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TAIL ORDERING AND ASYMPTOTIC EFFICIENCY OF RANK TESTS[1]

BY PHILIPPE CAPÉRAÀ

Université Laval

In this paper we consider a partial ordering that is "between" the stochastic ordering defined by Lehmann (1955) and an ordering associated with the monotone likelihood ratio property. A tail ordering deduced from it is applied to the comparison of the asymptotic efficiencies of rank tests in the two-sample problem. In particular, we show that the asymptotic relative efficiency of two rank tests preserve this tail ordering if one score function is "more convex" than the other.

1. Introduction. Partial orderings between univariate distributions are used extensively in mathematical statistics. We mention first the stochastic ordering introduced by Lehmann (1955), which is defined by $F >_i G$ (F is stochastically larger than G) if and only if $F(x) \leq G(x)$, $-\infty < x < +\infty$. An equivalent definition is $F >_i G$ if and only if $E[a(X)|F] \geq E[a(X)|G]$ for all nondecreasing functions a, which are integrable with respect to F and G, where $E[a(X)|F] = \int a(x)\,dF(x)$. A partial ordering associated with the monotone likelihood ratio is defined for two absolutely continuous distributions F and G by $F >_l G$ if and only if f/g is nondecreasing, where f and g are the densities of F and G, respectively.

This paper studies a partial ordering that is "between" the $>_i$ and $>_l$ orderings. It is defined by $F >_{(+)} G$ (F is uniformly stochastically larger than G in the positive direction) if and only if $(1 - G)/(1 - F)$ is nonincreasing, in the sense that the numerator vanishes if the denominator vanishes. Another similar ordering is $F >_{(-)} G$ if and only if G/F is nonincreasing, where defined. It is easy to prove the following implications: $F >_l G \Rightarrow F >_{(+)} G$ $(F >_{(-)} G) \Rightarrow F >_i G$ [Keilson and Sumita (1982)].

In Section 2, we show that $F >_{(+)} G$ can also be defined in terms of inequalities between expectations (with respect to F and G) of functions belonging to a well-specified set. In Section 3 a partial ordering that reflects the tail heaviness of distributions is deduced from $>_{(+)}$. An application of this tail ordering is made to the comparison of the asymptotic efficiencies of rank tests in the two-sample problem.

2. The main result. Let C be the set of nondecreasing functions defined on $\mathbb{R}$ and C_+ the set of nonnegative and nondecreasing real-valued functions

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defined on $\mathbb{R}$. Let F be a distribution function on $\mathbb{R}$ and

$$C_+(F) = \{b \in C_+; 0 < E[b(X)|F] < +\infty\}.$$

The partial ordering $>_0$ is then defined by $F >_0 G \Leftrightarrow E[a(X)b(X)|F] \geq 0$ for all functions $a \in C$ and $b \in C_+(G)$ such that $E[a(X)b(X)|G] = 0$ and $E[a(X)b(X)|F]$ is defined.

Theorem. *Let F and G be two distributions on $\mathbb{R}$. Then $F >_0 G \Leftrightarrow F >_{(+)} G$.*

Proof. Define the cdf F_b by [the integrals are over half-open intervals $(\alpha, \beta]$ or $(\alpha, +\infty)$, $-\infty \leq \alpha < \beta < +\infty$]

$$F_b(t) = \int_{-\infty}^{t} b(x)\,dF(x) \Big/ \int_{-\infty}^{+\infty} b(x)\,dF(x),$$

where $b \in C_+(F) \cap C_+(G)$. The cdf G_b is defined in a similar way.

We first show that $F_b >_i G_b$ for all functions $b \in C_+(F) \cap C_+(G)$ is a necessary and sufficient condition for $F >_0 G$ to hold. To prove the necessity part, suppose that there is a function $b \in C_+(F) \cap C_+(G)$ such that the distribution F_b is not stochastically larger than the distribution G_b. It follows that there exists a function $a \in C$ with

$$-\infty < E[a(X)|F_b] < E[a(X)|G_b] < +\infty.$$

If

$$a_1(x) = a(x) - E[a(X)|G_b], \qquad \forall\, x \in R,$$

then

$$E[a_1(X)|F_b] < E[a_1(X)|G_b] = 0.$$

This contradicts $F >_0 G$.

