The Annals of Statistics
1989, Vol. 17, No. 4, 1684–1692

BOUNDS FOR THE DISTRIBUTION OF THE GENERALIZED VARIANCE[1]

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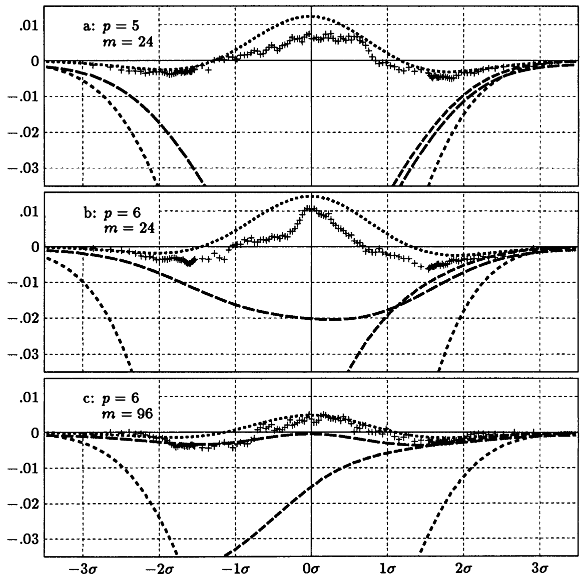
Let $D_{p,m}$ be the determinant of the sample covariance matrix for $m+p+1$ observations from a p-variate normal population having identity covariance matrix. We give bounds for the distribution of $D_{p,m}$ in terms of various chi-squared distribution functions. Let $F(\cdot|\nu)$ denote the chi-squared distribution function on ν degrees of freedom. We bound $P\{p(D_{p,m})^{1/p} > t\}$ above by $1 - F(t|p(m+1) + \frac{1}{2}(p-1)(p-2))$ and below by $1 - F(t|p(m+1))$. We give two more bounds involving chi-squared distributions. The proofs use a stochastic analog to the Gauss multiplication theorem.

1. Introduction and summary. The determinant of a $p \times p$ Wishart matrix can be represented as a product of p independent chi-squared variates whose degrees of freedom increase in arithmetic progression. Specifically, consider a Wishart matrix obtained as the corrected sum of cross-products matrix corresponding to a sample of $N = m + p + 1$ i.i.d. p-variate vectors having independent standard normal coordinates. Its determinant $D_{p,m}$ is then distributed as the product $\prod_{0 \le j \le p-1} X_j$, where the p mutually independent X_j have chi-squared distributions on $N - p + j = m + 1 + j$ degrees of freedom. See, for example, Anderson (1984). The distribution of $D_{p,m}$ in the case $p = 2$ was found by Wilks (1932). In our notation, $2(X_0 X_1)^{1/2}$ is distributed as $\chi^2_{2(m+1)}$.

Mathai and Rathie (1971) recover the density of $D_{p,m}$ by inverting the moment generating function of the logarithm of a generalized variance with a positive noncentrality parameter. By specializing their results to noncentrality parameter zero, they obtain the exact density of the generalized variance, which has the form $1/x$ times the Meijer G-function $G^{p,0}_{0,p}$. See their equation (3.2) for parameters and scaling. They also remark that the distribution function is expressible in terms of another G-function $G^{p,1}_{1,p+1}$.

There is, however, still a need for simple approximations to the distribution of $D_{p,m}$ when $p \ge 3$. Hoel (1937) suggests approximating the distribution of $p(D_{p,m})^{1/p}$ by that of the constant $(1 - (p-1)(p-2)(2N)^{-1})^{-1/p}$ times a chi-squared variate on $p(m+1)$ degrees of freedom. The constant and degrees of freedom are obtained by evaluating moments, letting $N \to \infty$ and retaining those terms which are $O(1/N)$. Hoel's suggestion has fallen into disfavor, as a result of work by Gnanadesikan and Gupta (1970) and Regier (1976). They approximate the logarithm of the generalized variance by a suitably chosen normal variate. Their work is based on that of Bartlett and Kendall (1946), who studied the approximate normality of log-transformed χ^2 variates. The latter

Received October 1986; revised December 1988.
[1]Supported in part by NSF Grant DMS-88-15106.
AMS 1980 *subject classifications.* Primary 62H10; secondary 33A15.
Key words and phrases. Gamma function, generalized variance, Gauss multiplication theorem.



