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COVARIANCE HYPOTHESES WHICH ARE LINEAR IN BOTH THE COVARIANCE AND THE INVERSE COVARIANCE

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It is proved in this paper that covariance hypotheses which are linear in both the covariance and the inverse covariance are products of models each of which consists of either (i) independent identically distributed random vectors which have a covariance with a real, complex or quaternion structure or (ii) independent identically distributed random vectors with a parametrization of the covariance which is given by means of the Clifford algebra. The models (i) are well known. For models (ii) we have found, under the assumption that the distribution is normal, the exact distributions of the maximum likelihood estimates and the likelihood ratio test statistics.

1. Introduction. The purpose of the present paper is to describe, for the family of normal distributions, the structure of those statistical hypotheses (models) which are linear in both the covariance and the inverse covariance.

In Section 2 we show that under such a hypothesis the problem of maximum likelihood estimation of the covariance has an explicit solution, and that these hypotheses are parametrized by Jordan algebras (quadratic subspaces). Based on the theory of Jordan algebras it is then possible to give a canonical form for such hypotheses. In Section 3 we show that the Jordan algebras are products of so-called simple Jordan algebras and in Section 4 we prove that the hypotheses can be decomposed into products of hypotheses parametrized by simple Jordan algebras. The simple Jordan algebras are classified according to their so-called degree. The only simple Jordan algebras of degree 1 is the set of real numbers, and corresponding to that we have the hypothesis under which the covariance is proportional to a known covariance. In Section 5 we give a canonical form for the hypotheses which are parametrized by simple Jordan algebras of degree greater than 3. It turns out that these hypotheses are equivalent to those consisting of independent identically distributed random vectors which have a covariance with a real, complex or quaternion structure, and the distributions of the maximum likelihood estimates are given by Wishart (1928), Goodman (1963) and Andersson (1975), respectively.

The statistical hypotheses which are parametrized by simple Jordan algebras of degree 2 are new, and in Section 6 we give a complete solution to the problems of maximum likelihood inference. It turns out that the distribution problems are not more complicated than those for the two-dimensional normal distribution. The mathematical structure of the hypotheses is closely related to the theory of spinors in quantum mechanics. Based on that theory, we give a canonical form for such hypotheses in Section 7.

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Finally, in Section 8 we discuss the extension of a covariance hypothesis with linear structure to a hypothesis which is linear in both the covariance and the inverse covariance. This extension clarifies the meaning of the relationship algebra introduced by James (1957). It also shows how the invariant normal models treated by Andersson (1975) are special cases of hypotheses parametrized by Jordan algebras.

2. Covariance hypotheses parametrized by Jordan algebras. Let V be an N-dimensional real vector space with an inner product $(\ ,\ )$. The vector space of linear mappings of V into itself is denoted $L(V)$. The trace and the determinant of an $A \in L(V)$ are denoted $\operatorname{tr} A$ and $\det A$, respectively. The adjoint linear mapping to A is denoted A'. It is defined by $(Ax, y) = (x, A'y)$ for all $x, y \in V$. A is called symmetric if $A' = A$, and the vector space of all symmetric linear mappings of V into itself is denoted $L_s(V)$. The identity mapping of V onto itself is denoted I. For $x, y \in V$ the linear mapping $z \to (y, z)x$ of V into itself is denoted xy'. We shall use the facts that $(Ax, y) = \operatorname{tr}(A(xy'))$ and $(x, z)(y, z) = \operatorname{tr}((zz')(xy'))$.

Let X be a normally distributed random vector with values in V and with expected value $EX = 0$. The covariance of X is the symmetric linear mapping Σ of V into itself such that $E((x, X)(y, X)) = (\Sigma x, y)$ for all $x, y \in V$. It follows from $(\Sigma x, y) = E((x, X)(y, X)) = E \operatorname{tr}((XX')(xy')) = \operatorname{tr}((EXX')(xy')) = ((EXX')x, y)$ that $\Sigma = EXX'$. If $(e_1, \ldots, e_N)$ is a basis of V and $(X_1, \ldots, X_N)$ the coordinates of X, then the matrix $B = (EX_iX_j)_{i,j}$ is called the covariance matrix of X. Let $S = ((e_i, e_j))_{i,j}$ be the matrix of the inner product. Then $\Sigma(e_i) = E((X, e_i)X) = \Sigma_k \Sigma_j (EX_k X_j)(e_j, e_i)e_k$, $i = 1, \ldots, N$. Hence the linear mapping Σ has matrix BS. The matrix of Σ is therefore the covariance matrix of X if and only if S is the identity matrix, i.e., if and only if the basis is orthonormal.

When the covariance Σ is positive definite, the distribution of X has density

$$(2\pi)^{-N/2} |\det \Sigma|^{-1/2} \exp\left(-\tfrac{1}{2}\operatorname{tr}\left(\Sigma^{-1}(xx')\right)\right),$$

w.r.t. the Lebesgue measure on V. By using the trace inner product $\operatorname{tr}(A_1A_2)$ on $L_s(V)$, it is seen that the family of normal distributions with mean vector 0 and positive definite covariance is a regular exponential family; see Barndorff-Nielsen (1978). The canonical parameter is $-\frac{1}{2}\Sigma^{-1}$, the canonical statistic is XX' and the mean value parameter is Σ.

Families of normal distributions having a linear covariance structure have received considerable attention. To define these families, or hypotheses, let L be an n-dimensional linear subspace of $L_s(V)$ such that

$$\Theta = \{\Sigma \in L \mid \Sigma \text{ is positive definite}\} \tag{1}$$

is nonempty. The hypothesis H: $\Sigma \in \Theta$ is said to be linear in the covariance. If $(G_1, \ldots, G_n)$ is any basis of L then H is the hypothesis that

$$\Sigma = \sum_{i=1}^{n} \sigma_i G_i, \tag{2}$$

where $\sigma_1, \ldots, \sigma_n$ are unknown real parameters such that Σ is positive definite.

Anderson (1969, 1970, 1973) has given the likelihood equations, an iterative method for solving these equations and the asymptotic distribution of the estimates. In order to obtain further results it seems necessary to consider less general hypotheses. From the point of view of the theory of exponential families it is natural to consider hypotheses which are linear in the canonical parameter [Barndorff-Nielsen (1978), Theorem 8.5].

Set $\Theta^{-1} = \{\delta \in L_s(V) | \delta^{-1} \in \Theta\}$. We shall say that the hypothesis H is linear in the inverse covariance, too, if there also exists a linear subspace M of $L_s(V)$ such that

$$\Theta^{-1} = \{\delta \in M | \delta \text{ is positive definite}\}. \tag{3}$$

In this case, H is a linear hypothesis in the canonical parameter. The likelihood equation for the mean value parameter Σ is the linear equation

$$\forall\, \delta \in M\colon \operatorname{tr}(\delta\Sigma) = \operatorname{tr}(\delta(XX')) \tag{4}$$

and the maximum likelihood estimate exists (and is unique) if and only if (4) has a solution $\Sigma \in \Theta$ [Barndorff-Nielsen (1978), Corollary 9.7]. It follows from Lemma 1 that $\dim L = \dim M$. Hence the equation has a unique solution $\Sigma \in L$. This solution is a linear function of XX', i.e., a quadratic function of X, which is complete, sufficient and unbiased for Σ. The maximum likelihood estimate exists if and only if the solution is positive definite. If the hypothesis is given by (2) and $(H_1, \dots, H_n)$ is any basis of M, then the likelihood equations for $\sigma_1, \dots, \sigma_n$ become

$$\sum_{i=1}^{n} \operatorname{tr}(H_j G_i)\sigma_i = \operatorname{tr}(H_j(XX')), \qquad j = 1, \dots, m.$$

It is thus seen that the problem of maximum likelihood estimation of the covariance has an explicit solution when the hypothesis is linear in both the covariance and the inverse covariance. We shall give a complete characterization of such hypotheses.

LEMMA 1. *Let Θ be given by* (1) *and suppose $I \in \Theta$. Then the hypothesis H: $\Sigma \in \Theta$ is linear in the inverse covariance, too, if and only if*

$$\forall\, A, B \in L\colon AB + BA \in L \tag{5}$$

and in this case $\Theta = \Theta^{-1}$.

