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VERY WEAK EXPANSIONS FOR SEQUENTIAL CONFIDENCE LEVELS[1]

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Asymptotic expansions are derived for a class of averages of the coverage probabilities for some sequential confidence bounds, when the data consist of i.i.d. observations from a one-parameter exponential family. These expansions show the effect of the optional stopping on the coverage probabilities quite clearly and provide a method for changing the confidence limits to reduce this effect.

1. Introduction. Let $X_1, X_2, \ldots$ denote i.i.d. random variables whose common distribution function F_ω depends on an unknown parameter $\omega \in \Omega$. Suppose that each F_ω has a finite mean $\theta = \theta(\omega)$ and a finite positive variance $\sigma^2 = \sigma^2(\omega)$, and consider the problem of setting confidence bounds for θ. For each integer $n \geq 1$, let $\hat{\theta}_n = \bar{X}_n = (X_1 + \cdots + X_n)/n$; and let $\hat{\sigma}_n^2 = \hat{\sigma}_n^2(X_1, \ldots, X_n)$, $n \geq 1$, denote a consistent sequence of positive estimators of σ^2. If n is a large, nonrandom sample size and if $X_1, \ldots, X_n$ are observed, then approximate confidence bounds may be determined from the approximate normality of the pivotal quantity $\sqrt{n}(\hat{\theta}_n - \theta)/\hat{\sigma}_n$ in a well-known manner; and the approximations may be refined in some cases by using Edgeworth expansions for the distribution of $\sqrt{n}(\hat{\theta}_n - \theta)/\hat{\sigma}_n$, as in Bhattacharya and Ghosh (1978) and Hall (1983).

Now suppose that a sequential sample is taken; that is, suppose that the fixed sample size n is replaced by a stopping time $t = t(X_1, X_2, \ldots)$; and consider the problem of setting approximate confidence bounds for θ when $X_1, \ldots, X_t$ are observed. This problem arises, in particular, when estimates are required following a sequential test, as in Siegmund (1978, 1980). Under modest conditions, developed by Anscombe (1952), $\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t$ may still be approximately normal, so that the (first-order) approximation to its distribution is unaffected by the optional stopping. Of course, the exact distribution of $\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t$ is affected by the optional stopping, but the effect disappears in the approximation. Siegmund (1978) expressed dissatisfaction with such approximations in a special case and proposed a (fairly complicated) alternative.

Here some very weak asymptotic expansions are determined for the distribution of $\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t$ and the coverage probabilities of some associated confidence bounds, in the case that F_ω, $\omega \in \Omega$, is a one-parameter exponential family. When compared to the asymptotic expansions for fixed sample sizes, these expansions allow one to determine the primary effect of the optional stopping on the

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distribution of $\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t$ in large samples. They also provide a method for correcting confidence bounds for the effect of optional stopping and other effects, such as skewness.

The expansions derived here are very weak in the following sense: instead of the coverage probability at a fixed but arbitrary ω, a collection of average coverage probabilities over ω in a neighborhood of a fixed but arbitrary ω_0 are considered.

The rationale for considering such averages is briefly as follows: average coverage probabilities are much simpler and give a better picture of the confidence level near a given ω_0 than does the value at ω_0; and, in repeated applications of any statistical procedure, the frequentist scenario, parameters may vary too. These points are developed in Section 5.

After some preliminaries in Section 2, the main result is presented in Section 3 and illustrated by examples in Section 4. Asymptotic expansions for posterior distributions are reviewed in Section 6 and used to prove the main result in Section 7. A refinement is developed in Sections 8 and 9.

There does not appear to be a great deal known about asymptotic expansions for coverage probabilities in the sequential case. Anscombe (1953) and Woodroofe (1977) treat a special case. There is current work by Keener (1984), Takahashi (1985), and Woodroofe and Keener (1985). Landers and Rogge (1976) develop bounds on the error of normal approximation for randomly stopped sums. None of these authors considers average coverage probabilities, however.

Average coverage probabilities have been considered by Stein (1981), who outlined a program for comparing the unconditional coverage probabilities which result from the use of different prior distributions. Many of the results presented here are in formal agreement with those in Section 2 of Stein's paper. Stein's approach is heuristic, and the application to sequential analysis is not considered.

Woodroofe (1985) computes average risks for sequential point estimation, using different methods.

2. Preliminaries. Let Ω denote a nondegenerate subinterval of $(-\infty, \infty)$ and let F_ω, $\omega \in \Omega$, denote a nondegenerate, one-parameter exponential family with natural parameter space Ω; that is, suppose

$$F_\omega(dx) = \exp[\omega x - \psi(\omega)]\,\Lambda(dx) \tag{1}$$

for $-\infty < x < \infty$ and $\omega \in \Omega$ for some nondegenerate, sigma-finite measure Λ on the Borel sets of $(-\infty, \infty)$ and that Ω consists of all $\omega \in (-\infty, \infty)$ for which $\int_{-\infty}^{\infty} e^{\omega x}\Lambda(dx) < \infty$. Then the function ψ is strictly convex and real analytic on the interior $\Omega^\circ = \text{int}(\Omega)$; and the mean and variance of F_ω are

$$\theta = \psi'(\omega) \quad \text{and} \quad \sigma^2 = \psi''(\omega) \tag{2}$$

for $\omega \in \Omega^\circ$, where $'$ denotes differentiation. See, for example, Lehmann (1959, Section 2.7).

Next, let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution F_ω under of probability measure P_ω for some unknown $\omega \in \Omega^\circ$. It is assumed that P_ω, $\omega \in \Omega^\circ$, are defined on a common probability space $(\mathscr{X}, \mathscr{B})$ and that the mapping from ω into $P_\omega(B)$ is Borel measurable (in ω) for all $B \in \mathscr{B}$.

Write $\Omega^\circ = (\underline{\omega}, \bar{\omega})$, where $-\infty \le \underline{\omega} < \bar{\omega} \le \infty$; let $\bar{\Omega} = [\underline{\omega}, \bar{\omega}]$ denote the closure of Ω° in $[-\infty, \infty]$; and let $\Theta = \psi'(\Omega^\circ) = (\underline{\theta}, \bar{\theta})$, where $-\infty \le \underline{\theta} < \bar{\theta} \le \infty$. Observe that ψ' is strictly increasing on Ω°, since $\sigma^2 > 0$ there. Let $\hat{\theta}_n = \bar{X}_n$ for $n \ge 1$, as above, and define $\hat{\omega}_n$, $n \ge 1$, as follows: if $\hat{\theta}_n \le \underline{\theta}$, then $\hat{\omega}_n = \underline{\omega}$; if $\underline{\theta} < \hat{\theta}_n < \bar{\theta}$, then $\psi'(\hat{\omega}_n) = \hat{\theta}_n$; and if $\hat{\theta}_n \ge \bar{\theta}$, then $\hat{\omega}_n = \bar{\omega}$. Thus, $\hat{\omega}_n$ may be an extended valued random variable for each $n \ge 1$. When $\hat{\theta}_n \in \Theta$, however, $\hat{\omega}_n$ is the M.L.E. of ω. Next, let Ω_n, $n \ge 1$, be an increasing sequence of compact subintervals of Ω° for which $\bigcup_{n=1}^{\infty} \Omega_n = \Omega^\circ$; let σ_n, $n \ge 1$, be positive continuous functions on $\bar{\Omega}$ for which $\sigma_n^2 = \psi''$ on Ω_n for each $n \ge 1$; and let $\hat{\sigma}_n = \sigma_n(\hat{\omega}_n)$ for $n \ge 1$. Thus, $\hat{\sigma}_n$, $n \ge 1$, is an asymptotically efficient sequence of positive estimators of σ.

Let $\mathscr{D}_n$ denote the sigma-algebra generated by $X_1, \ldots, X_n$ for each $n \ge 1$; and let t_a, $a \ge 1$, denote a family of stopping times w.r.t. $\mathscr{D}_n$, $n \ge 1$, which are almost surely finite w.r.t. P_ω for all $\omega \in \Omega^\circ$. It is assumed throughout that t_a, $a \ge 1$, satisfy the following two conditions: there is a continuous function κ on Ω° for which

$$\lim_{a \to \infty} E_\omega \left| \frac{a}{t_a} - \kappa(\omega) \right| = 0 \tag{3}$$

for a.e. $\omega \in \Omega^\circ$; and for every compact $\Omega_0 \subset \Omega^\circ$, there is an $\eta = \eta(\Omega_0) > 0$ for which

$$\lim_{a \to \infty} a^p \int_{\Omega_0} P_\omega\{t_a \le a\eta\}\, d\omega = 0 \tag{4}$$

for $p = \frac{1}{2}$ in Theorem 1 and $p = 1$ in Theorem 2. An additional condition is imposed in Theorem 2.

