The Annals of Statistics
1986, Vol. 14, No. 2, 487–501

CHARACTERIZATION OF EXTERNALLY BAYESIAN POOLING OPERATORS

BY CHRISTIAN GENEST, KEVIN J. MCCONWAY, AND MARK J. SCHERVISH

University of Waterloo, The Open University, and Carnegie-Mellon University

When a panel of experts is asked to provide some advice in the form of a group probability distribution, the question arises as to whether they should synthesize their opinions before or after they learn the outcome of an experiment. If the group posterior distribution is the same whatever the order in which the pooling and the updating are done, the pooling mechanism is said to be externally Bayesian by Madansky (1964). In this paper, we characterize all externally Bayesian pooling formulas and we give conditions under which the opinion of the group will be proportional to the geometric average of the individual densities.

1. Introduction. Let (Θ, μ) be a measure space and let Δ be the class of μ-measurable functions $f\colon \Theta \to [0, \infty)$ such that $f > 0$ μ-a.e. and $\int f\, d\mu = 1$. In the language of multiagent statistical decision theory (cf. Weerahandi and Zidek, 1981), a *pooling operator* is any function $T\colon \Delta^n \to \Delta$ which may be used to extract a "consensus" $T(f_1, \dots, f_n)$ from the different subjective opinions $f_1, \dots, f_n \in \Delta$ of the n members of a group. The current interest for pooling operators seems to stem from a theorem due to Wald (1939) concerning the optimality of Bayesian decision rules. When formulated in the context of a group decision problem, this theorem suggests that at least in the case where all the members of the group have the same utility function, it is generally preferable for them to agree on an "average opinion" $T(f_1, \dots, f_n)$ and to adopt that action which maximizes their common utility with respect to $T(f_1, \dots, f_n)$, rather than to take an "average decision" based on the optimal decisions of each of the individuals (see de Finetti, 1954).

A few years ago, Madansky (1964, 1978) suggested the use of pooling formulas, T, which have the following property:

$$(1.1)\qquad \begin{aligned} &T\left(lf_1\Big/\int lf_1\, d\mu, \dots, lf_n\Big/\int lf_n\, d\mu\right) \\ &\qquad = lT(f_1, \dots, f_n)\Big/\int lT(f_1, \dots, f_n)\, d\mu, \qquad \mu\text{-a.e.,} \end{aligned}$$

whenever $l\colon \Theta \to (0, \infty)$ is such that

$$(1.2)\qquad 0 < \int lf_i\, d\mu < \infty, \qquad i = 1, \dots, n.$$

Received October 1984; revised June 1985.
AMS 1980 *subject classifications*. Primary 62A99; secondary 39B40.
Key words and phrases. Consensus, expert opinions, external Bayesianity, logarithmic pool.

A function l which satisfies condition (1.2) above is hereafter called a *likelihood for* $(f_1, \dots, f_n)$. Implicit in the statement of (1.1) is that the integral $\int lT(f_1, \dots, f_n)\, d\mu$ is finite whenever l is a likelihood for $(f_1, \dots, f_n)$.

A different but obviously equivalent formulation of Madansky's condition would be to specify that

$$f_1/g_1 \propto \cdots \propto f_n/g_n \propto l \Rightarrow T(f_1, \dots, f_n)/T(g_1, \dots, g_n) \propto l. \tag{1.3}$$

Pooling operators which obey (1.1) or (1.3) are called *externally Bayesian*. By adopting an externally Bayesian formula, a group assures itself that when additional information l is perceived jointly, their collective opinion can be modified (using Bayes rule), producing the same result as if the pooling operator had been applied after each individual distribution has been revised. Raiffa (1968, pages 221–226) shows with reference to a concrete example how using a pooling formula which fails (1.1) may lead the members of the group to act strangely. In Raiffa's example, as it were, all the individuals try to increase the influence of their opinion on the consensus by insisting that it be computed before the outcome of an experiment is known. This happens because the members of the group know that whether the f_i's are updated or not, the consensus will be computed using the same weighted arithmetic average of their opinions, viz.

$$T(f_1, \dots, f_n) = \sum_{i=1}^{n} w_i f_i, \qquad \mu\text{-a.e.}, \tag{1.4}$$

a formula which violates (1.1) unless $w_i = 1$ for some i and $w_j = 0$ for all $j \neq i$.

To ensure that the order in which the pooling and updating are done is immaterial, Bacharach (1972) suggests that the consensus should be computed using a logarithmic pooling operator, viz.

$$T(f_1, \dots, f_n) = \prod_{i=1}^{n} f_i^{w_i} \Big/ \int \prod_{i=1}^{n} f_i^{w_i}\, d\mu, \qquad \mu\text{-a.e.}, \tag{1.5}$$

where $w_1, \dots, w_n$ are nonnegative weights such that $\sum_{i=1}^{n} w_i = 1$ as in (1.4). According to Bacharach, it is Peter Hammond who first observed that the operator (1.5) is externally Bayesian. In a recent article, Genest (1984b) has shown that this is also the only solution of the functional equation (1.1) when there exists a function G: $(0, \infty)^n \to (0, \infty)$ which is Lebesgue measurable and such that

$$T(f_1, \dots, f_n)(\theta) \propto G(f_1(\theta), \dots, f_n(\theta)), \qquad \mu\text{-a.e.}, \tag{1.6}$$

where the proportionality constant must be independent of θ. (A precise statement of this result is to be found in Section 2.) The formula (1.6) means that, except for a normalizing constant, the value of T at a particular θ depends on the f_i's only through their values at θ. The import of this theorem is still limited, however, especially because the proof given in Genest (1984b) does not apply when the space (Θ, μ) is purely atomic, an assumption which excludes the important case where Θ is finite or countable.

In this paper, we will provide all the solutions of the functional equation (1.1), that is, without restriction to those operators which obey (1.6) and without imposing regularity conditions on the space (Θ, μ). The form of the solutions is given in (2.1) below and is worked out explicitly in Section 3 for the case where individual opinions are represented as probabilities of a single event. When the function G in (1.6) is indexed by θ and (Θ, μ) can be partitioned into at least four nonnegligible sets, we show that all the solutions must be of the form

$$(1.7) \qquad T(f_1, \dots, f_n) = g \prod_{i=1}^{n} f_i^{w_i} \Big/ \int g \prod_{i=1}^{n} f_i^{w_i}\, d\mu, \qquad \mu\text{-a.e.},$$

g being an arbitrary bounded function on Θ and $w_1, \dots, w_n$ being arbitrary weights adding up to 1. Pooling operators of the form (1.7) have already been characterized by McConway (1978), but only in the case where the measure μ is purely atomic (thereby forcing Θ to be countable). Obviously, the weights in (1.7) can be negative, but only when Θ is finite.