Now we prove the sufficiency part. Let a be a function of C and b a function of $C_+(G)$. Suppose that $E[a(X)b(X)|G] = 0$ and $E[a(X)b(X)|F]$ is defined. To show that $E[a(X)b(X)|F]$ is nonnegative, as it was to be proved, we distinguish the three following cases: (i) $b \in C_+(F)$, (ii) $E[b(X)|F] = 0$ and (iii) $E[b(X)|F] = +\infty$. Only the last two cases can cause $b \notin C_+(F)$.

(i) Suppose that b is a function of $C_+(F)$. Since $F_b >_i G_b$, then

$$E[a(X)b(X)|G] = E[a(X)|G_b] = 0$$

implies $E[a(X)|F_b] \geq 0$.

(ii) Suppose, now, that $E[b(X)|F] = 0$, which implies $P[b(X) = 0|F] = 1$. Consider $b^*(\cdot) = b(\cdot) + 1$. The function b^* belongs to the set $C_+(F) \cap C_+(G)$ and hence $F_{b^*} >_i G_{b^*}$. Moreover, $E[b(X)|G] > 0$ implies $P[b(X) > 0|G] > 0$. Define the function a^* by

$$a^*(x) = \begin{cases} 0, & \text{if } b(x) = 0, \\ 1, & \text{if } b(x) > 0. \end{cases}$$

Then $a^* \in C$ and we have

$$0 = E[a^*(X)b^*(X)|F] < E[a^*(X)b^*(X)|G],$$

which contradicts $F_{b^*} >_i G_{b^*}$. Then the case (ii) $E[b(X)|F] = 0$ cannot occur.

(iii) Suppose that $E[b(X)|F] = +\infty$. First, assume $P[a(X) \leq 0|G] = 1$. For $c > 0$, let b_c be defined by

$$b_c(x) = \min\{b(x), c\}, \qquad \forall x \in \mathbb{R}.$$

Since b_c is a function of $C_+(F) \cap C_+(G)$ and that $E[a(X)b_c(X)|G] = 0$, it follows that $F_{b_c} >_i G_{b_c}$ and hence $E[a(X)b_c(X)|F] \geq 0$. Thus, by the monotone convergence theorem, we have

$$E[a(X)b(X)|F] = \lim_{c \to +\infty} E[a(X)b_c(X)|F] \geq 0.$$

Next assume that $P[a(X) > 0|G] > 0$. Since a and b are nondecreasing functions, we have

$$E[(a(X)b(X))^+|F] = +\infty,$$

where $(a(x)b(x))^+ = \max\{a(x)b(x), 0\}$, $-\infty < x < +\infty$. Moreover, we know that $E[a(X)b(X)|F]$ exists, thus we conclude that $E[a(X)b(X)|F] = +\infty$.

Finally, we prove that $F >_{(+)} G$ is a necessary and sufficient condition for $F_b >_i G_b$ to hold for all functions $b \in C_+(F) \cap C_+(G)$.

To prove the sufficiency it suffices to consider the functions $b \in C_+(F) \cap C_+(G)$, which are continuous from the right and verify $\lim_{x \to -\infty} b(x) = 0$. To see this, consider a function $b \in C_+(F) \cap C_+(G)$. By the Vitali–Carathéodory theorem [Rudin (1966), page 54] there exists a sequence of functions $\{g_n\}$ such that, $0 \leq g_n \leq b$, g_n is upper semicontinuous for all n and

$$\int_{-\infty}^{+\infty} (b(x) - g_n(x))\, d(F+G)(x) < \varepsilon_n,$$

with $\lim_{n \to +\infty} \varepsilon_n = 0$.