(Theorem 4, upper bound) minus Normal. ·········
(Theorem 4, lower bound) minus Normal: ········
(Theorem 8, lower bound) minus Normal: ‒ ‒ ‒ ‒
(Corollary 9, lower bound) minus Normal: ― ― ·
(simulated distribution) minus Normal: ++++

FIG. 1. *Deviations from the normal approximation.*

authors caution that the normal approximation may not fit well in the tails, a characteristic that is seen in Figure 1 to be shared by normal approximations to the generalized variance.

Our results are based on stochastic analogs to classical results in the theory of the gamma function. See Whittaker and Watson (1927), especially Sections 12.15 and 12.16. In light of the work of Mathai and Rathie, our results can be considered to yield bounds for $G^{p,1}_{1,p+1}$ in terms of the incomplete gamma function.

In Sections 3 and 4 we apply the results of Section 2 in order to approximate products of independent chi-squared variates by a power of a single chi-squared variate. (In a slight abuse of language we refer to any gamma distribution with scale 2 and shape α as "chi-squared on 2α degrees of freedom.")

The bounds are illustrated in Figure 1, plots (a) through (c). Plotted are $B(t) - (1 - \mathscr{N}(t))$ against quantiles of $\mathscr{N}(\cdot)$, where $B(\cdot)$ corresponds to one of the bounds derived below and $\mathscr{N}(\cdot)$ is the normal approximation discussed in Regier (1976) and in Gnanadesikan and Gupta (1970).

Also graphed are simulated empirical distributions of the generalized variance, evaluated at selected quantiles. The simulation was performed using the PC-SAS system. See Allen and Kalt (1985), especially page 78. For each figure, 16,000 realizations of $D_{p,m}$ were generated. For comparison, the Kolmogorov–Smirnov statistic's median is approximately 0.007.

The choices of p and m exemplify several features of the bounds. First, the $\chi^2_{p(m+1)+(p-1)(p-2)/2}$ distribution, which is the upper bound of Theorem 4, appears itself to be a good approximation. In some sense, we have turned the constant of Hoel's approximation—which for large N is almost $1 + (p-1)(p-2)/(2N_p)$—into the reciprocal of a beta-variate and multiplied to obtain $\chi^2_{p(m+1)+(p-1)(p-2)/2}$.

Second, the bounds are decidedly tighter for p even than for p odd. In moving from Figure 1(b) to 1(c), we have quadrupled m, while the maximum deviation between upper and lower bounds has decreased from 0.034 to 0.005—better than a fourfold increase in accuracy. For even p, because the approximating degrees of freedom are the same in upper and lower bounds and because the remainder term of Corollary 9 decreases as m^{-2}, the discrepancy between upper and lower bounds decreases as $o(m^{\varepsilon-3/2})$, for any $\varepsilon > 0$.

Third, the normal approximation puts too little mass in the right tail and too much mass in the left. See Gnanadesikan and Gupta's (1970) Figures 7–10 for comparison of the distribution of $\ln(\chi^2)$ with approximating normal distributions; similar behavior is apparent. Note that the normal baseline in our figures is outside the envelope determined by the upper and lower bounds of Theorem 4. The lack of good fit in the tails is of interest because of the role of tail behavior in ranking and selection problems.

Fourth, none of the bounds are dominated by any other. Further, the bounds appear to be tighter in the upper tail than in the lower tail of the distribution. Steyn (1978) presents an approximation based on the moment generating function that holds for all but a small fraction of the far left tail of the distribution. His approximation for the upper tail probability is less than a constant times $P\{\chi^2_{p(m+1)} > t\}$, which for large t is dominated by the bounds of Theorem 8.

2. An analog to the Gauss multiplication theorem. In this section we present two stochastic analogs of well-known results in the theory of the gamma function. Theorem 1 is the fundamental representation which we use throughout. The representation is essentially found in Bondesson (1978), who remarks that the "generalized normal distributions" can be represented as linear combinations of independent exponential random variables. Bondesson writes the characteristic function as an infinite product and so obtains our Theorem 1. Hall (1978) directly proves the special case of Theorem 1 for integer α in his study of the extreme value distribution. We provide a simple direct proof which is close in spirit to that of Hall.