Now consider the problem of setting approximate confidence bounds for θ when $X_1, \ldots, X_t$ are observed. (Here and below t is written for t_a to avoid second-order subscripts.) Let γ denote the desired confidence coefficient, and let c denote the γth quantile of the standard normal distribution Φ; that is, $0 < \gamma < 1$ and $\Phi(c) = \gamma$. Let b_n, $n \ge 1$, denote a sequence of continuous (real valued) functions on $\bar{\Omega}$ and define c_n and $\mathscr{I}_n$ by

$$c_n = c + \frac{1}{\sqrt{n}} b_n(\hat{\omega}_n) \tag{5}$$

and

$$\mathscr{I}_n = \left[\hat{\theta}_n - \frac{1}{\sqrt{n}} c_n \hat{\sigma}_n, \infty \right) \cap \Theta$$

for $n \ge 1$. The confidence intervals considered are of the form $\mathscr{I}_t$ for appropriate sequences b_n, $n \ge 1$. The confidence curves of such intervals are defined by

$$\gamma_a(\omega) = P_\omega\{\theta \in \mathscr{I}_t\} = P_\omega\{\sqrt{t}\,(\hat{\theta}_t - \theta)/\hat{\sigma}_t \le c_t\}$$

for $\omega \in \Omega^\circ$ and $a \ge 1$. As noted in Section 5, the behavior of $\gamma_a(\omega)$, $a \ge 1$, may be erratic, even in simple cases; but their averages are much better behaved. If ξ is a density on Ω°, then *the average coverage probability under* ξ is defined for

$a \geq 1$ by

$$\bar{\gamma}_a(\xi) = \int_\Omega \gamma_a(\omega)\xi(\omega)\, d\omega. \tag{6}$$

Then

$$\bar{\gamma}_a(\xi) = P^\xi\{\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t \leq c_t\},$$

where P^ξ denotes probability in the Bayesian model in which ω has prior density ξ and $X_1, X_2, \ldots$ are conditionally i.i.d. with common distribution F_ω, given ω. Here P^ξ is defined on the space $\Omega \times \mathscr{X}$, and the random variables $X_1, X_2, \ldots, t_a$, $a \geq 1$, etc., are injected into the larger space.

The focus here is on confidence bounds, as opposed to intervals, because expansions for intervals may be easily derived from those for bounds. In fact, expansions for intervals may be substantially simpler, since some of the bias terms may cancel.

3. Second-order expansions. It is assumed throughout the paper that the prior density ξ is of the following form: for some interger $q \geq 2$ and $\omega_0 < \omega_1$ in Ω°,

$$\xi(\omega) = (\omega - \omega_0)_+^q (\omega_1 - \omega)_+^q \xi_0(\omega) \quad \text{for } \omega \in \Omega^\circ, \tag{7}$$

where ξ_0 is positive and q times continuously differentiable on Ω° and $(x)_+ = \max(x, 0)$ for $-\infty < x < \infty$.

THEOREM 1. *Suppose that t_a, $a \geq 1$, satisfy* (3) *and* (4) *and that $b_n = b$ for all $n \geq 1$, where b is piecewise continuous on Ω°. Then*

$$\bar{\gamma}_a(\xi) = \gamma + \frac{1}{\sqrt{a}}\phi(c)\bar{\Gamma}_1(\xi) + o\left(\frac{1}{\sqrt{a}}\right) \quad \text{as } a \to \infty, \tag{8}$$

where

$$\bar{\Gamma}_1(\xi) = \int_\Omega \left[\sqrt{\kappa}\, b\xi + \sigma^{-1}\sqrt{\kappa}\,\xi' + \tfrac{1}{3}(c^2 - 1)\sqrt{\kappa}\,\psi_3\xi\right] d\omega,$$

$\psi_3 = \psi'''/\sigma^3$, *and* $\gamma = \Phi(c)$, *for all ξ of the form* (7).

Moreover, (8) *is valid, if b_n, $n \geq 1$, satisfy* (20), (21), *and* (22) *below.*

It is easy to describe the proof of Theorem 1. Let $\mathscr{D}_t$ denote the sigma-algebra generated by $X_1, \ldots, X_t$. Then

$$\begin{aligned}\bar{\gamma}_a(\xi) &= P^\xi\{\sqrt{t}(\theta - \hat{\theta}_t)/\hat{\sigma}_t \geq -c_t\} \\ &= \int P^\xi\{\sqrt{t}(\theta - \hat{\theta}_t)/\hat{\sigma}_t \geq -c_t | \mathscr{D}_t\}\, dP^\xi\end{aligned}$$

for all $a \geq 1$. Since posterior distributions are unaffected by optional stopping, the posterior probability may be expanded about a normal limit, using the (fixed sample size) results of Johnson (1967, 1970); and the expansion may be integrated term by term, as in Ghosh, Sinha, and Joshi (1982). The expansion (8) results

after some algebra and simple analysis. The details are presented in Sections 6 and 7.

Under additional modest conditions, the coefficient of $1/\sqrt{a}$ simplifies, and it is possible to make it vanish for all ξ by a proper choice of b.

COROLLARY 1. *If $\sqrt{\kappa}$ is absolutely continuous on (all compact subsets of) Ω°, then* (8) *holds with*

$$\bar{\Gamma}_1(\xi) = \int_\Omega \Gamma_1(\omega)\xi(\omega)\,d\omega, \tag{9}$$

where

$$\Gamma_1 = \sqrt{\kappa}\,b - \sigma^{-1}(\sqrt{\kappa})' + \tfrac{1}{6}(1 + 2c^2)\sqrt{\kappa}\,\psi_3.$$

PROOF. If $\sqrt{\kappa}$ is absolutely continuous, then

$$\int_\Omega \sigma^{-1}\sqrt{\kappa}\,\xi'\,d\omega = -\int_\Omega (\sigma^{-1}\sqrt{\kappa})'\xi\,d\omega = \int_\Omega \left[\tfrac{1}{2}\sqrt{\kappa}\,\psi_3 - \sigma^{-1}(\sqrt{\kappa})'\right]\xi\,d\omega$$

for ξ of the form (7). The corollary now follows from simple algebra. □

COROLLARY 2. *If $\sqrt{\kappa}$ is absolutely continuous and if $\sqrt{\kappa}\,b = \sigma^{-1}(\sqrt{\kappa})' - \frac{1}{6}(1 + 2c^2)\sqrt{\kappa}\,\psi_3$ a.e., then $\bar{\gamma}_a(\xi) = \gamma + o(1/\sqrt{a})$ as $a \to \infty$ for ξ of the form* (7).

Corollary 2 is obvious.

Corollary 1 may be paraphrased by asserting that $\gamma_a = \gamma + \phi(c)\Gamma_1/\sqrt{a} + o(1/\sqrt{a})$ as $a \to \infty$, very weakly (after integration w.r.t. a large class of densities). In particular, when $b_n = 0$ for all $n \geq 1$, Corollary 1 gives a very weak expansion for the distribution function of $\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t$,

$$P_\cdot\left\{\frac{\sqrt{t}(\hat{\theta}_t - \theta)}{\hat{\sigma}_t} \leq c\right\} = \Phi(c) - \frac{1}{\sqrt{a}}\phi(c)\left[\sigma^{-1}(\sqrt{\kappa})' - \tfrac{1}{6}(1 + 2c^2)\sqrt{\kappa}\,\psi_3\right] + o\left(\frac{1}{\sqrt{a}}\right) \tag{10}$$

as $a \to \infty$, very weakly. Relation (10) need not hold for any fixed $\omega \in \Omega^\circ$, however; see Section 5.

Relation (10) allows a simple determination of the primary effect of optional stopping on the distribution of $\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t$ for large a. Indeed, if t_a, $a \geq 1$, are nonrandom sample sizes, say $t_a = a$ for $a \geq 1$, then (10) holds with $\kappa = 1$ and $(\sqrt{\kappa})' = 0$. The difference between (10) with κ and (10) with $\kappa = 1$ arises from the optional stopping.

When $t_a = a$, $a \geq 1$, and the distributions F_ω, $\omega \in \Omega$, are sufficiently smooth, (10) holds for all $\omega \in \Omega^\circ$, with $\kappa = 1$; and letting $b = \frac{1}{6}(1 + 2c^2)\psi_3$ yields $\gamma_a(\omega) = \gamma + o(1/\sqrt{a})$ as $a \to \infty$ for all $\omega \in \Omega^\circ$. See Hall (1983). Discrete cases, such as the binomial, Poisson, and negative binomial distributions, are not considered in Hall (1983), but are covered by Theorem 1.

4. Examples. In this section, Theorem 1 is applied to set confidence bounds following truncated sequential probability ratio tests (S.P.R.T.) and repeated significance tests (R.S.T.). To simplify the notation it is assumed throughout that $0 \in \Omega^\circ$ and that ψ has been so normalized that $\psi(0) = 0 = \psi'(0)$.

