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ASYMPTOTIC THEORY OF A TEST FOR THE CONSTANCY OF REGRESSION COEFFICIENTS AGAINST THE RANDOM WALK ALTERNATIVE

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The LBI (locally best invariant) test is suggested under normality for the constancy of regression coefficients against the alternative hypothesis that one component of the coefficients follows a random walk process. We discuss the limiting null behavior of the test statistic without assuming normality under two situations, where the initial value of the random walk process is known or unknown. The limiting distribution is that of a quadratic functional of Brownian motion and the characteristic function is obtained from the Fredholm determinant associated with a certain integral equation. The limiting distribution is then computed by numerical inversion of the characteristic function.

1. Introduction. In this paper we are concerned with the model

$$\begin{aligned} y_t &= x_t\beta_t + z_t'\gamma + \varepsilon_t, \\ \beta_t &= \beta_{t-1} + u_t, \qquad t = 1, 2, \ldots, \end{aligned} \tag{1.1}$$

where (i) $\{y_t\}$ is a sequence of scalar observations, whereas $\{x_t\}$ and $\{z_t\}$ are scalar and $p \times 1$ nonstochastic, fixed sequences, respectively; (ii) $\{\varepsilon_t\}$ and $\{u_t\}$ are independent of each other and are i.i.d. with $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma_\varepsilon^2 > 0$, $E(u_t) = 0$ and $E(u_t^2) = \sigma_u^2 \geq 0$; (iii) $\{\beta_t\}$ starts with β_0, which is assumed to be a known or unknown constant, whereas γ is a $p \times 1$ unknown constant vector. The above model (1.1) belongs to a class of the so-called state space, or Kalman filter, models developed in control engineering for representing a stochastic behavior of a dynamical system (see, e.g., Jazwinski [7]). The model (1.1) with $\sigma_u^2 > 0$ is also regarded as representing coefficient instability in time-series regression and is often referred to as a varying coefficient regression model (see, e.g., Nicholls and Pagan [10]), where one component β_t of the coefficient vector $(\beta_t, \gamma')'$ varies over time following a random walk process.

The problem we deal with here is to test if β_t really exhibits variation following (1.1). To be more specific we consider testing for the constancy of β_t, which is equivalent to testing the hypothesis $\sigma_u^2 = 0$. In Section 2 we suggest the LBI (locally best invariant) test for

$$H_0\colon \rho = \sigma_u^2/\sigma_\varepsilon^2 = 0, \quad \text{against } H_1\colon \rho > 0, \tag{1.2}$$

assuming normality on $\{y_t\}$.

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The remaining parts of this paper concentrate on the asymptotic null distribution of the test statistic without assuming normality. Due to the invariance principle the problem essentially reduces to finding the distribution of a quadratic functional of Brownian motion. In Section 3 we introduce a homogeneous integral equation of the second kind and consider the associated Fredholm determinant, which yields the corresponding characteristic function. Section 4 gives some examples of the asymptotic distribution by making the regressors x_t and z_t in (1.1) more specific. It is of interest to notice that two situations where the initial value β_0 is known or unknown give different results. The relevant references in this field are Anderson and Darling [1], Varberg [11], Kac, Kiefer and Wolfowitz [8] and de Wet and Venter [3]. The distribution function can be obtained by applying Lévy's inversion formula numerically and upper percent points are tabulated in Section 5. The finite-sample distribution of the test statistic is also examined there by simulations.

Concluding remarks are given in Section 6, where it is noted that computerized algebra has been useful in the present work.

2. Locally best invariant test. In this section we derive the LBI test for the constancy of β_t by assuming both ε_t and u_t to be normal. Noting that $y_t = x_t\beta_0 + z_t'\gamma + x_t(u_1 + \cdots + u_t) + \varepsilon_t$, the observation vector $y = (y_1, \dots, y_T)'$ has the distribution

$$y \sim N\big(x\beta_0 + Z\gamma, \sigma_\varepsilon^2(I_T + \rho D_x A_T D_x)\big), \tag{2.1}$$

where I_T is the $T \times T$ identity matrix, $x = (x_1, \dots, x_T)'$, $\gamma = (\gamma_1, \dots, \gamma_p)'$, $Z = (z_1, \dots, z_T)'$, $D_x = \operatorname{diag}(x_1, \dots, x_T)$ and

$$A_T = ((\min(s,t))) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & T \end{bmatrix}.$$

Here we assume that $\operatorname{rank}(x, Z) = p + 1 < T$, which is the identifiability condition for β_0 and γ. Then we consider the testing problem (1.2) and suggest the LBI test under two situations where β_0 is known or unknown.

Let us consider first the case for β_0 unknown. From (2.1) the testing problem (1.2) is seen to be invariant under the group of transformations: $y \to ya + xb + Zc$, $\beta_0 \to a\beta_0 + b$, $\gamma \to a\gamma + c$, $\sigma_\varepsilon^2 \to a^2\sigma_\varepsilon^2$ and $\rho \to \rho$, where $0 < a \in R^1$, $b \in R^1$ and $c \in R^p$. The subsequent discussion follows Kariya [9]. Choose a $T \times (T - p - 1)$ matrix H such that $H'H = I_{T-p-1}$ and $HH' = I_T - (x, Z)((x, Z)'(x, Z))^{-1}(x, Z)'$ and put $w = H'y$. Then we have

$$w \sim N\big(0, \sigma_\varepsilon^2(I_{T-p-1} + \Phi(\rho))\big), \tag{2.2}$$

where $\Phi(\rho) = \rho H' D_x A_T D_x H$, and the statistic $s(w) = w/\|w\|$ is a maximal invariant. Let $P_\rho(\cdot)$ be the distribution of $s(w)$ and put

$$f_T(s(w)|\rho) = |I_{T-p-1} + \Phi(\rho)|^{-1/2}\left[\frac{w'\big(I_{T-p-1} + \Phi(\rho)\big)w}{w'w}\right]^{-(T-p-1)/2}, \tag{2.3}$$

which is the probability density of $P_\rho(\cdot)$ with respect to $P_0(\cdot)$. The rejection region of the LBI test is now obtained as

$$\left.\frac{\partial \log f_T(s(w)|\rho)}{\partial \rho}\right|_{\rho=0} > \text{constant} \tag{2.4}$$

(see Ferguson [5], page 235), which yields

$$R_T = \frac{y'MD_xA_TD_xMy}{y'My} > \text{constant}, \tag{2.5}$$

where

$$M = I_T - (x, Z)((x, Z)'(x, Z))^{-1}(x, Z)'. \tag{2.6}$$

When β_0 is known and assumed to be zero without any loss of generality, the LBI test is shown to have the rejection region (2.5) with

$$M = I_T - Z(Z'Z)^{-1}Z'. \tag{2.7}$$

In the following sections we concentrate on the derivation of the asymptotic null distribution of R_T in (2.5) as $T \to \infty$.