PROOF. Suppose Θ^{-1} is given by (3). Let us first prove $\Theta = \Theta^{-1}$. Let $A \in L$. For $t > 0$ sufficiently small $I - tA \in \Theta$ and $(I - tA)^{-1} = I + tA + t^2A^2 + \cdots \in \Theta^{-1} \subseteq M$. Hence $((I - tA)^{-1} - I)/t = A + tA^2 + \cdots \in M$. For $t \to 0$ it follows that $L \subseteq M$. By symmetry $M \subseteq L$. Hence $L = M$ and $\Theta = \Theta^{-1}$. Then $((I - tA)^{-1} - I - tA)/t^2 = A^2 + tA^3 + \cdots \in L$, and for $t \to 0$ it follows that $A^2 \in L$. By $AB + BA = (A + B)^2 - A^2 - B^2$ we have (5). For the converse, suppose L satisfies (5). By induction $A^n \in L$ for $A \in L$ and $n = 1, 2, \dots$. Let $A \in \Theta$. For $t > 0$ sufficiently small, $I - tA \in \Theta$ and $tA = (I - (I - tA))$. Since the eigenvalues of $I - tA$ are less than 1 it follows that $A^{-1} = t(I + (I - tA) +$

$(I - tA)^2 + \cdots) \in L$. Hence $\Theta \subseteq \Theta^{-1}$ and $\Theta^{-1} \subseteq (\Theta^{-1})^{-1} = \Theta$, and Θ^{-1} is given by (3) with $M = L$. □

REMARK. The condition $I \in \Theta$ is not essential. The covariances are identified with linear mappings by means of an arbitrary inner product, and one can just use one of the inverse covariances as the inner product. In fact, if the covariance of X is Σ then $E((\Sigma^{-1}x, X)(\Sigma^{-1}y, X)) = (\Sigma\Sigma^{-1}x, \Sigma^{-1}y) = (x, \Sigma^{-1}y) = (\Sigma^{-1}x, y)$. Thus the covariance of X w.r.t. the inner product $(\Sigma^{-1}\cdot, \cdot)$ is I.

Condition (5) expresses that L is a Jordan algebra of symmetric linear mappings. The theory of Jordan algebras is extensively treated in the mathematical literature [see Jackson (1968) or Braun and Koecher (1966)]. For the sake of convenience, we shall summarize the main facts in the next section.

Seely (1971, 1972) proved that the covariance hypothesis H admits a complete sufficient statistic if and only if L satisfies condition (5). (He called such a subspace a quadratic subspace, but in view of Lemma 1, a more appropriate term must be a Jordan algebra.)

Since then, condition (5) has appeared in numerous papers. It seems, however, to be overlooked that condition (5) is closely related to the normal distribution, while the condition that Θ^{-1} has a linear structure more generally expresses that the hypothesis is linear in the canonical parameter. In fact, consider a regular exponential family and a hypothesis which is linear in the mean value parameter. Then the following statements are equivalent: (a) The hypothesis is linear in the canonical parameter. (b) The maximum likelihood estimate of the mean value parameter is a linear function of the canonical statistics. (c) The hypothesis H admits a complete sufficient statistic. (d) There exists a uniformly minimum variance unbiased estimate of the mean value parameter. That (a) implies (b), (c) and (d) follows directly from the theory of exponential families just as in the case with the normal distribution treated previously. Although the proofs of the other implications simplify known proofs in the case of the normal distribution, we shall not give them here. As we shall do for the normal distribution, it would in our opinion be more valuable, for other exponential families, to characterize the hypotheses which are linear in both the mean value parameter and the canonical parameter.

3. The structure of Jordan algebras. A Jordan algebra over the set $\mathbb{R}$ of real numbers is a real vector space J with a composition $*$ such that $a * b = b * a$, $(\lambda a) * b = \lambda(a * b)$, $(a_1 + a_2) * b = a_1 * b + a_2 * b$ and $((a * a) * b) * a = (a * a) * (b * a)$ for $\lambda \in \mathbb{R}$ and $a, a_1, a_2, b \in J$.

For an associative algebra A, one can define a new composition by setting

$$a * b = \tfrac{1}{2}(ab + ba)$$

for $a, b \in A$, and it is easy to see that A with this composition is a Jordan algebra; it is denoted A^+. A Jordan algebra J is called special if there exists an associative algebra A such that J is isomorphic to a Jordan subalgebra of A^+. A

Jordan algebra J is called formally real if $a * a + b * b = 0$ implies $a = 0$ and $b = 0$. The following structure theorem is due to Jordan, von Neumann and Wigner (1934).

THEOREM 1. *Let J be a finite-dimensional, special and formally real Jordan algebra. Then J is isomorphic to a product $J_1 \times \cdots \times J_k$ of Jordan algebras, where each of the Jordan algebras J_i, $i = 1, \ldots, k$, is one of the following simple Jordan algebras*:

(i) *the set* $\mathbb{R}$ *of real numbers*;

(ii) $\mathbb{R} \times W$, *where W is a real vector space of dimension $m \geq 2$ with an inner product ϕ and composition*

$$(6) \qquad (\lambda_1, w_1) * (\lambda_2, w_2) = (\lambda_1\lambda_2 + \phi(w_1, w_2), \lambda_1 w_2 + \lambda_2 w_1);$$

(iii) *the vector space $H_r(D)$ of $r \times r$ Hermitian matrices over D, $r \geq 3$, where D is either the set $\mathbb{R}$ of real numbers, the set $\mathbb{C}$ of complex numbers or the set $\mathbb{H}$ of quarternions, and composition $A * B = \frac{1}{2}(AB + BA)$.*

PROOF. See Jacobson [(1968), page 205], Braun and Koecher [(1966), page 331] or Jordan, von Neumann and Wigner [(1934), page 63]. □

REMARK. All the simple Jordan algebras have an identity element. Hence J has an identity element; it is denoted 1.

The Jordan algebra $\mathbb{R}$ is said to be of degree 1, the simple Jordan algebras $\mathbb{R} \times W$, $\dim W \geq 2$, are said to be of degree 2 and dimension $1 + \dim W$, and the simple Jordan algebras $H_r(D)$, $D = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$, are said to be of degree r, $r \geq 3$. Two Jordan algebras of degree 2 are isomorphic if they have the same dimension. Apart from these cases none of the simple Jordan algebras mentioned in Theorem 1 is isomorphic. We shall see in Section 7 that the simple Jordan algebras of degree 2 and dimension 3, 4 and 6 are isomorphic to $H_2(D)$, $D = \mathbb{R}, \mathbb{C}$ *and* $\mathbb{H}$, respectively.

The vector space $L(V)$ is an associative algebra. Since $\frac{1}{2}(AB + BA)$ is symmetric when A and B are symmetric, $L_s(V)$ is a Jordan subalgebra of $L(V)^+$. Hence $L_s(V)$ is a special Jordan algebra, and it is clear that it is formally real. Any Jordan subalgebra of $L_s(V)$ is therefore special and formally real.

4. The decomposition of a hypothesis into hypotheses which are parametrized by simple Jordan algebras. As in Section 2, let X be a random vector which has a normal distribution with mean vector 0 and covariance Σ. Let H: $\Sigma \in \Theta$ be a hypothesis which satisfies the conditions in Lemma 1, i.e., $\Theta = \{\Sigma \in L | \Sigma \text{ is positive definite}\}$, where L is a Jordan subalgebra of $L_s(V)$. It follows from Theorem 1 that there exists a 1–1 Jordan algebra homomorphism

$$\tau\colon J \to L_s(V)$$

of a product $J = J_1 \times \cdots \times J_k$ of simple Jordan algebras onto L, i.e., τ is a 1–1 linear mapping such that $\tau(J) = L$, $\tau(1) = I$ and

$$\tau(a * b) = \tfrac{1}{2}(\tau(a)\tau(b) + \tau(b)\tau(a)) \quad \text{for } a, b \in J. \tag{7}$$

Set $K = \{a \in J | \tau(a) \text{ is positive definite}\}$. Then $\Theta = \tau(K)$ and the hypothesis H is parametrized by K and τ. It is clear that K is a convex cone in J. The problems are to determine K and to describe the structure of τ.

Theorem 2. *Let J and τ be as defined previously. Then there exists a unique decomposition $V = V_1 + \cdots + V_k$ of V into a sum of k pairwise orthogonal subspaces and* 1–1 *Jordan algebra homomorphisms τ_i: $J_i \to L_s(V_i)$, $i = 1, \dots, k$, such that*

$$\tau(a) = \begin{pmatrix} \tau_1(a_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_k(a_k) \end{pmatrix}, \tag{8}$$

$$a = (a_1, \dots, a_k) \in J = J_1 \times \cdots \times J_k.$$

Proof. Let $\varepsilon_i = (0, \dots, 1, \dots, 0) \in J_1 \times \cdots \times J_k$ with 1 the identity in J_i. Set $Q_i = \tau(\varepsilon_i)$ and $V_i = Q_i(V)$. By (7), $Q_i^2 = \tau(\varepsilon_i)\tau(\varepsilon_i) = \tau(\varepsilon_i * \varepsilon_i) = \tau(\varepsilon_i) = Q_i$. Hence Q_i is the orthogonal projection onto V_i. Since τ is linear, $I = \tau(1, \dots, 1) = \tau(\varepsilon_1 + \cdots + \varepsilon_k) = \tau(\varepsilon_1) + \cdots + \tau(\varepsilon_k) = Q_1 + \cdots + Q_k$, and it follows that we have an orthogonal decomposition. Let $A_i = \tau(0, \dots, a_i, \dots, 0)$, $a_i \in J_i$. Then by (7), $A_i = \tau((0, \dots, a_i, \dots, 0) * \varepsilon) = \frac{1}{2}(A_iQ_i + Q_iA_i)$. Hence $Q_iA_i = \frac{1}{2}(Q_iA_iQ_i + Q_iA_i)$ and $A_iQ_i = \frac{1}{2}(A_iQ_i + Q_iA_iQ_i)$ and we have $A_i = Q_iA_iQ_i$. There exists therefore a Jordan algebra homomorphism τ_i: $J_i \to L_s(V_i)$ such that $\tau(0, \dots, a_i, \dots, 0) = Q_i\tau_i(a_i)Q_i$, $a_i \in J_i$. Then

$$\tau(a_1, \dots, a_k) = \Sigma\tau(0, \dots, a_i, \dots, 0) = \Sigma Q_i\tau_i(a_i)Q_i$$

and this is equivalent to (8). Conversely, if $V = V_1 + \cdots + V_k$ is an orthogonal decomposition such that (8) holds, then it is obvious that $\tau(\varepsilon_i)$ is the orthogonal projection onto V_i. Hence the decomposition is uniquely determined by J and τ. □

Remark. Since $\tau(a)$ is positive definite if and only if $\tau_i(a_i)$, $i = 1, \dots, k$, are positive definite, then $K = K_1 \times \cdots \times K_k$, where $K_i = \{a_i \in J_i | \tau_i(a_i) \text{ is positive definite}\}$, $i = 1, \dots, k$.

