The Annals of Statistics
1985, Vol. 13, No. 3, 1140–1155

SOBOLEV TESTS FOR INDEPENDENCE OF DIRECTIONS

BY P. E. JUPP AND B. D. SPURR

University of St. Andrews

Two families of invariant tests for independence of random variables on compact Riemannian manifolds are proposed and studied. The tests are based on Giné's Sobolev norms which are obtained by mapping the manifolds into Hilbert spaces. For general compact manifolds, randomization tests are suggested. For the bivariate circular case, distribution-free tests based on uniform scores are considered.

1. Introduction. A problem of some interest in directional statistics is that of constructing useful tests for independence of two circular or spherical random variables. This problem extends naturally to that of producing tests for independence of random variables on general manifolds. Machinery based on Sobolev norms for comparing distributions on a compact Riemannian manifold was established by Giné (1975) and was used by him to introduce a large class of invariant tests of uniformity. This approach was employed by Wellner (1979) to construct two-sample tests and by Jupp and Spurr (1983) to provide tests of symmetry. In this paper, we employ a simple Hilbert space approach to Giné's Sobolev norms and use this machinery to produce two classes of invariant tests for independence of random variables on compact Riemannian manifolds.

The tests of independence of circular or spherical random variables which have been proposed so far fall into three types. Two of these regard points on the circle or sphere as unit vectors in the plane or 3-space. Tests of the first type measure correlation between the appropriate random unit vectors **U** and **V** by suitable functions of $\Sigma_{12}^* = E(\mathbf{UV}')$. Such correlation coefficients have been considered in the bivariate spherical case by Mackenzie (1957), Watson and Beran (1967), and Epp, Tukey and Watson (1971) and in a general setting by Stephens (1979). In the bivariate circular case, signed correlation coefficients of this type have been introduced by Rivest (1982) and Fisher and Lee (1983).

Tests of the second type consider functions of Σ_{12}, the covariance matrix of (**U**, **V**). Correlation coefficients of this kind were introduced by Downs (1974) and Johnson and Wehrly (1977) for circular variables, by Mardia and Puri (1978) for the circular and spherical cases, and by Jupp and Mardia (1980) for general manifolds. Tests of the third type are for the bivariate circular case only and use the empirical distribution function to derive rank tests which are distribution-free under the null hypothesis of independence. The appropriate correlation coefficients are due to Rothman (1971), Mardia (1975), and Fisher and Lee (1982, 1983).

In Section 2 of this paper, we introduce a class of randomization tests of

Received August 1983; revised February 1985.
AMS 1980 *subject classifications*. Primary 62H15, 62H20, 62G10, 62E20.
Key words and phrases. Consistency, correlation, directional data, independence, invariance, randomization tests, Riemannian manifolds, uniform scores.

independence on compact Riemannian manifolds. They are based on the Sobolev machinery and are in the tradition of the tests of the second type mentioned above. In Section 3, we apply this approach to uniform scores to obtain a class of distribution-free tests in the bivariate circular case. These tests are in the spirit of the third type above and include Rothman's (1971) test as well as a variant of the test of Mardia (1975). Proofs of the results are given in the Appendix.

2. Randomization tests of independence. An important tool in directional statistics is provided by the Sobolev seminorms introduced by Giné (1975). We now give a simple construction of these seminorms.

Let $\mathbb{X}$ be a compact Riemannian manifold. Then the Riemannian metric determines the uniform measure μ on $\mathbb{X}$. Denote by $L^2(\mathbb{X})$ the Hilbert space of real-valued functions on $\mathbb{X}$ which are square-integrable with respect to μ. On $L^2(\mathbb{X})$ we shall use exclusively the corresponding L^2 norm $\| \cdot \|_2$. The basic idea underlying Giné's approach is to map $\mathbb{X}$ into the vector space $L^2(\mathbb{X})$, thus transforming problems in directional statistics into multivariate problems (usually of infinite dimension). The norm $\| \cdot \|_2$ then gives rise to a Euclidean-type distance between probability measures on $\mathbb{X}$. The details are as follows.