3. Invariance principle and the Fredholm determinant. Under H_0 in (1.2) we have $R_T = \varepsilon'MD_xA_TD_xM\varepsilon/\varepsilon'M\varepsilon$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_T)'$ and the ε_t are i.i.d. with $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2) = \sigma_\varepsilon^2$. Since the null distribution of R_T does not depend on σ_ε^2, we put $\sigma_\varepsilon^2 = 1$. Then $\varepsilon'M\varepsilon/T$ converges in probability to 1. Therefore we consider the quadratic form in ε,

$$S_T = \varepsilon'MD_xA_TD_xM\varepsilon/c(T), \tag{3.1}$$

for some scaling factor $c(T)$. The following theorem gives the asymptotic distribution of S_T as $T \to \infty$. The theorem seems to be easier to apply to the present problem than the theorem of de Wet and Venter [3]. It is mainly due to the referee who has weakened the hypotheses of the authors' original theorem and kindly permitted the authors to use it in the present paper.

THEOREM 1. *Let $B_T = ((B_T(j,k)))$, $T = 1, 2, \dots,$ be $T \times T$ real symmetric matrices and assume that*

$$\lim_{T\to\infty} \max_{j,k} \left| B_T(j,k) - K\left(\frac{j}{T}, \frac{k}{T}\right)\right| = 0, \tag{3.2}$$

where $K(s,t)$ is a continuous and symmetric function on $[0,1] \times [0,1]$ and is positive definite in the sense

$$\int_0^1\int_0^1 K(s,t)f(s)f(t)\,ds\,dt \geq 0,$$

for all continuous functions $f(t)$ defined on $[0,1]$.

Let $D(\lambda)$ be the Fredholm determinant associated with the integral equation of the second kind,

$$f(t) = \lambda\int_0^1 K(s,t)f(s)\,ds. \tag{3.3}$$

Then the characteristic function of the limiting distribution of $\varepsilon' B_T \varepsilon / T$ is given by

$$\lim_{T\to\infty} E\left(e^{i\theta\varepsilon' B_T\varepsilon/T}\right) = \left(D(2i\theta)\right)^{-1/2}.$$

LEMMA 1. *Let $f(t)$ be a continuous real-valued function on $[0,1]$ such that $\int_0^1 (f(t))^2\,dt = \tau^2 > 0$. Then we have*

$$L\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{T} f\left(\frac{j}{T}\right)\varepsilon_j\right) \to N(0,\tau^2), \tag{3.4}$$

where $L(\cdot)$ denotes the probability law of $\cdot$.

PROOF. Since $f(t)$ is bounded for $t \in [0,1]$, the Lindeberg condition is satisfied. We have also

$$\operatorname{Var}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{T} f\left(\frac{j}{T}\right)\varepsilon_j\right) = \sum_{j=1}^{T}\frac{1}{T}\left(f\left(\frac{j}{T}\right)\right)^2 \to \int_0^1 (f(t))^2\,dt = \tau^2,$$

hence the conclusion (3.4) follows. □

LEMMA 2. *Let $f_1(t),\dots,f_n(t)$ be continuous real-valued functions on $[0,1]$ such that*

$$\int_0^1 f_k(t) f_l(t)\,dt = \delta_{kl} \qquad (\textit{Kronecker's delta}). \tag{3.5}$$

Then we have, for the limiting distribution of a random vector,

$$L\left(\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{T} f_l\left(\frac{j}{T}\right)\varepsilon_j\right)_{l=1,\dots,n}\right) \to N(0, I_n). \tag{3.6}$$

PROOF. Consider $f(t) = c_1 f_1(t) + \cdots + c_n f_n(t)$ for any real $c_1,\dots,c_n$ and apply Lemma 1. Then we have

$$L\left(\sum_{l=1}^{n} c_l \frac{1}{\sqrt{T}}\sum_{j=1}^{T} f_l\left(\frac{j}{T}\right)\varepsilon_j\right) \to N\left(0, \sum_{l=1}^{n} c_l^2\right),$$

in view of (3.5). Hence the conclusion (3.6) follows. □

LEMMA 3. *Let $H_T = ((H_T(j,k)))$ be a $T\times T$ real symmetric matrix such that*

$$|H_T(j,k)| \le \delta, \qquad j,k = 1,\dots,T.$$

Then we have

$$E\left(\left|\frac{1}{T}\varepsilon' H_T \varepsilon\right|\right) \le (1+\sqrt{2})\delta.$$

PROOF. Put

$$Q = \frac{1}{T}\varepsilon' H_T \varepsilon = \frac{1}{T}\sum_{j=1}^{T} H_T(j,j)\varepsilon_j^2 + \frac{2}{T}\sum_{1\le j<k\le T} H_T(j,k)\varepsilon_j\varepsilon_k$$
$$= Q_1 + Q_2.$$

Then we have clearly

$$E(|Q_1|) \le \delta \quad \text{and} \quad E\left(Q_2^2\right) \le \frac{4}{T^2}\frac{T(T-1)}{2}\delta^2 \le 2\delta^2;$$

hence we have

$$E(|Q|) \le E(|Q_1|) + E(|Q_2|) \le \delta + \sqrt{2}\,\delta = (1+\sqrt{2})\delta,$$

by using Schwarz's inequality. □

PROOF OF THEOREM 1. From the assumption (3.2) and Lemma 3, we have

$$\frac{1}{T}\varepsilon' B_T \varepsilon - \frac{1}{T}\sum_{j,k=1}^{T} K\left(\frac{j}{T},\frac{k}{T}\right)\varepsilon_j\varepsilon_k \to 0,$$

in probability; hence it suffices to consider the case $B_T(j,k) = K(j/T, k/T)$.