It may not always be reasonable to require a pooling formula to satisfy the criterion of Madansky (1964, 1978), viz. (1.1). Such would be the case if, for instance, the group itself were not required to make a decision, but rather were only asked to provide a group opinion to an external decision maker. In situations where a decision maker is present, French (1981, 1985) and Lindley (1985) have argued rightly that it would seem more reasonable for this decision maker to adopt the Bayesian approach and to treat the opinions of the members of the group as data which he/she would use to update his/her own subjective probability distribution. In certain circumstances, French and Lindley have shown how the formula $T(f_1, \dots, f_n)$ representing the decision maker's opinion after hearing the experts could well violate (1.1). In this paper, however, we address what French (1985) would call the "group decision problem," that in which there is no natural decision maker and the group is either unwilling or unable to provide one.

Whether it involves a decision maker or not, the problem of determining a sensible formula for representing the opinions of a group has received a lot of attention in recent years. Let us mention, for example, the papers of French (1980, 1981), Morris (1974, 1977), Winkler (1968, 1981), and Genest and Schervish (1985), all of which adopt the Bayesian viewpoint. References on the so-called "group problem" include Laddaga (1977), McConway (1981), Wagner (1982, 1984), and Genest (1984a, c). An extensive bibliography has recently been compiled on both versions of the problem by Genest and Zidek (1986).

2. Characterizing externally Bayesian pooling operators. First observe that the class Δ is nonempty if and only if the measure μ is σ-finite, and that the operator (1.7) is well-defined since

$$0 < \int g \prod_{i=1}^{n} f_i^{w_i}\, d\mu \le \|g\|_\infty \prod_{i=1}^{n} \left[\int f_i\, d\mu \right]^{w_i} = \|g\|_\infty < \infty$$

by Hölder's inequality. For convenience, we will assume that every singleton subset of Θ is measurable. The following theorem is a slight modification of a theorem of Genest (1984b).

THEOREM 2.1. *Let* (Θ, μ) *be a measure space, and suppose that* μ *is not purely atomic. Let also* T: $\Delta^n \to \Delta$ *be a pooling operator for which* (1.6) *holds. Then* T *is externally Bayesian if and only if it is logarithmic, i.e., if and only if there exist nonnegative weights* $w_1, \dots, w_n$ *such that* $\sum_{i=1}^n w_i = 1$ *for which* (1.5) *holds.*

PROOF. The proof is the same as in Genest (1984b), since the σ-field on which μ is defined *always* contains nonnegligible sets with arbitrarily small measure, except in the case where the measure μ is purely atomic. (See Halmos (1950), Exercise 1, page 174). □

In the following, our main objective is to generalize Theorem 2.1 by characterizing all the pooling operators which have property (1.1) without imposing condition (1.6), and without restricting the underlying measure space (Θ, μ). To do this, we first consider the case in which the "group" consists of a single expert, and we show that every externally Bayesian pooling operator is then of the form (1.7).

THEOREM 2.2. *Let* T: $\Delta \to \Delta$ *be a pooling operator. Then* T *is externally Bayesian if and only if there exists a bounded function* g: $\Theta \to [0, \infty)$ *such that* $g > 0$ μ*-a.e. and*

$$T(f) = gf \Big/ \int gf \, d\mu, \qquad \mu\text{-}a.e.,$$

for all μ*-densities* f *in* Δ.

PROOF. Let f and h be arbitrary in Δ. Set $g = T(h)/h$ and consider the likelihood function $l = f/h$. Since T is externally Bayesian, we have

$$T(f) = T\left(lh \Big/ \int lh \, d\mu\right) = lT(h) \Big/ \int lT(h) \, d\mu = gf \Big/ \int gf \, d\mu, \qquad \mu\text{-a.e.},$$

for all $f \in \Delta$. We also have $\int gf \, d\mu < \infty$ for all f, which implies that g is essentially bounded. (See, for example, Theorem 20.15 in Hewitt and Stromberg (1965).) The definition of T does not depend on the choice of h, since g is unique up to a constant multiple. □

In the case where $n > 1$, the basic idea consists of reducing the problem to the context of Theorem 2.2 by dividing the domain of T into equivalence classes in such a way that, given the value of T at one member of an equivalence class, the externally Bayesian property defines the value of T at all other members of that class.

DEFINITION 2.3. Two vectors $(f_1, \dots, f_n)$ and $(f_1^*, \dots, f_n^*)$ in Δ^n are said to be equivalent and we write $(f_1, \dots, f_n) \sim (f_1^*, \dots, f_n^*)$ if and only if

$$\forall i, j \, \exists \, c_{i,j} > 0 \quad \text{such that} \quad f_i/f_j = c_{i,j}(f_i^*/f_j^*), \qquad \mu\text{-a.e.}$$

It is obvious that $\sim$ is an equivalence relation on Δ^n and that two vectors of densities belong to the same equivalence class if and only if there exists a likelihood function $l: \Theta \to (0, \infty)$ such that

$$f_i^* = lf_i \Big/ \int lf_i \, d\mu, \qquad \mu\text{-a.e.},$$

for all $i = 1, \dots, n$. In the sequel, we use the Greek letter α to refer to an arbitrary element of the quotient-space $\mathscr{A} = \Delta^n/\sim$. For each $\alpha \in \mathscr{A}$, we denote $(f_1^\alpha, \dots, f_n^\alpha)$ an arbitrary but fixed vector of μ-densities in α. That such a representative can be chosen for each equivalence class α follows from the Axiom of Choice.

EXAMPLE 2.4. Let $(f_1^\alpha, \dots, f_n^\alpha)$ be a vector of exponential densities such that $f_i^\alpha(\theta) = \lambda_i \exp\{-\lambda_i \theta\}$ for $\theta > 0$, λ_i being a nonnegative parameter, $i = 1, \dots, n$. It is easy to see that a vector $(f_1, \dots, f_n)$ belongs to the same equivalence class as $(f_1^\alpha, \dots, f_n^\alpha)$ if and only if $\int f_1 \exp\{(\lambda_1 - \lambda_k)\theta\} \, d\theta < \infty$ for all $k = 1, \dots, n$, and $f_i \propto f_1 \exp\{(\lambda_1 - \lambda_i)\theta\}$, μ-a.e. for $i > 1$. The condition on f_1 means that its moment generating function is finite at each point $\lambda_1 - \lambda_k$, $k = 2, \dots, n$.

We are now in a position to state the main result of this section.

THEOREM 2.5. *Let T: $\Delta^n \to \Delta$ be an arbitrary pooling operator. Then T is externally Bayesian if and only if*

$$T(f_1, \dots, f_n) \propto b_\alpha v_\alpha f_1/f_1^\alpha, \qquad \mu\text{-a.e.}, \tag{2.1}$$

where (using the above notation) α represents the equivalence class of $(f_1, \dots, f_n)$, and for each α, b_α is some essentially bounded function and v_α is some function such that $v_\alpha \geq \max\{f_1^\alpha, \dots, f_n^\alpha\}$, μ-a.e.