THEOREM 1. *Let $\{Y_j\}$ be i.i.d. exponential with mean* 1 *and let* $\gamma = 0.5772\ldots$ *be the Euler–Mascheroni constant. Define G by the series*

$$\ln(G) = -\gamma + \sum_{j=0}^{\infty}\left(\frac{1}{j+1} - \frac{Y_j}{j+\alpha}\right). \tag{1}$$

Then G has a gamma distribution with shape parameter α and scale parameter 1.

PROOF. Let G be a gamma variate with shape α and scale 1. Choose Y_0 exponentially distributed, independent of G. Note that $G_1 = Y_0 + G$ is gamma $(\alpha + 1)$ independent of G/G_1. The latter has a beta distribution, represented as $U_0^{1/\alpha}$ where U_0 is uniform on $[0,1]$. Hence $G = U_0^{1/\alpha} \cdot G_1$. Iterate to obtain $G = G_{n+1} \cdot \prod_{j=0}^{n}(U_j)^{1/(\alpha+j)}$. Approximate $\ln(n)$ with the harmonic series, yielding

$$\ln(G) = -\gamma + \sum_{j=0}^{n}\left(\frac{1}{j+1} + \frac{\ln(U_j)}{j+\alpha}\right) + \ln\left(\frac{G_{n+1}}{n+1}\right) + o(1),$$

where the U_j are i.i.d. uniform $[0,1]$, G_n is gamma $(\alpha + n)$ and the remainder term $o(1)$ is due to approximating $\gamma + \ln(n)$ by the harmonic series. Finally, apply the weak law of large numbers to G_n/n. □

Note that (1) is a stochastic analog of a classical series for the digamma function. Theorem 2 is a stochastic analog of the Gauss multiplication theorem.

THEOREM 2. *Let G_k, $k = 0, \dots, p-1$, be independent gamma variates with shape parameters $\alpha + k/p$ and common scale parameter. Then $p(\prod_{k=0}^{p-1}G_k)^{1/p}$ is distributed as gamma with shape parameter $p\alpha$ and the same common scale parameter.*

PROOF. Without loss of generality take the common scale parameter to be 1. Write $Q = \prod_{k=0}^{p-1}G_k$. Use Theorem 1 to write the $\ln(Q)$ as the double sum

$$\sum_{k=0}^{p-1}\left(-\gamma + \sum_{j=0}^{\infty}\left(\frac{1}{j+1} - \frac{Y_{jk}}{j+\alpha+k/p}\right)\right),$$

where Y_{jk} are i.i.d. exponential with mean 1. Now divide by p and note that $pj + k$ runs over all nonnegative integers as j and k run through their values.

Hence we may represent $\ln(Q^{1/p})$ by

$$-\gamma + \kappa_p + \sum_{j=0}^{\infty}\left((j+1)^{-1} - Y_j/(j+p\alpha)^{-1}\right),$$

where the constant κ_p is $\sum_{k=0}^{p-1}\sum_{j=0}^{\infty}(1/(pj+p) - 1/(pj+k+1))$. Apply Theorem 1 again and then exponentiate to show that $Q^{1/p}$ is distributed as $\exp(\kappa_p)$ times a gamma variate with shape $p\alpha$ and scale 1. We show $\exp(\kappa_p) = 1/p$ by computing large moments of Q and using Stirling's formula for the gamma function. □

3. Stochastic inequalities. In this section, we apply Theorem 2 to prove that the distribution of the generalized variance lies strictly between two chi-squared distributions. The proof of the inequalities relies on the following technical lemma.

Lemma 3. *Let $c > \delta > 0$ and $t > 0$ be constants. Let Y_1 and Y_2 be i.i.d. exponential random variables. For all fixed c and t, $P\{Y_1/(c-\delta) + Y_2/(c+\delta) > t\}$ is monotone increasing in δ.*

Proof. The probability is a power series in δ having positive coefficients. □

We use $\succ_{\text{st}}$ for "stochastically greater than." Specifically, given X_1 and X_2, write $X_1 \succ_{\text{st}} X_2$ if $P\{X_1 > t\} \ge P\{X_2 > t\}$ for all t.