Corresponding to the orthogonal decomposition in Theorem 2, we have $X = (X_1, \dots, X_k)$. Under the hypothesis H the random vectors $X_1, \dots, X_k$ are independently distributed and X_i has a normal distribution with mean vector 0 and covariance $\tau_i(a_i)$, $a_i \in K_i$, $i = 1, \dots, k$. Thus the problems are reduced to considering hypotheses which are parametrized by simple Jordan algebras.

The only hypothesis parametrized by the simple Jordan algebra $\mathbb{R}$ is H: $\Sigma = aI$, where $a > 0$. In the next section we shall give a canonical form of a hypothesis which is parametrized by a simple Jordan algebra of degree $r \geq 3$. It turns out that these hypotheses are well known. The hypotheses which are parametrized by simple Jordan algebras of degree 2 are new, and we shall treat them in detail in Sections 6 and 7.

5. A canonical form of a hypothesis which is parametrized by a simple Jordan algebra of degree $r \geq 3$. Let $M_r(D)$ denote the algebra of $r \times r$ matrices over D, where $D = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. An element $A \in M_r(\mathbb{C})$ has the form $A = A_1 + A_2 i$, where $A_1, A_2 \in M_r(\mathbb{R})$, and the $2r \times 2r$ matrix over $\mathbb{R}$,

$$\begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix},$$

is called the real matrix of A. Similarly, an element $A \in M_r(\mathbb{H})$ has the form $A = A_1 + A_2 i + A_3 j + A_4 k$, where $A_1, A_2, A_3, A_4 \in M_r(\mathbb{R})$, and the $4r \times 4r$ matrix over $\mathbb{R}$,

$$\begin{pmatrix} A_1 & -A_2 & -A_3 & -A_4 \\ A_2 & A_1 & -A_4 & A_3 \\ A_3 & A_4 & A_1 & -A_2 \\ A_4 & -A_3 & A_2 & A_1 \end{pmatrix},$$

is called the real matrix of A. The real matrix of an $A \in M_r(\mathbb{R})$ is A itself. In any of the cases, $D = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the real matrix of an $A \in M_r(D)$ is denoted $\operatorname{re} A$. It is seen that $\operatorname{re}(AB) = (\operatorname{re} A)(\operatorname{re} B)$, $A, B \in M_r(D)$. Moreover, $\operatorname{re} A$ is symmetric if and only if A is Hermitian.

THEOREM 3. *Let τ: $H_r(D) \to L_s(V)$ be a 1–1 Jordan algebra homomorphism, $D = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and $r \geq 3$. Then there exists an orthonormal basis of V such that the matrix of $\tau(A)$ is*

$$(9) \qquad \operatorname{re} A \otimes I_n = \begin{pmatrix} \operatorname{re} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \operatorname{re} A \end{pmatrix}, \qquad A \in H_r(D),$$

where $n = N/(r \dim D)$.

PROOF. We shall say that a subspace V_0 of V is invariant if $\tau(A)(V_0) \subseteq V_0$ for all $A \in H_r(D)$. Since the linear mappings $\tau(A)$, $A \in H_r(D)$, are symmetric, it follows that the orthogonal complement to an invariant subspace is invariant. There exists therefore an orthogonal decomposition $V = V_1 + \cdots + V_n$ of V into a sum of minimal invariant subspaces; see Bourbaki [(1959), page 120]. Hence

$$\tau(A) = \begin{pmatrix} \tau_1(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_n(A) \end{pmatrix}, \qquad A \in H_r(D),$$

where τ_i: $H_r(D) \to L_s(V_i)$, $i = 1, \ldots, n$, are Jordan algebra homomorphisms.

Set $p = r \dim D$ and let i be fixed, $i = 1, \ldots, n$. It follows from Jacobson [(1968), page 143] that the Jordan algebra homomorphism τ_i can be uniquely extended to an algebra homomorphism η_i: $M_r(D) \to L(V_i)$, i.e., $\tau_i(A) = \eta_i(A)$, $A \in H_r(D)$. Since V_i is a minimal invariant subspace it follows from Bourbaki [(1958), page 49] that $\dim V_i = p$ and that there exists a basis $(e_{i,1}, \ldots, e_{i,p})$ of V_i such that the matrix of $\eta_i(A)$ is $\operatorname{re} A$, $A \in M_r(D)$. We shall find an orthonormal basis. Thus let $S = ((e_{i,j}, e_{i,k}))_{j,k}$ be the matrix of the restriction of the inner product to V_i. From $(x_i, \tau_i(A)y_i) = (\tau_i(A)x_i, y_i)$, $x_i, y_i \in V_i$, $A \in H_r(D)$, it follows that $S(\operatorname{re} A) = (\operatorname{re} A)S$, $A \in H_r(D)$. Then it is seen that $S = \operatorname{re}(cI_r)$, where $c \in D$ and I_r is the $r \times r$ identity matrix. Since S is symmetric and positive definite c is real and positive. Set $f_{i,j} = (1/\sqrt{c})\, e_{i,j}$, $j = 1, \ldots, p$. Then the basis $(f_{i,1}, \ldots, f_{i,p})$ is orthonormal, and the matrix of $\tau_i(A)$ is still $\operatorname{re} A$, $A \in H_r(D)$.

Since $V = V_1 + \cdots + V_n$ is an orthogonal decomposition it follows that $(f_{1,1}, \ldots, f_{1,p}, \ldots, f_{n,1}, \ldots, f_{n,p})$ is an orthonormal basis of V and that the matrix of $\tau(A)$ is given by (9). □

Remark. It can be seen that an $A \in H_r(D)$ is positive definite if and only if $\operatorname{re} A$ is positive definite. Hence it follows from (9) that the parameter set is the convex cone $K = \{A \in H_r(D) | A \text{ is positive definite}\}$, $D = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $r \geq 3$.

Since we have the canonical form (9) for an orthonormal basis, it follows that the hypothesis is equivalent to considering n independent identically distributed random vectors $X_1, \ldots, X_n$ of $p = r \dim D$ components such that X_i has a normal distribution with a covariance matrix, which is the real matrix of an $r \times r$ positive definite Hermitian matrix over D, $D = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The real case is classical; see Wishart (1928) or Anderson (1958). The complex case is treated by Goodman (1963) and Khatri (1965). Andersson (1975) treats the three cases simultaneously.

6. Maximum likelihood inference for a hypothesis which is parametrized by a simple Jordan algebra of degree 2. The hypothesis is given as described in the beginning of Section 4 with J a simple Jordan algebra of degree 2. Thus let W be a real vector space of dimension $m \geq 2$ with an inner product ϕ and τ: $\mathbb{R} \times W \to L_s(V)$ a 1–1 Jordan algebra homomorphism, i.e., τ is a 1–1 linear mapping such that

$$\tau(\lambda_1\lambda_2 + \phi(w_1, w_2), \lambda_1 w_2 + \lambda_2 w_1) = \tfrac{1}{2}\big(\tau(\lambda_1, w_1)\tau(\lambda_2, w_2) + \tau(\lambda_2, w_2)\tau(\lambda_1, w_1)\big) \tag{10}$$

for $(\lambda_1, w_1), (\lambda_2, w_2) \in \mathbb{R} \times W$. The random variable X has a normal distribution with density

$$(2\pi)^{-N/2} |\det \tau(\lambda, w)|^{-1/2} \exp\Big(-\tfrac{1}{2}\big(\tau(\lambda, w)^{-1} x, x\big)\Big), \tag{11}$$

and the parameter set is $K = \{(\lambda, w) \in \mathbb{R} \times W | \tau(\lambda, w) \text{ is positive definite}\}$.