Define the sequence $\{b_n\}$ by

$$b_n(x) = \sup_{a \geq 0} g_n(x-a) 1_{[-n, +\infty[}(x), \qquad \forall x \in \mathbb{R},$$

where 1_A is the indicator function. One verifies easily that b_n is nondecreasing, $\lim_{x \to -\infty} b_n(x) = 0$, $0 \leq b_n \leq b$ and

$$\lim_{n \to +\infty} \int_{-\infty}^{+\infty} (b(x) - b_n(x))\, d(F+G)(x) = 0.$$

Moreover, since each discontinuity of b_n is a discontinuity of g_n, this implies that b_n is continuous from the right. Thus there exists a sequence of functions $\{b_n\}$ such that $b_n \in C_+(F) \cap C_+(G)$ and is continuous from the right, with $\lim_{x \to -\infty} b_n(x) = 0$, $b = \lim_{n \to +\infty} b_n$ $(F+G)$-a.e. and $b_n \leq b$. By the dominated convergence theorem, if

$$F \underset{(+)}{>} G \text{ implies } F_{b_n} \underset{i}{>} G_{b_n},$$

then

$$F \underset{(+)}{>} G \text{ implies } F_b \underset{i}{>} G_b.$$

Let b be a function of $C_+(F) \cap C_+(G)$, which is continuous from the right and verifies $\lim_{x \to -\infty} b(x) = 0$. We can write

$$b(x) = \int_{-\infty}^{x} db(p)$$

and

$$\int_{-\infty}^{t} b(x)\, dF(x) = \int_{-\infty}^{t} \{F(t) - F(p^-)\}\, db(p).$$

Thus to prove the sufficiency we have to show

$$\begin{aligned} &\int_{-\infty}^{t} \{F(t) - F(p^-)\}\, db(p) \Big/ \int_{-\infty}^{+\infty} \{1 - F(p^-)\}\, db(p) \\ (1) \qquad &\le \int_{-\infty}^{t} \{G(t) - G(p^-)\}\, db(p) \Big/ \int_{-\infty}^{+\infty} \{1 - G(p^-)\}\, db(p) \\ &\Leftarrow F \underset{(+)}{>} G. \end{aligned}$$

Now

$$(2) \qquad F \underset{(+)}{>} G \Leftrightarrow 1 - G(p) = \{1 - F(p)\} l(p), \qquad \forall p \in \mathbb{R},$$

with l nonincreasing. Furthermore [see Keilson and Sumita (1982)],

$$(3) \quad F \underset{(+)}{>} G \Leftrightarrow \{G(t) - G(p)\}/\{1 - G(p)\} \ge \{F(t) - F(p)\}/\{1 - F(p)\},$$

for all (t, p), $t \ge p$.

Noticing that

$$\{F(t) - F(p)\}/\{1 - F(p)\} = m_t(p)$$

is nonincreasing in p, for all $t \le p$, then, by (2) and (3) the inequality in (1) is verified if

$$\begin{aligned} &\int_{-\infty}^{t} m_t(p^-)\{1 - F(p^-)\}\, db(p) \Big/ \int_{-\infty}^{+\infty} \{1 - F(p^-)\}\, db(p) \\ (4) \qquad &\le \int_{-\infty}^{t} m_t(p^-) l(p^-)\{1 - F(p^-)\}\, db(p) \\ &\div \int_{-\infty}^{+\infty} l(p^-)\{1 - F(p^-)\}\, db(p). \end{aligned}$$

Since m_t is nonincreasing, (4) holds, if for all t,

$$\begin{aligned} &\int_{-\infty}^{t} \{1 - F(p^-)\}\, db(p) \Big/ \int_{-\infty}^{+\infty} \{1 - F(p^-)\}\, db(p) \\ &\le \int_{-\infty}^{t} l(p^-)\{1 - F(p^-)\}\, db(p) \Big/ \int_{-\infty}^{+\infty} l(p^-)\{1 - F(p^-)\}\, db(p). \end{aligned}$$

This inequality, in turn, is satisfied because l is nonincreasing and, for all s, t in $\mathbb{R}$,

$$\begin{aligned} &\int_{-\infty}^{s} \{1 - F(p^-)\}\, db(p) \Big/ \int_{-\infty}^{+\infty} \{1 - F(p^-)\}\, db(p) \\ &\le \int_{-\infty}^{\min(s, t)} \{1 - F(p^-)\}\, db(p) \Big/ \int_{-\infty}^{t} \{1 - F(p^-)\}\, db(p). \end{aligned}$$

This proves the sufficiency part.