Let $0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues associated with the kernel $K(s,t)$ and $f_1(t), f_2(t), \dots$ be the corresponding eigenfunctions satisfying (3.5). Note that in the sequence $\{\lambda_l\}$ the same eigenvalue is listed according to its multiplicity, which is defined as the maximum number of linearly independent eigenfunctions corresponding to it. Since $K(s,t)$ is continuous, so are $f_1(t), f_2(t), \dots$; and Mercer's theorem asserts that

$$K(s,t) = \sum_{l=1}^{\infty}\frac{1}{\lambda_l} f_l(s) f_l(t),$$

with uniform convergence for $(s,t) \in [0,1]\times[0,1]$.

Let

$$K_n(s,t) = \sum_{l=1}^{n}\frac{1}{\lambda_l} f_l(s) f_l(t), \qquad n = 1,2,\dots,$$

and put $B_T^{(n)} = ((K_n(j/T, k/T)))$. Then we can conclude from Lemma 2 that

$$L\left(\frac{1}{T}\varepsilon' B_T^{(n)}\varepsilon\right) \to L\left(\sum_{l=1}^{n}\frac{1}{\lambda_l} v_l^2\right),$$

where $v_1, v_2, \dots$ are NID(0, 1), since

$$\frac{1}{T}\varepsilon' B_T^{(n)}\varepsilon = \sum_{l=1}^{n}\frac{1}{\lambda_l}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{T} f_l\left(\frac{j}{T}\right)\varepsilon_j\right)^2.$$

Applying Lemma 3 again, we have

$$L\left(\frac{1}{T}\varepsilon' B_T \varepsilon\right) \to L\left(\sum_{l=1}^{\infty}\frac{1}{\lambda_l} v_l^2\right).$$

The characteristic function of $\Sigma_{l=1}^{\infty} v_l^2/\lambda_l$ was found to be $(D(2i\theta))^{-1/2}$ by Anderson and Darling [1] and Varberg [11], where

$$D(\lambda) = \prod_{l=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_l}\right)$$

is the Fredholm determinant associated with the integral equation (3.3), which completes the proof. □

Because of Theorem 1 the limiting distribution of $\varepsilon' B_T \varepsilon / T$ does not depend on the common distribution of ε's as long as $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2) = 1$. This implies that the invariance principle holds in Donsker's sense (see Billingsley [2]), and we have

$$L\left(\sum_{l=1}^{\infty} \frac{1}{\lambda_l} v_l^2\right) = L\left(\int_0^1 \int_0^1 K(s, t)\, dw(s)\, dw(t)\right),$$

where $w(t)$ is Brownian motion with $E(w(t)) = 0$ and $E(w(s)w(t)) = \min(s, t)$.

Now we apply Theorem 1 putting $B_T = MD_x A_T D_x M / c(T)$. Because of the above invariance principle, we may assume that $\varepsilon_1, \varepsilon_2, \ldots$ are NID(0, 1) as far as the limiting distribution is concerned. If we factor A_T as $A_T = C_T' C_T$, where

$$C_T = \begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ 0 & & 1 \end{bmatrix},$$

then we have

$$\begin{array}{ccc} L\left(\dfrac{1}{T}\varepsilon' B_T \varepsilon\right) & \to & L\left(\displaystyle\int_0^1 \int_0^1 K(s, t)\, dw(s)\, dw(t)\right) \\ \parallel & & \\ L\left(\dfrac{1}{T}\varepsilon' B_T^* \varepsilon\right) & \to & L\left(\displaystyle\int_0^1 \int_0^1 K^*(s, t)\, dw(s)\, dw(t)\right), \end{array}$$

where we put $B_T^* = C_T D_x M D_x C_T' / c(T)$ and we assume that the conditions of Theorem 1 are also satisfied for B_T^* and $K^*(s, t)$.

The problem has now been reduced to obtaining the Fredholm determinant $D(\lambda)$ associated with the kernel $K(s, t)$ or $K^*(s, t)$, which is the main issue discussed in the next section.

4. Some examples of the asymptotic distribution. Here we consider the asymptotic distribution of (2.5) or (3.1) by specifying the regressors x_t: 1×1 and z_t: $p \times 1$ in (1.1). The following three cases are examined in this paper:

(A) $x_t = t^m$, $m > -1/2$, and $p = 0$;
(B) $x_t = 1$ for all t and $z_t = t^m$, $m = 1, 2, 3, 4$;
(C) $x_t = 1$ for all t and $z_t = (t, t^2)'$.

In each case we consider $B_T^* = C_T D_x M D_x C_T'/c(T)$ with M defined in (2.7) or (2.6) according to whether the initial value β_0 is known to be zero or unknown and find its uniform limit choosing $c(T)$ appropriately. For case (A) with $\beta_0 = 0$ we have, putting $c(T) = T^{2m+1}$,

$$B_T^*(j,k) = \sum_{l=\max(j,k)}^{T} \frac{x_l^2}{c(T)} = \sum_{l=\max(j,k)}^{T} \frac{(l/T)^{2m}}{T},$$

and thus the corresponding kernel is

$$K_1^*(s,t) = \frac{1}{2m+1}\left[1 - (\max(s,t))^{2m+1}\right]. \tag{4.1}$$

Similarly, the kernel $K_2^*(s,t)$ for case (A) with unknown β_0 is

$$K_2^*(s,t) = \frac{1}{2m+1}\left[(\min(s,t))^{2m+1} - (st)^{2m+1}\right], \tag{4.2}$$

putting $c(T) = T^{2m+1}$ as well. It is noted that the kernels (4.1) and (4.2) satisfy the requirements in Theorem 1 if $m > -1/2$. As for cases (B) and (C) the corresponding kernels have the form

$$K^*(s,t) = \min(s,t) + \sum_{k=1}^{r} \xi_k(s)\psi_k(t), \tag{4.3}$$

putting $c(T) = T$, where the conditions imposed on ξ_k and ψ_k will be described later.

In the following discussion we derive the Fredholm determinants for the previous kernels, thereby obtaining the asymptotic distributions. Case (A) is discussed in Section 4.1, whereas cases (B) and (C) are treated in Section 4.2.