PROOF. It is easy to see that for any fixed b_α and v_α, pooling operators of the form (2.1) are well-defined and externally Bayesian. The crux of the proof consists in showing that these are the *only* ones.

For each $\alpha \in \mathscr{A}$, denote $h_\alpha = T(f_1^\alpha, \dots, f_n^\alpha)$. For an arbitrary $(f_1, \dots, f_n)$ which is equivalent to $(f_1^\alpha, \dots, f_n^\alpha)$, consider the likelihood function $l = f_1/f_1^\alpha$. Since $f_i^\alpha/f_1^\alpha = c_{i,1} f_i/f_1$, μ-a.e., one has $\int lf_i^\alpha \, d\mu = c_{i,1} < \infty$ and $lf_i^\alpha / \int lf_i^\alpha \, d\mu = f_i$ for all $i = 1, \dots, n$. Since T is externally Bayesian, it follows that

$$T(f_1, \dots, f_n) \propto lT(f_1^\alpha, \dots, f_n^\alpha) = f_1 h_\alpha / f_1^\alpha, \qquad \mu\text{-a.e.},$$

and this remains true as long as $(f_1, \dots, f_n)$ belongs to the equivalence class α.

To complete the proof, assume that $v_\alpha \geq \max\{f_1^\alpha, \dots, f_n^\alpha\}$, μ-a.e. To show that $h_\alpha/v_\alpha = b_\alpha$ is essentially bounded, pick an arbitrary g in Δ and define $f_1 = gf_1^\alpha/v_\alpha$, μ-a.e. Since $\int f_1 f_i^\alpha/f_1^\alpha \, d\mu < \infty$, for all $i = 1, \dots, n$, we can define

$f_2, \dots, f_n \in \Delta$ such that $(f_1, \dots, f_n) \sim (f_1^\alpha, \dots, f_n^\alpha)$ by putting $f_i \propto f_1 f_i^\alpha / f_1^\alpha$ for $i \geq 2$. Then $T(f_1, \dots, f_n) \propto g h_\alpha / v_\alpha$ and hence

$$\int g h_\alpha / v_\alpha \, d\mu < \infty$$

for all $g \in \Delta$. The conclusion now follows from Theorem 20.15 in Hewitt and Stromberg (1965). □

At first glance, it may appear as though formula (2.1) is based solely on the opinion f_1 of the first expert, which is intriguing. In reality, however, the opinions of all the individuals influence the consensus since it is necessary to know them all in order to determine the equivalence class corresponding to the vector $(f_1, \dots, f_n)$. Note that each equivalence class is characterized by an arbitrary vector $(f_1^\alpha, \dots, f_n^\alpha)$ in the class or, equivalently, by a component f_k^α and the ratios f_i^α / f_k^α, which ratios are invariant (up to constant multiples) within a given class. For this reason, the operator (2.1) could also be written in the form

$$T(f_1, \dots, f_n) \propto b_\alpha v_\alpha f_k / f_k^\alpha, \qquad \mu\text{-a.e.},$$

or alternatively as

$$T(f_1, \dots, f_n) \propto b_\alpha v_\alpha \prod_{i=1}^{n} \left[f_i / f_i^\alpha \right]^{w_i}, \qquad \mu\text{-a.e.}, \tag{2.2}$$

provided that the weights w_i add up to 1. In principle, a different set of weights could even be chosen for each vector $(f_1, \dots, f_n)$, since the ratios f_i / f_i^α are equal to one another up to constant multiples. Another trivial observation is that every measurable function is bounded when Θ is finite. In this case, therefore, the requirement that b_α be bounded is vacuous.

Formula (2.1) is more general than the logarithmic opinion pool (1.7), but this operator is included as a particular case. To verify this, it suffices to choose the function b_α in (2.2) equal to $g \prod_{i=1}^{n} (f_i^\alpha)^{w_i} / v_\alpha$, where g is essentially bounded. This choice is legitimate since $\prod_{i=1}^{n} (f_i^\alpha)^{w_i} / v_\alpha \leq \prod_{i=1}^{n} (f_i^\alpha)^{w_i} / \max\{f_1^\alpha, \dots, f_n^\alpha\} \leq 1$, μ-a.e. More generally, the operator

$$T(f_1, \dots, f_n) = g_\alpha \prod_{i=1}^{n} f_i^{w_i(\alpha)} \Big/ \int g_\alpha \prod_{i=1}^{n} f_i^{w_i(\alpha)} \, d\mu, \qquad \mu\text{-a.e.},$$

is well-defined and is externally Bayesian when the functions g_α are essentially bounded and $\sum_{i=1}^{n} w_i(\alpha) = 1$. That is, the function g and the weights w_i in (1.7) may vary with the equivalence class to which the vector $(f_1, \dots, f_n)$ belongs without conflicting with (1.1) or (1.3).

In order to synthesize the group's opinions using a formula of the form (2.1), it will generally be necessary to first determine the equivalence class to which the observed vector of opinions belongs. In the following section, we will show what this involves in the specific case where each individual in the group is asked to provide his/her subjective probability for the realization of an event of interest.

3. The event case. In this section, we consider in detail the special case in which the space Θ consists of only two points, say $\Theta = \{0, 1\}$. This is the case in which each expert opinion can be thought of as the probability assigned to the event $E = \{\theta = 1\}$. In this case, the equivalence class to which a particular vector of probability assessments belongs is easy to construct. First note that each probability assessment consists of two positive numbers adding to unity, say p and $1 - p$. (The requirement that p be strictly between 0 and 1 is equivalent to the general assumption that the experts' densities are strictly positive almost everywhere.) A vector $\mathbf{p} = ([p_1, 1 - p_1], \ldots, [p_n, 1 - p_n])$ of such assessments is equivalent to another such vector $\mathbf{q} = ([q_1, 1 - q_1], \ldots, [q_n, 1 - q_n])$ in the sense of Definition 2.3 if and only if, for each $i = 2, \ldots, n$, we have a constant k_i such that

$$(3.1) \qquad p_i/p_1 = k_i q_i/q_1, (1 - p_i)/(1 - p_1) = k_i(1 - q_i)/(1 - q_1).$$