Lemma 2. *$\tau(\lambda, w)$ is invertible if and only if $\lambda^2 \neq \phi(w,w)$ and in this case*

$$\tau(\lambda, w)^{-1} = \left(\lambda^2 - \phi(w,w)\right)^{-1}\tau(\lambda, -w), \tag{12}$$

$$K = \left\{(\lambda, w) \in \mathbb{R} \times W | \lambda > 0, \lambda^2 > \phi(w,w)\right\}, \tag{13}$$

$$|\det \tau(\lambda, w)| = |\lambda^2 - \phi(w,w)|^{N/2}, \tag{14}$$

$$\operatorname{tr} \tau(\lambda, w) = N\lambda, \tag{15}$$

$$\operatorname{tr} \tau(\lambda_1, w_1)\tau(\lambda_2, w_2) = N\lambda_1\lambda_2 + N\phi(w_1, w_2). \tag{16}$$

Proof. (12): Since τ is linear we have $\tau(\lambda, w) = \tau(\lambda, 0) + \tau(0, w) = \lambda I + \tau(0, w)$ and $\tau(\lambda, -w) = \tau(\lambda, 0) + \tau(0, -w) = \lambda I - \tau(0, w)$. Hence $\tau(\lambda, w)$ and $\tau(\lambda, -w)$ commute, and it follows from (10) that $\tau(\lambda, w)\tau(\lambda, -w) = \tau(\lambda^2 - \phi(w,w), \lambda w + \lambda(-w)) = (\lambda^2 - \phi(w,w))I$.

(13): Since the set of positive definite linear mappings is an open and convex subset of $L_s(V)$, we have that $\tau(\lambda, w)$ is positive definite if and only if $\alpha\tau(\lambda, w) + (1-\alpha)I$ is invertible for all $\alpha \in [0,1]$. Now it follows from (12) that $\alpha\tau(\lambda, w) + (1-\alpha)I = \tau(\alpha\lambda + 1 - \alpha, \alpha w)$ is invertible if and only if $(\alpha\lambda + 1 - \alpha)^2 \neq \alpha^2\phi(w,w)$. Since $\phi(w,w) \geq 0$ this holds for all $\alpha \in [0,1]$ if and only if $\lambda > 0$ and $\lambda^2 > \phi(w,w)$.

(14) and (15): Let $u \in W$ be such that $\phi(u, w) = 0$ and $\phi(u,u) = 1$. It follows from (10) that

$$\begin{aligned}
&\tau(0,u)\tau(\lambda, w)\tau(0,u)^{-1} \\
&\quad = \left(2\tau(\phi(u,w), \lambda u) - \tau(\lambda, w)\tau(0,u)\right)\tau(0,u)^{-1} \\
&\quad = \left(2\lambda\tau(0,u) - \tau(\lambda, w)\tau(0,u)\right)\tau(0,u)^{-1} = 2\lambda I - \tau(\lambda, w) \\
&\quad = \tau(2\lambda, 0) - \tau(\lambda, w) = \tau(\lambda, -w).
\end{aligned}$$

Hence $\det \tau(\lambda, w) = \det \tau(\lambda, -w)$ and $\operatorname{tr} \tau(\lambda, w) = \operatorname{tr} \tau(\lambda, -w)$. Then, by (12), $(\det \tau(\lambda, w))^2 = (\lambda^2 - \phi(w,w))^N$ and $\operatorname{tr} \tau(\lambda, w) = \operatorname{tr} \frac{1}{2}(\tau(\lambda, w) + \tau(\lambda, -w)) = \operatorname{tr} \tau(\lambda, 0) = \operatorname{tr} \lambda I = \lambda N$.

(16) follows from (10) and (15). □

Let a hypothesis be given by

$$\Sigma = \sigma_0 I + \sum_{i=1}^{m} \sigma_i G_i,$$

where $G_1, \dots, G_m \in L_s(V)$ are known and $\sigma_0, \dots, \sigma_m$ are unknown real parameters [cf. (2)]. Suppose without restriction that $\operatorname{tr} G_i = 0$, $i = 1, \dots, m$. Then it follows from (10) and (15) that the hypothesis is parametrized by a simple Jordan algebra of degree 2 if and only if

$$G_i G_j + G_j G_i = (2/N)\left(\operatorname{tr} G_i G_j\right)I, \qquad i, j = 1, \dots, m.$$

In this case, $W = \mathbb{R}^m$, $\lambda = \sigma_0$, $w = (\sigma_1, \dots, \sigma_m)$ and the matrix of the inner product ϕ is $((1/N)\operatorname{tr} G_i G_j)_{i,j}$.

Lemma 3. *If x is an observation on X, the likelihood equations for $(\lambda, w) \in K$ are*

$$N\lambda = (x, x), \tag{17}$$

$$\forall\, u \in W\colon N\phi(u, w) = (\tau(0, u)x, x). \tag{18}$$

Let $(y, z) = (t_1(x), t_2(x)) \in \mathbb{R} \times W$ denote the solution to (17) *and* (18). *Then $(y, z) \in \overline{K} = \{(\lambda, w) \in \mathbb{R} \times W | \lambda \geq 0,\ \lambda \geq \phi(w, w)\}$ and the density* (11) *can be written*

$$(2\pi)^{-N/2}(\lambda^2 - \phi(w, w))^{-N/4}\exp\left(-\frac{N(\lambda y - \phi(w, z))}{2(\lambda^2 - \phi(w, w))}\right). \tag{19}$$

Proof. It follows from (4) and (12) that the likelihood equation is

$$\forall\, (c, u) \in \mathbb{R} \times W\colon \operatorname{tr} \tau(c, u)\tau(\lambda, u) = \operatorname{tr} \tau(c, u)(xx'). \tag{20}$$

Now $\operatorname{tr} \tau(c, u)(xx') = (\tau(c, u)x, x) = (\tau(c, 0)x, x) + (\tau(0, u)x, x) = c(x, x) + (\tau(0, u)x, x)$ and, by (16), $\operatorname{tr} \tau(c, u)\tau(\lambda, w) = Nc\lambda + N\phi(u, w)$. Hence (20) is equivalent to (17) and (18). Moreover,

$$N(cy + \phi(u, z)) = (\tau(c, u)x, x). \tag{21}$$

It follows from (13) that $\tau(c, u)$ is positive semidefinite if $(c, u) \in \overline{K}$. With $c = \phi(z, z)^{1/2}$ and $u = -z$ we have $N(\phi(z, z)^{1/2}y - \phi(z, z)) \geq 0$, i.e., $y \geq 0$ and $y^2 \geq \phi(z, z)$. Finally, (19) follows from (11), (12), (14) and (21). □

Remark. According to the general results in Section 2 the statistic $(Y, Z) = (t_1(X), t_2(X))$ is complete, sufficient and unbiased for (λ, w), and (Y, Z) is the maximum likelihood estimate of (λ, w) if $(Y, Z) \in K$.

We shall now introduce representations of a subgroup of the so-called Clifford group on the sample space V and on the parameter space W, respectively. The representations (or group actions) are used to find the distribution of (Y, Z). The theory of the Clifford group is well known and the reader is referred to Bourbaki [(1959), Section 9] or Chevalley [(1954), Chapter 2].

Let $C(W, \phi)$ denote the Clifford algebra of W and ϕ. It follows from Jacobson [(1968), page 75] that there exists a 1–1 Jordan algebra homomorphism σ: $\mathbb{R} \times W \to C(W, \phi)^+$ and an algebra homomorphism η: $C(W, \phi) \to L(V)$ such that

$$\forall\, (\lambda, w) \in \mathbb{R} \times W\colon \tau(\lambda, w) = \eta(\sigma(\lambda, w)). \tag{22}$$

Especially, we shall use that

$$\forall\, \lambda \in \mathbb{R}\colon \sigma(\lambda, 0) = \lambda 1, \tag{23}$$

$$\forall\, w_1, w_2 \in W\colon \tfrac{1}{2}(\sigma(0, w_1)\sigma(0, w_2) + \sigma(0, w_2)\sigma(0, w_1)) = \phi(w_1, w_2)1, \tag{24}$$

where 1 denotes the identity element in the Clifford algebra. The so-called main involution in the Clifford algebra is an antiautomorphism $s \to s^*$, $s \in C(W, \phi)$, characterized by $\sigma(\lambda, w)^* = \sigma(\lambda, w)$, $(\lambda, w) \in \mathbb{R} \times W$. Since $\eta(\sigma(\lambda, w)^*) =$

$\eta(\sigma(\lambda, w)) = \tau(\lambda, w) = \tau(\lambda, w)' = \eta(\sigma(\lambda, w))'$, it follows that

$$\forall\, s \in C(W, \phi): \eta(s^*) = \eta(s)'. \tag{25}$$

By means of the 1–1 linear mapping $w \to \sigma(0, w)$ of W into $C(W, \phi)$, we shall in the following consider W as a subspace of $C(W, \phi)$, i.e., we shall write w instead of $\sigma(0, w)$ and W instead of $\{\sigma(0, w) | w \in W\}$. Set

$$\Gamma = \{s \in C(W, \phi) | s^* = s^{-1}, \forall\, w \in W: sws^{-1} \in W\}. \tag{26}$$

It is clear that Γ is a group. If $s \in \Gamma$ the linear mapping $w \to sws^{-1}$ of W into itself is denoted $\chi(s)$. The group of all orthogonal linear mappings of W into itself is called the orthogonal group of ϕ and is denoted $O(\phi)$. The subgroup of all orthogonal linear mappings with determinant 1 is called the special orthogonal group of ϕ and is denoted $SO(\phi)$.