4.1. *Case* (A). Let us consider the integral equation

$$f(t) = \lambda \int_0^1 K^*(s,t) f(s)\, ds, \tag{4.4}$$

and first deal with $K^* = K_1^*$. The Fredholm determinant $D_1(\lambda)$ associated with K_1^* is obtained as follows. Suppose that a continuous function $f(t)$, not identically equal to zero, satisfies (4.4) with $K^* = K_1^*$ for some $\lambda > 0$. Then such $f(t)$ satisfies the homogeneous, differential equation

$$f''(t) - \frac{2m}{t} f'(t) + \lambda t^{2m} f(t) = 0, \tag{4.5}$$

with the boundary conditions

$$\lim_{t \to 0} \frac{f'(t)}{t^{2m}} = 0, \qquad f(1) = 0. \tag{4.6}$$

Conversely, given any $\lambda \neq 0$, (4.5) has a unique (except for a constant multiple) solution $f(t)$ satisfying the first condition in (4.6), which is given by

$$f(t) = t^{(2m+1)/2} J_{[1/2(m+1)]-1}\left(\frac{\sqrt{\lambda}}{m+1} t^{m+1}\right), \tag{4.7}$$

where $J_\nu(z)$ is the Bessel function of the first kind defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\Gamma(\nu + k + 1)}$$

(see Watson [12], page 40). The other boundary condition $f(1) = 0$ implies

$$J_{[1/2(m+1)]-1}\left(\frac{\sqrt{\lambda}}{m+1}\right) = 0; \tag{4.8}$$

and it can be shown that $f(t)$ in (4.7) is a solution to (4.4) with $K^* = K_1^*$ if λ $(\neq 0)$ satisfies (4.8).

Thus we have proved that $\lambda \neq 0$ is an eigenvalue of $K_1^*(s, t)$ if and only if (4.8) is satisfied. We now have the following theorem.

Theorem 2. *In case* (A) *with* $\beta_0 = 0$ *the limiting null c.f.* $\phi_1(\theta)$ *of* R_T/T^{2m+1} *is given by* $(D_1(2i\theta))^{-1/2}$, *where*

$$D_1(\lambda) = \Gamma\left(\frac{1}{2(m+1)}\right) J_{[1/2(m+1)]-1}\left(\frac{\sqrt{\lambda}}{m+1}\right) \Big/ \left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{[1/2(m+1)]-1} \tag{4.9}$$

is the Fredholm determinant associated with K_1^* *in* (4.1).

Proof. From Watson [12], page 498, we have

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{r_j^2}\right),$$

where $r_1 < r_2 < \cdots$ are the positive zeros of $J_\nu(z)$. Therefore, for $\lambda \neq 0$, (4.8) is equivalent to

$$0 = \prod_{j=1}^{\infty} \left(1 - \frac{\lambda}{(m+1)^2 r_j^2}\right) = \text{right-hand side of (4.9)},$$

which implies that all the zeros of (4.9) are simple. We have also shown that $f(t)$ satisfying (4.4) with $K^* = K_1^*$ is unique for every $\lambda_j = (m+1)^2 r_j^2$, $j = 1, 2, \ldots,$ up to a constant multiple, and thus $D_1(\lambda)$ in (4.9) is the Fredholm determinant associated with K_1^*, thereby completing the proof. □

Note. If we work with $K(s, t)$ instead of $K^*(s, t)$ in (4.1), then we have $K(s, t) = s^m t^m \min(s, t)$; and we arrive naturally at the same Fredholm determinant (4.9). In the following examples we work only with $K^*(s, t)$, which is simpler than $K(s, t)$.

The c.f. $\phi_1(\theta)$ corresponds to the limiting distribution of

$$\left\{\varepsilon_1^2 + 2^{2m}(\varepsilon_1+\varepsilon_2)^2 + \cdots + T^{2m}(\varepsilon_1+\cdots+\varepsilon_T)^2\right\}\Big/T^{2m+2},$$

or

$$\left\{(\varepsilon_1 + 2^m\varepsilon_2 + \cdots + T^m\varepsilon_T)^2 + (2^m\varepsilon_2+\cdots+T^m\varepsilon_T)^2 + \cdots + (T^m\varepsilon_T)^2\right\}\Big/T^{2m+2}.$$

In the case $m = 0$ we have $D_1(\lambda) = \cos\sqrt{\lambda} = \prod_{j=1}^{\infty}(1 - \lambda/((j-1/2)\pi)^2)$ and $\phi_1(\theta) = (\cos\sqrt{2i\theta})^{-1/2}$. For this case Erdös and Kac [4] gave an explicit expression for the distribution function $F_1(a)$, but we present a much simpler expression for $F_1(a)$ by using the method of Anderson and Darling [1],

$$F_1(a) = 2\sqrt{2}\sum_{j=0}^{\infty}\binom{-1/2}{j}\Phi\left(\frac{-2j-1/2}{\sqrt{a}}\right), \tag{4.10}$$

where $\Phi(\cdot)$ is the standard normal distribution.

For general m ($> -1/2$, $\neq 0$) it seems difficult to obtain the distribution function explicitly, but we do have that $\phi_1((m+1)\theta)$ converges to $(1-i\theta)^{-1/2}$ as $m \to \infty$. Therefore we obtain the following result.

COROLLARY 1. *The limiting null distribution of* $(m+1)R_T/T^{2m+1}$ *tends, as* $m \to \infty$, *to* $\chi^2(1)/2$.

From Corollary 1 it is of some interest to compare the limiting null distributions of $(m+1)R_T/T^{2m+1}$ with $\chi^2(1)/2$. On the basis of the Maclaurin expansion of $\phi_1((m+1)\theta)$, we obtain

COROLLARY 2. *The cumulants up to the fourth order of the limiting null distribution of* $(m+1)R_T/T^{2m+1}$ *are*

$$\kappa_1 = \tfrac{1}{2}, \qquad \kappa_2 = \frac{m+1}{2m+3}, \qquad \kappa_3 = \frac{8(m+1)^2}{(2m+3)(4m+5)},$$

$$\kappa_4 = \frac{24(m+1)^3(12m+17)}{(2m+3)^2(4m+5)(6m+7)}.$$

Case (A) with unknown β_0 proceeds in much the same way. We consider the integral equation (4.4) with $K^* = K_2^*$ in (4.2), which leads us to the differential equation (4.5) with the boundary conditions

$$f(0) = 0, \qquad f(1) = 0. \tag{4.11}$$

By the same argument as that leading to Theorem 2 we have the following theorem.