If we let $\mathscr{P}_i$ and $\mathscr{Q}_i$ be the odds ratios $p_i/(1 - p_i)$ and $q_i(1 - q_i)$, respectively, (3.1) simplifies to $\mathscr{P}_i/\mathscr{P}_1 = \mathscr{Q}_i/\mathscr{Q}_1$ for $i = 2, \ldots, n$. Hence two vectors of probability assessments for an event are equivalent if and only if the pairwise ratios of assessed odds within one set are equal to the corresponding ratios in the other set. For example, with $n = 2$, the two vectors ([0.5, 0.5], [0.7, 0.3]) and ([0.75, 0.25], [0.875, 0.125]) are equivalent since the odds ratio of the second expert is $7/3$ times as big as the odds ratio of the first expert in each case. The equivalence class $\alpha(\mathbf{p})$ to which a vector $\mathbf{p}$ of probability assessments belongs can be characterized by the $n - 1$ coordinates of the vector $\mathscr{P} = (\mathscr{P}_2/\mathscr{P}_1, \ldots, \mathscr{P}_n/\mathscr{P}_1)$. That is, the equivalence classes are in one-to-one correspondence with the $n - 1$ dimensional vectors of positive numbers. The equivalence class corresponding to a vector $(\alpha_2, \ldots, \alpha_n)$ contains all vectors of probability assessments such that $\mathscr{P}_i = \alpha_i \mathscr{P}_1$. A canonical representative $\mathbf{p}^\alpha$ can be chosen from each class α with $\mathscr{P}_1 = 1$, and we can identify α with the vector $(\alpha_2, \ldots, \alpha_n)$.

Next, we will look at what the externally Bayesian formulas are. Without loss of generality, we can assume that v_α in Theorem 2.5 is identically 1 for all α, since all densities are probabilities in this problem. It follows that each externally Bayesian formula can be represented as

$$(3.2) \qquad T(\mathbf{p}) = \left[\frac{b_\alpha(1)p_1}{b_\alpha(1)p_1 + b_\alpha(0)(1 - p_1)}, \frac{b_\alpha(0)(1 - p_1)}{b_\alpha(1)p_1 + b_\alpha(0)(1 - p_1)}\right],$$

where b_α is an arbitrary (bounded) function on Θ for each α, and $\alpha = (\mathscr{P}_2/\mathscr{P}_1, \ldots, \mathscr{P}_n/\mathscr{P}_1)$. Another way to express (3.2) is to say that the odds ratio for $T(\mathbf{p})$ is $\mathscr{P}_1 b_\alpha(1)/b_\alpha(0)$. Now it is trivial to see that we can assume $b_\alpha(0) = 1$, without loss of generality, by simply altering $b_\alpha(1)$. So the odds ratio for $T(\mathbf{p})$ is just $\mathscr{P}_1 b_\alpha(1)$. For example, if $b_\alpha(1) = 1$, then T is the dictatorship that simply follows expert 1. Or if $b_\alpha(1) = \alpha_i$, then T is the dictatorship that simply follows expert i. If $b_\alpha(1) = r\prod_{i=2}^n \alpha_i^{w_i}$ for arbitrary numbers w_i, $i = 2, \ldots, n$, and $r > 0$ then

$$(3.3) \qquad T(\mathbf{p}) = \left[\frac{q\prod_{i=1}^n p_i^{w_i}}{q\prod_{i=1}^n p_i^{w_i} + (1 - q)\prod_{i=1}^n (1 - p_i)^{w_i}}, \frac{(1 - q)\prod_{i=1}^n (1 - p)^{w_i}}{q\prod_{i=1}^n p_i^{w_i} + (1 - q)\prod_{i=1}^n (1 - p_i)^{w_i}}\right],$$

where $w_1 = 1 - \sum_{i=2}^{n} w_i$ and $q = r/(1 + r)$. Formula (3.3) is an analogue of (1.7) in the event case.

The form of the most general externally Bayesian rule (3.2) is very general. It is possible, for example, for $b_\alpha(1)$ to vary in an arbitrary fashion as a function of α. However, it would not usually be desirable for a small change in $\mathbf{p}$ to produce a large change in $T(\mathbf{p})$. It is easy to see that (3.2) is continuous as a function of $\mathbf{p}$ within each equivalence class α. In order for T to be continuous overall as a function of $\mathbf{p}$, all that is required is that $b_\alpha(1)$ be a continuous function of α (with the convention that $b_\alpha(0) = 1$). For example, we could let the weights w_i in (3.3) be continuous functions of α and obtain a generalized logarithmic pool with weights which depend on the degrees to which the experts differ.

4. The logarithmic opinion pool. Genest (1984b) conjectured that if an externally Bayesian pooling operator T: $\Delta^n \to \Delta$ were such that

$$T(f_1,\dots,f_n)(\theta) = G(\theta, f_1(\theta),\dots,f_n(\theta))\Big/\int G(\cdot, f_1,\dots,f_n)\,d\mu, \qquad \mu\text{-a.e.}, \tag{4.1}$$

for some $\mu \times$ Lebesgue measurable function G: $\Theta \times (0,\infty)^n \to (0,\infty)$, then T must be a "modified logarithmic opinion pool," viz.

$$T(f_1,\dots,f_n) = g\prod_{i=1}^{n} f_i^{w_i}\Big/\int g\prod_{i=1}^{n} f_i^{w_i}\,d\mu, \qquad \mu\text{-a.e.}, \tag{4.2}$$

where g is an arbitrary bounded function on Θ and $w_1,\dots,w_n$ are (not necessarily nonnegative) weights summing up to one. This result was actually proven by McConway (1978) in the case where Θ is countable and μ is a counting-type measure. In this section, we will extend this result by removing the restriction that the measure space should be purely atomic. Indeed, the only assumption which we will make here is that (Θ, μ) can be partitioned in at least four nonnegligible sets. We call a measure space which has this property *quaternary*, by analogy with the term *tertiary* introduced by Wagner (1982).

Our proof is a hybrid of that of Theorem 2.1 in Genest (1984b) and an argument which was developed by McConway (1978) for the countable case. In accordance with the convention adopted at the beginning of Section 2, each $\theta \in \Theta$ will either be an atom or will have measure zero. We begin by addressing the case in which the measure space contains atoms.

LEMMA 4.1. *Let (Θ, μ) be quaternary and let T: $\Delta^n \to \Delta$ be an externally Bayesian pooling operator. Suppose that (Θ, μ) contains at least two atoms and that there exists a $\mu \times$ Lebesgue measurable function G: $\Theta \times (0,\infty)^n \to (0,\infty)$ such that* (4.1) *holds for all $f_1,\dots,f_n \in \Delta$. Then for every pair of atoms (θ, η) in Θ^2, the identity*

$$\frac{T(f_1,\dots,f_n)(\theta)}{T(f_1,\dots,f_n)(\eta)} = \frac{T(h_1,\dots,h_n)(\theta)}{T(h_1,\dots,h_n)(\eta)} \tag{4.3}$$

holds for all densities $f_1, \dots, f_n$ *and* $h_1, \dots, h_n \in \Delta$ *for which*

$$f_i(\theta)/f_i(\eta) = h_i(\theta)/h_i(\eta), \qquad i = 1, \dots, n. \tag{4.4}$$

Before we present the proof of this lemma, let us mention parenthetically that the implication (4.4) ⇒ (4.3) is weaker but similar in spirit to the so-called axiom of relative propensity consistency (RPC) of Genest, Weerahandi, and Zidek (1984). In view of Lemma 4.1, it is not surprising, therefore, that their RPC condition should imply a logarithmic opinion pool, at least when the space (Θ, μ) is purely atomic.

