.

REMARK 2.3. It is obvious that the proof of Theorem 2.3 goes through under the following somewhat weaker conditions: $U'\in RV_{\gamma-1}$ and moreover:

(a) For $\gamma>0$: $\pm t^{-\gamma}U(t)\in\Pi$.
(b) For $\gamma<0$: $\pm t^{-\gamma}\{U(\infty)-U(t)\}\in\Pi$.
(c) For $\gamma=0$: There exist positive functions a_1 and a_2 such that for all $x>0$ [cf. Omey and Willekens (1987)],

$$\lim_{t\to\infty}\frac{U(tx)-U(t)-a_1(t)\log x}{a_2(t)}=\pm\frac{1}{2}\log^2 x.$$

REMARK 2.4. A third possibility is (2.1) with the limit function replaced by

$$\frac{x^{-\rho} - 1}{-\rho},$$

with $\rho > 0$. Then, provided $U' \in RV_{\gamma-1}$, necessarily (1.1) is true with $\gamma = 0$ and $a_n \equiv c_0 > 0$ for all n. The corresponding condition for F' is

$$\lim_{t\to\infty} \frac{e^{t+x}F'(t+x) - c}{e^t F'(t) - c} = e^{-\rho x}$$

for some positive constant c and all x. Set $F_0(t) := F(\log t)$. Then F_0 satisfies the conditions of Theorem 2.5(a) with $\gamma = 1$. Hence we can simply translate the result of Theorem 2.5 for this case and we get a polynomial rate for $m(n)$ like in Theorem 2.5 [cf. Cohen (1982), page 839].

REMARK 2.5. It is clear from the proof of Theorem 2.3 that if the right-hand side of (2.1) is $+\log x$ and if $m(n) \sim c \cdot n/g^{\leftarrow}(n)$ for some positive constant c $(n \to \infty)$, then $\sqrt{m}(\hat{\gamma} - \gamma)$ has asymptotically a normal distribution with the same variance but with mean $(c/2)^{1/2}$. The *sign* of the bias is the same as the sign in the right-hand side of (2.1). This may be of some help in finding an optimal choice for the sequence $m(n)$.

EXAMPLE 1. The distribution functions $F(x) := 1 - \exp(-x^{\alpha})$ satisfy the criterion of Theorem A.8 (Appendix) for all $\alpha > 0$, $\alpha \neq 1$.

EXAMPLE 2 (Normal distribution). Using the previous example for $\alpha = 2$ and Lemma A.2 we find that the normal distribution satisfies (A.25) with $f(t) = t^{-1}$ and $\alpha(t) = t^{-2}$. By Lemma A.1 the same relation also holds with $f(t) = f_0(t) = e^{t^2/2}\int_t^{\infty} e^{-s^2/2}\, ds$ since $\int_t^{\infty} e^{-s^2/2}\, ds = e^{-t^2/2}\{t^{-1} - t^{-3} + o(t^{-3})\}$, $t \to \infty$ [see Abramowitz and Stegun (1965), 26.2.12, page 932]. Hence (Theorem A.10) U satisfies (A.15) with a minus sign where $tU'(t) \sim \{U(t)\}^{-1}$ and $a(t) \sim \{U(t)\}^{-3}$ $(t \to \infty)$. It follows that here the function g from Theorem 2.3 satisfies $g(t) \sim 2t\log^2 t$ so that the theorem holds for sequences $m = m(n) \to \infty$ satisfying $m(n) = o(\log^2 n)$.

EXAMPLE 3 (Gamma distribution). The conditions of Theorem A.10 are easily checked using the expansion $(r \neq 1)$

$$\int_t^{\infty} s^{r-1}e^{-s}\, ds = e^{-t}\{t^{r-1} + (r-1)t^{r-2} + o(t^{r-2})\}.$$

EXAMPLE 4 (Cauchy distribution). The conditions of Theorem 2.5 are satisfied with $\rho = 2$ and $c = \pi^{-1}$. Then $g(t) \sim ct^5$ so that the theorem holds for sequences $m = m(n) \to \infty$ satisfying $m(n) = o(n^{4/5})$.

EXAMPLE 5. The distribution of $\exp(Z_1 + Z_2)$ with Z_1, Z_2 i.i.d. exponential satisfies Theorem 2.3 with $\gamma = 1$ and $a(t) = 1$.

EXAMPLE 6. For the *exponential* and *uniform* distributions we have $t^{1-\gamma}U'(t) \equiv 1$ so that the left-hand side of (2.3) is identically zero. It follows that the conclusion of Theorem 2.3 holds for *all* sequences $m = m(n) \to \infty$, $m(n)/n \to 0$ $(n \to \infty)$. The same is true for the *generalized Pareto distribution* $F_\gamma(x) = 1 - (1 + \gamma x)^{-1/\gamma}$, $\gamma \in \mathbb{R}$ and $1 + \gamma x \geq 0$.

EXAMPLE 7 (Extreme-value distribution: $G_\gamma(x) = \exp - (1 + \gamma x)^{-1/\gamma}$, $\gamma \in \mathbb{R}$). For $\gamma > 0$, condition (a) of Theorem 2.5 is satisfied with $c = \gamma^{-1-1/\gamma}$ and $\rho = \min(1, 1/\gamma)$ and for $\gamma < 0$, condition (b) is satisfied with $c = (-\gamma)^{-1-1/\gamma}$ and $\rho = -1/\gamma$. The theorem holds for sequences $m = m(n) \to \infty$ satisfying, respectively, $m(n) = o(n^{1-1/\gamma(1+\min(1,\gamma))})$ and $m(n) = o(n^{2/3})$. Note that for $\gamma = 0$ the tail of $G_0(x) = \exp(-\exp - x) \approx 1 - \exp(-x)$ is of the exponential type and so the conclusion of Theorem 2.3 holds for all sequences $m = m(n) \to \infty$, $m(n)/n \to 0$, $(n \to \infty)$. The same is true for the *logistic distribution*.

3. High quantile and endpoint estimation. The following theorem enables one to construct a confidence interval for a quantile x_p when $p = p_n \to 0$, $np_n \to \infty$ $(n \to \infty)$.

THEOREM 3.1. *Suppose F has a positive density F' so that U' exists. If $U' \in RV_{\gamma-1}$ (i.e., $F' \in RV_{-1-1/\gamma}$ for $\gamma > 0$, $1/F' \in \Gamma$ for $\gamma = 0$ and $F'(x^* - 1/x) \in RV_{1+1/\gamma}$ for $\gamma < 0$), then*

$$\sqrt{2m}\,\frac{X_{(n-m+1)}^{(n)} - U(1/p_n)}{X_{(n-m+1)}^{(n)} - X_{(n-2m+1)}^{(n)}} \tag{3.1}$$

is asymptotically normal with mean zero and variance, $2^{2\gamma+1}\gamma^2/(2^\gamma - 1)^2$ *provided $p_n \to 0$, $np_n \to \infty$ $(n \to \infty)$ and $m = m(n) := [np_n]$.*

The proof of Theorem 3.1 follows later in this section.

THEOREM 3.2. *Suppose the conditions of Theorems 2.3 or 2.5 hold with $\gamma < 0$. Then $x^* < \infty$ where $x^* = x^*(F) := \sup\{x | F(x) < 1\}$. Define*

$$\hat{x}_n^* := \frac{X_{(n-m+1)}^{(n)} - X_{(n-2m+1)}^{(n)}}{2^{-\hat{\gamma}_n} - 1} + X_{(n-m+1)}^{(n)}. \tag{3.2}$$

Then

$$\sqrt{2m}\,\frac{\hat{x}_n^* - x^*}{X_{(n-m+1)}^{(n)} - X_{(n-2m+1)}^{(n)}} \tag{3.3}$$

is asymptotically normal $(n \to \infty)$ with mean zero and variance

$$\frac{3\gamma^2 2^{2\gamma-1}}{(2^\gamma - 1)^6}.$$

The following auxiliary result is needed.

LEMMA 3.1. *Under the conditions of Theorems* 2.3 *or* 2.5 *the random vector*

$$(3.4)\quad \sqrt{2m}\left(\frac{V\left(E^{(n)}_{(n-2m+1)}\right)-V(\log(n/(2m)))}{V'(\log(n/(2m)))},\ \frac{V\left(E^{(n)}_{(n-m+1)}\right)-V\left(E^{(n)}_{(n-2m+1)}\right)}{2^{\gamma}V'\left(E^{(n)}_{(n-2m+1)}\right)}-\frac{1-2^{-\gamma}}{\gamma}\right)$$

is asymptotically standard normal. Moreover

$$(3.5)\quad \sqrt{2m}\left\{\frac{V'\left(E^{(n)}_{(n-2m+1)}\right)}{V'(\log(n/(2m)))}-1 -\gamma\cdot\frac{V\left(E^{(n)}_{(n-2m+1)}\right)-V(\log(n/(2m)))}{V'(\log(n/(2m)))}\right\}\to 0,\qquad n\to\infty$$

in probability.

PROOF.

$$(3.6)\quad \sqrt{2m}\,\frac{V\left(E^{(n)}_{(n-2m+1)}\right)-V(\log(n/(2m)))}{V'(\log(n/(2m)))} = \sqrt{2m}\int_0^{N_n/\sqrt{2m}}\frac{V'(\log(n/(2m))+s)}{V'(\log(n/(2m)))}\,ds$$

with

$$(3.7)\quad N_n := \sqrt{2m}\left\{E^{(n)}_{(n-2m+1)}-\log\frac{n}{2m}\right\}.$$

Also

$$(3.8)\quad \begin{aligned}&\sqrt{2m}\left\{\frac{V\left(E^{(n)}_{(n-m+1)}\right)-V\left(E^{(n)}_{(n-2m+1)}\right)}{2^{\gamma}V'\left(E^{(n)}_{(n-2m+1)}\right)}-\frac{1-2^{-\gamma}}{\gamma}\right\}\\ &\quad= 2^{-\gamma}\cdot\sqrt{2m}\int_0^{\log 2}\left\{\frac{V'\left(E^{(n)}_{(n-2m+1)}+s\right)}{V'\left(E^{(n)}_{(n-2m+1)}\right)}-e^{\gamma s}\right\}ds\\ &\qquad+2^{-\gamma}\sqrt{2m}\int_0^{Q_n/\sqrt{2m}}\frac{V'\left(E^{(n)}_{(n-2m+1)}+\log 2+s\right)}{V'\left(E^{(n)}_{(n-2m+1)}\right)}\,ds,\end{aligned}$$

with

$$(3.9)\quad Q_n := \sqrt{2m}\left\{E^{(n)}_{(n-m+1)}-E^{(n)}_{(n-2m+1)}-\log 2\right\}.$$

By (2.9), which is true also under the conditions of Theorem 2.5, the first term at the right-hand side of (3.11) tends to zero in probability. Clearly N_n and Q_n are independent and asymptotically standard normal. Both the right-hand side of (3.9) and the second term at the right-hand side of (3.11) converge to a standard normal distribution since (from the conditions of Theorems 2.3 or 2.5)

$$\lim_{t\to\infty} \frac{V'(t+s)}{V'(t)} = e^{\gamma s} \quad \text{for all } s. \tag{3.10}$$