LEMMA 4. *The linear mappings $\chi(s)$, $s \in \Gamma$, are orthogonal, the mapping $\chi\colon \Gamma \to O(\phi)$ is a group homomorphism and $\chi(\Gamma) \supseteq SO(\phi)$.*

PROOF. Let $s \in \Gamma$ and $w \in W$. It follows from (24) that

$$\begin{aligned}\phi(\chi(s)(w), \chi(s)(w))1 &= \chi(s)(w)\chi(s)(w) = sws^{-1}sws^{-1} = sw^2s^{-1}\\ &= s(\phi(w, w)1)s^{-1} = \phi(w, w)1.\end{aligned}$$

Hence $\chi(s)$ is orthogonal. It is clear that χ is a group homomorphism. Let $s \in W$ with $\phi(s, s) = 1$. Then it follows from (24) that $s^{-1} = s = s^*$ and, if $w \in W$, $sws^{-1} = (2\phi(s, w)1 - ws)s^{-1} = 2\phi(s, w)s - w \in W$, i.e., $s \in \Gamma$ and $\chi(s) = -\rho(s)$, where $\rho(s)$ denotes the reflection through the hyperplane orthogonal to s. Now it is well known that any orthogonal linear mapping can be written as a product of reflections. If $\pi \in SO(\phi)$ there exists, therefore, unit vectors $s_1, \dots, s_k \in W$ such that $\pi = \rho(s_1) \cdots \rho(s_k) = (-1)^k\chi(s_1) \cdots \chi(s_k) = (-1)^k\chi(s_1 \cdots s_k)$. Since $\det \pi = 1$ and $\det \rho(s_i) = -1$, $i = 1, \dots, k$, k must be even and $\pi = \chi(s_1 \cdots s_k)$. Thus $\pi \in \chi(\Gamma)$. □

The Clifford group of $O(\phi)$ is the group of invertible elements s of $C(W, \phi)$ such that $sws^{-1} \in W$ for every $w \in W$. Thus Γ is a subgroup of the Clifford group, and Lemma 4 is only a simple special case of Chevalley [(1954), Theorem II.3.1].

Let $O(V)$ denote the group of all orthogonal linear mappings of V into itself. If $s \in \Gamma$ then it follows from (25) and (26) that $\eta(s)' = \eta(s^*) = \eta(s^{-1}) = \eta(s)^{-1}$, i.e., $\eta(s) \in O(V)$. Moreover, the restriction $\eta\colon \Gamma \to O(V)$ is a group homomorphism. We shall see in Section 7 that it is induced by the classical spin representations of Γ. The representation $\chi\colon \Gamma \to O(\phi)$ is called the vector representation of Γ. The group Γ acts on the sample space V by $(s, x) \to \eta(s)(x)$ and on the parameter space $\mathbb{R} \times W$ by $(s, (\lambda, w)) \to (\lambda, \chi(s)(w))$. The hypothesis has the following important transformation property.

LEMMA 5. *Let $c \in \mathbb{R}$ and $s \in \Gamma$. If X has a normal distribution with covariance $\tau(\lambda, w)$, $(\lambda, w) \in K$, then cX has a normal distribution with covari-*

ance $\tau(c^2\lambda, c^2w)$ *and* $\eta(s)(X)$ *has a normal distribution with covariance* $\tau(\lambda, \chi(s)(w))$.

PROOF. The covariance of cX is $c^2\tau(\lambda, w) = \tau(c^2\lambda, c^2w)$. The covariance of $\eta(s)(X)$ is $\eta(s)\tau(\lambda, w)\eta(s)'$. By using (22), (25), (26) and the definition of χ we have

$$\begin{aligned}\eta(s)\tau(\lambda, w)\eta(s)' &= \eta(s)\eta(\sigma(\lambda, w))\eta(s^*) = \eta(s)\eta(\sigma(\lambda,0) + \sigma(0, w))\eta(s^{-1})\\ &= \eta\big(s\sigma(\lambda,0)s^{-1} + s\sigma(0, w)s^{-1}\big)\\ &= \eta(\sigma(\lambda,0) + \sigma(0, \chi(s)(w)))\\ &= \eta(\sigma(\lambda, \chi(s)(w))) = \tau(\lambda, \chi(s)(w)). \qquad \square\end{aligned}$$

Similarly, the solution $(y, z) = (t_1(x), t_2(x))$, $x \in V$, to the likelihood equations (17) and (18) has the following properties.

LEMMA 6. *Let* $c \in \mathbb{R}$ *and* $s \in \Gamma$. *Then*

$$t_1(cx) = c^2t_1(x), \qquad t_2(cx) = c^2t_2(x), \tag{27}$$

$$t_1(\eta(s)(x)) = t_1(x), \qquad t_2(\eta(s)(x)) = \chi(s)(t_2(x)) \quad \textit{for } x \in V. \tag{28}$$

PROOF. It is clear that (27) holds. Since $\eta(s)$ is orthogonal it follows that

$$t_1(\eta(s)(x)) = 1/N(\eta(s)(x), \eta(s)(x)) = 1/N(x, x) = t_1(x).$$

Let $u \in W$. By (18) we have that

$$\begin{aligned}N\phi(u, t_2(\eta(s)(x))) &= (\tau(0, u)\eta(s)(x), \eta(s)(x))\\ &= (\eta(s)'\tau(0, u)\eta(s)(x), x) \quad \text{(as in the proof of Lemma 5)}\\ &= \big(\tau(0, \chi(s^{-1})(u))(x), x\big)\\ &= N\phi\big(\chi(s^{-1})(u), t_2(x)\big) \quad \big[\text{since } \chi(s^{-1}) = \chi(s)^{-1} = \chi(s)'\big]\\ &= N\phi(u, \chi(s)(t_2(x))).\end{aligned}$$

Hence $t_2(\eta(s)(x)) = \chi(s)(t_2(x))$. $\square$

THEOREM 4. *If* $N/2 > m - 1$ *then* (Y, Z) *is the maximum likelihood estimate of* $(\lambda, w) \in K$ *with probability* 1 *and the distribution of* (Y, Z) *has density*

$$\frac{\big(y^2 - \phi(z, z)\big)^{N/4 - m/2 - 1/2}}{k(m, N)\big(\lambda^2 - \phi(w, w)\big)^{N/4}}\exp\left(-\frac{N(\lambda y - \phi(w, z))}{2(\lambda^2 - \phi(w, w))}\right), \quad (y, z) \in K, \tag{29}$$

w.r.t. the restriction of the Lebesgue measure to K, *where*

$$k(m, N) = \pi^{(m-1)/2}2^{N-1}N^{-N/2}\Gamma(N/4)\Gamma(N/4 - m/2 + 1/2).$$

If $N/2 = m - 1$ *then* $Y^2 = \phi(Z, Z)$ *and the maximum likelihood estimate never exists. The case* $N/2 < m - 1$ *cannot occur.*

Remark. We shall see in Theorem 6 that N has to be divisible by $\alpha(m)$, where $\alpha(m)$ is given in Table 1. Hence $N/2 = m - 1$ if and only if $(m, N) = (2,2)$, $(3,4)$, $(5,8)$ or $(9,16)$.

Proof. We shall first find the distribution of (Y, Z) when $w = 0$. Then the covariance of X is $\tau(\lambda, 0) = \lambda I$, $\lambda > 0$. The distribution of $Y = 1/N(X, X)$ is a χ^2-distribution with N degrees of freedom and scale parameter λ/N. Set $Z_1 = Z/Y$. It follows from (27) that the distribution of Z_1 does not depend on λ. Since Y is complete for λ we have that Y and Z_1 are independently distributed. It follows from Lemma 5 that the distribution of X is invariant under the transformations $\eta(s)$, $s \in \Gamma$. Hence it follows by (28) that the distribution of Z_1 is invariant under the transformations $\chi(s)$, $s \in \Gamma$. By Lemma 4 we have, therefore, that the distribution of Z_1 is invariant under all orthogonal transformations with determinant 1. Set $T = \phi(Z_1, Z_1)^{1/2} = \phi(Z, Z)^{1/2}/Y$ and $U = Z_1/T = Z/(YT)$. Then T and U are independently distributed and U is uniformly distributed on the unit sphere. Since $Y^2 \geq \phi(Z, Z)$ we have that $0 \leq T \leq 1$. The maximum likelihood estimate exists if and only if $T < 1$.