THEOREM 3. *In case* (A) *with unknown* β_0 *the limiting null c.f.* $\phi_2(\theta)$ *of* R_T/T^{2m+1} *is given by* $(D_2(2i\theta))^{-1/2}$, *where*

$$D_2(\lambda) = \Gamma\left(\frac{4m+3}{2(m+1)}\right)J_{1-[1/2(m+1)]}\left(\frac{\sqrt{\lambda}}{m+1}\right)\Big/\left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{1-[1/2(m+1)]} \tag{4.12}$$

is the Fredholm determinant associated with K_2^* *in* (4.2).

In the case $m = 0$ we have $D_2(\lambda) = \sin\sqrt{\lambda} / \sqrt{\lambda} = \prod_{j=1}^{\infty}(1 - \lambda/(j^2\pi^2))$ and $\phi_2(\theta) = (\sin\sqrt{2i\theta} / \sqrt{2i\theta})^{-1/2}$, which is also the c.f. for the limiting distribution of the Cramér–von Mises statistic, and the corresponding distribution function was given by Anderson and Darling [1].

Returning to general m ($> -1/2$, $\neq 0$), we note that

$$\lim_{m\to\infty} D_2\big((m+1)^2\lambda\big) = 2J_1(\sqrt{\lambda})/\sqrt{\lambda}$$

and obtain the following results.

COROLLARY 3. *The limiting null c.f. of* $(m+1)^2 R_T/T^{2m+1}$ *tends, as* $m \to \infty$, *to*

$$\left\{\frac{2J_1(\sqrt{2i\theta})}{\sqrt{2i\theta}}\right\}^{-1/2} = \left\{\sum_{k=0}^{\infty}\frac{(-i\theta/2)^k}{k!(k+1)!}\right\}^{-1/2}. \tag{4.13}$$

COROLLARY 4. *The cumulants up to the fourth order of the limiting null distribution of* $(m+1)^2 R_T/T^{2m+1}$ *are*

$$\kappa_1 = \frac{m+1}{2(4m+3)}, \qquad \kappa_2 = \frac{(m+1)^3}{(4m+3)^2(6m+5)},$$

$$\kappa_3 = \frac{8(m+1)^5}{(4m+3)^3(6m+5)(8m+7)},$$

$$\kappa_4 = \frac{24(m+1)^7(32m+27)}{(4m+3)^4(6m+5)^2(8m+7)(10m+9)}.$$

In Figure 1 the probability densities corresponding to $\phi_1((m+1)\theta)$ for $m = 0$, 2, 4 and ∞ are drawn whereas those corresponding to $\phi_2((m+1)^2\theta)$ are shown in Figure 2 for the same values of m. These were calculated by applying Fourier's inversion formula numerically. It is seen that the distribution for the former is shifted to the left as m becomes large and the shape is drastically changed for $m = \infty$, which is $\chi^2(1)/2$. In Figure 2 the distribution is shifted to the left quite smoothly as m becomes large. The percentiles of these distributions will be given in Section 5.

4.2. *Cases* (B) *and* (C). The kernel $K^*(s, t)$ treated here is of the form given in (4.3). We assume

(i) $\xi_k(s)$, $k = 1, \ldots, r$, and $\psi_k(t)$, $k = 1, \ldots, r$, are continuous and each set is linearly independent in the space $C[0,1]$.

If 1 and/or a linear function of t belong to the space of linear combinations of $\psi_k(t)$, $k = 1, \ldots, r$, then we can assume without loss of generality that they are the last one or two members of $\psi_k(t)$, in which case we can assume the linear independence of $\psi_k''(t)$, $k = 1, \ldots, q$, whereas $\psi_k''(t) = 0$, $k = q+1, \ldots, r$, with $0 \le r - q \le 2$, assuming the twice differentiability of ψ's. Thus we assume

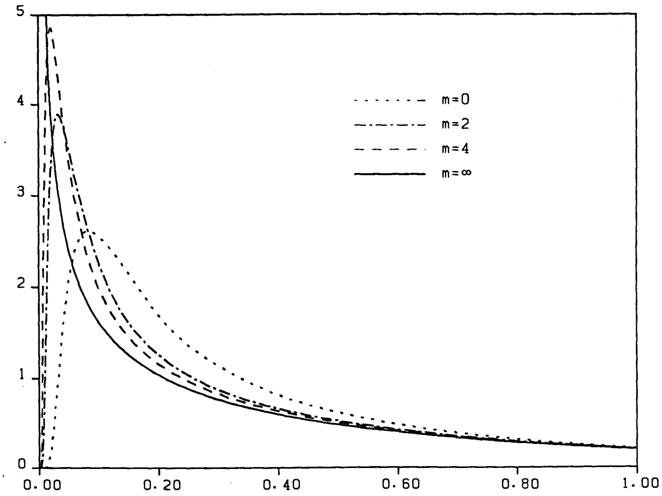

FIG. 1. *Limiting probability densities of* $(m + 1)R_T/T^{2m+1}$ *in Theorem* 2.

(ii) $\psi_k'(t)$, $k = 1,\ldots,r$, are continuous in $[0,1]$ and $\psi_k''(t)$, $k = 1,\ldots,r$, are continuous in $(0,1)$; furthermore, $\psi_k''(t)$, $k = 1,\ldots,q$, are linearly independent, whereas $\psi_k''(t) = 0$, $k = q+1,\ldots,r$, where $0 \le r - q \le 2$.