Furthermore

$$\begin{aligned}
&\sqrt{2m}\left\{\frac{V'\left(E^{(n)}_{(n-2m+1)}\right)}{V'(\log(n/2m))} - 1\right\} \\
&\quad = \sqrt{2m}\left[V'(\log(n/(2m))) + \left(E^{(n)}_{(n-2m+1)} - \log(n/(2m))\right)\right. \\
&\qquad \left. - e^{\gamma(E^{(n)}_{(n-2m+1)} - \log(n/(2m)))} V'(\log(n/(2m)))\right]\Big/\left[V'(\log(n/(2m)))\right] \\
&\qquad + \sqrt{2m}\left\{e^{\gamma(E^{(n)}_{(n-2m+1)} - \log(n/(2m)))} - 1\right\}.
\end{aligned}$$

The first term tends to zero by (2.9) and the second term is asymptotically γN_n. □

REMARK. Since in (1.1) we can take $b_n := V(\log n)$ and $a_n := V'(\log n)$, we can consider $\hat{b}_{n/m} := X^{(n)}_{(n-2m)}$ as an estimate of $b_{n/m}$ and $\hat{a}_{n/m} := \{X^{(n)}_{(n-m)} - X^{(n)}_{(n-2m)}\}\gamma/(2^\gamma - 1)$ as an estimate of $a_{n/m}$. Lemma 3.1 establishes joint asymptotic normality of

$$\sqrt{2m}\left(\frac{\hat{a}_{n/m}}{a_{n/m}} - 1, \frac{\hat{b}_{n/m} - b_{n/m}}{a_{n/m}}\right).$$

COROLLARY 3.1.

$$\lim_{n\to\infty} \frac{V\left(E^{(n)}_{(n-m+1)}\right) - V\left(E^{(n)}_{(n-2m+1)}\right)}{V'(\log(n/2m))} = \frac{2^\gamma - 1}{\gamma} \quad \textit{in probability}$$

and

$$\lim_{n\to\infty} V'\left(E^{(n)}_{(n-2m+1)}\right)\Big/V'\left(\log\frac{n}{2m}\right) = 1 \quad \textit{in probability}.$$

PROOF OF THEOREM 3.1.

$$\begin{aligned}
&\sqrt{2m}\,\frac{X^{(n)}_{(n-m+1)} - U(1/p_n)}{X^{(n)}_{(n-m+1)} - X^{(n)}_{(n-2m+1)}} \\
&\quad = \sqrt{2m}\left\{\frac{X^{(n)}_{(n-m+1)} - V(\log(n/m))}{V'(\log(n/m))} + \frac{V(\log(n/m)) - V(-\log p_n)}{V'(\log(n/m))}\right\} \\
&\qquad \times \frac{V'(\log(n/m))}{X^{(n)}_{(n-m+1)} - X^{(n)}_{(n-2m+1)}}.
\end{aligned}$$

Now we can use Lemma 3.1 and Corollary 3.1; further, since $V'(t+x) \sim e^{\gamma x} \cdot V'(t)$ locally uniformly ($t \to \infty$),

$$\sqrt{2m}\,\frac{V(\log(n/m)) - V(-\log p_n)}{V'(\log(n/m))}$$

$$= \sqrt{2m}\int_{\log(m/(np_n))}^{0} \frac{V'(s+\log(n/m))}{V'(\log(n/m))}\,ds \sim -\sqrt{2m}\left\{1 - \frac{m}{np_n}\right\} \to 0,$$

$$n \to \infty. \qquad \square$$

PROOF OF THEOREM 3.2.

$$\sqrt{2m}\left\{\frac{1}{2^{-\hat{\gamma}_n}-1} + \frac{V\left(E_{(n-m+1)}^{(n)}\right) - x^*}{V\left(E_{(n-m+1)}^{(n)}\right) - V\left(E_{(n-2m+1)}^{(n)}\right)}\right\}$$

$$= \sqrt{2m}\left\{\frac{1}{1-2^{\hat{\gamma}_n}} - \frac{1}{1-2^{\gamma}}\right\}$$

$$+\Bigg[\sqrt{2m}\,\frac{V\left(E_{(n-m+1)}^{(n)}\right) - V(\log(n/(2m)))}{V'(\log(n/(2m)))}$$

$$-\sqrt{2m}\left\{\frac{V(\infty) - V(\log(n/(2m)))}{V'(\log(n/(2m)))} + \frac{1}{\gamma}\right\}$$

$$+\sqrt{2m}\left\{\frac{V\left(E_{(n-m+1)}^{(n)}\right) - V\left(E_{(n-2m+1)}^{(n)}\right)}{2^{\gamma}V'\left(E_{(n-2m+1)}^{(n)}\right)} - \frac{1-2^{-\gamma}}{\gamma}\right\}$$

$$\times \frac{2^{\gamma}}{1-2^{\gamma}}\,\frac{V'\left(E_{(n-2m+1)}^{(n)}\right)}{V'(\log(n/(2m)))}$$

$$-\sqrt{2m}\left\{\frac{V'\left(E_{(n-2m+1)}^{(n)}\right)}{V'(\log(n/(2m)))} - 1\right\}\cdot\frac{1}{\gamma}\Bigg]$$

$$\times \frac{V'(\log(n/(2m)))}{V\left(E_{(n-m+1)}^{(n)}\right) - V\left(E_{(n-2m+1)}^{(n)}\right)}.$$

Let Q, R and S denote the limit random variables of the *independent* r.v.'s

$$Q_n := \sqrt{2m}\left\{E_{(n-m+1)}^{(n)} - E_{(n-2m+1)}^{(n)} - \log 2\right\},$$

$$R_n := \sqrt{4m}\left\{E_{(n-2m+1)}^{(n)} - E_{(n-4m+1)}^{(n)} - \log 2\right\},$$

$$S_n := \sqrt{4m}\left\{E_{(n-4m+1)}^{(n)} - \log\left(\frac{n}{4m}\right)\right\},$$

respectively, Q, R and S being i.i.d. with standard normal distribution. Then we obtain (cf. proof of Theorem 2.3)

$$\begin{aligned}
&\sqrt{2m}\left\{\frac{1}{1-2^{\hat{\gamma}_n}}-\frac{1}{1-2^{\gamma}}\right\}\\
&=\frac{\sqrt{2m}}{(1-2^{\hat{\gamma}_n})(1-2^{\gamma})}\{2^{\hat{\gamma}_n}-2^{\gamma}\}\\
&=\frac{\sqrt{2}}{(1-2^{\hat{\gamma}_n})(1-2^{\gamma})}\cdot\sqrt{m}\left\{\frac{V\big(E^{(n)}_{(n-m+1)}\big)-V\big(E^{(n)}_{(n-2m+1)}\big)}{V\big(E^{(n)}_{(n-2m+1)}\big)-V\big(E^{(n)}_{(n-4m+1)}\big)}-2^{\gamma}\right\}\\
&\sim\frac{\sqrt{2}}{(1-2^{\hat{\gamma}_n})}\frac{\gamma}{1-2^{\gamma}}\left\{\frac{2^{\gamma-1/2}Q_n-2^{-1}R_n}{1-2^{-\gamma}}\right\}.
\end{aligned}$$

Now we can use Lemma 3.2 with $N_n = 2^{-1/2}(R_n + S_n)$ and Corollary 3.1; further

$$\lim_{n\to\infty}\sqrt{2m}\left\{\frac{V(\infty)-V(\log(n/(2m)))}{V'(\log(n/(2m)))}+\frac{1}{\gamma}\right\}=0$$

by (2.9) and the proof of (a) ⇔ (b) of Theorem A.3 (Appendix). □

Finally we consider (1.9) with m fixed.

THEOREM 3.3. *Suppose the extreme-value condition* (1.1) *holds*, $p = p_n$ *and* $\lim_{n\to\infty} np_n = c \in (0, \infty)$. *Define* [*cf*. (1.9)]

$$\hat{x}_{p_n,n} := \frac{(m/(np_n))^{\hat{\gamma}_n}-1}{1-2^{-\hat{\gamma}_n}}\left\{X^{(n)}_{(n-m+1)}-X^{(n)}_{(n-2m+1)}\right\}+X^{(n)}_{(n-m+1)}.$$

Then for fixed $m > c$,

$$\frac{\hat{x}_{p_n,n}-x_{p_n}}{X^{(n)}_{(n-m+1)}-X^{(n)}_{(n-2m+1)}}\tag{3.11}$$

converges in distribution to the distribution of the random variable

$$\left[\left(\frac{m}{c}\right)^{\gamma}-2^{-\gamma}\right]\Big/(1-2^{-\gamma})+\left\{1-\left(\frac{Q_m}{c}\right)^{\gamma}\right\}\Big/\{e^{\gamma H_m}-1\},\tag{3.12}$$

where H_m *and* Q_m *are independent*, Q_m *has a standard gamma* $(2m+1)$-*distribution and* H_m *the distribution of* $\Sigma_{j=m+1}^{2m}Z_j/j$ *with* $Z_1, Z_2, \dots$ *i.i.d. standard exponential.*

REMARK 3.1. It can be shown by induction that H_m has the (beta-type) density function

$$f_1(x) := \frac{(2m)!}{m!(m-1)!}e^{-(m+1)x}(1-e^{-x})^{m-1},\qquad x\ge 0.$$

Furthermore, let f_2 be the density function of Q_m and G_m the distribution function of $\{1-(Q_m/c)^\gamma\}/\{e^{\gamma H_m}-1\}$ (in 3.12). Then

$$G_m(x) = \int_0^{ub(x;\gamma)} \int_{lb(x,y;\gamma)}^{\infty} f_2(z)\,dz\, f_1(y)\,dy, \qquad x \in \mathbb{R},$$

where

$$ub(x;\gamma) := \begin{cases} \infty, & \gamma = 0, \\ \dfrac{1}{\gamma}\log\left(1+\dfrac{1}{\min(-1,x)}\right), & \gamma < 0, \\ \dfrac{1}{\gamma}\log\left(1+\dfrac{1}{\max(0,x)}\right), & \gamma > 0 \end{cases}$$

and

$$lb(x,y;\gamma) := \begin{cases} c\cdot e^{-xy}, & \gamma = 0, \\ c\cdot[1+x(1-e^{\gamma y})]^{1/\gamma}, & \gamma \neq 0. \end{cases}$$

THEOREM 3.4. *Suppose the extreme-value condition* (1.1) *holds with* $\gamma < 0$. *Then* $x^* < \infty$, *where* $x^* = x^*(F) := \sup\{x | F(x) < 1\}$. *Define* $\hat{x}_n^*$ *as in* (3.2). *Then*

$$\frac{\hat{x}_n^* - x^*}{X_{(n-m+1)}^{(n)} - X_{(n-2m+1)}^{(n)}}$$

converges in distribution to the distribution of the random variable $(1-2^\gamma)^{-1} + \{e^{\gamma H_m}-1\}^{-1}$ *with* H_m *as in Theorem* 3.3.

The proofs are based on the following lemma and manipulations like those in the proof of Theorem 3.2. We omit the details.