Denote by $\mathscr{M}(\mathbb{X})$ the bounded Borel measures on $\mathbb{X}$ and by $\mathscr{P}(\mathbb{X})$ the Borel probability measures on $\mathbb{X}$. Let the Laplacian Δ of $\mathbb{X}$ have kth eigenspace E_k with eigenvalue σ_k, for $k = 0, 1, \cdots$, and let the functions $\{f_i\}$ be an orthonormal basis of $L^2(\mathbb{X})$ consisting of eigenfunctions of Δ. It is useful to describe a sequence $\{\alpha_k\}_{k=1}^{\infty}$ of real numbers as satisfying *condition C* on $\mathbb{X}$ if

$$\sup_k | \alpha_k \sigma_k^{s/2} | < \infty \quad \text{for some} \quad s > (\dim \mathbb{X})/2.$$

If $A = \{\alpha_k\}_{k=1}^{\infty}$ is a sequence satisfying condition C, then the function $\mathbf{t}\colon \mathbb{X} \to L^2(\mathbb{X})$ defined by

$$\mathbf{t}(x) = \textstyle\sum_{k=1}^{\infty} \alpha_k \sum_{f_i \in E_k} f_i(x)\, f_i$$

is continuous. A proof is given in the Appendix.

The function $\mathbf{t}$ gives rise to a function $\boldsymbol{\tau}\colon \mathscr{M}(\mathbb{X}) \to L^2(\mathbb{X})$ defined by

$$\boldsymbol{\tau}(\nu) = \int \mathbf{t}\, d\nu.$$

Note that if $\alpha_k \neq 0$ for all $k \geq 1$ then the restriction of $\boldsymbol{\tau}$ to $\mathscr{P}(\mathbb{X})$ is one-to-one. The L^2 norm $\| \cdot \|_2$ on $L^2(\mathbb{X})$ can be pulled back by $\boldsymbol{\tau}$ to a seminorm $\| \cdot \|_A$ on $\mathscr{M}(\mathbb{X})$ corresponding to the sequence A. Indeed $\| \cdot \|_A = \| \cdot \|$ is defined by

$$\| \nu \|^2 = \left\| \int \mathbf{t}\, d\nu \right\|_2^2 = \| \boldsymbol{\tau}(\nu) \|_2^2 \quad \nu \in \mathscr{M}(\mathbb{X}). \tag{2.1}$$

An alternative expression is

$$\| \nu \|^2 = \textstyle\sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_k} \left(\displaystyle\int f_i\, d\nu \right)^2 \quad \nu \in \mathscr{M}(\mathbb{X}). \tag{2.2}$$

Sobolev seminorms were used in Giné's (1975) tests of uniformity to measure the distance between the empirical and the uniform distributions. They were used by Wellner (1979) to assess the distance between the empirical distributions in the two-sample case and by Jupp and Spurr (1983) to measure the distance of the empirical distribution from its symmetrized version. In order to construct tests for independence in an analogous manner, it is useful to consider products of Sobolev seminorms. Let $\mathbb{X}$ and $\mathbb{Y}$ be compact Riemannian manifolds and let $\|\cdot\|_A$ and $\|\cdot\|_B$ be Sobolev seminorms on $\mathscr{M}(\mathbb{X})$ and $\mathscr{M}(\mathbb{Y})$ given by sequences $A = \{\alpha_k\}_{k=1}^{\infty}$ and $B = \{\beta_\ell\}_{\ell=1}^{\infty}$ satisfying condition C on $\mathbb{X}$ and $\mathbb{Y}$, respectively. Define $\mathbf{u}$: $\mathbb{Y} \to L^2(\mathbb{Y})$ by

$$\mathbf{u}(y) = \textstyle\sum_{\ell=1}^{\infty} \beta_\ell \sum_{g_j \in F_\ell} g_j(y) g_j$$

where F_ℓ is the ℓth eigenspace of Δ on $\mathbb{Y}$ and $\{g_j\}$ is an orthonormal basis of $L^2(\mathbb{Y})$ consisting of eigenfunctions of Δ. Then the product seminorm $\|\cdot\|_{A\otimes B} = \|\cdot\|_A \otimes \|\cdot\|_B$ on $\mathscr{M}(\mathbb{X} \times \mathbb{Y})$ is defined by

$$\|\nu\|_{A\otimes B}^2 = \left\| \int \mathbf{t} \otimes \mathbf{u}\, d\nu \right\|_2^2 \qquad \nu \in \mathscr{M}(\mathbb{X} \times \mathbb{Y}) \tag{2.3}$$