To find the distribution of T, let $u \in W$ with $\phi(u, u) = 1$ and set $Z_2 = \phi(UT, u) = \phi(Z_1, u) = \phi(Z, u)/Y$. It follows from (17) and (18) that $Z_2 = (\tau(0, u)X, X)/(X, X)$. Hence

$$\begin{aligned}(1 - Z_2)/2 &= (1/2(I - \tau(0, u))X, X)/(X, X)\\ &= (\tau(1/2, -u/2)X, X)/(X, X).\end{aligned}$$

By (10) we have that $\tau(1/2, -u/2)^2 = \tau(1/2, -u/2)$. Hence $\tau(1/2, -u/2)$ is an orthogonal projection onto a subspace of V of dimension $\operatorname{tr} \tau(1/2, -u/2) = N/2$ [cf. (15)]. We have, therefore, that $(1 - Z_2)/2$ has a beta distribution with $(N/2, N/2)$ degrees of freedom. By a simple transformation it is seen that Z_2^2 has a beta distribution with $(1, N/2)$ degrees of freedom. Since U is uniformly distributed on the unit sphere, we have that $\phi(U, u)^2$ has a beta distribution with $(1, m - 1)$ degrees of freedom. Let $a > 0$. From $EZ_2^{2a} = E\phi(UT, u)^{2a} = E(T^{2a}E(\phi(U, u)^{2a}|T)) = ET^{2a}E(\phi(U, u)^{2a}$ we obtain

$$ET^{2a} = \frac{\Gamma(N/4 + 1/2)\Gamma(m/2 + a)}{\Gamma(N/4 + 1/2 + a)\Gamma(m/2)}. \tag{30}$$

Since $ET^2 \leq 1$ it follows that $N/2 \geq m - 1$. If $N/2 = m - 1$ then $ET^{2a} = 1$, $a > 0$, and $T = 1$ with probability 1. Since T depends continuously on X and the support of X is V, it follows that $T = 1$ in this case.

Assume that $N/2 > m - 1$. Then it follows from (30) that T^2 has a beta distribution with $(m, N/2 - m + 1)$ degrees of freedom. Hence the distribution of T has density

$$c(m, N)t^{m-1}(1 - t^2)^{N/4 - m/2 - 1/2}, \qquad 0 \leq t \leq 1,$$

where $c(m, N) = 2/B(m/2, N/4 - m/2 + 1/2)$. The distribution of $Z_1 = TU$ therefore has density

$$(c(m, N)/mK(m))(1 - \phi(z_1, z_1))^{N/4 - m/2 - 1/2}, \qquad \phi(z_1, z_1) \leq 1,$$

where $K(m) = \pi^{m/2}/\Gamma((m+2)/2)$ is the Lebesgue measure of the set $\{z_1 \in W | \phi(z_1, z_1) \le 1\}$. Since Y has a χ^2-distribution with N degrees of freedom and scale parameter λ/N, it is seen by a simple transformation that the distribution of $(Y, Z) = (Y, YZ_1)$ has a density which is given by (29) with $w = 0$.

Now for an arbitrary parameter $w \in W$, $\lambda^2 > \phi(w, w)$. Let $f(y, z, \lambda, w)$ and $g(y, z, \lambda, w)$ denote (19) and (29), respectively. By (19), the distribution of X has a density which depends only on (y, z). Hence the distribution of (Y, Z) has density $g(y, z, \lambda, 0) f(y, z, \lambda, w)/f(y, z, \lambda, 0) = g(y, z, \lambda, w)$. □

The convex cone $K = \{(\lambda, w) \in \mathbb{R} \times W | \lambda > 0,\ \lambda^2 > \phi(w, w)\}$ is called a spherical cone. The next theorem shows that the statistical analysis of the distributions (29) on K is particularly simple. Thus the distributions may also be useful in analysing directional data.

It is possible to extend the actions of Γ to a larger group such that the action on K becomes transitive. In fact, we could just take the group generated by $\{\sigma(\lambda, w) | \lambda^2 \ne \phi(w, w)\}$. Since

$$\tau(\lambda, w)\tau(y, z)\tau(\lambda, w) = \tau\big(\lambda^2 y + 2\lambda\phi(z, w) + y\phi(w, w), (2\lambda y + 2\phi(w, z))w + (\lambda^2 - \phi(w, w))z\big), \qquad (y, z) \in K,$$

it is straightforward to generalize Lemmas 5 and 6. Hence the distributions (29) could be found by using the uniqueness of an invariant measure on K. This would, however, not simplify the proof of Theorem 6. In this connection it shall be mentioned that the so-called irreducible self-dual homogeneous convex cones are either the set of positive real numbers, the spherical cones or the cones of positive definite matrices given in the remark to Theorem 3; see, e.g., Hertneck (1962) or Vinberg (1963).

Let W_0 be a subspace of W of dimension k, $0 < k < m$. It follows from (6) that $\mathbb{R} \times W_0$ is a subalgebra of the Jordan algebra $\mathbb{R} \times W$. We shall consider the hypothesis H_0: $w \in W_0$. Let q denote the orthogonal projection onto W_0 and set $K_0 = \{(\lambda, w) \in \mathbb{R} \times W_0 | \lambda > 0,\ \lambda^2 > \phi(w, w)\}$.

Theorem 5. *Suppose $N/2 > m - 1$. Then the maximum likelihood estimate of (λ, w) under H_0 is $(Y, q(Z))$ and the likelihood ratio test statistic for testing H_0 is*

$$Q = \left(\frac{Y^2 - \phi(Z, Z)}{Y^2 - \phi(q(Z), q(Z))} \right)^{N/4}. \tag{31}$$

Under the hypothesis H_0 the statistics $(Y, q(Z))$ and Q are independently distributed. The distribution of $(Y, q(X))$ is given by (29) with W, m and K replaced by W_0, k and K_0, respectively, and $Q^{4/N}$ has a beta distribution with $(N/2 - m + 1, m - k)$ degrees of freedom.

Proof. Under the hypothesis H_0 the likelihood equations for $(\lambda, w) \in K_0$ are given by (17) and (18) with W replaced by W_0. Let $u \in W_0$. Then $(\tau(0, u)X, X) = N\phi(u, Z) = N\phi(u, q(Z))$. Hence $(Y, q(Z))$ is the maximum likelihood estimate of (λ, w). It follows from (19) that the density (11) can be written

$$(2\pi)^{-N/2}(\lambda^2 - \phi(w,w))^{-N/4}\exp\left(-\frac{N(\lambda y - \phi(w, q(z)))}{2(\lambda^2 - \phi(w,w))}\right)$$

and that Q is given by (31). Let Y, T and U be as in the proof of Theorem 4. Set $T_1 = \phi(q(U), q(U))^{1/2}$ and $U_1 = q(U)/T_1$. Then $q(Z) = q(YTU) = YTq(U) = YTT_1U_1$ and $Q^{4/N} = (1 - T^2)/(1 - T^2T_1^2)$. We shall first find the distributions when $w = 0$. Then Y, T and U are independently distributed, Y has a χ^2-distribution with N degrees of freedom, T^2 has a beta distribution with $(m, N/2 - m + 1)$ degrees of freedom and U is uniformly distributed on the unit sphere in W. By a well known result we have that U_1 and T_1 are independently distributed, that U_1 is uniformly distributed on the unit sphere in W_0 and that T_1^2 has a beta distribution with $(k, m - k)$ degrees of freedom. By a simple transformation it is seen that $T^2T_1^2$ and $(1 - T^2)/(1 - T^2T_1^2)$ are independent variables which have beta distributions with $(k, N/2 - k + 1)$ and $(N/2 - m + 1, m - k)$ degrees of freedom, respectively. Hence $Q^{4/N}$ and $(Y, q(Z)) = (Y, YTT_1U_1)$ are independently distributed, and the distribution of $(Y, q(Z))$ is found as in the proof of Theorem 4. Since the density depends only on $(y, q(z))$, the results for an arbitrary parameter $w_0 \in W_0$, $\lambda^2 > \phi(w_0, w_0))$ follows as in the proof of Theorem 4. □

REMARK. The likelihood ratio test statistic for testing the hypothesis that $w = 0$ is $Q = (1 - T^2)^{N/4}$ and $1 - T^2$ has a beta distribution with $(N/2 - m + 1, m)$ degrees of freedom.