PROOF. First observe that if two vectors of μ-densities $(f_1, \dots, f_n)$ and $(h_1, \dots, h_n)$ satisfy (4.4) and belong to the same equivalence class, the conclusion is immediate from (2.1) in Theorem 2.5. Hence, the proof will be complete if we can show that whenever there exist $f_1, \dots, f_n$, $h_1, \dots, h_n$ in Δ such that $f_i(\theta)/f_i(\eta) = h_i(\theta)/h_i(\eta)$, $i = 1, \dots, n$, such densities exist which belong to the same equivalence class. To do this, first set $x_i = f_i(\theta)$, $y_i = f_i(\eta)$, $x_i^* = h_i(\theta)$, and $y_i^* = h_i(\eta)$ so that $x_i/x_i^* = y_i/y_i^* = t_i$, say, for $i = 1, \dots, n$. Since (Θ, μ) is σ-finite and quaternary, there must exist a partition of Θ into four sets A_j with $0 < \mu(A_j) < \infty$, for $j = 1, 2, 3$, $\mu(A_4) > 0$, and such that $x_i\mu(A_1) + y_i\mu(A_2) < 1$, and $x_i^*\mu(A_1) + y_i^*\mu(A_2) < 1$, for all $i = 1, \dots, n$. In particular, we can take $A_1 = \{\theta\}$ and $A_2 = \{\eta\}$. We will construct densities $\bar{f}_1, \dots, \bar{f}_n$ and a likelihood l for $(\bar{f}_1, \dots, \bar{f}_n)$ such that $\bar{f}_i(\theta) = x_i$, $\bar{f}_i(\eta) = y_i$ and $\bar{h}_i(\theta) = x_i^*$, $\bar{h}_i(\eta) = y_i^*$, where $\bar{h}_i = l\bar{f}_i/\int l\bar{f}_i\, d\mu$, $i = 1, \dots, n$.

Denote $\gamma_i = x_i\mu(A_1) + y_i\mu(A_2)$, and note that $0 < \gamma_i < \min\{t_i, 1\}$ for each i. This is true because

$$x_i\mu(A_1) + y_i\mu(A_2) = t_i[x_i^*\mu(A_1) + y_i^*\mu(A_2)] < t_i,$$

for all $i = 1, \dots, n$. Choose $0 < \lambda$, $\xi < \infty$ such that

$$\lambda < \min_{i=1,\dots,n}\left\{[1 - \gamma_i]^{-1}(t_i - \gamma_i)\right\} \le \max_{i=1,\dots,n}\left\{[1 - \gamma_i]^{-1}(t_i - \gamma_i)\right\} < \xi.$$

Fixing $h \in \Delta$ an arbitrary density, now define

$$\begin{aligned}\bar{f}_i = x_i\mathscr{I}(A_1) + y_i\mathscr{I}(A_2) &+ \frac{t_i - \gamma_i - \lambda(1 - \gamma_i)}{\mu(A_3)(\xi - \lambda)}\mathscr{I}(A_3)\\ &+ \frac{\xi(1 - \gamma_i) - (t_i - \gamma_i)}{R(\xi - \lambda)}h\mathscr{I}(A_4),\end{aligned} \tag{4.5}$$

where $R = \int\mathscr{I}(A_4)h\, d\mu > 0$ and, in general, $\mathscr{I}(A)$ denotes the indicator of the set A. Clearly, $\int\bar{f}_i\, d\mu = 1$ and $\bar{f}_i(\theta) = x_i$, $\bar{f}_i(\eta) = y_i$, $i = 1, \dots, n$. To generate the $\bar{h}_i$'s, consider the likelihood

$$l = \mathscr{I}(A_1) + \mathscr{I}(A_2) + \xi\mathscr{I}(A_3) + \lambda\mathscr{I}(A_4). \tag{4.6}$$

It is easy to see that $\int l\bar{f}_i\, d\mu = t_i$, so that $\bar{h}_i(\theta) = x_i^*$ and $\bar{h}_i(\eta) = y_i^*$, $i = 1, \dots, n$. □

Lemma 4.1 shows that if $T\colon \Delta^n \to \Delta$ is externally Bayesian and satisfies (4.1) for some function G on $\Theta \times (0, \infty)^n$, then for all pairs of atoms (θ, η) in Θ^2, there must exist a Lebesgue measurable function $Q(\theta, \eta)\colon (0, \infty)^n \to (0, \infty)$ such that for all $f_1, \dots, f_n \in \Delta$,

$$\frac{T(f_1, \dots, f_n)(\theta)}{T(f_1, \dots, f_n)(\eta)} = Q(\theta, \eta)\left[\frac{f_1(\theta)}{f_1(\eta)}, \dots, \frac{f_n(\theta)}{f_n(\eta)}\right]. \tag{4.7}$$

We will now derive a more specific form for the right-hand side of (4.7).

LEMMA 4.2. *In addition to the conditions of Lemma* 4.1, *suppose that* (Θ, μ) *contains at least three atoms. Then, there exist constants* $v_1, \dots, v_n$, *such that for all* $x_1, \dots, x_n > 0$ *and every pair of atoms* (θ, η) *in* Θ^2, *we have*

$$Q(\theta, \eta)(x_1, \dots, x_n) = Q(\theta, \eta)(\mathbf{1}) \prod_{i=1}^{n} x_i^{v_i}, \tag{4.8}$$

where $\mathbf{1}$ *denotes the n-dimensional vector* $(1, \dots, 1)$.