Theorem 4 provides upper and lower bounds for the distribution of (a power transformation of) the generalized variance. The idea is to perturb the degrees of freedom of the factor chi-squared distributions in order to apply Theorem 2.

Theorem 4. *Let X_k be distributed as chi-squared $(m+1+k)$ for $k = 0,\dots,p-1$. It follows that*

$$\chi^2_{p(m+1)+(p-1)(p-2)/2} \succ_{\text{st}} p\left(\prod_{k=0}^{p-1} X_k\right)^{1/p} \succ_{\text{st}} \chi^2_{p(m+1)}. \tag{2}$$

Proof. The case $p = 1$ is trivial; the case $p = 2$ is exactly Wilks' (1932) result. This proof generalizes that of Theorem 2. For $p \ge 3$, write D for the product $\prod_{k=0}^{p-1} X_k$. Note that the set of degrees of freedom belonging to the X_j is symmetrically centered around $\nu = (m+1+(p-1)/2)$. Choose Y_{jk} i.i.d. exponential with mean 1. Now apply (1) to obtain the representation

$$\ln(D) = p\ln(2) - p\gamma + \sum_{k=0}^{p-1}\sum_{j=0}^{\infty}\left(\frac{1}{j+1} - \frac{Y_{jk}}{j+(m+1+k)/2}\right) \tag{3}$$

and apply Lemma 3 to the pairs $\{Y_{jk}, Y_{j,\,p-(k+1)}\}$. Hence $\prod_{k=0}^{p-1} X_k' \succ_{\text{st}} \ln(D)$, where X_k' has a gamma distribution with scale 2 and shape $\nu/2 + k/p$. The upper inequality now follows from Theorem 2.

For the second inequality, let $\alpha = (m+1)/2$. Apply Theorem 1 to bound $\ln(\prod_{k=0}^{p-1} X_j)$ below by $p\ln(2) - p\gamma + \sum_{k=0}^{p-1}\sum_{j=0}^{\infty}((j+1)^{-1} - Y_{jk}/(j+\alpha+k/p)^{-1})$. Hence, from Theorem 2, $D \succ_{\text{st}} D^*$, where $p(D^*)^{1/p} \sim \chi^2_{p(m+1)}$. □

4. Refining the lower bound. In this section we analyze the error of approximation in using $\chi^2_{p(m+1)+(p-1)(p-2)/2}$, the upper bound of Theorem 4, to approximate the distribution of $p(D_{p,m})^{1/p}$. The degrees of freedom in the proposed approximation are very reminiscent of the constant in Hoel (1937). One might approximate the distribution of $p(D_{p,m})^{1/p}$ by the distribution of the stochastically larger $\chi^2_{p(m+1)+(p-1)(p-2)/2}$. The latter, however, may be realized as the quotient X/B, where $X \succ_{\text{st}} \chi^2_{p(m+1)}$ and B is distributed as beta $(p(m+1)/2, (p-1)(p-2)/4)$, independent of X.

Note that $1/E\{B\} = 1 + (p-1)(p-2)/(2p(m+1))$. If p and m are both moderately large, the distribution of B will be concentrated near its mean and so $1/B$ will tend to be near $1/E\{B\}$, which is approximately Hoel's (1937) scaling constant $(1 - \frac{1}{2}(p-1)(p-2)/(m+1+p))^{-1/p}$. For example, when $p = 4$ and $m = 12$, Hoel's constant is 1.050 and $1/E\{B\} = 1.058$.

The series of approximations suggests that the chi-squared approximation should be about as good as Hoel's when the latter applies and should be substantially better in the extreme right tails, where the chi-squared approximation puts less mass. To improve the lower bound, we plan to pick a good value for r, to approximate

$$\text{(4)} \qquad \frac{1}{p}\ln(D_{p,m}) - \ln(2) + \gamma = \frac{1}{p}\sum_{j=0}^{\infty}\sum_{k=0}^{p-1}\left(\frac{1}{j+1} - \frac{Y_{jk}}{\alpha + j + k/2}\right)$$

by the slightly larger sum

$$\text{(5)} \quad \frac{1}{p}\sum_{j+k/2<r}\left(\frac{1}{j+1} - \frac{Y_{jk}}{\alpha + r + k/2}\right) + \frac{1}{p}\sum_{j+k/2\geq r}\left(\frac{1}{j+1} - \frac{Y_{jk}}{\alpha + j + k/2}\right)$$

and then to bound (5) from below by a sum to which we apply Theorem 2.