EXAMPLE 1 (S.P.R.T.). Let $\delta^- < 0 < \delta^+$ denote a conjugate pair—that is, $\psi(\delta^-) = \psi(\delta^+)$; and consider the problem of testing $\omega \leq \delta^-$ vs. $\omega \geq \delta^+$. Then the sequence of likelihood ratios of δ^+ to δ^- is $L_n = \exp(\delta S_n)$, $n \geq 1$, where $\delta = \delta^+ - \delta^-$ and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. If $\varepsilon > 0$, then

$$t_a = \inf\{n \geq 1: |S_n| > a \text{ or } n > a/\varepsilon\} \tag{11}$$

is the stopping time of a truncated S.P.R.T. of $\omega = \delta^-$ vs. $\omega = \delta^+$ for each $a \geq 1$. It is clear that (3) holds with $\kappa(\omega) = \max\{\varepsilon, |\theta|\}$; and it is easily verified that (4) holds with $p = 1$. Thus, Theorem 1 is applicable.

There is natural interest in the (limiting) function b which makes the coefficient of $1/\sqrt{a}$ vanish in (9)—namely, $b_0 = (\sqrt{\kappa})'/\sigma\sqrt{\kappa} - \frac{1}{6}(1 + 2c^2)\psi_3$. This function is undefined and discontinuous where $\theta = \pm\varepsilon$; and it may be desirable to smooth this discontinuity with an appropriate sequence b_n, $n \geq 1$.

EXAMPLE 2 (R.S.T.). Now consider the problem of testing $\omega = 0$. Let $\Lambda_n = \sup_\Omega[\omega S_n - n\psi(\omega)]$, $n \geq 1$, denote the log likelihood ratio statistics; and let $0 < \delta_0 < \delta_1 < \infty$. Then

$$t_a = \inf\{n \geq a/\delta_1: \Lambda_n > a \text{ or } n > a/\delta_0\}$$

defines the stopping time of an R.S.T. for each $a \geq 1$. It is easily seen that (3) holds with $\kappa(\omega) = \min\{\delta_1, \max[\delta_0, \omega\theta - \psi(\omega)]\}$ for $\omega \in \Omega^\circ$; and (4) holds for any $p > 0$, since $t_a \geq a/\delta_1$ for $a \geq 1$. Thus, Theorem 1 is again applicable.

As above, there is interest in the function $b_0 = (\sqrt{\kappa})'/\sigma\sqrt{\kappa} - \frac{1}{6}(1 + 2c^2)\psi_3$ of Corollary 2. This function is undefined and possibly discontinuous where $\omega\theta - \psi(\omega) = \delta_0$ or δ_1, but is bounded on compact $\Omega_0 \subset \Omega^\circ$. It may be smoothed by an appropriate sequence b_n, $n \geq 1$.

Similarly, Theorem 1 may be applied to set confidence bounds following tests which use Anderson's (1960) triangular region or Schwarz's (1962) asymptotic shapes.

In order to apply Theorem 1 and its corollaries to the untruncated S.P.R.T., the more general conditions (20), (21), and (22) (below) must be used. See Example 3, below.

5. Average vs. real confidence. Asymptotic expansions for the confidence curves γ_a, $a \geq 1$, appear to be more elusive than the simple expansions of Theorems 1 and 2 (below). Woodroofe and Keener (1985) give some; but these exploit the form of the stopping times involved. The difficulty in obtaining expansions for the distributions of randomly stopped sums is noted by Siegmund (1985, Section 1.6), in particular.

Even where obtainable, asymptotic expansions for γ_a may be more complicated and less usable than those for $\bar{\gamma}_a$. For simplicity, this comparison is

developed in the context of a symmetric S.P.R.T. about a normal mean. The qualitative points made apply more generally, however.

Suppose that $X_1, X_2, \ldots$ are i.i.d. normally distributed random variables with unknown mean θ, $-\infty < \theta < \infty$, and unit variance, in which case $\omega = \theta$ in (1) and (2). Then, for the untruncated S.P.R.T. [(11) with $\varepsilon = 0$], it is possible to derive an asymptotic expansion for γ_a as $a \to \infty$. Let

$$u(\theta, r) = P_\theta\{S_k \geq r, \text{ for all } k \geq 1\}$$

for $-\infty < r < \infty$ and $\theta > 0$. Further, let $-\infty < c < \infty$, and let $N = N(\theta, a, c)$ and $f = f(\theta, a, c)$ denote the integral and fractional parts of the quantity $a/\theta + c^2/2\theta^2 - (c/2\theta^2)\sqrt{(4a\theta + c^2)}$ for $a \geq 1$ and $\theta > 0$. Then by Theorem 2 of Woodroofe and Keener (1985)

$$P_\theta\{\sqrt{t}(\hat{\theta}_t - \theta) \leq c\} = \Phi(c) - \frac{1}{\sqrt{N}}\phi(c)\int_{-\infty}^{0}[1 - u(\theta, r)]\,dr + \frac{1}{\sqrt{N}}\phi(c)\left[\theta f - \sum_{k=1}^{\infty}\int_{(k-f)\theta}^{\infty} u(\theta, r)\,dr\right] + o\left(\frac{1}{\sqrt{N}}\right) \tag{12}$$

as $a \to \infty$ for $-\infty < c < \infty$ and $\theta > 0$.

Observe that the fractional part in (12) oscillates wildly as $a \to \infty$. In view of this, changing c slightly in order to make the coefficient of $1/\sqrt{N}$ disappear in (12) appears to be a delicate question with a possibly oscillatory answer. By contrast, the simple approximation (10) holds for all $\theta > 0$, with $\kappa(\omega) = |\theta|$; and Corollary 2 provides a method to change c slightly in order to make the coefficient of $1/\sqrt{a}$ vanish. Thus, the average confidence levels $\bar{\gamma}_a$, $a \geq 1$, are much simpler than the confidence curves γ_a, $a \geq 1$.

The oscillatory behavior of the fractional part f is lost when γ_a is replaced by $\bar{\gamma}_a$. This may be an advantage in some cases. For example, if a list of values of coverage probabilities on a grid of θ-values is desired, then the average coverage probability near points on the grid may give a better picture of overall behavior than the value at points on the grid, precisely because the oscillations have been smoothed.

In addition to simplicity, average confidence levels may provide a better measure of frequentist properties than do confidence curves. To see how, suppose that an experiment produces an outcome Y, possibly a vector, from which a confidence set $C(Y)$ for an unknown parameter ω is to be constructed; and let $\gamma(\omega) = P_\omega\{\omega \in C(Y)\}$ denote the coverage probability when ω is the state of nature. Suppose now that the experiment is repeated N times with parameters ω_i and outcomes Y_i, $i = 1, \ldots, N$. If it is assumed that the parameters are drawn independently from an unknown distribution G, then the expected relative frequency of coverage is

$$\bar{\gamma}(G) = \int_\Omega \gamma(\omega)G(d\omega);$$

and this is also a first approximation to the actual frequency of coverage by the law of large numbers. Thus having good frequentist properties requires $\bar{\gamma}(G)$ to

be large for all G of interest. Requiring $\bar{\gamma}(G) \geq \gamma$ for all G is equivalent to requiring $\gamma(\omega) \geq \gamma$ for all ω, the conventional formulation. However, if the G's of interest are all smooth, and if an approximation is allowed to replace the inequality, then the two conditions may be quite different, as illustrated above. In such cases, the average confidence levels $\bar{\gamma}(G)$ seem more directly related to relative frequencies than do the confidence curves.

6. Expansions for posterior distributions. Recall that ξ denotes a density of the form (7), and $\Omega_0 = [\omega_0, \omega_1]$.

Let $\omega_n^* = \sqrt{n}\,\hat{\sigma}_n(\omega - \hat{\omega}_n)$ for $n \geq 1$. If g is a bounded measurable function on $(-\infty, \infty)$, say $|g| \leq 1$, and if $\hat{\omega}_n \in \Omega^\circ$, then

$$E^\xi\left[g(\omega_n^*)|\mathscr{D}_n\right] = \mathscr{C}_n(g)/\mathscr{C}_n(1),$$

where

$$\mathscr{C}_n(g) = \int g(z)\exp\left\{-n\left[\psi\left(\hat{\omega}_n + \frac{z}{\sqrt{n}\,\hat{\sigma}_n}\right) - \psi(\hat{\omega}_n) - \frac{\hat{\theta}_n z}{\sqrt{n}\,\hat{\sigma}_n}\right]\right\}\xi\left(\hat{\omega}_n + \frac{z}{\sqrt{n}\,\hat{\sigma}_n}\right)dz$$

for bounded measurable g and 1 denotes the constant function. If $\hat{\omega}_n \in \Omega_0^\circ$, then it is straightforward to expand ψ and ξ in Taylor series about $\hat{\omega}_n$ and perform the formal division. To state the result, let $\xi_j = \xi^{(j)}/\sigma^j\xi$ and $\psi_j = \psi^{(j)}/\sigma^j$ for $j = 1, 2, \ldots$, where (j) denotes the jth derivative; also, let $m_j = \int_{-\infty}^{\infty} z^j\,d\Phi(z)$ denote the jth moment of the standard normal distribution and let $Q_j(g) = \int_{-\infty}^{\infty}(z^j - m_j)g(z)\,d\Phi(z)$ for bounded measurable g and $j \geq 1$. If $\omega_0 < \hat{\omega}_n < \omega_1$, then

$$E^\xi\left[g(\omega_n^*)|\mathscr{D}_n\right] = \int_{-\infty}^{\infty} g\,d\Phi + \frac{1}{\sqrt{n}}\left[Q_1(g)\xi_1(\hat{\omega}_n) - \tfrac{1}{6}Q_3(g)\psi_3(\hat{\omega}_n)\right] + \frac{1}{n}R_n^0, \tag{13}$$

where R_n^0 is a remainder term; and, if $q \geq 3$, in (7), then

$$R_n^0 = \tfrac{1}{2}Q_2(g)\xi_2(\hat{\omega}_n) - \tfrac{1}{6}Q_4(g)\psi_3(\hat{\omega}_n)\xi_1(\hat{\omega}_n) - \tfrac{1}{24}Q_4(g)\psi_4(\hat{\omega}_n) + \tfrac{1}{72}Q_6(g)\psi_3(\hat{\omega}_n)^2 + \frac{1}{\sqrt{n}}R_n^1, \tag{14}$$

where R_n^1 is another remainder term. See Johnson (1967, 1970).