LEMMA 3.2. *Let* $E_{(1)}^{(n)} \le E_{(2)}^{(n)} \le \cdots \le E_{(n)}^{(n)}$ *be standard exponential order statistics. Then for* $n \to \infty$ *and* m *fixed the random vector*

$$\left(E_{(n-m+1)}^{(n)} - E_{n-2m+1)}^{(n)}, E_{(n-2m+1)}^{(n)} - \log n\right)$$

converges in distribution to the distribution of $(H_m, -\log Q_m)$ *where* H_m *and* Q_m *are independent and have the probability distributions mentioned in Theorem* 3.3.

REMARK 3.2. Note that, although m remains constant, the number of order statistics involved in the definition of $\hat{\gamma}_n$ [appearing in (3.2)] should go to infinity in order to guarantee consistency.

REMARK 3.3. We do not enter here into the question of how to choose m in an optimal way.

TABLE 1

Survey of the experiments and some results. m_1 is the number of upper order statistics involved in the estimation of γ, n is the sample size, γ is the extreme value index of the distribution, $\bar{\hat{\gamma}}_n$ is the average of 5000 estimates of γ for given m_1, σ is the theoretical standard error for given γ and m_1 (cf. Theorem 2.3), $\sigma(\hat{\gamma}_n)$ is the standard error of the 5000 estimates of γ for given m_1, nl is the number of estimates below $\bar{\hat{\gamma}}_n - 2\sigma(\hat{\gamma}_n)$ and nr is the number of estimates above $\bar{\hat{\gamma}}_n + 2\sigma(\bar{\gamma}_n)$.

			$m_1 = 40$					$m_1 = 80$					$m_1 = 120$				
Distribution	n	γ	$\bar{\hat{\gamma}}_n$	σ	$\sigma(\hat{\gamma}_n)$	nl	nr	$\bar{\hat{\gamma}}_n$	σ	$\sigma(\hat{\gamma})$	nl	nr	$\bar{\hat{\gamma}}_n$	σ	$\sigma(\hat{\gamma}_n)$	nl	nr
1. Exp(1)	1000	0	0.006	0.570	0.579	149	104	−0.004	0.403	0.408	127	101	−0.005	0.329	0.328	115	114
2. Uniform(0, 1)	1000	−1	−1.028	0.559	0.560	124	85	−1.023	0.395	0.392	119	104	−1.023	0.323	0.328	127	108
3. Normal(0, 1)	1000	0	−0.164	0.570	0.574	133	100	−0.190	0.403	0.397	128	95	−0.220	0.329	0.320	124	107
4. Normal(0, 1)[a]	1,000,000	0	−0.065	0.403	0.404	144	96	−0.056	0.285	0.281	113	120	−0.066	0.233	0.229	128	119
5. GPD(1, 1)	1000	1	1.025	0.684	0.703	116	123	1.006	0.484	0.481	124	112	0.998	0.395	0.399	120	111
6. GPD(−0.19, 40)	1000	−0.19	−0.215	0.559	0.575	132	96	−0.196	0.396	0.402	135	91	−0.199	0.323	0.333	115	101
7. GPD(−0.19, 40)	216	−0.19	−0.216	0.559	0.581	135	100	−0.200	0.396	0.404	124	110	−0.196	0.323	0.327	108	112

[a]Note that for experiment 4, m_1 is, respectively, 80, 160 and 240.

Remark 3.4. Theorem 2.1 only implies consistency of the estimate $\hat{x}_{p_n, n}$ for x_{p_n} in the trivial case $\gamma < 0$. It is doubtful if in general a consistent estimate is possible within the present setup. By exploiting the fact that $\sqrt{2m}(H_m - \log 2)$ and $\{Q_m - (2m + 1)\}/\sqrt{2m+1}$ are asymptotically standard normal for $m \to \infty$, one can show, however, that for $\gamma < \frac{1}{2}$ the expression (3.12) converges to zero, but for $\gamma = \frac{1}{2}$ it converges to a normal distribution and for $\gamma > \frac{1}{2}$ it diverges $(m \to \infty)$.

Concluding remarks. The main difference between the present approach to estimating γ and previous ones is that we avoid applying maximum likelihood methods in an approximate model. Such methods, employed by Cohen (1982) and Smith (1987) lead to not explicitly known estimators, the need to estimate an extra (scale) parameter and specific problems with the solution of the equations for $\gamma \leq -\frac{1}{2}$. It is not clear whether the maximum likelihood estimators are consistent under the single condition (1.1).

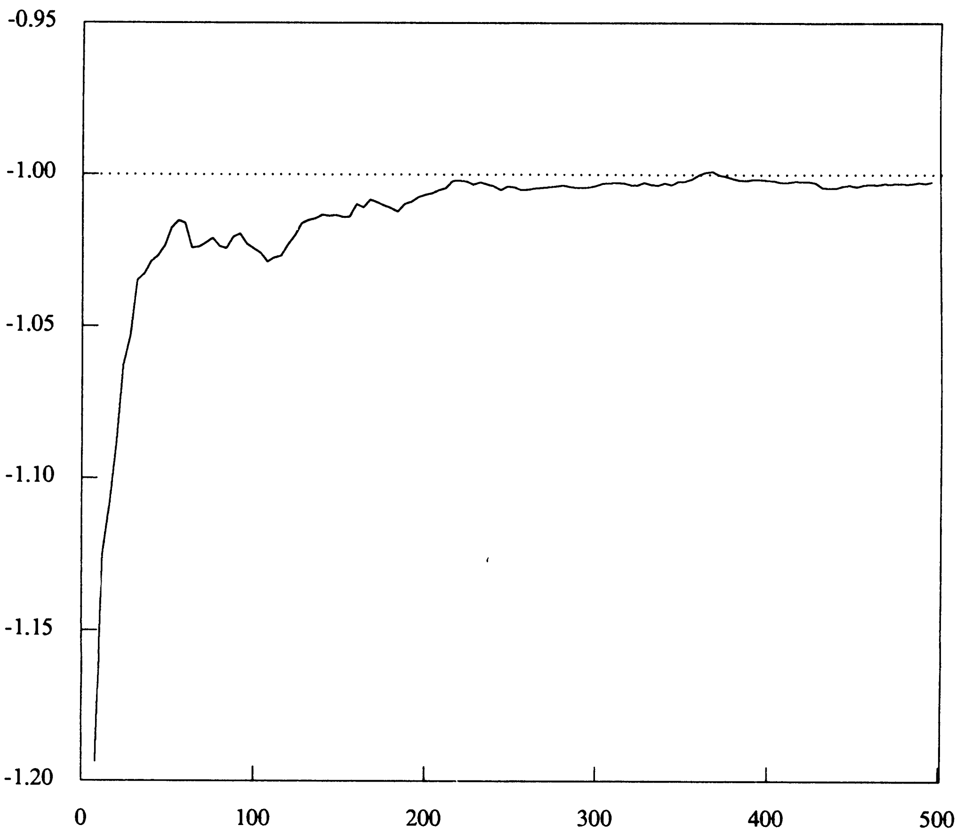

Fig. 1. *Uniform*$(0,1)$, γ *against* m_1.

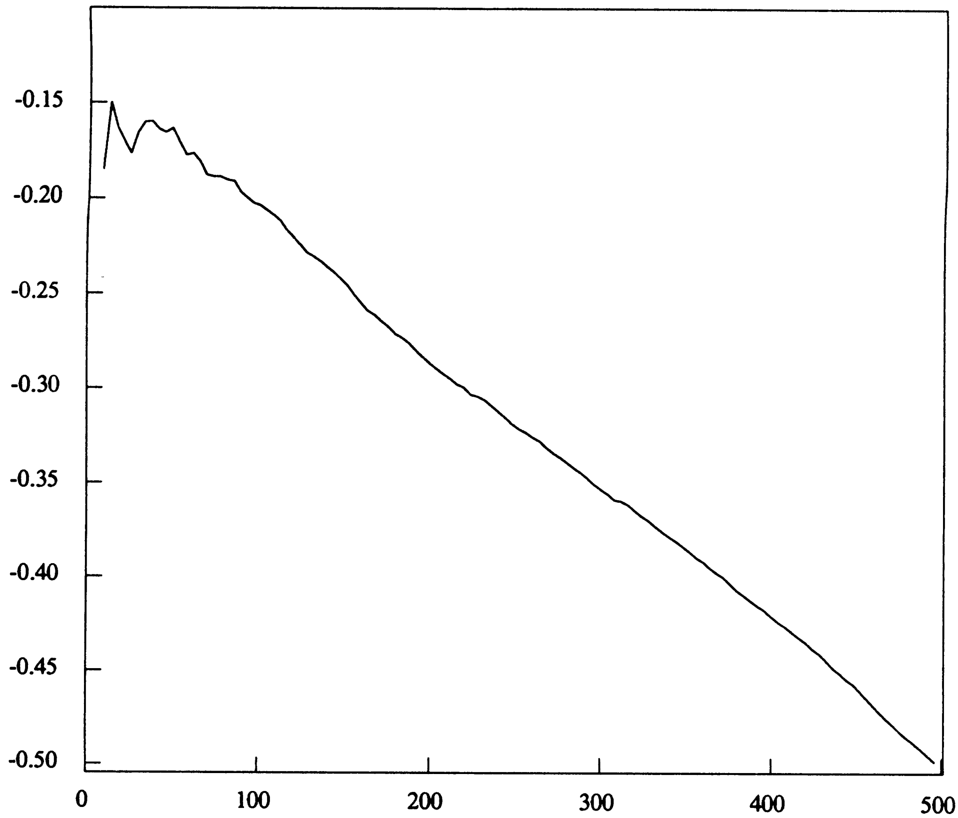

FIG. 2. *Normal*(0, 1), γ *against* m_1.

Our second order conditions are comparable with those of Smith in the case $\gamma > 0$ and $-\frac{1}{2} < \gamma < 0$, but much more general in other cases. Apart from avoiding the maximum likelihood procedure we also do not use explicitly the so-called generalized Pareto approximation. Our approach is basically nonparametric. We hope to have shown that the remark on the top of page 1176 of Smith (1987) is somewhat premature.

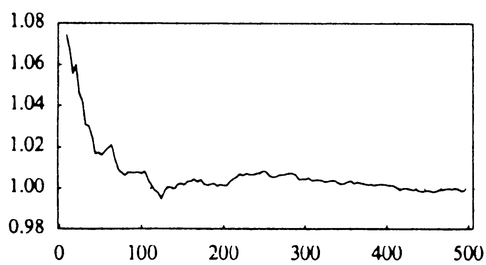

FIG. 3. GPD(1, 1), γ *against* m_1.

Some of the remarks above also apply to the difference between our approach to the estimation of the endpoint of the distribution in the case $\gamma < 0$ and the approach based on maximum likelihood methods in an approximate model given by Hall (1982).

4. Simulation results and an application. The simulation experiments of this section illustrate how the theoretical results of Sections 2 and 3 work out in practice. We also apply our methods to the sequence of observed high tide water levels at the Dutch island Terschelling.

For several distributions, samples of order statistics are generated using the *sequential method*. Let $U_1, U_2, \dots, U_n$ be independent $U(0,1)$. Then a sample of uniform order statistics is generated by $U_{(n)}^{(n)} := U_1^{1/n}$ and $U_{(n-k+1)}^{(n)} := U_{(n-k+2)}^{(n)} \cdot \{U_k\}^{1/(n-k+1)}$, $k = 2, 3, \dots, n$. Taking $X_{(i)}^{(n)} := F^{-1}(U_{(i)}^{(n)})$, where F^{-1} denotes the inverse distribution function, then gives a sample of order statistics from a distribution function F. This method is particularly useful in our situation where only a small subset of the upper order statistics is needed.