PROOF. In the same manner as Genest, Weerahandi and Zidek (1984), define new functions $NQ(\theta, \eta)\colon (0, \infty)^n \to (0, \infty)$ by

$$NQ(\theta, \eta)(x_1, \dots, x_n) = \frac{Q(\theta, \eta)(x_1, \dots, x_n)}{Q(\theta, \eta)(1, \dots, 1)}$$

for all atoms $\theta \neq \eta$. Let θ, η, and ζ be three distinct atoms in Θ and pick $\varepsilon > 0$ small enough that there exist densities in Δ which assume any of the values ε, εx_i, or ε / y_i at any of these three atoms. Writing $\boldsymbol{\varepsilon} = (\varepsilon, \dots, \varepsilon)$, $\boldsymbol{x} = (x_1, \dots, x_n)$, and $\mathbf{y} = (y_1, \dots, y_n)$, and assuming that all operations on vectors are performed componentwise, we have

$$\begin{aligned} NQ(\theta, \eta)(\mathbf{x}\mathbf{y}) &= \frac{G(\theta, \varepsilon\mathbf{x}) / G(\eta, \varepsilon/\mathbf{y})}{G(\theta, \boldsymbol{\varepsilon}) / G(\eta, \boldsymbol{\varepsilon})} \\ &= \frac{G(\zeta, \boldsymbol{\varepsilon}) / G(\eta, \varepsilon/\mathbf{y})}{G(\zeta, \boldsymbol{\varepsilon}) / G(\eta, \boldsymbol{\varepsilon})} \, \frac{G(\theta, \varepsilon\mathbf{x}) / G(\zeta, \boldsymbol{\varepsilon})}{G(\theta, \boldsymbol{\varepsilon}) / G(\zeta, \boldsymbol{\varepsilon})} \\ &= NQ(\theta, \zeta)(\mathbf{x}) NQ(\zeta, \eta)(\mathbf{y}) \end{aligned}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in $(0, \infty)^n$. The argument now proceeds exactly as that beginning at (2.1) of Genest, Weerahandi, and Zidek (1984), except that in our case, the nonmeasurable solutions are automatically eliminated because G, Q, and hence NQ were assumed to be Lebesgue measurable. It follows that (4.8) holds for all $x_1, \dots, x_n > 0$ and all pairs of atoms θ and η. □

To complete the proof of (4.2) for atoms, fix ζ an atom in Θ and choose ε strictly between 0 and $1/\mu(\zeta)$. For all atoms $\theta \in \Theta$, now define $g(\theta) = Q(\theta, \zeta)(\mathbf{1}) G[\zeta, \boldsymbol{\varepsilon}] \varepsilon^{-v}$, where $v = \sum_{i=1}^{n} v_i$. Then for all atoms θ, we find

$$G(\theta, x_1, \dots, x_n) = g(\theta) \prod_{i=1}^{n} x_i^{v_i},$$

for all $0 < x_1, \ldots, x_n < 1/\mu(\theta)$, which implies that

$$(4.9) \qquad T(f_1, \ldots, f_n)(\theta) = g(\theta) \prod_{i=1}^{n} f_i(\theta)^{v_i} \Big/ \int g \prod_{i=1}^{n} f_i^{v_i} \, d\mu$$

for all atoms θ.

Next, we derive a formula similar to (4.9) for those θ which are not atoms.

LEMMA 4.3. *Let T: $\Delta^n \to \Delta$ be externally Bayesian and assume that* (4.1) *holds for some $\mu \times$ Lebesgue measurable function G: $\Theta \times (0, \infty)^n \to (0, \infty)$. Assume also that (Θ, μ) is not purely atomic. Let N be the complement of the set of atoms. Then*

$$(4.10) \qquad T(f_1, \ldots, f_n)(\theta) = g(\theta) \prod_{i=1}^{n} f_i(\theta)^{w_i} \Big/ \int G(\cdot, f_1, \ldots, f_n) \, d\mu, \qquad \mu\text{-a.e. on } N,$$

for some nonnegative weights $w_1, \ldots, w_n \in \mathbb{R}$ adding up to unity.

PROOF. Define a new function NG: $\Theta \times (0, \infty)^n \to (0, \infty)$ by

$$NG(\theta, z_1, \ldots, z_n) = G(\theta, z_1, \ldots, z_n)/G(\theta, 1, \ldots, 1)$$

for all $z_1, \ldots, z_n \in (0, \infty)$. It will be enough to show that $NG(\theta)(z_1, \ldots, z_n)$ is a function of the z_i's only, say $NG(z_1, \ldots, z_n)$. For, once this is done, we can define a new pooling operator T^*: $\Delta^n \to \Delta$ by

$$T^*(f_1, \ldots, f_n)(\theta) = NG(f_1(\theta), \ldots, f_n(\theta)) \Big/ \int NG(f_1, \ldots, f_n) \, d\mu.$$

It is easy to see that T^* is externally Bayesian and of the form (1.6) with G replaced by NG. We can then apply Theorem 2.1 to conclude that

$$NG(z_1, \ldots, z_n) = \prod_{i=1}^{n} z_i^{w_i}$$

for some nonnegative constants $w_1, \ldots, w_n$ adding up to one. Letting $g(\theta) = G(\theta, 1, \ldots, 1)$ for all θ in N, we arrive at (4.10).

To see that $NG(\theta, z_1, \ldots, z_n)$ is a function of the z_i's alone, we proceed along the same lines as in the proof of Lemma 4.1. Given $z_1, \ldots, z_n > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \min_{i=1,\ldots,n}\{1/2, 1/2z_i\}$ and let A_j, $1 \le j \le 4$, be a partition of N such that $0 < \mu(A_j) < \varepsilon$ for $j = 1, 2$; $0 < \mu(A_3) < \infty$ and $\mu(A_4) > 0$. That such a partition of N exists follows from the fact that (Θ, μ) is σ-finite and nonatomic on N.

Next, define $\bar{f}_i$ as in (4.5) using $x_i = y_i = 1$ and $t_i = 1/z_i$, $i = 1, \ldots, n$. If the likelihood for $(\bar{f}_1, \ldots, \bar{f}_n)$ is defined by (4.6), we have $\int l\bar{f}_i \, d\mu = 1/z_i$, $i = 1, \ldots, n$. Letting $\bar{h}_i = l\bar{f}_i/\int l\bar{f}_i \, d\mu$, we have that $\bar{f}_i(\theta) = 1$ and $\bar{h}_i = z_i$ for $\theta \in A_1 \cup A_2$ and $i = 1, \ldots, n$. Now, since T is externally Bayesian, we know that

$$\frac{T(\bar{h}_1, \ldots, \bar{h}_n)(\theta)}{l(\theta) T(\bar{f}_1, \ldots, \bar{f}_n)(\theta)}$$

is constant μ-almost everywhere. Also, since $l(\theta) = 1$ on $A_1 \cup A_2$ and since (4.1) holds μ-almost everywhere on N, it follows that

$$NG(\theta, z_1, \dots, z_n) = \frac{T(\bar{h}_1, \dots, \bar{h}_1)(\theta)}{l(\theta)T(\bar{f}_1, \dots, \bar{f}_n)(\theta)}$$

on $A_1 \cup A_2$. Hence $NG(\theta, z_1, \dots, z_n)$ is essentially constant as a function of θ on $A_1 \cup A_2$. Denote the constant value $NG(z_1, \dots, z_n)$. To see that NG is essentially constant on N as a function of θ, assume to the contrary that there exists a subset B of N with $\mu(B) > 0$ and such that $NG(\theta, z_1, \dots, z_n) > (<) NG(z_1, \dots, z_n)$ for almost all θ in B. Since μ is nonatomic on N, choose a subset of B with positive measure at most ε. Let this new set be A_2, and repeat the above construction, keeping A_1 the same as before. The conclusion still holds that $NG(\theta, z_1, \dots, z_n)$ is essentially constant on $A_1 \cup A_2$ in contradiction to the assumption that $NG(\theta, z_1, \dots, z_n) > (<) NG(z_1, \dots, z_n)$ on A_2. □

We are now in a position to prove the main result of this section.