There is special interest here in the case that g is the indicator of an interval $[-c, \infty)$, where $-\infty < c < \infty$. In this case $Q_j(c)$ is written for $Q_j(g)$. Letting $\phi = \Phi'$ denote the standard normal density, it is easily verified that $Q_1(c) = \phi(c)$, $Q_2(c) = -c\phi(c)$, $Q_3(c) = (2 + c^2)\phi(c)$, $Q_4(c) = -(3c + c^3)\phi(c)$, and $Q_6(c) = -(15c + 5c^3 + c^5)\phi(c)$ for $-\infty < c < \infty$.

The following bounds for R_n^0 and R_n^1, $n \geq 1$, are adapted from Ghosh, Sinha, and Joshi (1982). Let A_n denote the event

$$A_n = \left\{\omega_0 + \frac{\log n}{\sqrt{n}} \leq \hat{\omega}_n \leq \omega_1 - \frac{\log n}{\sqrt{n}}\right\} \tag{15}$$

for $n \geq 1$. Then a careful examination of the Taylor series expansions and formal division show the existence of constants $K_\iota = K_\iota(\xi)$, depending on ξ but not on n, g, or $\mathscr{D}_n$ (if $|g| \leq 1$), for which

$$\left|R_n^\iota I_{A_n}\right| \leq K_\iota\left[(\hat{\omega}_n - \omega_0)_+^{-2-\iota} + (\omega_1 - \hat{\omega}_n)_+^{-2-\iota}\right] \tag{16}$$

for $n > 2$ and $i = 0, 1$, where I_A denotes the indicator of an event A.

LEMMA 1. *Let ξ and A_n, $n \geq 1$, be as in* (7) *and* (15) *with $q \geq 2$. Then*: (i)

$$P^\xi\left(\bigcup_{k=n}^{\infty} A_k'\right) \leq K\left(\frac{\log n}{\sqrt{n}}\right)^{q+1}$$

for all $n \geq 2$ for some constant K, which is independent of n; and (ii)

$$E^\xi\left\{\sup_{n \geq 2}\left[(\hat{\omega}_n - \omega_0)_+^{-q} + (\omega_1 - \hat{\omega}_n)_+^{-q}\right] I_{A_n}\right\} < \infty.$$

PROOF. Let $\Omega_0 = [\omega_0, \omega_1]$ denote the support of ξ. Then an easy exercise using Bernstein's inequality and simple properties of ψ' show the existence of $\varepsilon_0 = \varepsilon_0(\Omega_0) > 0$, $\delta_0 = \delta_0(\Omega_0) > 0$, and $0 < K_0 = K_0(\Omega_0) < \infty$ for which

$$P_\omega\{|\hat{\omega}_n - \omega| \geq \varepsilon\} \leq K_0 \exp\left[-n\delta_0\varepsilon^2\right] \tag{17}$$

for all $0 < \varepsilon \leq \varepsilon_0$, $\omega \in \Omega_0$, and $n \geq 1$. Let $\varepsilon_n = \log n/\sqrt{n}$ for $n \geq 1$; and observe that ε_n is decreasing in $n \geq 9$. Let $m \geq 9$ be so large that $\varepsilon_n \leq \varepsilon_0$ for all $n \geq m$. If $n \geq m$ and if $\hat{\omega}_k \leq \omega_0 + \varepsilon_k$ for some $k \geq n$, then either $\omega \leq \omega_0 + 2\varepsilon_n$ or $|\hat{\omega}_k - \omega| \geq \varepsilon_k$ for some $k \geq n$. So, for $n \geq m$,

$$\begin{aligned} P\{\hat{\omega}_k \leq \omega_0 + \varepsilon_k, \exists\, k \geq n\} &\leq \int_{\omega_0}^{\omega_0 + 2\varepsilon_n} \xi(\omega)\, d\omega \\ &\quad + \int_{\Omega_0} P_\omega\{|\hat{\omega}_k - \omega| \geq \varepsilon_k, \exists\, k \geq n\}\xi(\omega)\, d\omega, \end{aligned} \tag{18}$$

which is the order $(\log n/\sqrt{n})^{q+1}$ as $n \to \infty$, by (7) and (17). Inequality (18) and its dual (at ω_1) combine to prove (i).

The proof of (ii) is similar. Let m be as above; let $x > 1/\varepsilon_m$; and let $l = l(x)$ be the largest integer $n \geq m$ for which $\varepsilon_n > 1/x$. If $n \geq m$, A_n occurs, and $(\hat{\omega}_n - \omega_0)_+^{-1} > x$, then $\varepsilon_n < 1/x$ and, therefore $n > l$. Thus, if $\sup_{n \geq m}(\hat{\omega}_n - \omega_0)_+^{-1} I_{A_n} > x$, then either $\omega \leq \omega_0 + 2\varepsilon_l$ or $|\hat{\omega}_n - \omega| > \varepsilon_n$ for some $n > l$. The probability of the latter event may be estimated as in (18) and is easily seen to be of order $1/x^{q+1}$ as $x \to \infty$. Assertion (ii) then follows from this result and its dual (at ω_1), since $(\hat{\omega}_n - \omega_0)_+^{-1}$ and $(\omega_1 - \hat{\omega}_n)_+^{-1}$ are bounded on A_n for each fixed $n \geq 2$. □

7. Proof of Theorem 1. The conditions on b_n, $n \geq 1$, in Theorem 1 are described next. Let t_a, $a \geq 1$, denote stopping times which satisfy (3) and (4) with $p = \frac{1}{2}$; let $\Omega_0 = [\omega_0, \omega_1] \subset \Omega^\circ$ be compact; and define $B_a = B_a(\Omega_0)$ by

$$B_a = \left\{t_a \geq \eta a,\ \omega_0 + \frac{\log t}{\sqrt{t}} \leq \hat{\omega}_t \leq \omega_1 - \frac{\log t}{\sqrt{t}}\right\} \tag{19}$$

for $a \geq 1$, where $\eta = \eta(\Omega_0)$ is as in (4). Let P^{Ω_0} denote P^ξ when ξ is the uniform density on Ω_0, and observe that $P^\xi \leq KP^{\Omega_0}$ for some $K = K(\xi)$ for all ξ of the form (7) with support Ω_0. In Theorem 1, it is assumed that

$$\left(\frac{a}{t}\right)^p |b_t(\hat{\omega}_t)| \cdot I_{B_a},\ a \geq 1, \text{ are unif. int. w.r.t. } P^{\Omega_0} \tag{20}$$

and

$$\operatorname{ess\,sup} |b_t(\hat{\omega}_t)/t| \cdot I_{B_a} \to 0 \pmod{P^{\Omega_0}} \tag{21}$$

as $a \to \infty$ for all compact $\Omega_0 \subset \Omega^\circ$ and $p = \frac{1}{2}$. In addition, it is assumed that there is a measurable function b on Ω° for which

$$b_t(\hat{\omega}_t) \to b(\omega) \quad \text{in } P_\omega\text{-probability} \tag{22}$$

as $a \to \infty$ for a.e. $\omega \in \Omega^\circ$. These conditions have been formulated to obtain reasonable generality and to simplify the proofs of Theorems 1 and 2. They are not especially elegant. The conditions are satisfied if b_n, $n \geq 1$, converge to a continuous limit uniformly on compact subintervals of Ω°; but they are substantially more general. Example 3 illustrates the interplay between t_a, $a \geq 1$, and b_n, $n \geq 1$, in (19) and (20).

The proof of Theorem 1 is given next. The reader may wish to review its statement in Section 3.

PROOF OF THEOREM 1. Fix a density ξ of the form (7); let $\Omega_0 = [\omega_0, \omega_1]$ denote its support; and define B_a, $a \geq 1$, by (19). Then $P^\xi(B_a') = o(1/\sqrt{a})$ as $a \to \infty$ by (4) and Lemma 1. So, $\bar{\gamma}_a(\xi) = P^\xi\{\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t \leq c_t, B_a\} + o(1/\sqrt{a})$ as $a \to \infty$.