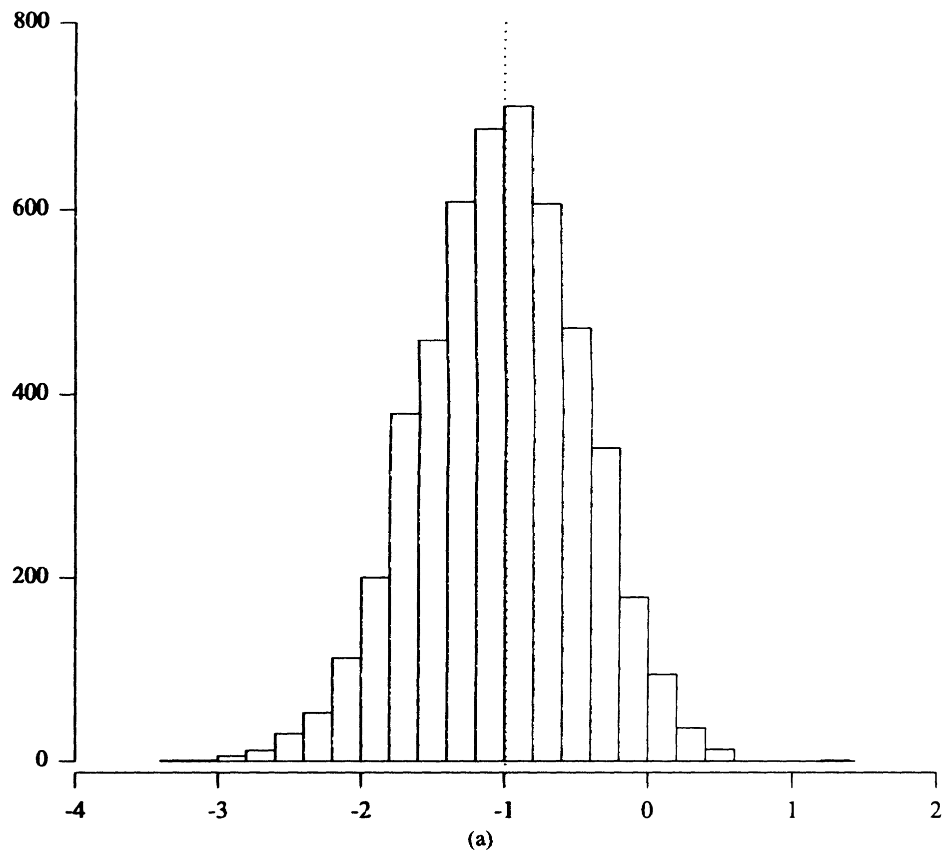

FIG. 4. (*a*) *Uniform*(0, 1), $m_1 = 40$.

The following distributions are considered: standard exponential, uniform(0, 1), normal(0, 1) and the generalized Pareto distribution, GPD(γ, σ), for several values of γ and σ. Recall that the distribution function of the generalized Pareto distributions (related to the extreme-value distributions) is defined by $F(x) = 1 - (1 + \gamma x/\sigma)^{-1/\gamma}$, $\gamma \in \mathbb{R}$, $\sigma > 0$ and $1 + \gamma x/\sigma \geq 0$. A survey of the experiments is given in Table 1.

The construction of order statistics described above was repeated 5000 times for each probability distribution. For most distributions (Except 4 and 7) we chose the sample size n to be 1000 and constructed a total of $k = 500$ upper order statistics using the random number generator GGUBFS of the IMSL package.

4.1. *Consistency of* $\hat{\gamma}_n$. Let m_1 be the number of upper order statistics used for estimating γ (hence $m_1/4$ is the number m from Theorem 2.3). In Figures 1, 2 and 3 the average of the 5000 estimates of γ is plotted against m_1, $m_1 =$

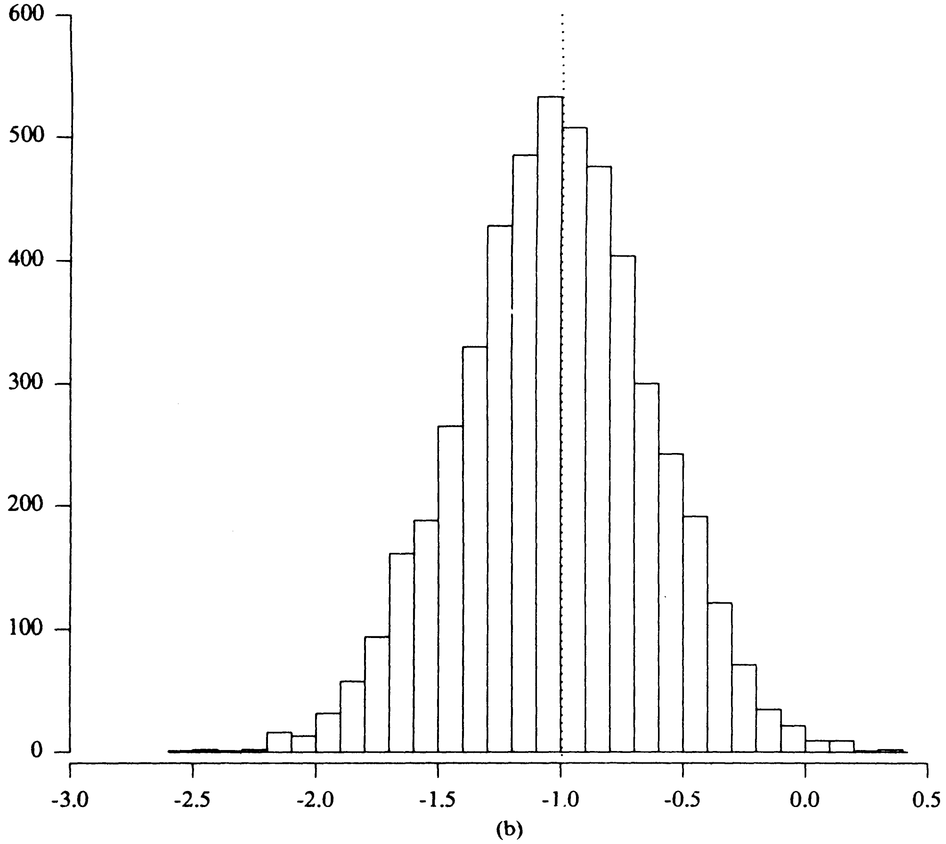

FIG. 4. (*b*) *Uniform*(0, 1), $m_1 = 80$.

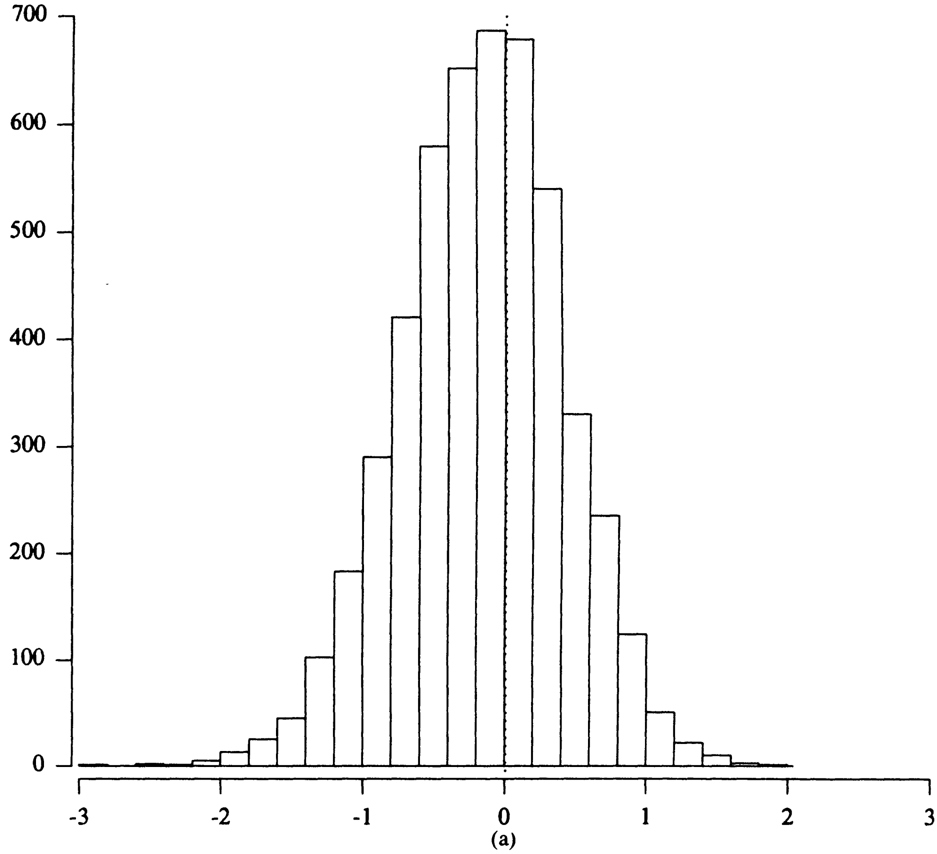

FIG. 5. (*a*) *Normal*(0, 1), $m_1 = 40$.

$4, 8, \ldots, 500$, for the uniform, normal and GPD(1, 1) distributions. The dotted line indicates the true value of γ.

One would expect that if m_1 is too small, the variance of the estimates is large because of the large variation of the few upper order statistics involved in the estimation of γ. With respect to the disappointing results for the normal distribution (Figure 2) we remark that the convergence to the limiting distribution is known to be slow [Hall (1979)].

4.2. *Asymptotic normality of* $\hat{\gamma}_n$. In Figures 4a, 5a and 6a histograms of 5000 estimates of γ are given for $m_1 = 40$, and in Figures 4b, 5b, and 6b for $m_1 = 80$. The dotted vertical lines again indicate the true value of γ.

