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DISCUSSION

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My comments center on three topics: the resampling algorithm of Section 7 as a bootstrap algorithm; criteria for assessing performance of a confidence set; and robustifying jackknife or bootstrap estimates for variance and bias. It will be apparent that I do not accept several of Wu's conclusions, particularly those concerning the bootstrap. The implied criticism does not diminish the paper's merit in advancing jackknife theory for the heteroscedastic linear model.

1. The bootstrap idea is a statistical realization of the simulation concept: one fits a plausible probability model to the data and acts thereafter as though the fitted model were true. Suppose that the errors $\{e_i\}$ in the linear model (2.1) are independent and that the c.d.f. of e_i is $F(\cdot/\sigma_i)$, where F has mean zero and variance one. Consistent estimates of the $\{\sigma_i\}$ and of F are not available, in general. Nevertheless, let $\hat{\sigma}_{n,i}$ be an esimate of σ_i, such as $\hat{\sigma}_{n,i} = |r_i|(1 - w_i)^{-1/2}$ or $\hat{\sigma}_{n,i} = |r_i|(1 - n^{-1}k)^{-1/2}$, and let $\hat{F}_n$ be any c.d.f. with mean zero and variance one. The fitted model here is the heteroscedastic linear model parametrized by the estimates $\hat{\beta}_n, \{\hat{\sigma}_{n,i}\}$ and $\hat{F}_n$. The appropriate bootstrap algorithm, which I will call the heteroscedastic bootstrap, draws samples from this fitted model.

Section 7 of the paper describes just this resampling procedure, without recognizing it as a bootstrap algorithm suitable for the heteroscedastic linear model. The two bootstrap algorithms that are discussed critically in Section 2 are not even intended for the heteroscedastic linear model. The first is designed for the homoscedastic linear model; the second for linear predictors based on multivariate i.i.d. samples (Freedman (1981)).

Let $B_n(\beta, \{\sigma_i\}, F)$ and $V_n(\beta, \{\sigma_i\}, F)$ be the bias and variance of $g(\hat{\beta}_n)$ under the heteroscedastic model described in the preceding paragraphs. The appropriate heteroscedastic bootstrap estimates of bias and variance are then $B_n(\hat{\beta}_n, \{\hat{\sigma}_{n,i}\}, \hat{F}_n)$ and $V_n(\hat{\beta}_n, \{\hat{\sigma}_{n,i}\}, \hat{F}_n)$, respectively. They satisfy

$$B_n(\hat{\beta}_n, \{\hat{\sigma}_{n,i}\}, \hat{F}_n) \simeq 2^{-1}\mathrm{tr}(g''(\hat{\beta}_n)\hat{v}),$$
$$V_n(\hat{\beta}_n, \{\hat{\sigma}_{n,i}\}, \hat{F}_n) \simeq g'(\hat{\beta}_n)^T \hat{v} g'(\hat{\beta}_n),$$

where

$$\hat{v} = (X^TX)^{-1} \sum_1^n \hat{\sigma}_{n,i}^2 x_i x_i^T (X^TX)^{-1},$$

provided higher-order terms are negligible. For suitably chosen $\{\hat{\sigma}_{n,i}^2\}$, these two bootstrap estimates agree, to the first order, with Wu's jackknife estimates of bias and variance associated with $v_{J(1)}$ or $v_{H(1)}$.

2. An unusual feature of the heteroscedastic bootstrap algorithm is its use of inconsistent estimates $\{\hat{\sigma}_{n,i}\}$ and $\hat{F}_n$ to fit the probability model. Does this really work? The answer is a qualified yes, because of the central limit theorem and relative stability. Consider the particular linear model $y_i = \beta + e_i$, where the $\{e_i\}$ are independent and e_i has c.d.f. $F(\cdot/\sigma_i)$, as in comment 1. Assume additionally that $\max_{1 \le i \le n} \sigma_i^2 / \sum_1^n \sigma_i^2 \to 0$ as n increases. Let $H_n(\{\sigma_i\}, F)$ denote the distribution of $(\sum \hat{\sigma}_{n,i}^2)^{-1/2} n(\hat{\beta}_n - \beta)$.

The following triangular array weak convergence holds. Suppose $\{G_n\}$, G are any c.d.f.'s with mean zero and variance one such that $G_n \Rightarrow G$. Suppose $\{\sigma_{n,i}^2: 1 \le i \le n\}$ is any sequence of sets of variances such that $\max_{1 \le i \le n} \sigma_{n,i}^2 / \sum_1^n \sigma_{n,i}^2 \to 0$. Then $H_n(\{\sigma_{n,i}\}, G_n) \Rightarrow \Phi$, the standard normal distribution.

Returning to the bootstrap, suppose the variance estimates $\{\hat{\sigma}_{n,i}^2\}$ are such that $\max_{1 \le i \le n} \hat{\sigma}_{n,i}^2 / \sum_1^n \hat{\sigma}_{n,i}^2 \to 0$ in probability. (Either of the choices $\sigma_{n,i}^2 = r_i^2/(1 - w_i)$ or $\hat{\sigma}_{n,i}^2 = r_i^2/(1 - n^{-1}k)$ will do here). Suppose the c.d.f. estimates $\{\hat{F}_n\}$ have mean zero, variance one and converge weakly, in probability, to a c.d.f. with the same first two moments. (For instance, $\hat{F}_n = G$ for every n, where G is an arbitrary c.d.f. with the required moments.) By virtue of the previous paragraph, the heteroscedastic bootstrap distribution $H_n(\{\hat{\sigma}_{n,i}\}, \hat{F}_n) \Rightarrow \Phi$ in probability. Moreover, $H_n(\{\sigma_i\}, F)$, the actual distribution of $(\sum \hat{\sigma}_{n,i}^2)^{-1/2} n(\hat{\beta}_n - \beta)$, also converges weakly to Φ.