The necessity is proved by considering $F_b >_i G_b$ with b defined by

$$b(x) = \begin{cases} 0, & \text{for } x \leq p, \\ 1, & \text{for } x > p. \end{cases}$$

Then by (3) the result follows. □

REMARK. A partial ordering equivalent to $>_{(-)}$ is obtained if, in the definition of $>_0$, one replaces b is a nonnegative and nondecreasing function by b is a nonnegative and nonincreasing function.

The following corollary completes the results obtained by Bickel and Lehmann (1975), Lemmas 1 and 2, pages 1060–1061.

COROLLARY. *Let F and G be two distributions on $\mathbb{R}$. Then $F >_{(+)} G$ is a necessary and sufficient condition for*

$$\frac{\int_{-\infty}^{+\infty}\alpha(x)\,dF(x)}{\int_{-\infty}^{+\infty}\beta(x)\,dF(x)} \geq \frac{\int_{-\infty}^{+\infty}\alpha(x)\,dG(x)}{\int_{-\infty}^{+\infty}\beta(x)\,dG(x)} \tag{5}$$

to hold for all functions α and β, integrable with respect to F and G, such that β is nonnegative, α/β and β are nondecreasing.

PROOF. Writing

$$e = \int_{-\infty}^{+\infty}\alpha(x)\,dG(x) \Big/ \int_{-\infty}^{+\infty}\beta(x)\,dG(x),$$

we have

$$(5) \Leftrightarrow \int_{-\infty}^{+\infty}(\alpha(x) - e\beta(x))\,dF(x) \geq 0.$$

Then the result follows from the theorem above, with $a(x) = \alpha(x)/\beta(x) - e$ and $b(x) = \beta(x)$. □

3. Application. This section is concerned with partial orderings on symmetric distributions (with respect to zero), which reflect the relative heaviness of tails, and their applications to the comparison of rank tests. These partial orderings are scale-free. The first ordering of that type is the s-ordering $<_s$ of van Zwet (1964) who defined $F <_s G$ by $(f \circ F^{-1})(u)/(g \circ G^{-1})(u)$ nondecreasing for $u \in (\frac{1}{2}, 1)$. Another ordering of distributions (r-ordering $<_r$), which reflects the property of the heaviness of a tail was introduced by Lawrence (1975). He stated that G has heavier tails than F ($F <_r G$) if $G^{-1}(u)/F^{-1}(u)$ is nondecreasing for $u \in (\frac{1}{2}, 1)$.

This ordering is weaker than van Zwet's. In Rivest (1982) and Loh (1984a), there are many examples of distributions ordered by the r-ordering. In particular, if t_n is the t-distribution with n degrees of freedom, then $t_n <_r t_m$ if $m < n$. These orderings have found several applications in mathematical statistics [see, for example, Bickel and Lehmann (1975), Singh (1977), Yanagimoto and Sibuya (1980), Benjamini (1983) and Loh (1984b)].

In this section we are concerned with the comparison of the asymptotic relative efficiencies of pairs of rank tests T and T' (ARE($T, T'|F$)) for the location parameter in the two-sample problem. van Zwet has compared the Wilcoxon rank sum test (W) to the normal-score test ($N - S$). He proved that $F <_s G \Rightarrow \text{ARE}(W, N - S|F) \le \text{ARE}(W, N - S|G)$. For the same problem Gastwirth (1970) obtained more general results on the comparison of AREs of rank tests, using an ordering stronger than the s-ordering. Hájek (1969) also shed some light on this problem. It appears from these papers that a good rank test for a light (heavy) tail distribution places more (less) weight on the extreme ranks than on the central ones. We now give necessary and sufficient conditions for these results to hold.