The distributions of $\hat{\gamma}_n$ for uniform (0, 1) and GPD(1, 1) seem to be more or less symmetric about γ, but the estimates in Figure 5 are too low. This is illustrated numerically in Table 1, where for $m_1 = 40$, 80 and 120 the average of the 5000 estimates of γ are given, indicated by $\bar{\hat{\gamma}}_n$. Also given are the theoretical standard error, indicated by σ (cf. Theorem 2.3 with the true value for γ and m_1

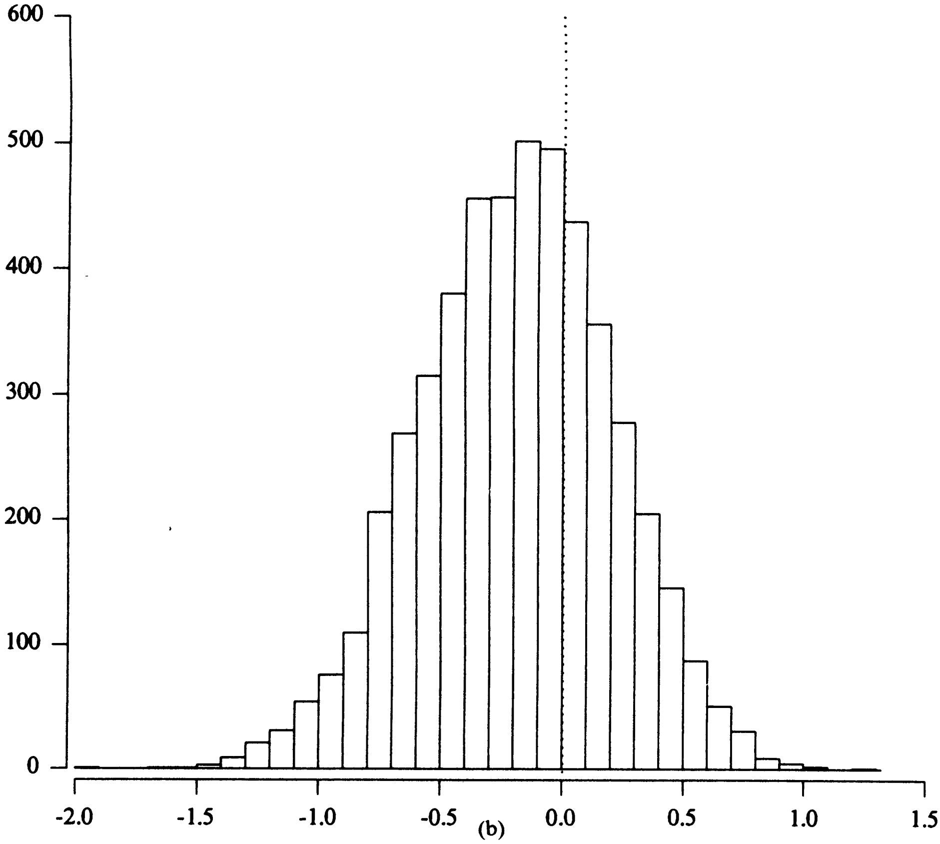

FIG. 5. (*b*) *Normal*(0, 1), $m_1 = 80$.

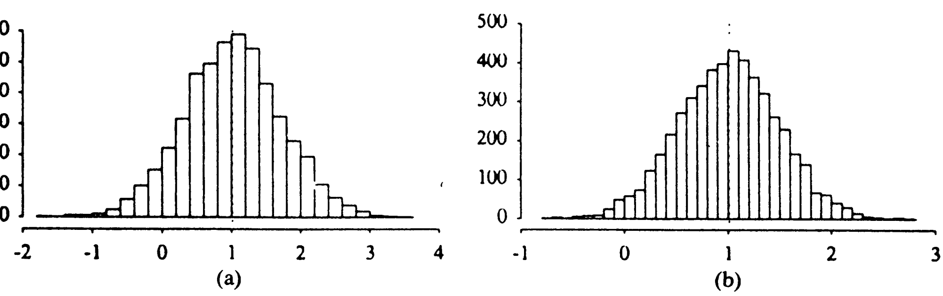

FIG. 6. (*a*) GPD(1, 1), $m_1 = 40$. (*b*) GPD(1, 1), $m_1 = 80$.

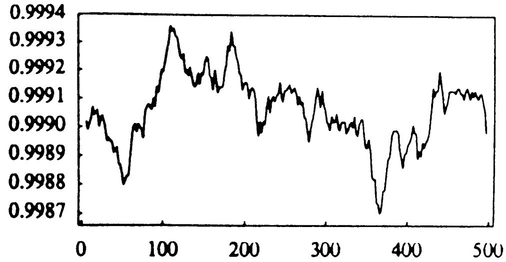

FIG. 7. *Uniform*(0, 1), *quantile against* m_2.

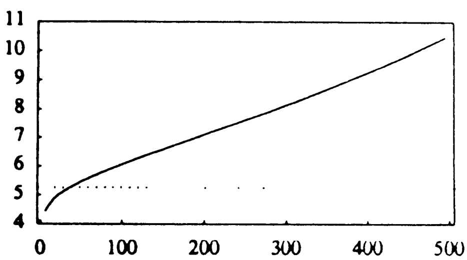

FIG. 8. *Normal*(0, 1), *quantile against* m_2.

for m substituted), and the standard error of the 5000 estimates of γ, which is indicated by $\sigma(\hat{\gamma}_n)$. To give an impression about the skewness of the estimates, the number of estimates below $\bar{\hat{\gamma}}_n - 2\sigma(\hat{\gamma}_n)$ (nl) and the number of estimates above $\bar{\hat{\gamma}}_n + 2\sigma(\hat{\gamma}_n)$ (nr) are given.

Note that for asymptotic normality in the case of the normal distribution we need to have $m(n) = o(\log^2 n)$ $(n \to \infty)$, therefore in experiment 4 the sample size n was taken as 10^6. The improvement of the estimates is remarkable (Table 1).

4.3. *Large quantile estimation.* Let m_2 be the number of upper order statistics used in formula (1.9) (hence $m_2/2$ is the number m from Theorem 3.3).

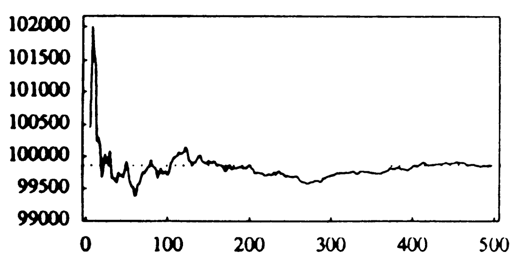

FIG. 9. GPD(1, 1), *quantile against* m_2.

A large quantile of the distributions is estimated with $p_0 = 10^{-4}$ and $k = 10$ [cf. (1.4)]. In order to make the computational work tractable, we have substituted γ for $\hat{\gamma}_n$ in (1.9).

We consider the case that m_2 is fixed (cf. Theorem 3.3 and Remark 3.2). Nevertheless we give estimates for the quantile for several values of m_2 in order to get an impression of the stability of the estimation when m_2 varies.

Figures 7, 8 and 9 show that the estimation in case of the uniform(0, 1) and GPD distributions seems to be reasonable for m_2 not too small. Again one can see that in the case of the normal distribution, convergence seems to be slow.

4.4. *An application to high tide water levels.* Terschelling is one of the islands at the Dutch coast. High tide water levels are available from 1932 until 1985. The data before 1932 cannot be used since in that year the situation around Terschelling was changed dramatically by the closure of the Zuider Zee.

In order to transform the original sequence of high tide water levels into an (approximate) i.i.d. sequence, only observations were used that are above a

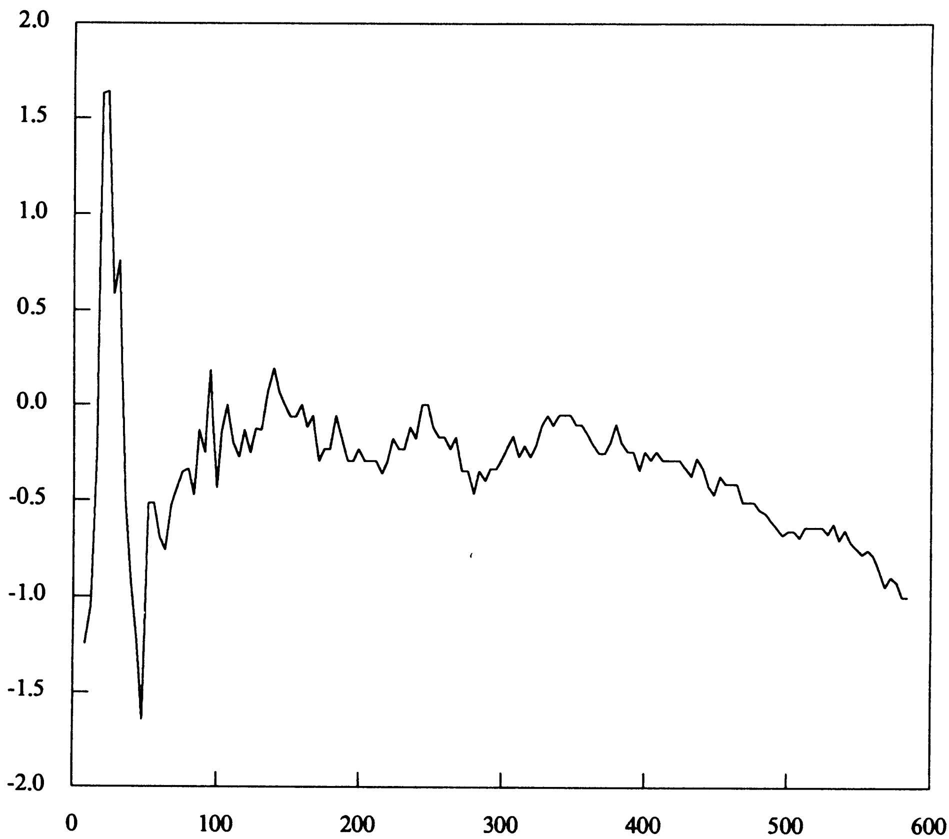

FIG. 10. γ *against* m_1.

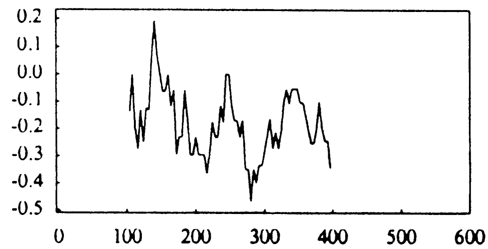

FIG. 11. γ *against* m_1.

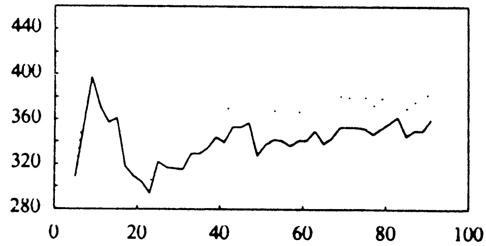

FIG. 12. *Quantile against* m_2, $\gamma = -0.16933$.

certain threshold, sufficiently apart from each other in time and that fall within the winter season. After this selection procedure, 588 observations remain.

In Figure 10 the estimates of γ are plotted against m_1, $m_1 = 4, 8, \dots, 588$. It seems that for m_1 neither too small nor too high, the estimates of γ are more or less constant. In Figure 11 the same estimates are plotted but now only for $m_1 = 100, 104, \dots, 400$. One can see that there are some remarkable fluctuations, but for most values of m_1 the estimates seem to be negative. This implies $x^* < \infty$, i.e., the distribution has a finite right endpoint.

Figure 12 shows the estimation of a large quantile for several m_2, with $p_0 = 10^{-3}$, $k = 10$, $c = 0.1727$, $\hat{\gamma}_n = -0.16933$ and $m_1 = 268$ [cf. (1.4), (1.9) and Theorem 3.3]. The dotted curve indicates for each m_2 a 5% one-sided confidence interval (cf. Theorem 3.3 and Remark 3.1). In spite of the fact that the estimates seem to be reasonably constant when m_2 changes, provided m_2 is not too small, it turns out that a small change in $\hat{\gamma}_n$ causes a big change in the estimation of a large quantile.

APPENDIX

Analytical results. The conditions in Theorem 2.3 are phrased in terms of U, the inverse function of $1/(1-F)$. The aim of this section is to formulate these conditions in terms of the distribution function F itself and its density. The main result here is actually in terms of F alone but this result is not immediately applicable for Theorem 2.3. It is given for completeness and since it will be useful in other contexts.