It is convenient to first consider ω, where $\psi'(\omega) = \theta$. Recall that $\omega_n^* = \sqrt{n}\,\hat{\sigma}_n(\omega - \hat{\omega}_n)$ for $n \geq 1$, and define Δ_a by

$$\Delta_a = P^\xi\{\omega_t^* \geq -c_t, B_a\} \tag{23}$$

for $a \geq 1$. Let $\mathscr{D}_t$ denote the sigma-algebra of events which are determined prior to time t. Then, since posterior distributions are unaffected by optional stopping,

$$\begin{aligned}\Delta_a &= \int_{B_a} P^\xi\{\omega_t^* \geq -c_t | \mathscr{D}_t\}\, dP^\xi \\ &= \int_{B_a} \Phi(c_t)\, dP^\xi + \frac{1}{\sqrt{a}} \int_{B_a} \sqrt{\left(\frac{a}{t}\right)} \left[Q_1(c_t)\xi_1(\hat{\omega}_t) - \tfrac{1}{6}Q_3(c_t)\psi_3(\hat{\omega}_t)\right] dP^\xi \\ &\quad + \frac{1}{a}\int_{B_a}\left(\frac{a}{t}\right) R_t^0\, dP^\xi\end{aligned} \tag{24}$$

for $a \geq 1$. See (13). The latter three terms are considered separately.

By (16) and Lemma 1, the last integral in (24) remains bounded as $a \to \infty$; so, the final term is of order $1/a$ as $a \to \infty$. Next, $c_t \to c$ and

$$\sqrt{(a/t)}\, Q_1(c_t)\xi_1(\hat{\omega}_t) \to \sqrt{\kappa}(\omega) Q_1(c)\xi_1(\omega) = \phi(c)\sqrt{\kappa}(\omega)\xi'(\omega)/\sigma(\omega)\xi(\omega)$$

in P^{ξ}-probability as $a \to \infty$; and

$$\sqrt{(a/t)}\,|Q_1(c_t)|\,|\xi_1(\hat{\omega}_t)| \le K\left[(\hat{\omega}_t - \omega_0)_+^{-1} + (\omega_1 - \hat{\omega}_t)_+^{-1}\right]$$

on B_a for all $a \ge 1$ for some constant K. So,

$$\int_{B_a} \sqrt{\left(\frac{a}{t}\right)}\, Q_1(c_t)\xi_1(\hat{\omega}_t)\, dP^{\xi} \to \phi(c)\int_{\Omega} \sqrt{\kappa}\,\sigma^{-1}\xi'\, d\omega$$

as $a \to \infty$ by the Dominated Convergence Theorem and Lemma 1. Similarly,

$$\int_{B_a} \sqrt{\left(\frac{a}{t}\right)}\, Q_3(c_t)\psi_3(\hat{\omega}_t)\, dP^{\xi} \to (2 + c^2)\phi(c)\int_{\Omega} \sqrt{\kappa}\,\psi_3\xi\, d\omega$$

as $a \to \infty$. To estimate the first term of the right side of (24), expand $\Phi(c_t)$ in a Taylor series about c as $\Phi(c_t) = \Phi(c) + \phi(c_t^{\#})(c_t - c) = \gamma + \phi(c_t^{\#})b_t(\hat{\omega}_t)/\sqrt{t}$, where $|c_t^{\#} - c| \le |c_t - c|$. Thus,

$$\begin{aligned}\int_{B_a} \Phi(c_t)\, dP^{\xi} &= \gamma + \frac{1}{\sqrt{a}}\int_{B_a} \sqrt{\left(\frac{a}{t}\right)}\,\phi(c_t^{\#})b_t(\hat{\omega}_t)\, dP^{\xi} + o\left(\frac{1}{\sqrt{a}}\right)\\ &= \gamma + \frac{1}{\sqrt{a}}\phi(c)\int_{\Omega} \sqrt{\kappa}\, b\xi\, d\omega + o\left(\frac{1}{\sqrt{a}}\right)\end{aligned}$$

as $a \to \infty$, since $P^{\xi}(B_a') = o(1/\sqrt{a})$ and since (20) and (22) imply the convergence of the integrals. Substituting these three limits into (24) yields the following asymptotic expansion for Δ_a,

$$\Delta_a = \gamma + \frac{1}{\sqrt{a}}\phi(c)\int_{\Omega}\left[\sqrt{\kappa}\, b\xi + \sigma^{-1}\sqrt{\kappa}\,\xi' - \tfrac{1}{6}(2 + c^2)\sqrt{\kappa}\,\psi_3\xi\right] d\omega + o\left(\frac{1}{\sqrt{a}}\right) \tag{25}$$

as $a \to \infty$.

It remains to relate Δ_a to $\bar{\gamma}_a(\xi)$. Let v denote the inverse function to ψ', so that $\omega = v(\theta)$. If $\hat{\omega}_n \in \Omega_0$ and n is sufficiently large, then $\sqrt{n}\,(\hat{\theta}_n - \theta)/\hat{\sigma}_n \le c_n$ iff $\theta \ge \hat{\theta}_n - c_n\hat{\sigma}_n/\sqrt{n}$ iff $\omega \ge v(\hat{\theta}_n - c_n\hat{\sigma}_n/\sqrt{n})$; and

$$\begin{aligned} v\left(\hat{\theta}_n - \frac{c_n\hat{\sigma}_n}{\sqrt{n}}\right) &= v(\hat{\theta}_n) - v'(\hat{\theta}_n)\frac{c_n\hat{\sigma}_n}{\sqrt{n}} - \frac{1}{2}v''(\theta_n^{\#})\frac{c_n^2\hat{\sigma}_n^2}{n}\\ &= \hat{\omega}_n - \frac{1}{\sqrt{n}\,\hat{\sigma}_n}\left[c_n - \frac{1}{2\sqrt{n}}v''(\theta_n^{\#})c_n^2\hat{\sigma}_n^3\right]\end{aligned}$$

for some intermediate point $\theta_n^{\#}$ between $\hat{\theta}_n$ and $\hat{\theta}_n - c_n\hat{\sigma}_n/\sqrt{n}$. So, if B_a occurs and a is sufficiently large, then $\sqrt{t}\,(\hat{\theta}_t - \theta)/\hat{\sigma}_t \le c_t$ iff $\omega_t^* \ge -c_t^{\circ}$, where c_n°, $n \ge 1$, are defined by (5) with b_n, $n \ge 1$, replaced by b_n°, $n \ge 1$, where

$$b_n^{\circ}(\hat{\omega}_n) = b_n(\hat{\omega}_n) - \tfrac{1}{2}v''(\theta_n^{\#})c_n^2\hat{\sigma}_n^3$$

for $\hat{\omega}_n \in \Omega_0$ and $n \ge 1$. The functions b_n°, $n \ge 1$, may be extended to $\bar{\Omega}$ in a piecewise constant fashion. It follows that $\bar{\gamma}_a(\xi) = \Delta_a^{\circ} + o(1/\sqrt{a})$ as $a \to \infty$, where Δ_a° is defined by (23) with b_n, $n \ge 1$, replaced by b_n°, $n \ge 1$. If b_n, $n \ge 1$, satisfy conditions (20)–(22) with limit b, then it is easily verified that b_n°, $n \ge 1$, satisfy (20)–(22) with limit $b^{\circ} = b + \frac{1}{2}c^2\psi_3$. Relation (8) now follows from replacing b_n, $n \ge 1$, by b_n°, $n \ge 1$, in (25). □

The need for the more general conditions on b_n, $n \geq 1$, is illustrated by the following example.

EXAMPLE 3 (S.P.R.T). The stopping time of an untruncated S.P.R.T. is of the form (11) with $\varepsilon = 0$. Then $\kappa = |\theta|$ and the function b_0 which makes the coefficient of $1/\sqrt{a}$ disappear has a nonintegrable discontinuity at $\omega = 0$. It is possible to construct an increasing sequence Ω_n, $n \geq 1$, of compact subsets of Ω° for which $\bigcup_{n=1}^{\infty} \Omega_n = \Omega^\circ$ and continuous function b_n, $n \geq 1$, for which $|b_n| \leq \min\{\sqrt{n}, |b_0|\}$ on $\bar{\Omega}$ and $b_n = b_0$ on Ω_n for all $n \geq 1$. Any such sequence satisfies (20)–(22). To see this let Ω_0 denote any compact subinterval of Ω° and observe that $|b_n(\omega)| \leq K_1/|\theta|$ for all $\omega \in \Omega_0$ and $n \geq 1$ for some constant $K_1 = K_1(\Omega_0)$. Thus, $|b_t(\hat{\omega}_t)| \leq K_1/|\hat{\theta}_t| \leq K_1 t_a/a$ and, therefore, $\sqrt{(a/t)}\,|b_t(\hat{\omega}_t)| \leq K_1\sqrt{(t/a)}$ on B_a for all $a \geq 1$. Next, using Wald's Lemma and the fact that $E_\omega[\sup_{n \geq 1} X_n^2/n]$ is bounded w.r.t. $\omega \in \Omega_0$, it is easily seen that there is a constant K_2 for which $E_\omega[t_a/a] \leq K_2/|\theta|$ for all $\omega \in \Omega_0$. Thus,

$$\int_{B_a}\left[\sqrt{\left(\frac{a}{t}\right)}\,|b_t(\hat{\omega}_t)|\right]^{3/2} dP^{\Omega_0} \leq K_1^{3/2} E^{\Omega_0}\left[\left(\frac{t}{a}\right)^{3/4}\right] \leq K_3 \int_{\Omega_0} |\theta|^{-3/4}\, d\omega < \infty$$

for all $a \geq 1$ for some constant K_3. Conditions (20)–(22) follow.