Thus, confidence intervals for β obtained by referring $(\sum_1^n \hat{\sigma}_{n,i}^2)^{-1/2} |\hat{\beta}_n - \beta|$ to quantiles of its heteroscedastic bootstrap distribution have correct asymptotic coverage probabilities. In the absence of trustworthy estimates for F and $\{\sigma_i\}$, there is no reason to expect that such confidence intervals are superior in any way to those based on the normal approximation.

3. In some regression models, certain bootstrap confidence sets have a theoretical edge. Suppose we fit the line $y = \beta_1 + \beta_2 x$ to the pairs $\{(x_i, y_i): 1 \le i \le n\}$, which are assumed to be i.i.d. with certain finite moments. Let $\hat{\beta}_n$ be the least squares estimate of $\beta = (\beta_1, \beta_2)^T$. For the real-valued function $g(\beta)$, consider one-sided confidence sets of the form $\{t \in R: t > g(\hat{\beta}_n) - n^{-1/2}\hat{c}_n(\alpha)\}$.

Let $\hat{\sigma}_n$ be an asymptotically efficient estimate of Av arg $(\hat{\beta}_n)$—possibly a jackknife estimate (Beran (1984)). Three choices of critical value $\hat{c}_n(\alpha)$ that give the confidence set asymptotic coverage probability $1 - \alpha$ are

(a) The normal approximation: $\hat{c}_n(\alpha) = \hat{\sigma}_n \Phi^{-1}(1 - \alpha)$.
(b) The simple bootstrap: $\hat{c}_n(\alpha)$ is a $(1 - \alpha)$th quantile of the pertinent bootstrap distribution for $n^{1/2}\{g(\hat{\beta}_n) - g(\beta)\}$. Resampling here is from the empirical distribution of the $\{(x_i, y_i)\}$.
(c) The studentized bootstrap: $\hat{c}_n(\alpha) = \hat{\sigma}_n \hat{d}_n(\alpha)$, where $\hat{d}_n(\alpha)$ is a $(1 - \alpha)$th quantile of the bootstrap distribution for $n^{1/2}\{g(\hat{\beta}_n) - g(\beta)\}/\hat{\sigma}_n$. The resampling scheme is that of (b).

The coverage probability error (CPE) is $O(n^{-1/2})$ in cases (a) and (b), but is $O(n^{-1})$ in case (c) (cf. Abramovitch and Singh (1985) for discussion of (c)). In (a), neither bias correction of $g(\hat{\beta}_n)$ nor use of the t-distribution quantile will reduce the order of the CPE, except in special situations. Several authors have recently developed techniques, including iterated bootstrapping, for reducing the order of CPE even beyond that achieved in case (c).

Similar theoretical developments seem possible and worthwhile for heteroscedastic linear models in which the vector $\{\log(\sigma_i): 1 \le i \le n\}$ is constrained to be within a specified finite-dimensional subspace. The analysis of bootstrapping in k-sample models by Abramovitch and Singh (1985) is a first step in this direction.

4. Coverage probability is only one aspect of confidence set performance. Notions of average or median length of a confidence interval, used in Section 10, overlook possible miscentering of the confidence interval. More cogent performance criteria exist. Suppose C_n is a confidence set for the parameter θ. Let $l(t, \theta)$ be an appropriate loss function and define the risk of C_n to be

$$\rho_n = E\left[\sup\{l(t, \theta): t \in C_n\}\right].$$

(The median or other quantiles of the distribution of the supremum could be considered instead.) The goal is to minimize ρ_n by choice of the confidence set C_n, subject to the coverage probability constraint on C_n.

For example, if θ is real-valued, C_n is the interval $(c_{n,L}, c_{n,U})$, and $l(t, \theta) = |t - \theta|$, then

$$\rho_n = E\left[|2^{-1}(c_{n,L} + c_{n,U}) - \theta|\right] + E\left[2^{-1}(c_{n,U} - c_{n,L})\right].$$

This risk measures the average miscentering of C_n as well as the average length. The intuitive supposition, that the interval C_n should be centered at an asymptotically efficient estimate of θ, is supported by a local asymptotic minimax analysis of ρ_n (Beran and Millar (1985)).

5. Both jackknife and bootstrap estimates of variance involve the variance functional, which is not weakly continuous. Consequently, weak convergence of the bootstrap distribution to a normal distribution need not entail convergence of the bootstrap estimate of variance to the asymptotic variance. Nor does weak convergence to normality of the empirical distribution of the centered pseudovalues guarantee corresponding convergence of the jackknife estimate of variance. The situation begs for robustification—replacement of the variance functional by a scale equivariant functional that equals variance at normal distributions, but is weakly continuous there while retaining high asymptotic efficiency. One possibility is a standardized trimmed variance.

A similar argument exists for replacing the mean functional by a symmetrically trimmed mean (say) in bootstrap and jackknife estimates of bias.

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Professor Wu is to be congratulated on a very interesting paper that advances our knowledge of jackknife methods and illustrates some problems of heteroscedastic data. Of course, Professor Wu's paper does not demonstrate a superiority of the jackknife over the bootstrap and is not intended as such. The bootstrap is a more general method. The bootstrap philosophy is to estimate the probability distribution of the data as accurately as possible and then find or approximate the sampling distribution of the relevant statistic under this estimated distribution. We agree with this philosophy. The present paper does a great service in underscoring the need for care about assumptions, both in this specific case and in statistics in general.

The robustness of the jackknife variance estimator to nonconstant variance is an interesting and potentially useful property, but what is its real importance for statistical practice? To answer this question we need to ask, "What types of heteroscedasticity can we expect in practice and what should be done about

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