The relation to be studied is $\pm t^{1-\gamma}U'(t) \in \Pi$. In order to avoid duplication, we only consider this relation with the + sign in the *proofs*. First we consider the case $\gamma > 0$. Without loss of generality we suppose that $F(0) = 0$.

THEOREM A.1. *Suppose U has a positive derivative U' and $\gamma > 0$. Equivalent are*:

(A.1) $$\pm t^{1-\gamma}U'(t) \in \Pi.$$

(A.2) $$\pm \{-U(t) + \gamma^{-1}tU'(t)\} \in RV_\gamma.$$

(A.3) $$\pm (t^{-\gamma}U(t))' \in RV_{-1}.$$

(A.4) $$\pm t^{1+1/\gamma}F'(t) \in \Pi.$$

PROOF [see de Haan (1977)]. (A.1) ⇔ (A.2):

$$\int_0^1 s^{\gamma-1}\log s\, ds \leftarrow \int_0^1 \frac{(ts)^{1-\gamma}U'(ts) - t^{1-\gamma}U'(t)}{a(t)} s^{\gamma-1}\, ds = \frac{U(t) - \gamma^{-1}tU'(t)}{t^\gamma a(t)}.$$

(A.2) ⇔ (A.3): Obvious.

(A.2) ⇔ (A.4): Replacing t by $1/(1 - F(s)) \in RV_{1/\gamma}$ in (A.2) yields

$$\gamma^{-1}\frac{1 - F(s)}{F'(s)} - s \in RV_1 \quad \text{i.e.,} \quad \gamma^{-1}(1 - F(s)) - sF'(s) \in RV_{-1/\gamma}.$$

This is a relation like (A.2) for U. The equivalence of this relation and (A.4) and also the converse implication are proved as in the first part of the proof. □

Relation (A.3) of Theorem A.1 implies $\pm t^{-\gamma}U(t) \in \Pi$. The latter relation can also be translated for F even when there is no derivative. That is the content of the next theorem.

THEOREM A.2. *Equivalent are, for $\gamma > 0$*:

(A.5) $$\pm t^{-\gamma}U(t) \in \Pi.$$

(A.6) $$\mp t^{1/\gamma}\{1 - F(t)\} \in \Pi.$$

PROOF. This is a slight generalization of de Haan and Resnick (1979), Theorem 1. □

Next we consider the case $\gamma < 0$.

THEOREM A.3. *Let $\gamma < 0$ and suppose U has a positive derivative U'. Equivalent are, with $U(\infty) := \lim_{t\to\infty}U(t)$*:

(A.7) $$\pm t^{1-\gamma}U'(t) \in \Pi.$$

(A.8) $$U(\infty) < \infty \quad \textit{and} \quad \pm\{U(\infty) - U(t) + \gamma^{-1}tU'(t)\} \in RV_\gamma.$$

(A.9) $$U(\infty) < \infty \quad \textit{and} \quad \pm(t^{-\gamma}\{U(\infty) - U(t)\})' \in RV_{-1}.$$

(A.10) $$U(\infty) < \infty \quad \textit{and} \quad \mp t^{-1-1/\gamma}F'(U(\infty) - t^{-1}) \in \Pi.$$

PROOF. (A.7) ⇔ (A.8):

$$\int_1^\infty s^{\gamma-1}\log s\,ds \leftarrow \int_1^\infty \frac{(ts)^{1-\gamma}U'(ts) - t^{1-\gamma}U'(t)}{a(t)} s^{\gamma-1}\,ds$$

$$= \frac{U(\infty) - U(t) + \gamma^{-1}tU'(t)}{t^\gamma a(t)}.$$

(A.8) ⇔ (A.9): Obvious.

(A.8) ⇒ (A.10): Write $U(\infty) - U(t) = s$ with $U(\infty) - U(t) \in RV_\gamma$. Then $t = U^{\leftarrow}(U(\infty) - s)$ and $U^{\leftarrow}(U(\infty) - s^{-1}) = 1/\{1 - F(U(\infty) - s^{-1})\} \in RV_{-1/\gamma}$. Replacing t by $U^{\leftarrow}(U(\infty) - s^{-1})$ in (A.8) yields

$$-s^{-1} - \gamma^{-1}\frac{U^{\leftarrow}(U(\infty) - s^{-1})}{(U^{\leftarrow})'(U(\infty) - s^{-1})} = -s^{-1} - \gamma^{-1}\frac{1 - F(U(\infty) - s^{-1})}{F'(U(\infty) - s^{-1})} \in RV_{-1}$$

$$\text{using}\quad (U^{\leftarrow})' = \frac{F'}{(1-F)^2}.$$

Since

$$F'(U(\infty) - s^{-1}) = \frac{\{1 - F(U(\infty) - s^{-1})\}^2}{U'(1/\{1 - F(U(\infty) - s^{-1})\})} \in RV_{1+1/\gamma},$$

we finally obtain

$$-\left\{s^{-1}F'(U(\infty) - s^{-1}) + \gamma^{-1}\int_s^\infty F'(U(\infty) - u^{-1})u^{-2}\,du\right\}$$

$$= -s^{-1}F'(U(\infty) - s^{-1}) - \gamma^{-1}\{1 - F(U(\infty) - s^{-1})\} \in RV_{1/\gamma}.$$

Now use de Haan (1977). The implication (A.10) ⇒ (A.8) is proved in an analogous way. □

Here again there is a result that does not involve derivatives. We omit the proof.

THEOREM A.4. *Equivalent are, for* $\gamma < 0$:

(A.11) $$\pm t^{-\gamma}\{U(\infty) - U(t)\} \in \Pi.$$

(A.12) $$U(\infty) < \infty \quad \textit{and} \quad \mp t^{-1/\gamma}\{1 - F(U(\infty) - t^{-1})\} \in \Pi.$$

The case $\gamma = 0$ is considerably more complicated. We start with a theorem on U.

THEOREM A.5. *Suppose U has a positive derivative U′. Equivalent are*:

(A.13)
$$\pm\, tU'(t) \in \Pi(a).$$

(A.14)
$$\pm\left\{tU'(t) - U(t) + \frac{1}{t}\int_0^t U(s)\,ds\right\} \sim a(t),$$

$t \to \infty$ *where a is slowly varying*.

(A.15)
$$\frac{U(tx) - U(t) - tU'(t)\log x}{a(t)} \to \pm\frac{1}{2}\log^2 x,$$

$t \to \infty$, *for* $x > 0$, *where a is a positive function*.

PROOF. (A.13) ⇔ (A.14):

$$\frac{U(t) - (1/t)\int_0^t U(s)\,ds - tU'(t)}{a(t)} = \int_0^1 \frac{txU'(tx) - tU'(t)}{a(t)}\,dx$$
$$\to \int_0^1 \log x\,dx = -1, \qquad t \to \infty.$$

(A.13) ⇒ (A.15): For $x > 0$ and $t \to \infty$,

$$\frac{U(tx) - U(t) - tU'(t)\log x}{a(t)} = \int_1^x \frac{tyU'(ty) - tU'(t)}{a(t)}\frac{dy}{y}$$
$$\to \int_1^x \frac{\log y}{y}\,dy, \qquad t \to \infty.$$

(A.15) ⇒ (A.13): [cf. Omey and Willekens (1987)]. For $x, y > 0$,

$$\frac{U(txy) - U(ty) - U(tx) + U(t)}{a(t)} = \frac{U(txy) - U(t) - tU'(t)\log(xy)}{a(t)}$$
$$- \frac{U(ty) - U(t) - tU'(t)\log y}{a(t)}$$
$$- \frac{U(tx) - U(t) - tU'(t)\log x}{a(t)}$$
$$\to \log x \cdot \log y.$$

It follows that for all $x > 1$ the function $U(tx) - U(t)$ is in $\Pi(a(t)\log x)$ for $t \to \infty$. Hence $a \in RV_0$. Now for $t \to \infty$,

$$\frac{(\log xy)^2}{2} \leftarrow \frac{U(txy) - U(t) - t\log(xy)U'(t)}{a(t)}$$
$$= \frac{U(txy) - U(ty) - ty\log xU'(ty)}{a(ty)} \cdot \frac{a(ty)}{a(t)}$$
$$+ \frac{U(ty) - U(t) - t\log yU'(t)}{a(t)} + \log x\frac{tyU'(ty) - tU'(t)}{a(t)}.$$

Since everything else converges, the last term must also converge, hence $tU'(t) \in \Pi(a)$. □

After these preliminary statements on U we show what the translation to the inverse function is going to be in the nice case when one can work with derivatives. This serves as an introduction to the general results given afterwards.

Let Q be a three times differentiable function. Then

$$Q(t+x) - Q(t) = xQ'(t) + \frac{x^2}{2}Q''(t) + \frac{x^3}{6}Q'''(t) + \cdots .$$

If $Q'(t) > 0$ and $Q''(t)/Q'(t) \to 0$, then all terms except the first one are asymptotically negligible: $Q''(t)/Q'(t) \to 0$ implies $Q'(t+x)/Q'(t) \to 1$ $(t \to \infty)$ for all x and hence $\{Q(t+x) - Q(t)\}/Q'(t) \to x$ $(t \to \infty)$ locally uniformly for all x (just integrate). This is basically Π-variation. Suppose next that $Q''(t) > 0$ and $Q'''(t)/Q''(t) \to 0$ $(t \to \infty)$. Then all terms except the first two ones are asymptotically negligible: $Q'''(t)/Q''(t) \to 0$ implies $Q''(t+x)/Q''(t) \to 1$ $(t \to \infty)$ for all x and hence

$$(*) \qquad \{Q(t+x) - Q(t) - xQ'(t)\}/Q''(t) \to x^2/2, \qquad t \to \infty$$

(use the just mentioned result for Q' instead of Q and integrate).

Now let P be the inverse function of Q. Let $x^* := \sup\{x | P(x) < \infty\}$. We expand P as follows (we still suppose $Q' > 0$, hence $P' > 0$),

$$P\left(t + \frac{x}{P'(t)}\right) - P(t) = x + \frac{x^2}{2}\frac{P''(t)}{\{P'(t)\}^2} + \frac{x^3}{6}\frac{P'''(t)}{\{P'(t)\}^3} + \cdots .$$

If $P''(t)/\{P'(t)\}^2 \to 0$, then $P'(t + x/P'(t))/P'(t) \to 1$ $(t \uparrow x^*)$ locally uniformly, hence $P(t + x/P'(t)) - P(t) \to x$ $(t \uparrow x^*)$ locally uniformly. This is basically Γ-variation. Suppose next $P''(t) > 0$, $P''(t)/\{P'(t)\}^2 \to 0$ and $P'''(t)/\{P''(t) \cdot P'(t)\} \to 0$ $(t \uparrow x^*)$. Then

$$\log P''\left(t + \frac{x}{P'(t)}\right) - \log P''(t) = \frac{x}{P'(t)} \frac{P'''(t + x\theta/P'(t))}{P''(t + x\theta/P'(t))}$$

$$\sim \frac{xP'''(t + x\theta/P'(t))}{P'(t + x\theta/P'(t))P''(t + x\theta/P'(t))} \to 0,$$

$$t \uparrow x^*, \text{ locally uniformly,}$$

where $\theta = \theta(t, x) \in [0,1]$ and we can prove (see Theorem A.8 below)

$$(**) \qquad \frac{P(t + x/P'(t)) - P(t) - x}{-P''(t)/\{P'(t)\}^2} \to \frac{x^2}{2}, \qquad t \uparrow x^*, \text{ locally uniformly.}$$

Note that the joint statements $Q''(t)/Q'(t) \to 0$ and $Q'''(t)/Q''(t) \to 0$ $(t \to \infty)$ are equivalent to the statements $P''(t)/\{P'(t)\}^2 \to 0$ and $P'''(t)/\{P''(t) \cdot P'(t)\} \to 0$ $(t \uparrow x^*)$.