where $\mathbf{t} \otimes \mathbf{u}$: $\mathbb{X} \times \mathbb{Y} \to L^2(\mathbb{X}) \otimes L^2(\mathbb{Y}) \subset L^2(\mathbb{X} \times \mathbb{Y})$ and $\|\cdot\|_2$ is the L^2 norm on $L^2(\mathbb{X} \times \mathbb{Y})$. An alternative expression is

$$\|\nu\|_{A\otimes B}^2 = \textstyle\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \alpha_k^2 \beta_\ell^2 \sum_{f_i \in E_k} \sum_{g_j \in F_\ell} \left\{ \int_{\mathbb{X}\times\mathbb{Y}} f_i(x) g_j(y)\, d\nu(x, y) \right\}^2 \qquad \nu \in \mathscr{M}(\mathbb{X} \times \mathbb{Y}). \tag{2.4}$$

For distributions ν of random variables (X, Y) on $\mathbb{X} \times \mathbb{Y}$, the hypothesis of independence is H_0: $\nu = \nu_1 \otimes \nu_2$, where $\nu_1 \otimes \nu_2$ is the product of the marginal distributions ν_1 and ν_2. This suggests that we should test H_0 by measuring the distance from ε, the empirical distribution of a sample of size n, to $\varepsilon_1 \otimes \varepsilon_2$, the product of its marginals. For any product seminorm $\|\cdot\|_{A\otimes B}$, we define

$$T_n = n \|\varepsilon - \varepsilon_1 \otimes \varepsilon_2\|_{A\otimes B}^2. \tag{2.5}$$

Our tests compare the observed value of T_n with its null distribution conditional on the marginals $(\varepsilon_1, \varepsilon_2)$ and the null hypothesis of independence is rejected for large values of T_n. The statistic T_n is analogous to that used in the nonparametric test of independence introduced by Hoeffding (1948) and by Blum, Kiefer and Rosenblatt (1961) based on the multivariate empirical distribution function. The null distribution of T_n conditional on $(\varepsilon_1, \varepsilon_2)$ can be calculated in $O(n!)$ operations. Enumeration is feasible for small values of n. For larger values we recommend sampling from this permutation distribution. The tests depend on the sequences $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_\ell\}_{\ell=1}^{\infty}$. If only a few α_k and β_ℓ are nonzero, then T_n is easily computed. However, if many of these coefficients are nonzero then the test is consistent against a wide class of alternatives. (See Theorem 2.1.)

An alternative derivation of T_n is the following. Given a sample $(x_1, y_1), \cdots, (x_n, y_n)$ of points in $\mathbb{X} \times \mathbb{Y}$, consider its image $(\mathbf{t}(x_1), \mathbf{u}(y_1)), \cdots, (\mathbf{t}(x_n), \mathbf{u}(y_n))$ in

$L^2(\mathbb{X}) \times L^2(\mathbb{Y})$. A natural measure of dependence between X and Y based on this sample is the sample covariance S_{12} of $\mathbf{T} = \mathbf{t}(X)$ and $\mathbf{U} = \mathbf{u}(Y)$ defined by

$$S_{12} = n^{-1} \sum_{i=1}^n \mathbf{t}(x_i) \otimes \mathbf{u}(y_i) - (n^{-1} \sum_{i=1}^n \mathbf{t}(x_i)) \otimes (n^{-1} \sum_{i=1}^n \mathbf{u}(y_i)).$$

Then it is reasonable to regard X and Y as independent if $\| S_{12} \|_2^2$ is small, where $\| \cdot \|_2^2$ is the L^2 norm on $L^2(\mathbb{X} \times \mathbb{Y})$. In fact, we have

$$T_n = n \| S_{12} \|_2^2. \tag{2.6}$$

A more useful formula for computing T_n is

$$\begin{aligned} T_n = {} & n^{-1} \sum_{i=1}^n \sum_{j=1}^n h_A(x_i, x_j) h_B(y_i, y_j) \\ & - 2n^{-2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n h_A(x_i, x_j) h_B(y_i, y_k) \\ & + n^{-3} \{\sum_{i=1}^n \sum_{j=1}^n h_A(x_i, x_j)\}\{\sum_{i=1}^n \sum_{j=1}^n h_B(y_i, y_j)\} \end{aligned} \tag{2.7}$$