First, we define a new ordering on the tails of distributions that have square integrable densities on $\mathbb{R}$ (assumption A_1). Let F^* be the distribution on $(\frac{1}{2}, 1)$ defined by

$$F^*(t) = \int_{1/2}^{t} (f \circ F^{-1})(u)\,du \Big/ \int_{1/2}^{1} (f \circ F^{-1})(u)\,du, \qquad t \in \left(\tfrac{1}{2}, 1\right),$$

and let G^* be the corresponding distribution for G. Then the relation $<_{*+}$ defined by

$$F <_{*+} G \Leftrightarrow F^* >_{(+)} G^*,$$

is a scale-free partial ordering weaker than the s-ordering since $F <_s G \Leftrightarrow F^* >_l G^*$.

Let $\mathscr{A}$ be the set of rank tests T with a score function ϕ_T, antisymmetric with respect to $\frac{1}{2}$, nondecreasing and convex on $(\frac{1}{2}, 1)$. We assume that ϕ_T is a square-integrable essentially nonconstant function on $(0, 1)$ and has a derivative α_T a.e. Let F be the distribution of the combined sample, which is assumed to have a finite Fisher information (assumption A_2). We can write after some manipulations [see, for example, Puri and Sen (1971), Chapter 3, or Lehmann (1983), Chapter 5] the squared efficacy of $T \in \mathscr{A}$ as

$$k(T, F)\left\{\int_{1/2}^{1} \alpha_T(u)(f \circ F^{-1})(u)\,du\right\}^2,$$

where $k(T, F)$ is a positive constant depending on T and F. Let $\mathscr{B} = \{(T, T') \in \mathscr{A} \times \mathscr{A} | \alpha_T(u) = c(u)\alpha_{T'}(u),\ c(u)$ positive and nondecreasing for $u \in (\frac{1}{2}, 1)\}$.

We note that if $\alpha_T(u) = c(u)\alpha_{T'}(u)$ with $c(\cdot)$ positive and nondecreasing, there exists a nondecreasing convex function k such that

$$\phi_T(u) = (k \circ \phi_{T'})(u). \tag{6}$$

We interpret this relationship as ϕ_T is "more convex" than $\phi_{T'}$, because (6) implies

$$\frac{d^2\phi_T(u)}{du^2} \Big/ \frac{d\phi_T(u)}{du} \ge \frac{d^2\phi_{T'}(u)}{du^2} \Big/ \frac{d\phi_{T'}(u)}{du},$$

if the second derivatives exist. Now, $[d^2\phi_T(u)/du^2]/[d\phi_T(u)/du]$ can be considered as a scale-free measure of convexity for ϕ.

PROPOSITION 1. *Let F and G be two symmetric distributions with respect to zero, satisfying the assumptions* A_1 *and* A_2. *Then*

$$\mathrm{ARE}(T, T'|F) \geq \mathrm{ARE}(T, T'|G), \qquad \forall\, (T, T') \in \mathscr{B} \Leftrightarrow F <_{*+} G.$$

The proof of this proposition is analogous to the proof of the corollary of Section 2.