To relate this to our problem, let $P := -\log(1 - F)$ hence $Q = U \circ \exp$. Relation $(*)$ is the same as relation (A.15) from Theorem A.5. Note that $P(t + x/P'(t)) - P(t) \to x$ $(t \uparrow x^*)$ means $\{1 - F(t + xf_0(t))\}/\{1 - F(t)\} \to e^{-x}$ $(t \uparrow x^*)$ with $f_0(t) = \{1 - F(t)\}/F'(t)$. Hence $(**)$ can be translated as follows [note $f_0'(t) \to 0$ $(t \uparrow x^*)$],

$$\frac{[1 - F(t + xf_0(t))]/[1 - F(t)] - e^{-x}}{f_0'(t)}$$

$$\sim \frac{e^{-x}\log\{[1 - F(t + xf_0(t))]e^{x}/(1 - F(t))\}}{f_0'(t)}$$

$$= \frac{-P(t + x/P'(t)) + P(t) + x}{f_0'(t)} \cdot e^{-x} \to \frac{x^2}{2}e^{-x},$$

$t \uparrow x^*$, locally uniformly.

We shall see that this is basically the relation we get in the general case.

We now work in an order different from what we did for $\gamma \neq 0$ and start with deriving the result with no differentiability assumption. The differentiable case will then be quite obvious.

THEOREM A.6. *Suppose Q is nondecreasing and $P = Q^{\leftarrow}$. Equivalent are:*

$$\text{(A.16)}\qquad \frac{Q(t + x) - Q(t) - xa_1(t)}{a_2(t)} \to \pm\frac{x^2}{2}, \qquad t \to \infty, \textit{ for all } x,$$

where a_1 and a_2 are positive functions.

$$\text{(A.17)}\qquad \frac{P(t + xf(t)) - P(t) - x}{\alpha(t)} \to \mp\frac{x^2}{2}, \; t \to \infty, \textit{ for all } x \textit{ locally uniformly},$$

where f and α are positive functions and $\alpha(t) \to 0$ $(t \uparrow x^*)$.

REMARK. If (A.16) holds, then (A.17) is true with $f(t) := a_1(P(t))$ and $\alpha(t) := a_2(P(t))/a_1(P(t))$. If (A.17) holds, then (A.16) is true with $a_1(t) := f(Q(t))$ and $a_2(t) := \alpha(Q(t)) \cdot f(Q(t))$.

PROOF. (A.16) $\Rightarrow$ (A.17): For $\varepsilon > 0$ and all x,

$$Q(P(t) + x) - t \geq \{Q(P(t) + x) - Q(P(t))\}$$
$$- \left\{Q\left(P(t) + \epsilon\frac{a_2(P(t))}{a_1(P(t))}\right) - Q(P(t))\right\}.$$

A similar upper inequality is obtained, hence by the local uniformity in (A.16) and because $a_2(P(t))/a_1(P(t)) \to 0$ $(t \to \infty)$,

$$\lim_{t\to\infty} \frac{Q(P(t)+x) - t - xa_1(P(t))}{a_2(P(t))} = \frac{x^2}{2} \tag{A.18}$$

locally uniformly and in particular

$$Q(P(t)) - t = o\big(a_2(P(t))\big), \qquad t \to \infty. \tag{A.19}$$

Also, with $\alpha(t) = a_2(P(t))/a_1(P(t))$,

$$Q\left(P(t) + x - \frac{x^2}{2}\alpha(t)\right) - t - \left\{x - \frac{x^2}{2}\alpha(t)\right\}a_1(P(t)) \sim \frac{x^2}{2}a_2(P(t)),$$

hence locally uniformly for $\varepsilon > 0$ and t sufficiently large,

$$\left\{Q\left(P(t) + x - \frac{x^2}{2}\alpha(t)\right) - t - xa_1(P(t))\right\}/a_2(P(t)) \le \varepsilon.$$

Then also $P(t) + x - (x^2/2)\alpha(t) \le P(t + xa_1(P(t)) + \varepsilon a_2(P(t)))$ or [by substituting $y = x + \varepsilon\alpha(t)$],

$$P(t + ya_1(P(t))) - P(t) - y \ge -\varepsilon\alpha(t) - \frac{(y + \varepsilon\alpha(t))^2}{2}\alpha(t).$$

A similar lower inequality is readily derived. Relation (A.17) follows.

(A.17) ⇒ (A.16): The proof follows the same line. We omit the details. □

COROLLARY A.1. *If condition* (A.17) *of Theorem* A.6 *holds, then* $\alpha(t + xf(t)) \sim \alpha(t)$ *locally uniformly* $(t \uparrow x^*)$.

PROOF. Since $\alpha(t) \to 0$, $P(t + xf(t)) - P(t) \to x$ locally uniformly $(t \uparrow x^*)$. We must prove (cf. the first part of the proof of Theorem A.2) that $a_i(P(t) + xf(t)) \sim a_i(P(t))$ locally uniformly for $i = 1, 2$. Now $a_i(t + x) \sim a_i(t)$ locally uniformly, hence $a_i(P(t + xf(t)) - P(t) + P(t)) \sim a_i(x + P(t)) \sim a_i(P(t))$ [cf. Omey and Willekens (1987)]. □

COROLLARY A.2. *If condition* (A.17) *of Theorem* A.6 *holds, then* $\{f(t + xf(t)) - f(t)\}/\{\alpha(t)f(t)\} \to \pm x$ *locally uniformly* $(t \uparrow x^*)$.

PROOF. Replace t in (A.17) by $t + yf(t) \to \infty$ for some real y. Then $(t \uparrow x^*)$

$$\begin{aligned}
-\frac{x^2}{2} \leftarrow & \left[P\left(t + \left\{y + x\frac{f(t + yf(t))}{f(t)}\right\}\cdot f(t)\right)\right.\\
& \left.-P(t) - \left\{y + x\frac{f(t+yf(t))}{f(t)}\right\}\right]\Big/[\alpha(t)]\\
& \times\frac{\alpha(t)}{\alpha(t + yf(t))} - \frac{P(t + yf(t)) - P(t) - y}{\alpha(t)}\cdot\frac{\alpha(t)}{\alpha(t + yf(t))}\\
& + \frac{x\alpha(t)}{\alpha(t + yf(t))}\left\{\frac{f(t + yf(t))}{f(t)} - 1\right\}\Big/\alpha(t).
\end{aligned}$$

Since every other term converges, the last term must also converge, thus giving the statement of the corollary. □

LEMMA A.1. *Let* $P := -\log(1-F)$. *The function* P *satisfies* (A.17) *of Theorem* A.6 *if and only if*

$$\left[\frac{1-F(t+xf(t))}{1-F(t)} - e^{-x}\right] \Big/ [\alpha(t)] \to \pm\frac{x^2}{2}e^{-x}, \tag{A.20}$$

$t \uparrow x^*$, *locally uniformly.*

Moreover (A.20) *holds with* f *replaced by* g *and* $(x^2/2)e^{-x}$ *replaced by* $(\pm x^2/2 - cx)e^{-x}$ *if and only if* $\{g(t)-f(t)\}/\{\alpha(t)f(t)\} \to c\ (t\uparrow x^*\}$.

PROOF. Suppose (A.17) holds. Since $\alpha(t) \to 0$, $P(t+xf(t)) - P(t) \to x$ $(t\uparrow x^*)$ locally uniformly, i.e., $\{1-F(t+xf(t))\}/\{1-F(t)\} \to e^{-x}$ $(t\uparrow x^*)$ locally uniformly. Hence

$$\begin{aligned}\frac{1-F(t+xf(t))}{1-F(t)}e^{x} - 1 &\sim \log\left\{\frac{1-F(t+xf(t))}{1-F(t)}e^{x}\right\}\\ &= -P(t+xf(t)) + P(t) + x.\end{aligned}$$

The converse is proved similarly. Now suppose (A.20) holds:

$$\begin{aligned}&\frac{P(t+xg(t)) - P(t) - x}{\alpha(t)}\\ &= \left[P\left(t + \left\{\frac{xg(t)}{f(t)}\right\}\cdot f(t)\right) - P(t) - x\frac{g(t)}{f(t)}\right]\Big/[\alpha(t)]\\ &\quad + x\cdot\left[\frac{g(t)}{f(t)} - 1\right]\Big/[\alpha(t)].\end{aligned}$$

Since the first term on the right converges, the convergence of the other terms implies each other. □

REMARK. f can be called the scale function and α the reference function for $1-F$.

THEOREM A.7. *If* $P := -\log(1-F)$ *satisfies* (A.17), *then*

$$\left[\frac{1-F(t+xf_1(t))}{1-F(t)} - e^{-x}\right]\Big/[\alpha(t)] \to \pm\left(\frac{x^2}{2} - x\right)e^{-x},$$

$t\uparrow x^*$, *locally uniformly,*

with $f_1(t) := [\int_t^{x^*} 1 - F(s)\,ds]/[1-F(t)]$.

PROOF. Write $U := (1/(1-F))^{\leftarrow}$ as before and $Q := U\circ\exp$. Then Q satisfies (A.16). As in Omey and Willekens (1987) one sees that

$$\frac{U(tx) - U(t) - DU(t)\log x}{a_2(t)} \to \frac{1}{2}\log^2 x - \log x$$

locally uniformly ($t \to \infty$), with $DU(t) := t\int_t^{\infty} U(s)\,ds/s^2 - U(t) = \int_1^{\infty}\{U(ty) - U(t)\}\,dy/y^2$. It follows that locally uniformly ($t \uparrow x^*$)

$$\left[\frac{1 - F(t + xDU(1/(1 - F(t))))}{1 - F(t)} - e^{-x}\right]\Big/[\alpha(t)] \to \left(\frac{x^2}{2} - x\right)e^{-x}.$$

Now

$$\begin{aligned} DU\left(\frac{1}{1 - F(t)}\right) &= \frac{1}{1 - F(t)}\int_{1/[1-F(t)]}^{\infty} U(s)\frac{ds}{s^2} - U\left(\frac{1}{1 - F(t)}\right) \\ &= \frac{1}{1 - F(t)}\int_t^{\infty} y\,dF(y) - U\left(\frac{1}{1 - F(t)}\right) \\ &= \frac{1}{1 - F(t)}\int_t^{\infty} y\,dF(y) - t + o(\alpha(t)f(t)), \end{aligned}$$

using (A.19) in the last equality. The result now follows from Lemma A.1. □

COROLLARY A.3. *Under the conditions of Theorem A.7,*

$$\frac{f_1(t + xf_1(t)) - f_1(t)}{\alpha(t)f_1(t)} \to \pm x \quad \textit{locally uniformly}, \ t \uparrow x^*.$$

PROOF. Corollary A.2. □

LEMMA A.2. *If* (A.20) *holds for* F, *then the same relation holds with* $1 - F$ *replaced by* $1 - F_1(x) := \max(0, \int_x^{x^*}\{1 - F(u)\}\,du)$ *and* f *replaced by* $f_1(t) := [\int_t^{x^*} 1 - F(s)\,ds]/[1 - F(t)]$.