7. A canonical form of a hypothesis which is parametrized by a simple Jordan algebra of degree 2. We shall still consider the hypothesis given at the beginning of Section 6. The purpose of this section is to give a concrete representation of the mapping τ. It follows from (22) that $\tau = \eta\sigma$, where σ: $\mathbb{R} \times W \to C(W, \phi)^+$ is a Jordan algebra homomorphism and η: $C(W, \phi) \to L(V)$ is an algebra homomorphism. The Clifford algebra is determined only up to an algebra isomorphism. It follows from Chevalley [(1954), page 66] that a concrete representation of $C(W, \phi)$ is given by Table 1. We shall say we are in case (b) if $m = 5, 9, 13, 17, \ldots$ and in case (a) otherwise. Thus we can suppose that σ is represented by a Jordan algebra homomorphism

$$\sigma_{0,m}\colon \mathbb{R} \times W \to M_r(D) \tag{32}$$

in case (a) and a Jordan algebra homomorphism

$$\sigma_m = (\sigma_{1,m}\sigma_{2,m})\colon \mathbb{R} \times W \to M_r(D) \times M_r(D) \tag{33}$$

in case (b), where r and D are given in Table 1. In case (b) it is clear that the mappings $\sigma_{i,m}$: $\mathbb{R} \times W \to M_r(D)$, $i = 1, 2$, are also Jordan algebra homomor-

TABLE 1

m	$C(W,\phi)$	r	D	$\alpha(m) = r \dim D$
$8a+1$	$M_r(\mathbb{R}) \times M_r(\mathbb{R})$	2^{4a}	$\mathbb{R}$	2^{4a}
$8a+2$	$M_r(\mathbb{R})$	2^{4a+1}	$\mathbb{R}$	2^{4a+1}
$8a+3$	$M_r(\mathbb{C})$	2^{4a+1}	$\mathbb{C}$	2^{4a+2}
$8a+4$	$M_r(\mathbb{H})$	2^{4a+1}	$\mathbb{H}$	2^{4a+3}
$8a+5$	$M_r(\mathbb{H}) \times M_r(\mathbb{H})$	2^{4a+1}	$\mathbb{H}$	2^{4a+3}
$8a+6$	$M_r(\mathbb{H})$	2^{4a+2}	$\mathbb{H}$	2^{4a+4}
$8a+7$	$M_r(\mathbb{C})$	2^{4a+3}	$\mathbb{C}$	2^{4a+4}
$8a+8$	$M_r(\mathbb{R})$	2^{4a+4}	$\mathbb{R}$	2^{4a+4}

phisms. Since the elements $\sigma(\lambda, w)$, $(\lambda, w) \in \mathbb{R} \times W$, generate the Clifford algebra, it follows that $\sigma_{1,m}$ and $\sigma_{2,m}$ are different.

LEMMA 7. *Let σ: $\mathbb{R} \times W \to M_r(D)$ be a Jordan algebra homomorphism, $D = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and $r = 2, 3, 4 \dots$. Then there exists an algebra isomorphism ρ of $M_r(D)$ such that $\rho(\sigma(\lambda, w))$ is Hermitian for any $(\lambda, w) \in \mathbb{R} \times W$.*

PROOF. Let $e_1, \dots, e_m$ be an orthonormal basis of W and set $G_i = \operatorname{re} \sigma(0, e_i)$, $i = 1, \dots, m$. Since σ is a Jordan algebra homomorphism, we have that $G_i^2 = \operatorname{re} \sigma(0, e_i)^2 = \operatorname{re} \sigma(\phi(e_i, e_i), 0) = I$ and that $G_iG_j + G_jG_i = \operatorname{re} 2\sigma(\phi(e_i, e_j), 0) = 0$ for $i \neq j$. Hence we have that the group $\mathscr{G}$ generated by the invertible matrices $I, G_1, \dots, G_m$ consists of the 2^{m+1} matrices $\pm I$ and $\pm G_{i_1} \cdots G_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq m$, $k = 1, \dots, m$. Set $B = \Sigma G'G$, where the summation is over the group $\mathscr{G}$. Then B is a positive definite matrix, which is the real matrix of an element in $M_p(D)$. Since $G_i^{-1} = G_i$ we have that $BG_i = \Sigma G'(GG_i) = \Sigma(GG_i^{-1})'G = \Sigma(GG_i)'G = \Sigma G_i'G'G = G_i'B$. Hence $G_i' = BG_iB^{-1}$. Let $F = B^{1/2}$, i.e., F is the uniquely determined symmetric matrix such that $F^2 = B$. It can be shown that F is the real matrix of an element in $M_r(D)$. Then $F(\operatorname{re} A)F^{-1}$ is also the real matrix of an element in $M_r(D)$. Hence there exists a linear mapping ρ of $M_r(D)$ onto itself such that $\operatorname{re} \rho(A) = F(\operatorname{re} A)F^{-1}$, $A \in M_r(D)$. It is clear that ρ is an algebra isomorphism. Moreover, $(\operatorname{re} \rho(\sigma(0, e_i)))' = (FG_iF^{-1})' = F^{-1}G_i'F = F^{-1}BG_iB^{-1}F = F^{-1}F^2G_iF^{-2}F = FG_iF^{-1} = \operatorname{re} \rho(\sigma(0, e_i))$, $i = 1, \dots, m$. Since $e_1, \dots, e_m$ is a basis of W, it follows that $\operatorname{re} \rho(\sigma(\lambda, w))$ is symmetric for any $(\lambda, w) \in \mathbb{R} \times W$. □

Since the Clifford algebra is determined only up to an algebra isomorphism, it follows from Lemma 7 that we can suppose that the Jordan algebra homomorphisms $\sigma_{i,m}$, $i = 0, 1, 2$, given by (32) and (33) map $\mathbb{R} \times W$ into $H_r(D)$. Then σ is represented by a Jordan algebra homomorphism

$$\sigma_{0,m}: \mathbb{R} \times W \to H_r(D) \tag{34}$$

in case (a) and a Jordan algebra homomorphism

$$(\sigma_{1,m}, \sigma_{2,m}): \mathbb{R} \times W \to H_r(D) \times H_r(D) \tag{35}$$

in case (b), where D and r are given in Table 1. This shows that there does exist at least one statistical model for any simple Jordan algebra of degree 2.

THEOREM 6. *Let* $\tau: \mathbb{R} \times W \to L_s(V)$ *be a Jordan algebra homomorphism and let* $\sigma: \mathbb{R} \times W \to C(W, \phi)$ *be represented by* (34) *in case* (*a*) *and by* (35) *in case* (*b*). *Then there exists an orthonormal basis of* V *such that the matrix of* $\tau(\lambda, w)$ *is*

$$\operatorname{re} \sigma_{0,m}(\lambda, w) \otimes I_n = \begin{pmatrix} \operatorname{re} \sigma_{0,m}(\lambda, w) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \operatorname{re} \sigma_{0,m}(\lambda, w) \end{pmatrix} \tag{36}$$

in case (*a*) *and*

$$\begin{pmatrix} \operatorname{re} \sigma_{1,m}(\lambda, w) \otimes I_{n_1} & 0 \\ 0 & \operatorname{re} \sigma_{2,m}(\lambda, w) \otimes I_{n_2} \end{pmatrix} \tag{37}$$

in case (*b*), $(\lambda, w) \in \mathbb{R} \times W$, *where* $n = N/\alpha(m)$ *and* $n_1 + n_2 = N/\alpha(m)$, *respectively, and* 0 *here denotes a matrix of zeros.*

PROOF. The proof is analogous to the proof of Theorem 3 except for the following changes: In case (b) we obtain an algebra homomorphism $\rho_i: M_r(D) \times M_r(D) \to L(V_i)$ and the matrix of $\rho_i(A_1, A_2)$ is either $\operatorname{re} A_1$ or $\operatorname{re} A_2$, $A_1, A_2 \in M_r(D)$. For the matrix S we obtain $(\operatorname{re} \sigma_{i,m}(\lambda, w))S = S(\operatorname{re} \sigma_{i,m}(\lambda, w))$, $(\lambda, w) \in \mathbb{R} \times W$, where i is either 0, 1 or 2. Since the elements $\sigma(\lambda, w)$, $(\lambda, w) \in \mathbb{R} \times W$, generate $C(W, \phi)$ it follows that $(\operatorname{re} A)S = S(\operatorname{re} A)$, $A \in M_r(D)$. □

It can be seen that the matrices $\eta(s)$, $s \in C(W, \phi)$, have the form $\rho_0(s) \otimes I_n$ in case (a) and the form

$$\begin{pmatrix} \rho_1(s) \otimes I_{n_1} & 0 \\ 0 & \rho_2(s) \otimes I_{n_2} \end{pmatrix}$$

in case (b). The restrictions of the ρ's to the group Γ are called the spin representations of Γ. Hence the group homomorphism $\eta: \Gamma \to O(V)$ is said to be induced by the spin representations.