Note that the sums (4) and (5) differ only in the denominators of the variates Y_{jk}. In both sums, all these denominators are of the form $\alpha + l$ or $\alpha + l + \frac{1}{2}$, for l a nonnegative integer. Of particular importance to the analysis is the number of variates in (4) and (5) associated with a particular denominator. A straightforward counting argument yields Lemma 5, presented without proof.

LEMMA 5. *Let* $r = \frac{1}{2}\lfloor p/2 \rfloor - \frac{1}{2}$. *The denominator count for the two sums given in* (4) *and* (5) *is given in Table* 1, *for* l *a nonnegative integer.*

The pattern of counts almost lets us use the stochastic version of the Gauss multiplication theorem—Theorem 2. To this end, we need to spread the denominators to an arithmetic progression with increment $1/p$. This is the substance of Lemma 6.

LEMMA 6. *Let* $r = \frac{1}{2}\lfloor p/2 \rfloor - \frac{1}{2}$. *Let* X *be a random variable with the same distribution as the sum* (5). *If* p *is even,* $X \succ_{\text{st}} \ln(G)$, *where* G *is gamma*

TABLE 1

Parity	Denominator	Occurrences in (4)	Occurrences in (5)
p even	$\alpha + l$	$\min\{l+1, \lfloor (p+1)/2 \rfloor\}$	$\lfloor p/2 \rfloor$
	$\alpha + l + \frac{1}{2}$	$\min\{l+1, \lfloor p/2 \rfloor\}$	$\lfloor p/2 \rfloor$
p odd	$\alpha + l$	$\min\{l+1, \lfloor (p+1)/2 \rfloor\}$	$\lfloor p/2 \rfloor$ if $l < \lfloor p-1 \rfloor/2$; $\lfloor (p+1)/2 \rfloor$ if $l \geq \lfloor (p-1)/2 \rfloor$
	$\alpha + l + \frac{1}{2}$	$\min\{l+1, \lfloor p/2 \rfloor\}$	$\lfloor p/2 \rfloor$

distributed with scale $1/p$ *and shape* $\frac{1}{2}(p\alpha + \frac{1}{2}(p-1)(p-2))$. *If* p *is odd,* $X \succ_{\text{st}} \ln(G)$, *where* G *is gamma distributed with scale* $1/p$ *and shape* $\frac{1}{2}(p\alpha + \frac{1}{2}(p-1)(p-2) - p/2)$.

PROOF. If p is odd, augment (5) with a few more exponentials to make the pattern of denominator counts either $\{0,\dots,0,(p+1)/2,(p-1)/2,(p+1)/2,\dots\}$ or $\{0,\dots,0,(p-1)/2,(p+1)/2,(p-1)/2,\dots\}$. Regardless of the parity of p, use Lemma 3 to spread the denominators to an arithmetic progression, to show

$$X \succ_{\text{st}} \frac{1}{p}\sum_{j=0}^{\infty}\sum_{k=0}^{p-1}\left(\frac{1}{j+1} - \frac{Y_{jk}}{\alpha + r - \theta + k/p}\right),$$

where $\theta = (\frac{1}{2}p - 1)/(2p)$ if p is even and $\theta = (\frac{1}{2}(p+1) - 1)/(2p)$ if p is odd. Both the augmentation and spreading make the sum stochastically smaller because all the exponentials have negative coefficients. Now apply Theorem 2. □

There are two ways to transform (4) to the form required by Lemma 6. We use a Radon–Nikodym derivative and take expectations to prove Theorem 7. Alternatively, we subtract (5) from (4) and bound the remainder to prove Theorem 8.