PROOF.

$$\begin{aligned} &\left\{\frac{1 - F_1(t + xf_1(t))}{1 - F_1(t)} - e^{-x}\right\}\Big/\alpha(t) \\ &\quad = \frac{f_1(t + xf_1(t))}{f_1(t)}\left\{\frac{1 - F(t + xf_1(t))}{1 - F(t)} - e^{-x}\right\}\Big/\alpha(t) \\ &\qquad + e^{-x}\frac{\{f_1(t + xf_1(t)) - f_1(t)\}}{\{\alpha(t)f_1(t)\}}. \end{aligned}$$

Use Corollary A.3. □

Next we proceed to give sufficient conditions in terms of derivatives.

THEOREM A.8. *Suppose* F *is three times differentiable and* $F' > 0$. *Set* $f_0 := (1 - F)/F'$. *Suppose* f_0' *is of constant sign and* $f_0'(t) \to 0$ ($t \uparrow x^*$). *If*

(A.21) $$f_0''(t)f_0(t)/f_0'(t) \to 0, \qquad t \uparrow x^*,$$

then

(A.22) $$f_0'(t + xf_0(t))/f_0'(t) \to 1 \quad \textit{locally uniformly}, \ t \uparrow x^*.$$

If (A.22), *then*

(A.23) $$\frac{f_0(t + xf_0(t)) - f_0(t)}{f_0'(t) f_0(t)} \to x \quad \textit{locally uniformly}, \ t \uparrow x^*.$$

If (A.23), *then*

(A.24) $$\left[\frac{1 - F(t + xf_0(t))}{1 - F(t)} - e^{-x}\right] \Big/ [f_0'(t)] \to \frac{x^2}{2} e^{-x}$$

locally uniformly, $t \uparrow x^*$.

PROOF. $f_0'(t) \to 0$ implies $f_0(t)/t \to 0$ if $x^* = \infty$ and $f_0(t)/(x^* - t) \to 0$ if $x^* < \infty$, hence

$$\frac{f_0(t + xf_0(t))}{f_0(t)} - 1 = \int_0^x f'(t + uf_0(t))\, du \to 0 \quad \text{locally uniformly}.$$

Using this we find

$$\begin{aligned}\log f_0'(t + xf_0(t)) - \log f'(t) &= xf_0(t) \frac{f_0''(t + x\theta f_0(t))}{f_0'(t + x\theta f_0(t))} \\ &\sim x \frac{f_0(t + x\theta f_0(t)) f_0''(t + x\theta f_0(t))}{f_0'(t + x\theta f_0(t))}\end{aligned}$$

for some $\theta = \theta(t, x) \in [0, 1]$. Hence (A.21) implies (A.22). Further (A.23) follows from (A.22) by integrating both sides of (A.22) over $x \in [0, y]$ and (A.24) follows from (A.23) by integrating both sides of the relation $\{1 - f_0(t)/f_0(t + xf_0(t))\}/f_0'(t) \to x$ over $x \in [0, y]$ and using Lemma A.1. □

Necessary and sufficient conditions are contained in the next theorem.

THEOREM A.9. *Set* $F_0 := F$ *and* $1 - F_i(t) := \max\{0, \int_t^{x^*} (1 - F_{i-1}(u))\, du\}$ *for* $i = 1, 2, \ldots$. *Also set* $f_i(t) := \{1 - F_i(t)\}/\{1 - F_{i-1}(t)\}$ *for* $i = 1, 2, \ldots$. *Equivalent are*:

For some positive functions f *and* α, $\alpha(t) \to 0$ $(t \uparrow x^*)$,

(A.25) $$\left[\frac{1 - F(t + xf(t))}{1 - F(t)} - e^{-x}\right] \Big/ [\alpha(t)] \to \pm \frac{x^2}{2} e^{-x}$$

locally uniformly, $t \uparrow x^*$.

(A.26) f_2' *is of positive (negative) sign*, $f_3'(t) \sim f_2'(t)$ *and* $f_3(t) \sim f_2(t)$, $t \uparrow x^*$.

(A.27) $$\frac{f_1(t + xf_1(t)) - f_1(t)}{\alpha(t) f_1(t)} \to \pm x,$$

locally uniformly $(t \uparrow x^*)$ *for some positive function* $\alpha(t) \to 0$, $t \uparrow x^*$.

$$1 - F(t) = c \cdot g_1(t) \exp - \int_q^t \frac{ds}{g_2(s)} \text{ with } c \text{ a positive constant,}$$

(A.28) $q := \min(0, x^*(F) - 1)$, g_1, g_2 *positive*, g_2 *satisfying* (A.27) *and* $1 - g_1(t)/g_2(t) = o(\alpha(t))$ $(t \to \infty)$.

REMARK. The derivatives are to be taken in the Radon–Nikodym sense, if necessary.

PROOF. (A.25) $\Rightarrow$ (A.26): Note that $\alpha(t) \to 0$ $(t \uparrow x^*)$ implies $f_3(t) \sim f_2(t)$ [de Haan (1970), Theorem 2.5.2]. Now

$$\left[\frac{1 - F_1(t + xf_1(t))}{1 - F_1(t)} - e^{-x}\right] \Big/ [\alpha(t)] \to \frac{x^2}{2} e^{-x}, \qquad t \uparrow x^*, \text{ locally uniformly}$$

by Theorem A.7 and Lemma A.2. But then according to Theorem A.7 also

$$\left[\frac{1 - F_1(t + xf_2(t))}{1 - F_1(t)} - e^{-x}\right] \Big/ [\alpha(t)] \to \left(\frac{x^2}{2} - x\right) e^{-x}.$$

Hence $[f_2(t) - f_1(t)]/[\alpha(t) f_1(t)] \to 1$ $(t \uparrow x^*)$ by Lemma A.1.

Repeating this reasoning with $1 - F$ replaced by $1 - F_1$ and $1 - F_1$ replaced by $1 - F_2$, we also get

$$\frac{f_3(t) - f_2(t)}{\alpha(t) f_2(t)} \to 1, \qquad t \uparrow x^*.$$

Now note that $f_i'(t) = -1 + f_i(t)/f_{i-1}(t)$ for $i \geq 2$.

(A.26) $\Rightarrow$ (A.27):

$$\frac{f_3'' f_3}{f_3'} = 2\frac{f_3}{f_2} + \frac{f_3}{f_2} \cdot \frac{f_2}{f_1} \frac{f_1 - f_2}{f_3 - f_2} - \frac{f_3}{f_2} \cdot \frac{f_2}{f_1} \to 0, \qquad t \uparrow x^*.$$

Hence $f_3'(t + xf_3(t)) \sim f_3'(t)$ locally uniformly $(t \uparrow x^*)$ by Theorem A.8. Using $f_3' = -1 + f_3/f_2$ gives

$$\frac{f_3(t + xf_3(t)) - f_2(t + xf_3(t))}{f_3'(t) f_3(t)} \to 1 \quad \text{locally uniformly, } t \uparrow x^*,$$

and hence (using Theorem A.8 again)

$$\frac{f_2(t + xf_3(t)) - f_2(t)}{f_3'(t) f_3(t)} = \frac{f_2(t + xf_3(t)) - f_3(t + xf_3(t))}{f_3'(t) f_3(t)} + \frac{f_3(t + xf_3(t)) - f_3(t)}{f_3'(t) f_3(t)} + \frac{f_3(t) - f_2(t)}{f_3'(t) f_3(t)} \to x$$

locally uniformly, $t \uparrow x^*$.

In exactly the same way one then obtains

$$\frac{f_1(t + xf_1(t)) - f_1(t)}{f_2'(t)f_1(t)} \sim \frac{f_1(t + xf_3(t)) - f_1(t)}{f_3'(t)f_3(t)} \to x, \qquad t \uparrow x^*.$$

(A.27) $\Rightarrow$ (A.28): Take $cg_1 = g_2 = f_1$.

(A.28) $\Rightarrow$ (A.25): Define $P(t) := \int_q^t (ds)/[g_2(s)]$. Straightforward calculation gives

$$\frac{P(t + xg_2(t)) - P(t) - x}{-\alpha(t)} \to \frac{1}{2}x^2 \text{ locally uniformly, } t \uparrow x^*,$$

i.e., with

$$1 - F_*(t) := \exp - \left(\exp\left(- \int_q^t \frac{ds}{g_2(s)} \right) \right),$$

$$\left[\frac{1 - F_*(t + xg_2(t))}{1 - F_*(t)} - e^{-x} \right] \Big/ [\alpha(t)] \to \frac{1}{2}x^2 e^{-x} \text{ locally uniformly, } t \uparrow x^*.$$

Next use a decomposition like the one in the proof of Lemma A.2 to obtain

$$\frac{1 - F(t + xg_2(t))}{1 - F(t)} - e^{-x} \sim \alpha(t)\left(\frac{x^2}{2} - x \right) e^{-x}.$$

Finally apply Lemma A.1. □

COROLLARY A.4. *If* (A.25) *holds, then* (A.25) *also holds with* α *replaced by* $|f_2/f_1 - 1| = |f_2'|$ *(see the first part of the proof of Theorem A.*9) *and* f *replaced by* $\{f_1\}^2/f_2$ *(see Theorem* A.7 *and Lemma* A.1).

Finally we turn back to the question of how to translate the condition $tU'(t) \in \Pi$ into a condition for the distribution function F and its derivative F'.

THEOREM A.10. *Set* $F_0 := F$, $1 - F_i(t) := \max\{0, \int_t^{x^*}(1 - F_{i-1}(u))\,du\}$, $f_0 = \{1 - F\}/F'$ *and* $f_i := \{1 - F_i(t)\}/\{1 - F_{i-1}(t)\}$ *for* $i = 1,2,\dots$. *Equivalent are*:

(A.29) $$\pm\, tU'(t) \in \Pi.$$

(A.30) $$\left[\frac{1 - F(t + xf_0(t))}{1 - F(t)} - e^{-x} \right] \Big/ [\beta(t)] \to \pm \frac{x^2}{2} e^{-x} \text{ locally uniformly,}$$
$t \uparrow x^*$, *for some positive function* $\beta(t) \to 0$, $t \uparrow x^*$.

(A.31) $f_1'(t)$ *is of constant sign and* $f_2'(t) \sim f_1'(t) \to 0$, $t \uparrow x^*$.

(A.32) $$\frac{f_0(t + xf_0(t)) - f_0(t)}{\alpha(t) f_0(t)} \to \pm x \text{ locally uniformly,}$$
$t \uparrow x^*$, *for some positive function* $\alpha(t) \to 0$, $t \uparrow x^*$.

The proof of this theorem will not be given since it follows exactly the same lines as the proof in the general case, but uses Theorem A.5 relation (A.15) to get f_0 in relation (A.30). The proof here is actually easier since inversion is very simple.

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