8. A lemma. This section contains a technical lemma which is needed to obtain higher order expansions. As above, elegance has been sacrificed for generality in the statement of some conditions. Let

$$\Delta h(\hat{\theta}, \theta) = \sup\{|h(s) - h(\hat{\theta})| : |s - \hat{\theta}| \leq |\theta - \hat{\theta}|\} \tag{26}$$

for functions h defined on Θ.

LEMMA 2. *Let t_a, $a \geq 1$, denote stopping times which satisfy* (3) *and* (4) *with $p = 1$. Let $\Omega_0 = [\omega_0, \omega_1]$ be a compact subinterval of Ω°; let $\Theta_0 = \psi'(\Omega_0)$; and let g be a continuously differentiable function on Θ_0° for which*

$$|g'(\theta)| \leq K\left[(\omega - \omega_0)_+^{-2} + (\omega_1 - \omega)_+^{-2}\right] \tag{27}$$

for all $\theta = \psi'(\omega) \in \Theta_0^\circ$ for some constant K. Define B_a, $a \geq 1$, by (19) *and let U_n, $n \geq 1$, be uniformly bounded and $\mathscr{D}_n$, $n \geq 1$, measurable. Then*

$$\int_{B_a} U_t \sqrt{\left(\frac{a}{t}\right)}\,[g(\theta) - g(\hat{\theta}_t)]\, dP^\xi = o(1/\sqrt{a}) \tag{28}$$

as $a \to \infty$ for all ξ of the form (7) *with support Ω_0 and $q \geq 3$.*

Moreover, (28) *continues to hold if g is replaced by a sequence g_n, $n \geq 1$, of continuously differentiable functions for which*

$$\left\{\int_{B_a}\left[\sqrt{\left(\frac{a}{t}\right)}\,|g_t'(\hat{\theta}_t)|\right]^{3/2} dP^{\Omega_0}\right\}^{2/3} = o(\sqrt{a}) \tag{29}$$

and

$$\int_{B_a}\left[\sqrt{\left(\frac{a}{t}\right)}\,\Delta g_t'(\hat\theta_t,\theta)\right]^{\alpha} dP^{\Omega_0} = o(1) \tag{30}$$

as $a \to \infty$ *for some* $\alpha > 1$, *where* P^{Ω_0} *is as in* (21).

PROOF. If g is continuously differentiable on Θ_0°, then the integrand in (28) may be written $U_t\sqrt{(a/t)}\,[g(\theta) - g(\hat\theta_t)] = I_a + II_a$, where $I_a = U_t\sqrt{(a/t)}\;g'(\hat\theta_t)(\theta - \hat\theta_t)$ and $II_a = U_t\sqrt{(a/t)}\,[g'(\theta_t^{\#}) - g'(\hat\theta_t)](\theta - \hat\theta_t)$ for some intermediate point $\theta_t^{\#}$ on B_a for all $a \geq 1$. A simple integration by parts shows that $E^{\xi}(\theta - \hat\theta_t|\mathscr{D}_t) = (1/t)E^{\xi}(\sigma\xi_1|\mathscr{D}_t)$ on B_a for $a \geq 1$. So,

$$\left|\int_{B_a} I_a\,dP^{\xi}\right| \leq \left|\int_{B_a} U_t\sqrt{\left(\frac{a}{t}\right)}\,g'(\hat\theta_t)\left(\frac{1}{t}\right)E^{\xi}(\sigma\xi_1|\mathscr{D}_t)\,dP^{\xi}\right|$$

$$\leq K_1 a^{-1}\left\{\int_{B_a}\left[\sqrt{\left(\frac{a}{t}\right)}\,|g'(\hat\theta_t)|\right]^{3/2} dP^{\xi}\right\}^{2/3}\left\{\int_{\Omega}|\xi_1|^3\xi\,d\omega\right\}^{1/3}$$

for all $a \geq 1$ for some constant K_1; and this is of smaller order of magnitude than $1/\sqrt{a}$ under both sets of assumptions on g, by Lemma 1 under the first set. Let $\alpha > 1$ and $\beta > 1$ satisfy $1/\alpha + 1/\beta = 1$. Then

$$\sqrt{a}\int_{B_a}|II_a|\,dP^{\xi} \leq K_2\left\{\int_{B_a}\left[\sqrt{\left(\frac{a}{t}\right)}\,|g'(\theta_t^{\#}) - g'(\hat\theta_t)|\right]^{\alpha} dP^{\xi}\right\}^{1/\alpha}$$

$$\times\left\{\int \sup_{n\geq\eta a}\left[\sqrt{a}\,|\hat\theta_n - \theta|\right]^{\beta} dP^{\xi}\right\}^{1/\beta}$$

for all $a \geq 1$ for some constant K_2. The second factor remains bounded as $a \to \infty$ for any $\beta > 1$. If (27) holds, then the first factor approaches zero as $a \to \infty$ for any $\alpha < \frac{3}{2}$ by Lemma 1 and the Dominated Convergence Theorem; and if (30) holds, then the first factor approaches zero for some $\alpha > 1$. □

9. Higher-order expansions. There are three important differences between Theorem 2, which computes $\bar\gamma_a(\xi)$ up to $o(1/a)$, and Theorem 1: the algebra is substantially more cumbersome; the analysis is slightly more complicated, since the coefficients of 1 and $1/\sqrt{a}$ must be computed more accurately; and some additional conditions are needed to justify the more delicate calculations.

In Theorem 2 it is assumed that the stopping times t_a, $a \geq 1$, satisfy (3) and (4) with $p = 1$. In addition, it is assumed that

$$\int_{\Omega_0}\left|E_{\omega}\left\{\left[\sqrt{\left(\frac{a}{t}\right)} - \sqrt{\kappa(\omega)}\right]I_{[t\geq\eta a]}\right\}\right|^2 \frac{d\omega}{\kappa(\omega)} = o(1/a) \tag{31}$$

as $a \to \infty$ with $\eta = \eta(\Omega_0)$ as in (4) for all compact $\Omega_0 \subset \Omega^\circ$. Further, it is

assumed that the functions b_n, $n \geq 1$, of (5) are of the form

$$b_n = b_{1n} + \frac{1}{\sqrt{n}} b_{2n}, \tag{32}$$

where b_{1n}, $n \geq 1$, and b_{2n}, $n \geq 1$, both satisfy conditions (21) and (22), with limits denoted by b_1 and b_2, and b_{1n}^2, $n \geq 1$, and b_{2n}, $n \geq 1$, both satisfy condition (20) with $p = 1$. Finally, letting $g_n(\theta) = b_{1n}(\omega)$ for $\theta = \psi'(\omega) \in \Theta$, it is assumed that g_n, $n \geq 1$, satisfy (29) and (30).

THEOREM 2. *Suppose that t_a, $a \geq 1$, and b_n, $n \geq 1$, satisfy the conditions imposed in the previous paragraph. Then*

$$\bar{\gamma}_a(\xi) = \gamma + \frac{1}{\sqrt{a}}\phi(c)\bar{\Gamma}_1(\xi) + \frac{1}{a}\phi(c)\bar{\Gamma}_2(\xi) + o\left(\frac{1}{a}\right)$$

as $a \to \infty$ for all ξ of the form (7) *with $q \geq 3$. Here $\bar{\Gamma}_1(\xi)$ is as in* (8) *and*

$$\begin{aligned}\bar{\Gamma}_2(\xi) = &\int_\Omega \left[\tfrac{1}{8}(c - c^3)\psi_4 - \tfrac{1}{72}(4c^5 - 31c^3 + 15c)\psi_3^2\right]\kappa\xi\, d\omega \\ &+ \int_\Omega \left[\tfrac{1}{6}(3c - 2c^3)\sigma^{-1}\psi_3\xi' - \tfrac{1}{2}c\sigma^{-2}\xi'' - c\sigma^{-1}b_1\xi'\right]\kappa\, d\omega \\ &+ \int_\Omega \left[\tfrac{1}{3}(3c - c^3)b_1\psi_3 + b_2 - \tfrac{1}{2}cb_1^2\right]\kappa\xi\, d\omega.\end{aligned} \tag{33}$$