REMARK 1. Another interesting expression for the Pitman asymptotic efficiency allows us to highlight the s-ordering in the comparison of rank tests. We have [see, for example, Hájek and Šidák (1967), page 268]

$$\begin{aligned} &\mathrm{ARE}(T, T'|F) \geq \mathrm{ARE}(T, T'|G) \\ &\Leftrightarrow \frac{\int_{1/2}^{1} \phi_T(u)(-f'/f) \circ F^{-1}(u)\, du}{\int_{1/2}^{1} \phi_{T'}(u)(-f'/f) \circ F^{-1}(u)\, du} \\ &\qquad \geq \frac{\int_{1/2}^{1} \phi_T(u)(-g'/g) \circ G^{-1}(u)\, du}{\int_{1/2}^{1} \phi_{T'}(u)(-g'/g) \circ G^{-1}(u)\, du}. \end{aligned}$$

We consider now $\mathscr{A}'$, the set of the locally most powerful rank tests for the strongly unimodal distributions. These rank tests have a nondecreasing score function. Let

$$\mathscr{B}' = \{(T, T') \in \mathscr{A}' \times \mathscr{A}' | \phi_T(u) = c(u)\phi_{T'}(u),$$
$$c(u) \text{ positive and nondecreasing for } u \in (\tfrac{1}{2}, 1)\}.$$

PROPOSITION 2. *Let F and G be two unimodal symmetric distributions with respect to zero, satisfying the assumptions* A_1 *and* A_2 *and such that* $\lim_{u \to 1}(f \circ F^{-1})(u) = \lim_{u \to 1}(g \circ G^{-1})(u) = 0$. *Then*

$$\mathrm{ARE}(T, T'|F) \geq \mathrm{ARE}(T, T'|G), \qquad \forall\, (T, T') \in \mathscr{B}' \Leftrightarrow F \underset{s}{<} G.$$

PROOF. Note that

$$1 - \int_{1/2}^{t} (-f'/f) \circ F^{-1}(u)\, du \Big/ \int_{1/2}^{1} (-f'/f) \circ F^{-1}(u)\, du = (f \circ F^{-1})(t)/f(0),$$

and apply the corollary of Section 2. □

This result is more general than Gastwirth's (1970) because the s-ordering is weaker than his information ordering.

REMARK 2. For the Wilcoxon rank sum test we have $\alpha_w(u) = 1$, $0 < u < 1$. Then

PROPOSITION 3. *Let F and G be two symmetric distributions with respect to zero, satisfying the assumptions* A_1 *and* A_2. *Then* $F^* >_i G^*$ *is a necessary and sufficient condition to have*

$$\mathrm{ARE}(T, W|F) \geq \mathrm{ARE}(T, W|G), \qquad \forall\, T \in \mathscr{A}.$$

The proof follows from the definition of stochastic ordering. This result extends van Zwet's result (1964) on the comparison of the Wilcoxon rank sum test to the normal-score test.

REMARK 3. It seems difficult to compare the r-ordering and the ordering defined by $<_{*+}$. The former is more intuitive and easier to verify than the latter. On the other hand, some distributions, which are intuitively comparable with respect to their tails heaviness, are ordered by the latter but not by the former. For example, we know [see Loh (1984a)] that the double exponential (F) and Cauchy (G) distributions are not r-ordered. Since

$$(f \circ F^{-1})(u) = (1 - u) \quad \text{and} \quad (g \circ G^{-1})(u) = 1/\pi \cos^2 \pi\left(u - \tfrac{1}{2}\right),$$

we have

$$F <_{*+} G \text{ if and only if } \pi(1-u)\sin \pi\left(u - \tfrac{1}{2}\right) - \cos \pi\left(u - \tfrac{1}{2}\right) \le 0, \quad \text{for } u \in \left(\tfrac{1}{2}, 1\right).$$

This last inequality is satisfied because $\pi(1-u)\sin \pi(u - \frac{1}{2}) - \cos \pi(u - \frac{1}{2})$ is nondecreasing for $u \in (\frac{1}{2}, 1)$ and its maximum value is zero on this interval. Thus F and G are ordered by $<_{*+}$.

REMARK 4. It is possible to apply the corollary of Section 2 to other problems than to the one of the location parameter in the two-sample problem. For example, similar results can be obtained for the one-sample problem.

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DÉPARTEMENT DE MATHÉMATIQUES,
STATISTIQUE ET ACTUARIAT
UNIVERSITÉ LAVAL
QUÉBEC, QUÉBEC G1K 7P4
CANADA