where

$$h_A(x_r, x_s) = \langle \mathbf{t}(x_r), \mathbf{t}(x_s) \rangle = \sum_{k=1}^\infty \alpha_k^2 \sum_{f_i \in E_k} f_i(x_r) f_i(x_s)$$

and

$$h_B(y_r, y_s) = \langle \mathbf{u}(y_r), \mathbf{u}(y_s) \rangle = \sum_{\ell=1}^\infty \beta_\ell^2 \sum_{g_j \in F_\ell} g_j(y_r) g_j(y_s)$$

with $\langle \cdot, \cdot \rangle$ denoting L^2 inner products. Note that (2.7) is analogous to formula (4.2) of Wellner (1979) with h_A and h_B identical to his h.

These tests based on T_n are invariant under separate isometries of $\mathbb{X}$ and of $\mathbb{Y}$. This is a simple consequence of the isometry-invariance of the Sobolev seminorms. Note that the randomization test for independence of spherical variables considered by Watson and Beran (1967) and by Epp, Tukey and Watson (1971) is invariant under a common rotation of the spheres but not invariant under separate rotations.

In general, $\| \nu \|_{A \otimes B} = 0$ does not imply that ν is uniform on $\mathbb{X} \times \mathbb{Y}$. However, for A and B with $\alpha_k \neq 0$ and $\beta_k \neq 0$, for all k, it is easily shown that $\nu = \nu_1 \otimes \nu_2$ if and only if $\| \nu - \nu_1 \otimes \nu_2 \|_{A \otimes B} = 0$. This prompts the following consistency result which is proved in the Appendix.

THEOREM 2.1. *The sequence of tests based on T_n conditional on $(\varepsilon_1, \varepsilon_2)$ is consistent against an alternative ν if and only if $\| \nu - \nu_1 \otimes \nu_2 \|_{A \otimes B}^2 > 0$. In particular, a sequence of tests is consistent against all alternatives if and only if $\alpha_k \neq 0$ and $\beta_k \neq 0$ for all k.*

The asymptotic distributions of T_n under local and under fixed alternatives are given in the next two theorems which follow from the central limit theorem in the Hilbert space $L^2(\mathbb{X} \times \mathbb{Y})$. We denote by $Z^{(\nu)}(f)$ the Gaussian process indexed by $f \in L^2(\mathbb{X} \times \mathbb{Y}, \nu)$ with mean zero and covariance structure given by

$$\operatorname{Cov}(Z^{(\nu)}(f), Z^{(\nu)}(g)) = \int_{\mathbb{X}\times\mathbb{Y}} \left(f - \int_{\mathbb{X}\times\mathbb{Y}} f \, d\nu\right)\left(g - \int_{\mathbb{X}\times\mathbb{Y}} g \, d\nu\right) d\nu.$$

Also, $\to_{w^*}$ and $\to_d$ denote, respectively, convergence in the weak (star) topology of $\mathscr{P}(\mathbb{X} \times \mathbb{Y})$ and convergence in distribution.

THEOREM 2.2 (Local alternatives). *Let $\{\nu_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{P}(\mathbb{X} \times \mathbb{Y})$ satisfying $\nu_n \to_{w^*} \nu$ for some ν with independent marginals on $\mathbb{X}$ and $\mathbb{Y}$. Suppose that*

$$\lim_{n\to\infty} n^{1/2} \int_{\mathbb{X}\times\mathbb{Y}} f_i(x) g_j(y)\, d(\nu_n - \nu_{n_1} \otimes \nu_{n_2})(x, y) = d_{ij}$$

for $f_i \in E_k$, $g_j \in F_\ell$ with

$$\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \alpha_k^2 \beta_\ell^2 \sum_{f_i \in E_k} \sum_{g_j \in F_\ell} d_{ij}^2 < \infty,$$

where ν_{n_1} and ν_{n_2} are the marginals of ν_n on $\mathbb{X}$ and $\mathbb{Y}$. Then if T_n is generated by random sampling from ν_n we have

$$T_n \to_d \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \alpha_k^2 \beta_\ell^2 \sum_{f_i \in E_k} \sum_{g_j \in F_\ell} (Z^{(\nu)}(f_i g_j) + d_{ij})^2.$$