THEOREM 4.4. *Let (Θ, μ) be a quaternary measure space and let T: $\Delta^n \to \Delta$ be an externally Bayesian pooling operator. If there exists a $\mu \times$ Lebesgue measurable function G: $\Theta \times (0, \infty)^n \to (0, \infty)$ such that* (4.1) *holds for all vectors of opinions $(f_1, \dots, f_n)$ in Δ^n, then T is of the form* (1.7), *i.e.*,

$$(4.11) \qquad T(f_1, \dots, f_n) = g \prod_{i=1}^{n} f_i^{w_i} \Big/ \int g \prod_{i=1}^{n} f_i^{w_i}\, d\mu, \qquad \mu\text{-a.e.},$$

for some essentially bounded function g: $\Theta \to (0, \infty)$ and some constants $w_1, \dots, w_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} w_i = 1$. Moreover, the weights w_i are nonnegative unless Θ is finite or there does not exist a countably infinite partition of (Θ, μ) into nonnegligible sets.

PROOF. If (Θ, μ) does not contain any atoms, (4.11) is immediate from Lemma 4.3. If (Θ, μ) is purely atomic, then (4.11) derives easily from (4.9) with $w_i = v_i$, $i = 1, \dots, n$. It is straightforward to see that $\sum_{i=1}^{n} w_i = 1$ from the fact that T is externally Bayesian.

More difficult is the case in which μ has atoms but is not purely atomic. In this case, we can use Lemma 4.3 to obtain the result on the set N, the complement of the set of atoms of μ. Label the atoms $\theta_1, \theta_2, \dots$ and let $G_i(x_1, \dots, x_n)$ denote $G(\theta_i, x_1, \dots, x_n)$ for all $x_1, \dots, x_n$ strictly between 0 and $1/\mu(\theta_i)$. From the definition of externally Bayesian, we have

$$(4.12) \qquad \frac{l(\theta)G(\theta, f_1(\theta), \dots, f_n(\theta))}{G(\theta, h_1(\theta), \dots, h_n(\theta))} = \text{constant a.e. } \mu,$$

whenever h_i is proportional to lf_i for all i. From (4.10), we have that on N, the left-hand side of (4.12) equals $\prod_{i=1}^{n} t_i^{w_i}$, where t_i is the integral of lf_i for each i.

Now fix $t_1, \dots, t_n$ and pick a single atom θ_j. Let ε be small enough so that ε / t_i is strictly between 0 and $1/\mu(\theta_j)$ for each i. Let $A_1 = \{\theta_j\}$ and construct the same densities as in the proof of Lemma 4.1, setting $x_i = \varepsilon$ and $x_i^* = \varepsilon/t_i$ for each i. (Here A_2, A_3, A_4, and the y_i's and y_i^* are arbitrary sets and values satisfying the restrictions described in the proof of Lemma 4.1.) Also construct the same likelihood l as in that proof. Using the fact that (4.12) holds on all of Θ, we have

$$(4.13) \qquad \frac{G_j(\varepsilon,\dots,\varepsilon)}{G_j(\varepsilon/t_1,\dots,\varepsilon/t_n)} = \prod_{i=1}^{n} t_i^{w_i}.$$

It follows directly from (4.13) that for all $z_1,\dots,z_n$ between 0 and $1/\mu(\theta_j)$, $G_j(z_1,\dots,z_n) = G_j(\varepsilon,\dots,\varepsilon)\varepsilon^{-1}\prod_{i=1}^{n} z_i^{w_i}$. By letting $g(\theta_j) = G_j(\varepsilon,\dots,\varepsilon)\varepsilon^{-1}$, we have proven (4.11).

Finally, note that the weights are automatically nonnegative provided that μ is not purely atomic. If (Θ,μ) is purely atomic but includes a countably infinite number of atoms, it is fairly easy to construct densities $f_1,\dots,f_n$ which will make the integral

$$(4.14) \qquad \int g \prod_{i=1}^{n} f_i^{w_i}\, d\mu$$

infinite and lead to a contradiction unless all the weights are nonnegative. Of course, (4.14) is always finite when Θ is finite and μ is some sort of counting measure, so negative weights cannot be ruled out in this case. Similarly, g must be essentially bounded, or else there exists f such that (4.14) is infinite when all of the f_i are equal to f (c.f. Theorem 20.15 of Hewitt and Stromberg, 1965). □

Unless the group of experts is reporting its opinions to an outside decision maker who chooses to aggregate them using (4.9), it is difficult to interpret g, let alone offer advice as to how it could be selected. To some extent, the same applies to the interpretation and determination of the weights, although some heuristics are available (for example, see Winkler, 1968 or Genest and Schervish, 1985). In fact, even if the pooling operator (4.9) is adopted by a decision maker, it is not so clear what g and the weights stand for. In particular, we should guard from concluding too hastily that the function g represents the decision maker's "prior." After all, there is no reason why a prior density should necessarily be bounded, nor is every bounded function a possible prior density.

One way around the choice of g in (4.9) would be to insist that the pooling operator T preserves unanimity. In general, an aggregation procedure T: $\Delta^n \to \Delta$ preserves unanimity if and only if $T(f,\dots,f) = f$ for all $f \in \Delta$. As the following corollary indicates, this is enough to reduce (4.9) to an ordinary logarithmic opinion pool.

COROLLARY 4.5. *Let* (Θ,μ) *be a quaternary measure space and let* T: $\Delta^n \to \Delta$ *be an externally Bayesian pooling operator which preserves unanimity. If there exists a* $\mu \times$ *Lebesgue measurable function* G: $\Theta \times (0,\infty)^n \to (0,\infty)$ *such that* (4.1) *holds for all vectors of opinions* $(f_1,\dots,f_n)$ *in* Δ^n, *then* T *is a* *logarithmic opinion pool, i.e.,*

$$T(f_1,\dots,f_n) = \prod_{i=1}^{n} f_i^{w_i} \Big/ \int \prod_{i=1}^{n} f_i^{w_i}\, d\mu, \qquad \mu\text{-a.e.},$$

for some arbitrary reals $w_1,\dots,w_n$ *adding up to* 1. *Moreover, the weights* w_i *are nonnegative unless* Θ *is finite or there does not exist a countable partition of* (Θ,μ) *into nonnegligible sets.*

5. Discussion. The purpose of this paper has been two-fold. On the one hand we have provided a characterization of all externally Bayesian pooling operators in Theorem 2.5. The form of these operators is quite general, as it was derived under the sole assumption that all densities are strictly positive almost everywhere. On the other hand, when the space Θ is assumed to be quaternary and the operator is required to satisfy a "locality" condition (4.1), we show that the operator must be logarithmic in the sense of (1.7). This latter result does not apply to the cases in which Θ consists of merely two or three atoms.