Then the integral equation with the kernel (4.3) is equivalent to the nonhomogeneous differential equation

$$f''(t) + \lambda f(t) = \lambda \sum_{k=1}^{q} a_k \psi_k''(t), \tag{4.14}$$

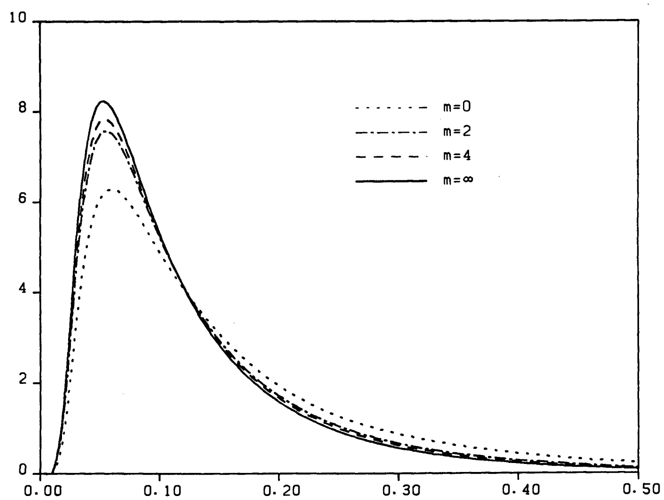

FIG. 2. *Limiting probability densities of* $(m + 1)^2R_T/T^{2m+1}$ *in Theorem* 3.

with the boundary conditions

$$f(0) = \lambda \sum_{k=1}^{r} a_k \psi_k(0), \tag{4.15}$$

$$f'(1) = \lambda \sum_{k=1}^{r} a_k \psi_k'(1), \tag{4.16}$$

where

$$a_k = \int_0^1 \xi_k(s) f(s)\, ds, \qquad k = 1, \dots, r. \tag{4.17}$$

The Fredholm determinant associated with the above $K^*(s, t)$ is obtained as follows, by modifying the technique of Kac, Kiefer and Wolfowitz [8]. For given $\lambda \neq 0$ the general solution to (4.14) is

$$f(t) = c_1 \cos\sqrt{\lambda}\, t + c_2 \sin\sqrt{\lambda}\, t + \sum_{k=1}^{q} a_k g_k(t), \tag{4.18}$$

where $g_k(t)$ is a special solution of $g_k''(t) + \lambda g_k(t) = \lambda \psi_k''(t)$. Note that (ii) implies that $\cos\sqrt{\lambda}\, t$, $\sin\sqrt{\lambda}\, t$ and $g_k(t)$, $k = 1, \dots, q$, are linearly independent.

Substituting $f(t)$ from (4.18), we regard (4.15), (4.16) and (4.17) as a system of $r + 2$ linear homogeneous equations in $a_1, \dots, a_r$, c_1 and c_2. Then it can be shown that $\lambda \neq 0$ is an eigenvalue of $K^*(s, t)$ if and only if the system has a nontrivial solution. We now have the following theorem.

THEOREM 4. *Let K^* be given by* (4.3) *and assume* (i) *and* (ii). *Let $M(\lambda)$ be the $(r + 2) \times (r + 2)$ coefficient matrix of the system of linear homogeneous equations in $a_1, \dots, a_r$, c_1 and c_2 given by* (4.15), (4.16) *and* (4.17) *for $f(t)$ in* (4.18). *Further assume*

(iii) $$\det M(\lambda) = \alpha \lambda^{\beta} \prod_{j=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_j}\right)^{l_j},$$

(iv) $$\operatorname{rank} M(\lambda_j) = r + 2 - l_j, \qquad j = 1, 2, \dots,$$

where α ($\neq 0$) and β are constants, l_j are positive integers and λ_j, $j = 1, 2, \dots$, are the nonzero solutions to $\det M(\lambda) = 0$. Then the Fredholm determinant $D(\lambda)$ associated with $K^*(s, t)$ is

$$D(\lambda) = \frac{1}{\alpha \lambda^{\beta}} \det M(\lambda) = \prod_{j=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_j}\right)^{l_j}. \tag{4.19}$$

As an illustration let us take case (B) with $\beta_0 = 0$. Then $B_T^* = C_T M C_T'/T$ with M in (2.7), and letting $T \to \infty$, we have, for $m > 0$,

$$K_3^*(s, t) = \min(s, t) + 1 - s - t - \frac{2m + 1}{(m + 1)^2}(1 - s^{m+1})(1 - t^{m+1}). \tag{4.20}$$

We put

$$\xi_1(s) = \frac{2m+1}{m+1}(1 - s^{m+1}), \qquad \xi_2(s) = 1, \qquad \xi_3(s) = -s,$$

$$\psi_1(t) = -\frac{1}{m+1}(1 - t^{m+1}), \qquad \psi_2(t) = 1 - t, \qquad \psi_3(t) = 1.$$

The general solution corresponding to (4.18) is

$$f(t) = c_1 \cos\sqrt{\lambda}\, t + c_2 \sin\sqrt{\lambda}\, t + a_1 m \sqrt{\lambda} \int_{\mu}^{t} s^{m-1} \sin\sqrt{\lambda}\,(t-s)\, ds,$$

where μ is any constant. Evaluating the determinant of the 5×5 coefficient matrix $M(\lambda)$ for $m = 1, 2, 3, 4$, we obtain the following theorem.

Theorem 5. *In case* (B) *with* $\beta_0 = 0$ *the limiting null c.f.* $\phi_3(\theta)$ *of* R_T/T *is given by* $(D_3(2i\theta))^{-1/2}$, *where*

$$\begin{aligned} D_3(\lambda) &= \frac{3}{\lambda^{3/2}}(\sin\sqrt{\lambda} - \sqrt{\lambda}\cos\sqrt{\lambda}), && m = 1,\\ &= \frac{20}{\lambda^{5/2}}\left(-2\sqrt{\lambda} + (1+\lambda)\sin\sqrt{\lambda} + \sqrt{\lambda}\left(1 - \frac{\lambda}{3}\right)\cos\sqrt{\lambda}\right), && m = 2,\\ &= \frac{126}{\lambda^{7/2}}\left(\left(2 - 2\lambda + \frac{\lambda^2}{2}\right)\sin\sqrt{\lambda} + \sqrt{\lambda}\left(-2 + \frac{4\lambda}{3} - \frac{\lambda^2}{10}\right)\cos\sqrt{\lambda}\right), && m = 3,\\ &= \frac{864}{\lambda^{9/2}}\left(\sqrt{\lambda}(-12 + 2\lambda) + \left(6 + 6\lambda - 2\lambda^2 + \frac{\lambda^3}{6}\right)\sin\sqrt{\lambda}\right.\\ &\qquad \left. + \sqrt{\lambda}\left(6 - 4\lambda + \frac{7\lambda^2}{10} - \frac{\lambda^3}{42}\right)\cos\sqrt{\lambda}\right), && m = 4, \end{aligned}$$

is the Fredholm determinant associated with K_3^* *in* (4.20).