The representations (34) and (35) are not uniquely determined. Representations for small values of m can be found by induction by using the method in the proof of Theorem II.2.5 in Chevalley (1954). Since the correctness of a representation can easily be checked by the conditions that $\operatorname{re} \sigma_{i,m}(\lambda, 0) = \lambda I$ and $(\operatorname{re} \sigma_{i,m}(0, w))^2 = \phi(w, w)I$, we shall only give the results. For $W = \mathbb{R}^m$ and ϕ the usual inner product on $\mathbb{R}^m$ (i.e., for an orthonormal basis of W), we have found the following representations for $2 \leq m \leq 9$ [i.e., $\alpha(m) \leq 16$]:

$$\operatorname{re} \sigma_{0,2}(\lambda, (w_1, w_2)) = \begin{pmatrix} \lambda + w_1 & w_2 \\ w_2 & \lambda - w_1 \end{pmatrix}$$

or

$$\operatorname{re} \sigma_{0,2}((a+b)/2, ((a-b)/2, c)) = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \tag{38}$$

$$\operatorname{re} \sigma_{0,3}((a+b)/2, ((a-b)/2, c, f)) = \begin{pmatrix} a & c & 0 & f \\ c & b & -f & 0 \\ 0 & -f & a & c \\ f & 0 & c & b \end{pmatrix}, \tag{39}$$

$$\operatorname{re} \sigma_{1,5}((a+b)/2, ((a-b)/2, c, f, g, h))$$
$$= \begin{pmatrix} a & c & 0 & f & 0 & g & 0 & h \\ c & b & -f & 0 & -g & 0 & -h & 0 \\ 0 & -f & a & c & 0 & h & 0 & -g \\ f & 0 & c & b & -h & 0 & g & 0 \\ 0 & -g & 0 & -h & a & c & 0 & f \\ g & 0 & h & 0 & c & b & -f & 0 \\ 0 & -h & 0 & g & 0 & -f & a & c \\ h & 0 & -g & 0 & f & 0 & c & b \end{pmatrix}, \tag{40}$$

$$\sigma_{2,5}(\lambda, (w_1, w_2, w_3, w_4, w_5)) = \sigma_{1,5}(\lambda, (w_1, w_2, w_3, w_4, -w_5)),$$

$$\sigma_{0,4}(\lambda, (w_1, w_2, w_3, w_4)) = \sigma_{1,5}(\lambda, (w_1, w_2, w_3, w_4, 0)),$$

$$\operatorname{re} \sigma_{1,9}((a+b)/2, ((a-b)/2, c, d, f, i, g, j, h, k))$$
$$= \begin{pmatrix} A_1 & B & C & D \\ B' & A_2 & -D & C \\ C' & D & A_2 & -B \\ -D & C' & -B' & A_1 \end{pmatrix}, \tag{41}$$

where

$$A_1 = \begin{pmatrix} a & c & 0 & d \\ c & b & -d & 0 \\ 0 & -d & a & c \\ d & 0 & c & b \end{pmatrix}, \quad A_2 = \begin{pmatrix} a & c & 0 & -d \\ c & b & d & 0 \\ 0 & d & a & c \\ -d & 0 & c & b \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -f & 0 & i \\ f & 0 & -i & 0 \\ 0 & -i & 0 & -f \\ i & 0 & f & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -g & 0 & j \\ g & 0 & -j & 0 \\ 0 & -j & 0 & -g \\ j & 0 & g & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 0 & h & 0 & k \\ -h & 0 & -k & 0 \\ 0 & k & 0 & -h \\ -k & 0 & h & 0 \end{pmatrix},$$

$$\sigma_{2,9}(\lambda(w_1,\dots,w_8,w_9)) = \sigma_{1,9}(\lambda,(w_1,\dots,w_8,-w_9)),$$
$$\sigma_{0,6}(\lambda,(w_1,\dots,w_6)) = \sigma_{1,9}(\lambda,(w_1,\dots,w_6,0,0,0)),$$
$$\sigma_{0,7}(\lambda,(w_1,\dots,w_7)) = \sigma_{1,9}(\lambda,(w_1,\dots,w_7,0,0)),$$
$$\sigma_{0,8}(\lambda,(w_1,\dots,w_8)) = \sigma_{1,9}(\lambda,(w_1,\dots,w_8,0)).$$

It is seen from (38), (39) and (40) that the simple Jordan algebras of degree 2 and dimensions 3, 4 and 6 are isomorphic to the Jordan algebras $H_2(D)$, $D = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, respectively. Furthermore, it follows from Theorem 6 that Theorem 3 holds also with $r = 2$ and $D = \mathbb{R}$ or $D = \mathbb{C}$. Since $\sigma_{1,5}$ and $\sigma_{2,5}$ are different it also follows that Theorem 3 is false with $r = 2$ and $D = \mathbb{H}$. By the linear mappings (38), (39) and (40) the distributions (29) with $m = 2$, 3 and 5 are transformed into the two-dimensional real, complex and quaternion Wishart distributions, respectively.

The statistical models considered in this section seem rather peculiar and we have not found a statistical interpretation for them. It shall be noticed, however, that Jordan algebras were first introduced in an attempt to formulate the foundations of quantum mechanics. Thus the given covariance matrices closely correspond to so-called Dirac matrices which are used to describe the spin of a particle. It is also seen from Lemma 5 that the distribution of the observation X transforms in a way which is similar to the way the so-called wave functions transform. (A wave function f: $W \to V$ transforms into the function $w \to \eta(s)f(\chi(s)^{-1}(w))$ under the orthogonal transformation $\chi(s)$; see Varadarajan [(1970), Chapter 12].)

8. Extension of a covariance hypothesis with linear structure. As we have seen in Section 2, the covariance hypotheses which are parametrized by Jordan algebras are the only covariance hypotheses with linear structure for which there exists a complete, sufficient and unbiased estimator. On the other hand, Theorems 2, 3 and 6 show that these hypotheses are very restrictive. For most covariance hypotheses with linear structure one has, therefore, the problem to choose between various estimators of the covariance. This problem can in some cases be reduced by considering an extension of the hypotheses.

Thus, let L be a linear subspace of $L_s(V)$ and suppose $I \in L$. Set $\Theta = \{\Sigma \in L | \Sigma \text{ is positive definite}\}$ and consider the hypothesis H: $\Sigma \in \Theta$. Let L_1 be the Jordan subalgebra of $L_s(V)$ generated by L and set $\Theta_1 = \{\Sigma \in L_1 | \Sigma \text{ is positive definite}\}$. Then the hypothesis H_1: $\Sigma \in L_1$ is the smallest hypothesis which includes H and which is linear in both the covariance and the inverse covariance. Hence one can first estimate Σ under the hypothesis H_1. This estimate is sufficient and unbiased. The problem is therefore reduced to estimating a parameter of dimension dim L from a statistic of dimension dim L_1.

The case where the elements of L commute has received considerable attention. In this case the Jordan algebra L_1 is isomorphic to a product of the simple Jordan algebras $\mathbb{R}$. Hence the statistical problem is equivalent to considering $\dim L_1$ independent random variables, which have Γ-distributions with known degrees of freedom, and to estimating the scale parameters under a linear hypothesis of dimension $\dim L$. In the simplest case where $\dim L = 2$ and $\dim L_1 = 3$ one has three random variables X_1, X_2, X_3 with scale parameters β_1, β_2, $\beta_1 + \beta_2$, $\beta_1 > 0$ and $\beta_2 > 0$, and it can be seen that the likelihood function may have two local maxima.

Let L_0 be the associative subalgebra of $L(V)$ generated by L. Set $L_2 = L_0 \cap L_s(V)$ and $\Theta_2 = \{\Sigma \in L_2 | \Sigma \text{ is positive definite}\}$. It is clear that L_2 is a Jordan subalgebra of $L_s(V)$ and that $L \subseteq L_2$. Hence H_2: $\Sigma \in L_2$ is also a hypothesis which includes H and which is linear in both the covariance and the inverse covariance. The Jordan algebra L_2 is isomorphic to a product $J_1 \times \cdots \times J_k$ of simple Jordan algebras, and the structure of the hypothesis H_2 is given by Theorems 2, 3 and 6. Theorem 6 can, however, be simplified in this case. If J_i is a simple Jordan algebra of degree 2, then it follows that J_i is of dimension 3, 4 or 6. [The mapping (34) is only an isomorphism if m is 2 or 3, and the mappings $\sigma_{1,m}$ and $\sigma_{2,m}$ given by (35) are only isomorphisms if $m = 5$.] Hence J_i is $H_2(D)$, where $D = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Moreover, if $J_i = H_2(\mathbb{H})$ it follows that (37) holds only with $n_1 = 0$ or $n_2 = 0$. [The mapping (35) with $m = 5$ is not an isomorphism.] We can say, therefore, that the structure of the hypothesis H_2 is given by Theorems 2 and 3 with $r \geq 2$.

If the hypothesis H is a model of an experimental design, then the algebra L_0 is the relationship algebra introduced by James (1957). Thus the role of the relationship algebra seems to be that it gives an extension of the hypothesis to a hypothesis which is linear in both the covariance and the inverse covariance.

Let $\mathscr{G}$ be a subgroup of the group $O(V)$ of all orthogonal linear mappings of V into itself. Andersson (1975) considers the hypothesis H_3: $\Sigma \in \Theta_3$, where $\Theta_3 = \{\Sigma \in L_3 | \Sigma \text{ is positive definite}\}$ and $L_3 = \{\Sigma \in L_s(V) \mid \forall\ G \in \mathscr{G}\colon G\Sigma G' = \Sigma\}$. The hypothesis H_3 is said to be given by invariance under a group action. It is clear that $L_3 = L_4 \cap L_s(V)$, where $L_4 = \{A \in L(V) | \forall\ G \in \mathscr{G}\colon GA = AG\}$ is an associative subalgebra of $L(V)$. Hence it follows from the preceding considerations that the structure theorem in Andersson (1975) is a special case of Theorems 2, 3 and 6, and that a hypothesis parametrized by a simple Jordan algebra of degree 2 cannot be given by invariance under a group action except in the cases where the Jordan algebra is $H_2(D)$, $D = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$.

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