If the space Θ contains only two points, then it is trivial to see that every pooling operator satisfies (4.1). In this case, one can easily construct externally Bayesian operators which are not logarithmic. One such example is constructed from (3.2) as follows. Let $b_\alpha(0) = 1$ and let $b_\alpha(1) = \max\{1, \alpha_2,\dots,\alpha_n\}$. It is easy to see that $T(\mathbf{p})$ equals $[\max_i\{p_i\}, 1 - \max_i\{p_i\}]$, which is externally Bayesian, satisfies (4.1), and is clearly not logarithmic. If (Θ,μ) consists of only three atoms, it is not known whether logarithmic opinion pools are the only externally Bayesian operators which satisfy (4.1). Theorem 2.5 still holds in this case, however. The case of one atom is left to the reader.

The theorems of this paper shed some light on the mechanics of externally Bayesian behavior. If, however, a decision maker wishes to treat the opinions of a group of experts as data, little is known about the implications of the externally Bayesian criterion for the modeling process. In the Bayesian model proposed by Lindley (1985), conditions are given under which the decision maker's posterior distribution would be externally Bayesian, but his conditions are not easily interpretable.

Acknowledgments. Thanks are due to M. H. DeGroot (Carnegie-Mellon University), C. Sundberg (University of Tennessee), S. Dharmadhikari (Southern Illinois University, Carbondale), and J. V. Zidek (University of British Columbia) for fruitful discussions in the course of writing this article. The first author's investigation was supported in part by an N.S.E.R.C. postdoctoral fellowship which was held at Carnegie-Mellon University. Part of the second author's work was performed at the University of Washington, which generously provided research facilities.

REFERENCES

BACHARACH, MICHAEL (1972). Scientific disagreement. Unpublished manuscript.

DE FINETTI, BRUNO (1954). Media di decisioni e media di opinioni. *Bull. Inst. Internat. Statist.* **34** 144–157.

FRENCH, SIMON (1980). Updating of belief in the light of someone else's opinion. *J. Roy. Statist. Soc. Ser. A* **143** 43–48.

FRENCH, SIMON (1981). Consensus of opinion. *European J. Oper. Res.* **7** 332–340.

FRENCH, SIMON (1985). Group consensus probability distributions: A critical survey. In *Bayesian Statistics* **2** (J. M. Bernardo, et al., eds.) 183–201. North-Holland, Amsterdam.

GENEST, CHRISTIAN (1984a). A conflict between two axioms for combining subjective distributions. *J. Roy. Statist. Soc. Ser. B* **46** 403–405.

GENEST, CHRISTIAN (1984b). A characterization theorem for externally Bayesian groups. *Ann. Statist.* **12** 1100–1105.

GENEST, CHRISTIAN (1984c). Pooling operators with the marginalization property. *Canad. J. Statist.* **12** 153–163.

GENEST, CHRISTIAN and SCHERVISH, MARK J. (1985). Modeling expert judgments for Bayesian updating. *Ann. Statist.* **13** 1198–1212.

GENEST, CHRISTIAN and ZIDEK, JAMES V. (1986). Combining probability distributions: A critique and an annotated bibliography (with discussion). *Statist. Sci.* **1** 114–148.

GENEST, C., WEERAHANDI, S. and ZIDEK, J. V. (1984). Aggregating opinions through logarithmic pooling. *Theory and Decision* **17** 61–70.

HALMOS, PAUL R. (1950). *Measure Theory*. Van Nostrand, New York.

HEWITT, EDWIN and STROMBERG, KARL (1965). *Real and Abstract Analysis*. Springer, New York.

LADDAGA, ROBERT (1977). Lehrer and the consensus proposal. *Synthese* **36** 473–477.

LINDLEY, DENNIS V. (1985). Reconciliation of discrete probability distributions. In *Bayesian Statistics* **2** (J. M. Bernardo, et al., eds.) 375–390. North-Holland, Amsterdam.

MADANSKY, ALBERT (1964). Externally Bayesian groups. Technical Report RM-4141-PR, RAND Corporation.

MADANSKY, ALBERT (1978). Externally Bayesian groups. Unpublished manuscript, University of Chicago.

MCCONWAY, KEVIN J. (1978). The combination of experts' opinions in probability assessment: Some theoretical considerations. Ph.D. thesis, University College London.

MCCONWAY, KEVIN J. (1981). Marginalization and linear opinion pools. *J. Amer. Statist. Assoc.* **76** 410–414.

MORRIS, PETER A. (1974). Decision analysis expert use. *Management Sci.* **20** 1233–1241.

MORRIS, PETER A. (1977). Combining expert judgments: a Bayesian approach. *Management Sci.* **23** 679–693.

RAIFFA, HOWARD (1968). *Decision Analysis: Introductory Lectures on Choices under Uncertainty.* Addison-Wesley, Reading, Mass.

WAGNER, CARL G. (1982). Allocation, Lehrer models, and the consensus of probabilities. *Theory and Decision* **14** 207–220.

WAGNER, CARL G. (1984). Aggregating subjective probabilities: Some limitative theorems. *Notre Dame J. Formal Logic* **25** 233–240.

WALD, ABRAHAM (1939). Contributions to the theory of statistical estimation and testing hypotheses. *Ann. Math. Statist.* **10** 299–326.

WEERAHANDI, SAMARADASA and ZIDEK, JAMES V. (1981). Multi-Bayesian statistical decision theory. *J. Roy. Statist. Soc. Ser. A* **144** 85–93.

WINKLER, ROBERT L. (1968). The consensus of subjective probability distributions. *Management Sci.* **15** B61–B75.

WINKLER, ROBERT L. (1981). Combining probability distributions from dependent information sources. *Management Sci.* **27** 479–488.

CHRISTIAN GENEST
DEPARTMENT OF STATISTICS
AND ACTUARIAL SCIENCE
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO N2L 3G1
CANADA

KEVIN J. MCCONWAY
DEPARTMENT OF STATISTICS
THE OPEN UNIVERSITY
MILTON KEYNES MK7 6AA
ENGLAND

MARK J. SCHERVISH
DEPARTMENT OF STATISTICS
CARNEGIE-MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213