PROOF. Fix a density ξ of the form (7) with $q \geq 3$ and support $\Omega_0 = [\omega_0, \omega_1] \subset \Omega^\circ$; and define B_a, $a \geq 1$, by (19). Then $P^\xi(B_a') = o(1/a)$ as $a \to \infty$ by (4) and Lemma 1. So, $\bar{\gamma}_a(\xi) = P^\xi\{\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t \leq c_t,\ B_a\} + o(1/a)$ as $a \to \infty$, as in the proof of Theorem 1. Define Δ_a, $a \geq 1$, by (23). Then, also as in the proof of Theorem 1,

$$\Delta_a = \beta_0(a) + \frac{1}{\sqrt{a}}\beta_1(a) + \frac{1}{a}\beta_2(a) + \left(\frac{1}{\sqrt{a}}\right)^3\beta_3(a), \tag{34}$$

where

$$\beta_0(a) = \int_{B_a} \Phi(c_t)\, dP^\xi,$$

$$\beta_1(a) = \int_{B_a} \sqrt{\left(\frac{a}{t}\right)}\left[Q_1(c_t)\xi_1(\hat{\omega}_t) - \tfrac{1}{6}Q_3(c_t)\psi_3(\hat{\omega}_t)\right] dP^\xi,$$

$$\begin{aligned}\beta_2(a) = \int_{B_a} \left(\frac{a}{t}\right)\Big[&\tfrac{1}{2}Q_2(c_t)\xi_2(\hat{\omega}_t) - \tfrac{1}{6}Q_4(c_t)\psi_3(\hat{\omega}_t)\xi_1(\hat{\omega}_t) \\ &- \tfrac{1}{24}Q_4(c_t)\psi_4(\hat{\omega}_t) + \tfrac{1}{72}Q_6(c_t)\psi_3(\hat{\omega}_t)^2\Big]\, dP^\xi,\end{aligned}$$

and

$$\beta_3(a) = \int_{B_a} \left(\frac{a}{t}\right)^{3/2} R_t^1\, dP^\xi$$

for $a \geq 1$. See (13), (14), and (16).

By (16) and Lemma 1, $\beta_3(a) = O(1)$ as $a \to \infty$. So, the final term in (34) is $o(1/a)$ as $a \to \infty$. Next, the Law of Large Numbers, the Dominated Convergence

Theorem, and Lemma 1 combine to show that

$$\beta_2(a) \to \beta_2^0 = \tfrac{1}{2}Q_2(c)\int_\Omega \kappa\xi_2\xi\,d\omega - \tfrac{1}{6}Q_4(c)\int_\Omega \kappa\psi_3\xi_1\xi\,d\omega$$
$$- \tfrac{1}{24}Q_4(c)\int_\Omega \kappa\psi_4\xi\,d\omega + \tfrac{1}{72}Q_6(c)\int_\Omega \kappa\psi_3^2\xi\,d\omega$$

as $a \to \infty$, as in the proof of Theorem 1. For β_1, expand $Q_1(c_t)$ and $Q_3(c_t)$ in Taylor series about c to find that

$$\beta_1(a) = Q_1(c)\int_{B_a}\sqrt{\left(\frac{a}{t}\right)}\,\xi_1(\hat\omega_t)\,dP^\xi - \tfrac{1}{6}Q_3(c)\int_{B_a}\sqrt{\left(\frac{a}{t}\right)}\,\psi_3(\hat\omega_t)\,dP^\xi$$
$$+\frac{1}{\sqrt{a}}\int_{B_a}\left(\frac{a}{t}\right)b_t(\hat\omega_t)\left[Q_1'(c_t')\xi_1(\hat\omega_t) - \tfrac{1}{6}Q_3'(c_t'')\psi_3(\hat\omega_t)\right]dP^\xi$$
$$= \beta_{11}(a) - \beta_{12}(a) + \frac{1}{\sqrt{a}}\beta_2^1(a), \quad \text{say,}$$

where c_t' and c_t'' are intermediate points. As above, it is easily seen that

$$\beta_2^1(a) \to \beta_2^1 = Q_1'(c)\int_\Omega \kappa b_1\xi_1\xi\,d\omega - \tfrac{1}{6}Q_3'(c)\int_\Omega \kappa b_1\psi_3\xi\,d\omega.$$

Let

$$\beta_{11}^0 = Q_1(c)\int_\Omega \sqrt{\kappa}\,\xi_1\xi\,d\omega \quad \text{and} \quad \beta_{12}^0 = \tfrac{1}{6}Q_3(c)\int_\Omega \sqrt{\kappa}\,\psi_3\xi\,d\omega.$$

Then

$$\beta_{11}(a) - \beta_{11}^0 = Q_1(c)\int_{B_a}\sqrt{\left(\frac{a}{t}\right)}\left[\xi_1(\hat\omega_t) - \xi_1(\omega)\right]dP^\xi$$
$$+Q_1(c)\int_{B_a}\left[\sqrt{\left(\frac{a}{t}\right)} - \sqrt{\kappa}\,(\omega)\right]\xi_1(\omega)\,dP^\xi - Q_1(c)\int_{B_a'}\sqrt{\kappa}\,\xi_1\,dP^\xi \tag{35}$$

for all $a \geq 1$. Here the first term on the right is $o(1/\sqrt{a})$ as $a \to \infty$, by Lemma 2; the second is $o(1/\sqrt{a})$ as $a \to \infty$ by (31); and the third is $o(1/\sqrt{a})$ as $a \to \infty$ by Schwarz's Inequality, since $P^\xi(B_a') = o(1/a)$ as $a \to \infty$ and $\sqrt{\kappa}\,\xi_1$ is square integrable w.r.t. P^ξ. This shows that $|\beta_{11}(a) - \beta_{11}^0| = o(1/\sqrt{a})$ as $a \to \infty$; and a similar argument shows that $|\beta_{12}(a) - \beta_{12}^0| = o(1/\sqrt{a})$, as $a \to \infty$ too. Finally, consider $\beta_0(a)$, $a \geq 1$. Expanding $\Phi(c_t)$ in a Taylor series about c and using $P^\xi(B_a') = o(1/a)$ yields

$$\beta_0(a) = \Phi(c) + \frac{1}{\sqrt{a}}\phi(c)\int_{B_a}\sqrt{\left(\frac{a}{t}\right)}\,b_{1t}(\hat\omega_t)\,dP^\xi$$
$$+\frac{1}{a}\int_{B_a}\left[\phi(c)\left(\frac{a}{t}\right)b_{2t}(\hat\omega_t) - \tfrac{1}{2}c_t^{\#}\phi\left(c_t^{\#}\right)\left(\frac{a}{t}\right)b_t^2(\hat\omega_t)\right]dP^\xi + o\left(\frac{1}{a}\right)$$
$$= \gamma + \frac{1}{\sqrt{a}}\beta_1^1(a) + \frac{1}{a}\beta_2^2(a) + o\left(\frac{1}{a}\right), \quad \text{say,}$$

as $a \to \infty$ for some intermediate point $c_t^{\#}$ between c_t and c. As in the proof of Theorem 1, conditions (20)–(22) yield

$$\beta_2^2(a) \to \beta_2^2 = \phi(c)\int_\Omega \kappa\left[b_2 - \tfrac{1}{2}cb_1^2\right]\xi\, d\omega$$

as $a \to \infty$. Let $\beta_1^1 = \phi(c)\int_\Omega \sqrt{\kappa}\, b_1 \xi\, d\omega$. Then $|\beta_1^1(a) - \beta_1^1| = o(1/\sqrt{a})$ as $a \to \infty$ by an argument similar to that of (35), using the second set of conditions in Lemma 2 and the integrability of $\kappa b_1^2 \xi$.