Proof. We consider the case $m = 2$. The other cases can be treated similarly. It may be shown that, for some constants α ($\neq 0$) and β, $\det M(\lambda)$ is $\alpha\lambda^{\beta}$ multiplied by

$$G(z) = -2z + (1 + z^2)\sin z + \left(z - \frac{z^3}{3}\right)\cos z,$$

where $z = \sqrt{\lambda}$. Note that every zero of $G(z)$ is real because nonzero $z^2 = \lambda$ is an eigenvalue associated with $K_3^*(s,t)$ in (4.20). Let r_j, $j = 1, 2, \ldots$, be the positive zeros of $G(z)$. Then the rank of the 5×5 coefficient matrix $M(r_j^2)$ is 4, which implies that $l_j = 1$ in assumption (iv) of Theorem 4. It may also be checked that every nonzero solution of $G(z) = 0$ is simple by showing that there exists no nonzero solution common to $G(z) = 0$ and $G'(z) = 0$. Finally, we show that (iii) of Theorem 4 holds. The function $G^*(z) = 20G(z)/z^5$ is even and analytic for all z with $G^*(0) = 1$ and with the zeros $\pm r_j$, $j = 1, 2, \ldots$. Furthermore, $G^{*\prime}(z)/G^*(z)$ is bounded on the squares C_k, $k = 1, 2, \ldots$, with the vertices at

$2k\pi(\pm 1 \pm i)$. Hence, by a theorem of Weierstrass (see Whittaker and Watson [13], page 137), we have the infinite product expression for $G^*(z)$, resulting in (iii) with $l_j = 1$, $j = 1, 2, \ldots$, which establishes the theorem. □

For case (B) with unknown β_0 we have $B_T^* = C_T M C_T'/T$ with M in (2.6), and letting $T \to \infty$ leads to

$$K_4^*(s,t) = \min(s,t) + t\left(-\frac{(m+1)^2}{m^2}s + \frac{2m+1}{m^2}s^{m+1}\right) + \frac{2m+1}{m^2}t^{m+1}\left(s - s^{m+1}\right). \tag{4.21}$$

Proceeding in the same way as before, we obtain the following theorem.

Theorem 6. *In case* (B) *with unknown* β_0 *the limiting null c.f.* $\phi_4(\theta)$ *of* R_T/T *is given by* $(D_4(2i\theta))^{-1/2}$, *where*

$$\begin{aligned}
D_4(\lambda) &= \frac{12}{\lambda^2}\left(2 - \sqrt{\lambda}\sin\sqrt{\lambda} - 2\cos\sqrt{\lambda}\right), & m&=1,\\
&= \frac{45}{\lambda^3}\left(\sqrt{\lambda}\left(1 - \frac{\lambda}{3}\right)\sin\sqrt{\lambda} - \lambda\cos\sqrt{\lambda}\right), & m&=2,\\
&= \frac{224}{\lambda^4}\left(4 - 2\lambda + \sqrt{\lambda}\left(-2 + \frac{4\lambda}{3} - \frac{\lambda^2}{10}\right)\sin\sqrt{\lambda}\right.\\
&\qquad \left. + \left(-4 + 2\lambda - \frac{\lambda^2}{2}\right)\cos\sqrt{\lambda}\right), & m&=3,\\
&= \frac{1350}{\lambda^5}\left(\sqrt{\lambda}\left(6 - 4\lambda + \frac{7\lambda^2}{10} - \frac{\lambda^3}{42}\right)\sin\sqrt{\lambda}\right.\\
&\qquad \left. + \left(-6\lambda + 2\lambda^2 - \frac{\lambda^3}{6}\right)3c\sqrt{\lambda}\right), & m&=4,
\end{aligned}$$

is the Fredholm determinant associated with K_4^* *in* (4.21).

As for the last case (C) the kernel K_5^* for $\beta_0 = 0$ is

$$K_5^*(s,t) = \min(s,t) + \tfrac{1}{9} - s + 2s^2 - \tfrac{10}{9}s^3 - t + 2t^2(1 - 6s^2 + 5s^3) + 10t^3\left(-\tfrac{1}{9} + s^2 - \tfrac{8}{9}s^3\right), \tag{4.22}$$

whereas the kernel K_6^* for β_0 unknown is

$$K_6^*(s,t) = \min(s,t) + t(-9s + 18s^2 - 10s^3) + 6t^2(3s - 8s^2 + 5s^3) + 10t^3(-s + 3s^2 - 2s^3). \tag{4.23}$$

Theorem 7. *In case* (C) *the limiting null c.f.'s* $\phi_5(\theta)$ *of* R_T/T *for* $\beta_0 = 0$ *and* $\phi_6(\theta)$ *of* R_T/T *for unknown* β_0 *are given by* $(D_5(2i\theta))^{-1/2}$ *and* $(D_6(2i\theta))^{-1/2}$,

TABLE 1
Upper percent points of the limiting distribution

Tail Prob.	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = \infty$
	1. Case for $(m+1)R_T/T^{2m+1}$ in Theorem 2 [c.f. $(m+1)K_1^*$]					
0.100	1.19582	1.26250	1.28938	1.30392	1.31303	1.35277
0.050	1.65574	1.76929	1.81462	1.83905	1.85433	1.92073
0.010	2.78746	3.01575	3.10636	3.15508	3.18551	3.31745
0.001	4.48646	4.88668	5.04519	5.13034	5.18351	5.41378
	2. Case for $(m+1)^2R_T/T^{2m+1}$ in Theorem 3 [c.f. $(m+1)^2K_2^*$]					
0.100	0.34730	0.29090	0.27572	0.26867	0.26460	0.24939
0.050	0.46136	0.38306	0.36207	0.35233	0.34671	0.32576
0.010	0.74346	0.61125	0.57592	0.55955	0.55010	0.51494
0.001	1.16786	0.95484	0.89800	0.87168	0.85651	0.80003

respectively, where

$$D_5(\lambda) = \frac{960}{\lambda^4}\left(2 + \lambda + \sqrt{\lambda}\left(-2 - \frac{\lambda}{3}\right)\sin\sqrt{\lambda} + \left(-2 + \frac{\lambda^2}{12}\right)\cos\sqrt{\lambda}\right),$$

$$D_6(\lambda) = \frac{8640}{\lambda^4}\left(2 + \frac{\lambda}{3} + \sqrt{\lambda}\left(-2 + \frac{\lambda}{12}\right)\sin\sqrt{\lambda} + \left(-2 + \frac{2\lambda}{3}\right)\cos\sqrt{\lambda}\right)$$

are the Fredholm determinants associated with K_5^* in (4.22) *and K_6^* in* (4.23), *respectively.*