THEOREM 7. *Let* $p \ge 4$. *Write* $r = \frac{1}{2}\lfloor p/2 \rfloor - \frac{1}{2}$ *and let*

$$c_{p,m} = \prod_{j+k/2<r}\left(\frac{m+1+2j+k}{m+1+2r+k}\right).$$

Then, for all $t > 0$,

$$P\left\{p\left(D_{p,m}\right)^{1/p} > t\right\} \ge \begin{cases} c_{p,m}P\left\{\chi^2_{p(m+1)+\frac{1}{2}(p-1)(p-2)} > t\right\} & \text{if } p \text{ is even}, \\ c_{p,m}P\left\{\chi^2_{p(m+1)+\frac{1}{2}(p-1)(p-2)-\frac{1}{2}p} > t\right\} & \text{if } p \text{ is odd}. \end{cases}$$

PROOF. Write $\alpha = (m+1)/2$. Write $D_{p,m}$ as a product of chi-squared variates and write A for the event

$$\left\{\frac{1}{p}\ln\left(D_{p,m}\right) > t\right\} = \left\{-\gamma + \ln(2) + \frac{1}{p}\sum_{j=0}^{\infty}\sum_{k=0}^{p-1}\left(\frac{1}{j+1} - \frac{Y_{jk}}{\alpha + j + k/2}\right) > t\right\},$$

where the Y_{jk} are i.i.d. exponential with mean 1. Let L_{jk} be the Radon–Nikodym derivative of the exponential distribution with mean $\theta_{jk} = (\alpha + j + k/2)/(\alpha + r + k/2)$ relative to the original exponential distribution of Y_{jk}. Specifically, $L_{jk} = \theta_{jk}^{-1}\exp((1 - \theta_{jk}^{-1})Y_{jk})$.

From Lemmas 5 and 6,

$$\begin{aligned} P\left\{\frac{1}{p}\ln\left(D_{p,m}\right) > t\right\} &= E\left\{I_A \prod_{j+k/2<r} \left(L_{jk}L_{jk}^{-1}\right)\right\} \\ &> E\left\{\prod_{j+k/2<r} \theta_{jk}L_{jk}I_A\right\} \\ &> c_{p,m}P\left\{\ln\left(p^{-1}\chi_\nu^2\right) > t\right\}, \end{aligned}$$

where ν is as specified in the statement of the theorem and in Lemma 6. □

To obtain our final lower bound, we analyze directly the difference between the sums (4) and (5). The bound suggests that the simple upper bound obtained in the previous section should be quite sharp for even p. Note that the remainder term R in the next theorem is positively correlated with the approximating chi-squared variate X.

THEOREM 8. *Let $p \geq 4$. Write $r = \frac{1}{2}\lfloor p/2 \rfloor - \frac{1}{2}$, $\rho = 2\lfloor p^2/16 \rfloor$ and $\alpha = (m+1)/2$. Let $\nu = p(m+1) + \frac{1}{2}(p-1)(p-2)$ if p is even and $\nu = p(m+1) + \frac{1}{2}(p-1)(p-2) - \frac{1}{2}p$ if p is odd. There exist random variables $X \sim \chi_\nu^2$ and $R \sim \chi_\rho^2$ such that*

$$p\left(D_{p,m}\right)^{1/p} \succ_{\text{st}} \exp\left(-\frac{r}{2p\alpha(\alpha+r)}R\right) \cdot X.$$

PROOF. We can approximate the generalized variance from below by writing its logarithm as a sum like (4) and then decomposing the sum into an approximating term of form (5) and a remainder of the form

$$-\frac{1}{p}\sum_{j+k/2<r} Y_{jk}\left(\frac{1}{\alpha+j+k/2} - \frac{1}{\alpha+r+k/2}\right) > -\frac{r}{p\alpha(\alpha+r)}\sum_{j+k/2<r} Y_{jk}.$$

There are fewer than $\lfloor p^2/16 \rfloor$ terms in the latter sum of i.i.d. exponential variates. The distribution of the sum is $\frac{1}{2}\chi_\rho^2$. Now use Lemma 6 to further reduce the approximating term. □

The following is an immediate consequence of the above approximation.

COROLLARY 9. *Let $p \geq 4$ and let ν and ρ be as in Theorem 8. For any $\theta > 0$,*

$$(6)\quad P\left\{p\left(D_{p,m}\right)^{1/p} > t\right\} > P\left\{\exp\left(-\frac{\theta r}{2p\alpha(\alpha+r)}\right)\chi_\nu^2 > t\right\} - P\left\{\chi_\rho^2 > \theta\right\}.$$

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