Let $\beta_1 = \beta_1^0 + \beta_1^1$ and $\beta_2 = \beta_2^0 + \beta_2^1 + \beta_2^2$, where $\beta_1^0 = \beta_{11}^0 - \beta_{12}^0$. Then

$$\Delta_a = \Phi(c) + \frac{1}{\sqrt{a}}\beta_1 + \frac{1}{a}\beta_2 + o\left(\frac{1}{a}\right) \tag{36}$$

as $a \to \infty$. To complete the proof, let v denote the inverse function to ψ'. If a is sufficiently large and if B_a occurs, then it is easily seen that $\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t \le c_t$ iff $\omega_t^* = \sqrt{t}\,\hat{\sigma}_t(\omega - \hat{\omega}_t) \ge -c_t^\circ$, where c_n°, $n \ge 1$, are defined by (5) and (32), but with b_{1n}, $n \ge 1$, and b_{2n}, $n \ge 1$, replaced by b_{1n}°, $n \ge 1$, and b_{2n}°, $n \ge 1$, where

$$b_{1n}^\circ(\hat{\omega}_n) = b_{1n}(\hat{\omega}_n) + \tfrac{1}{2}c^2\psi_3(\hat{\omega}_n)$$

and

$$b_{2n}^\circ(\hat{\omega}_n) = b_{2n}(\hat{\omega}_n) + \psi_3(\hat{\omega}_n)\left[cb_n(\hat{\omega}_n) + \frac{1}{2\sqrt{n}}b_n^2(\hat{\omega}_n)\right] + \tfrac{1}{6}v'''\left(\theta_n^{\#}\right)c_n^3\hat{\sigma}_n^4$$

for some intermediate point $\theta_n^{\#}$ between $\hat{\theta}_n$ and $\hat{\theta}_n - c_n\hat{\sigma}_n/\sqrt{n}$ for $\hat{\omega}_n \in \Omega_0$ and sufficiently large n. If b_{1n}, $n \ge 1$, and b_{2n}, $n \ge 1$, satisfy the conditions of the theorem with limits b_1 and b_2, then it is easily seen that b_{1n}°, $n \ge 1$, and b_{2n}°, $n \ge 1$, satisfy these conditions too, but with limits $b_1^\circ = b_1 + c^2\psi_3/2$ and $b_2^\circ = b_2 + c\psi_3 b_1 - c^3\psi_4/6 + c^3\psi_3^2/2$. Let Δ_a°, $a \ge 1$, denote Δ_a, $a \ge 1$, when c_n, $n \ge 1$, are replaced by c_n°, $n \ge 1$. Then $\bar{\gamma}_a(\xi) = \Delta_a^\circ + o(1/a)$ as $a \to \infty$. The theorem now follows from (36) and simple (if tedious) algebra.

Recall that $\bar{\Gamma}_1(\xi)$ may be written in the form (9), if $\sqrt{\kappa}$ is absolutely continuous, and that $\bar{\Gamma}_1(\xi) = 0$ for all ξ of the form (7), if $\sqrt{\kappa}$ is absolutely continuous and $\sqrt{\kappa}\, b = \sigma^{-1}(\sqrt{\kappa})' - \frac{1}{6}(1 + 2c^2)\sqrt{\kappa}\psi_3$ a.e. Analogous results hold for $\bar{\Gamma}_2$. □

COROLLARY 3. *Suppose that $\sqrt{\kappa}$ is absolutely continuous on Ω°, that $(\sqrt{\kappa})'$ is locally square integrable, and that $\sqrt{\kappa}\, b_1 = \sigma^{-1}(\sqrt{\kappa})' + \sqrt{\kappa}\, g$, where g is absolutely continuous. Then there is a function Γ_2 on Ω° for which*

$$\bar{\Gamma}_2(\xi) = \int_\Omega \Gamma_2(\omega)\xi(\omega)\, d\omega \tag{37}$$

for all densities ξ of the form (7).

PROOF. If $\sqrt{\kappa}$ is absolutely continuous on Ω°, then so is κ and $\kappa' = 2\sqrt{\kappa}(\sqrt{\kappa})'$; and, if $\sqrt{\kappa}\, b_1 = \sigma^{-1}(\sqrt{\kappa})' + \sqrt{\kappa}\, g$, where g is absolutely continuous, then the

middle line in (33) may be written

$$\int_\Omega \left[\tfrac{1}{6}(3c - 2c^3)\sigma^{-1}\psi_3\kappa\xi' - \tfrac{1}{2}c\sigma^{-2}(\kappa\xi')' - c\sigma^{-1}\kappa g\xi'\right] d\omega$$

(38)

$$= -\int_\Omega \left\{\tfrac{1}{6}(3c - 2c^3)(\sigma^{-1}\psi_3\kappa)' + \tfrac{1}{2}c\left[\kappa(\sigma^{-2})'\right]' - c(\sigma^{-1}\kappa g)'\right\}\xi\, d\omega.$$

The corollary follows directly. □

The expression for Γ_2 is complicated and not especially enlightening. However, the following two special cases are of interest.

COROLLARY 4. *Suppose that $\sqrt{\kappa}$ is absolutely continuous and that $(\sqrt{\kappa})'$ is locally square integrable. If $\sqrt{\kappa}\, b_1 = \sigma^{-1}(\sqrt{\kappa})' - \frac{1}{6}(1 + 2c^2)\sqrt{\kappa}\psi_3$ and $\kappa b_2 = \frac{1}{24}(c + 3c^3)\kappa\psi_4 + \frac{1}{72}(4c - 7c^3)\kappa\psi_3^2 - \frac{5}{12}c\sigma^{-1}\kappa\psi_3 + \frac{1}{2}c\sigma^{-2}(\sqrt{\kappa})'^2$, then $\bar{\Gamma}_1(\xi) = \bar{\Gamma}_2(\xi) = 0$ for all ξ of the form (7).*

COROLLARY 5. *Suppose that κ is continuously differentiable and that κ' is absolutely continuous. If $b_1 = b_2 = 0$, then*

$$\Gamma_2 = \tfrac{1}{24}(3c + 5c^3)\kappa\psi_4 - \tfrac{1}{72}(15c + 17c^3 + 4c^5)\kappa\psi_3^2 + \tfrac{1}{6}(3c + 2c^3)\sigma^{-1}\kappa'\psi_3 - \tfrac{1}{2}\sigma^{-2}\kappa''.$$

PROOFS. Corollary 3 follows by setting $g = -\frac{1}{6}(1 + 2c^2)\psi_3$ in Corollary 2. Corollary 4 follows from an integration by parts which is similar to that described in (38).

Of course, Corollary 5 gives a very weak asymptotic expansion for the distribution function of $\sqrt{t}(\hat{\theta}_t - \theta)/\hat{\sigma}_t$ as $a \to \infty$. Unfortunately, the condition that κ' be absolutely continuous is violated in many examples, including Examples 1 and 2.

The corrections described in Corollary 4 are applicable in Examples 1 and 2, after some smoothing. They are not applicable in Example 3, since $(\sqrt{\kappa})'$ is not square integrable near zero in the example. □

10. Concluding remarks. The condition that t_a be stopping times w.r.t. $\mathscr{D}_n$, $n \geq 1$, may be relaxed. It is only necessary that $P_\omega\{t_a \geq n\}$ be independent of ω for all n and a.

Ghosh et al. (1982) consider asymptotic expansions for posterior distributions when the prior density ξ decays exponentially at the endpoints of its support, as well as ξ of the form (7). It seems likely that Theorems 1 and 2 hold for such ξ too.

There is no statement of uniformity w.r.t. ξ in Theorems 1 and 2; but it would be desirable to have one. In particular, when $\bar{\gamma}_a(\xi)$ is thought of as an average of γ_a near a given point ω, it would be desirable to allow the support of ξ to shrink to ω as $a \to \infty$.

If F_ω is the normal distribution with mean $\theta = \omega$ and unit variance for $-\infty < \omega < \infty$, then $\psi'' = 1$ and $\psi_j = 0$ for all $j \geq 3$. So, the function b of

Corollary 2 is simply $b = \kappa'/2\kappa$ and the confidence interval described there may be written $\mathscr{I}_t = [\tilde{\theta}_t - c/\sqrt{t}, \infty)$, where $\tilde{\theta}_t = \hat{\theta}_t - b_t(\hat{\theta}_t)/t$, in this case. Since b does not depend on c, the sequence b_n, $n \geq 1$, may be chosen independently of c (if at all), and the term $b_t(\hat{\theta}_t)/t$ may be regarded as a correction for bias. In the context of Example 2, it agrees with the bias correction suggested by Siegmund (1978).

In the normal case with $\theta = \omega > 0$ and $t_a = \inf\{n \geq 1: S_n > a\}$ for $a \geq 1$, the very weak expansions of Theorem 1 agree with stronger (fixed θ) expansions for an analogous problem with Brownian motion. To see how, let $B(s)$, $0 \leq s < \infty$, denote Brownian motion with drift $\theta > 0$ and unit variance, both per unit time; let $\tau = \tau_a = \inf\{s \geq 0: B(s) > a\}$ for $a \geq 1$; and let $B_a^{\#} = [B(\tau_a) - \mu\tau_a]/\sqrt{\tau_a}$ for $a \geq 1$. If $-\infty < z < \infty$ and a is sufficiently large, then it is easily seen that $B_a^{\#} \leq z$ iff $\tau_a \geq m$, where m solves the equation $a - \mu m = z\sqrt{m}$. Using a well-known formula for the distribution function of τ_a [for example, Siegmund (1984), Equation (3.15)] and some simple analysis,

$$P_\theta\{B_a^{\#} \leq z\} = P_\theta\{\tau_a \geq m\} = \Phi\left(\frac{a - m\mu}{\sqrt{m}}\right) - e^{2a\mu}(1 - \Phi)\left(\frac{a + m\mu}{\sqrt{m}}\right)$$

$$= \Phi(z) - \frac{1}{2\sqrt{(a\mu)}}\phi(z) + o\left(\frac{1}{\sqrt{a}}\right),$$

where $o(1/\sqrt{a})$ is uniform on compacts in $-\infty < z < \infty$. The same relation was obtained very weakly for the discrete problem in Theorem 1.

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