The percentiles of the limiting distributions will be given in the next section. These limiting distributions are all unimodal though not drawn here to save space.

5. Numerical and simulation results. Using Lévy's inversion formula numerically, the upper 10, 5, 1 and 0.1 percent points are tabulated in Tables 1 and 2 for the limiting null distributions of the following statistics:

Table 1: $(m+1)R_T/T^{2m+1}$ in Theorem 2 and $(m+1)^2R_T/T^{2m+1}$ in Theorem 3;

Table 2: R_T/T in Theorems 5, 6 and 7.

In carrying out the numerical integration we have used Simpson's formula, the upper limit of the integral and the number of subintervals being chosen so that the error of the resulting value is at most one unit in the last decimal.

We also examine finite sample properties of the statistic R_T under H_0. In Figure 3 the "exact" distribution functions of $(m+1)^2R_T/T^{2m+1}$ in Theorem 3 are drawn for $(m, T) = (0, 30)$, $(4, 30)$, $(4, 50)$ and $(4, 100)$ together with the corresponding limiting distributions as $T \to \infty$. The "exact" distributions were obtained by simulations based on 10,000 replications. As is seen from Figure 3, the sampling distribution for $m = 0$ is quite close to the limiting distribution

TABLE 2
Upper percent points of the limiting distribution of R_T/T

Tail Prob.	$m = 1$	$m = 2$	$m = 3$	$m = 4$	
	1. Case for Theorem 5 (c.f. K_3^*)				**3. Theorem 7 (c.f. K_5^*)**
0.100	0.19270	0.30223	0.42137	0.52002	0.09035
0.050	0.24810	0.40511	0.57380	0.71276	0.11050
0.010	0.38533	0.65895	0.94917	1.18721	0.15939
0.001	0.59245	1.04038	1.51287	1.89959	0.23301
	2. Case for Theorem 6 (c.f. K_4^*)				**4. Theorem 7 (c.f. K_6^*)**
0.100	0.11922	0.13049	0.15147	0.17181	0.07146
0.050	0.14789	0.16416	0.19394	0.22254	0.08595
0.010	0.21775	0.24724	0.29917	0.34826	0.12048
0.001	0.32308	0.37287	0.45802	0.53776	0.17183

even for $T = 30$ and thus the latter may be used as an approximation for moderate sample sizes. When $m = 4$, the approximation, however, is not good enough. It seems that the sample size should be more than 100 for good approximation. Though not shown here, the situation is almost the same for the distributions of $(m + 1)R_T/T^{2m+1}$ in Theorem 2. As for the distributions of R_T/T in Theorems 5, 6 and 7 it was also observed that T greater than 100 is desirable for the approximation to be accurate.

The test statistics described above take the form of $W = V/U$, where W, U and V are all positive and W is independent of U, which converges in probability to 1 under H_0. W and V have the same limiting distribution, but it was found,

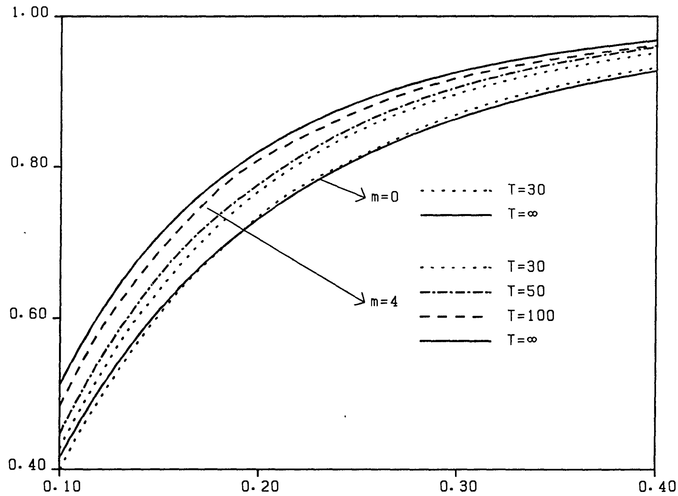

FIG. 3. *Sampling distribution functions of $(m + 1)^2 R_T/T^{2m+1}$ in Theorem 3.*

though not shown here, that the sampling distribution of V is closer to the limiting distribution than that of W. The distribution of W is more concentrated than that of V or the limiting distribution. This fact is partly seen in Figure 3, where the sampling distribution for $m = 0$ crosses the limiting distribution at some point around 0.2 and becomes numerically larger after that point. The sampling distributions for $m = 4$ are all located below the limiting distribution in Figure 3, but the former cross the latter eventually. This may be explained by the fact that $\text{Cov}(U, V) > 0$ and $\text{Var}(W) < \text{Var}(V)/E^2(U) \sim \text{Var}(V)$.

6. Concluding remarks. We have shown that the LBI test statistic derived under normality converges in distribution, without assuming normality, to a quadratic functional of Brownian motion and that the c.f. for the functional can be obtained from the Fredholm determinant. Some examples were also shown on how to obtain the Fredholm determinant, which is usually computationally burdensome. It might be mentioned that computerized algebra is useful for this purpose. Actually we have used the computer package REDUCE developed by Hearn [6] to check our results. It was also effectively used to obtain cumulants as given in Corollaries 2 and 4.

In this paper we have examined only cases where one component of the regression coefficients is subject to vary over time. The extension to more general cases has not been done yet, but will be possible by the present approach.

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