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UNIQUENESS OF THE SPATIAL MEDIAN

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The purpose of this note is to show that if a probability measure on a Euclidean space is not concentrated on a line, then its spatial median is unique.

Statement of the result. Let E be a Euclidean space with Euclidean distance $\|\cdot\|$ and let P be a probability measure on E. Any vector of E minimizing the function $\phi(\alpha) = \int(\|x - \alpha\| - \|x\|)P(dx)$ is called a spatial median of P [see also Pollard (1984), page 28]. The existence of a spatial median follows from the tightness of P.

If P puts all its mass on a line in E, then the spatial median reduces to the (one-dimensional) median which is generally not unique. It turns out that this is the only situation where nonuniqueness can arise. We have indeed the following

Theorem. *If P is not concentrated on a line in E, then P has only one spatial median.*

Haldane (1948) proved the uniqueness in the special case where P is the uniform distribution on a finite set of noncollinear points in a plane. Ducharme and Milasevic (1987) have investigated statistical properties of the spatial median in the analysis of directional data.

Proof of the theorem. Suppose there are two medians $\alpha_1 \neq \alpha_2$ of P and let l be the line passing through α_1 and α_2. For each $0 < \lambda < 1$ and each $x \in E \setminus l$ we have

$$\|x - \lambda\alpha_1 - (1-\lambda)\alpha_2\| - \|x\| < \lambda(\|x - \alpha_1\| - \|x\|) + (1-\lambda)(\|x - \alpha_2\| - \|x\|).$$

Since $\operatorname{supp}(P) \not\subset l$, this entails

$$\phi(\lambda\alpha_1 + (1-\lambda)\alpha_2) < \lambda\phi(\alpha_1) + (1-\lambda)\phi(\alpha_2) = \min_{\alpha \in E}\phi(\alpha),$$

which is a contradiction. □

Note as a corollary that if P is symmetric with respect to the origin, then $\min\phi(\alpha) = \phi(0) = 0$.

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Note added in proof. Just before the proofreading of this communication, we became aware of some work by Kemperman (1987) which generalizes our result to strictly convex Banach spaces. Some other properties of the median are also given in Valadier (1984).

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