THEOREM 2.3. (Fixed alternatives). *For random samples from $\nu \in \mathscr{P}(\mathbb{X} \times \mathbb{Y})$ with*

$$\| \nu - \nu_1 \otimes \nu_2 \|^2 > 0, \quad n^{-1/2}(T_n - n \| \nu - \nu_1 \otimes \nu_2 \|^2) \to_d N(0, \mathrm{Var}_\nu(w))$$

where

$$\begin{aligned} w(x, y) &= 2\langle E_\nu[\mathbf{v}], \mathbf{v}(x, y)\rangle \\ &= 2 \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \alpha_k^2 \beta_\ell^2 \sum_{f_i \in E_k} \sum_{g_j \in F_\ell} \left\{ \int_{\mathbb{X}\times\mathbb{Y}} f_i g_j \, d(\nu - \nu_1 \otimes \nu_2) \right\} \\ &\qquad \cdot (f_i(x) - E_\nu[f_i])(g_j(y) - E_\nu[g_j]) \end{aligned}$$

with $\mathbf{v}(x, y) = (\mathbf{t}(x) - E_\nu[\mathbf{t}]) \otimes (\mathbf{u}(y) - E_\nu[\mathbf{u}])$.

The compact manifolds most frequently arising as sample spaces are the p-dimensional spheres, S^p in $\mathbb{R}^{p+1}$. When both $\mathbb{X}$ and $\mathbb{Y}$ are spheres, two of the above tests are of particular interest. One is the quick test obtained by taking $\alpha_1 = \beta_1 = 1$, $\alpha_k = \beta_k = 0$ for $k > 1$, and so using $h_A(\mathbf{x}_i, \mathbf{x}_j) = h_B(\mathbf{x}_i, \mathbf{x}_j) = (p + 1)\mathbf{x}_i' \mathbf{x}_j$ in (2.7). This corresponds to using the seminorm employed in the Rayleigh test of uniformity (Mardia, 1972, page 133). We have

$$(2.8) \quad T_n = n(p + 1)^2 \mathrm{tr}(S_{12} S_{12}') = n(p + 1)^2 \{\mathrm{tr}(S_{12}^* S_{12}^{*\prime}) - 2\bar{\mathbf{x}}' S_{12}^* \bar{\mathbf{y}} + \bar{R}_X^2 \bar{R}_Y^2\}$$

where $S_{12}^* = n^{-1} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{y}_i'$, $\bar{R}_X$, $\bar{R}_Y$ are the lengths of the sample means $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$ of the random unit vectors $\mathbf{X}$, $\mathbf{Y}$, and $S_{12} = S_{12}^* - \bar{\mathbf{x}}\bar{\mathbf{y}}'$. Note that the asymptotic distribution of T_n given by Theorem 2.2 is generally complicated by the dependence of the $Z^{(\nu)}(f_i g_j)$. The other test, which is consistent against all alternatives, uses

$$h_A(\mathbf{x}_i, \mathbf{x}_j) = h_B(\mathbf{x}_i, \mathbf{x}_j)$$
$$= c\{1 - 2\pi^{-1}\psi_{i,j}\} + d\{1 - p2^{-1}[\Gamma(\alpha + 1/2)/\Gamma(\alpha + 1)]^2 \sin \psi_{i,j}\}$$

where $\psi_{i,j} = \cos^{-1}(\mathbf{x}_i' \mathbf{x}_j)$, $\mathbb{X} = S^p$, $\alpha = (p-1)/2$, and c and d are arbitrary positive constants. This test is obtained by taking

$$\alpha_{2k} = \beta_{2k} = \left\{\frac{dp(2k-1)}{8\pi(2k+p)}\right\}^{1/2} \frac{\Gamma(\alpha + 1/2)\Gamma(k - 1/2)}{\Gamma(k + \alpha + 1/2)},$$

$$\alpha_{2k-1} = \beta_{2k-1} = \frac{c^{1/2} 2^{p-1} \Gamma(\alpha + 1)\Gamma(\alpha + k)(2k-2)!}{\pi(k-1)!(2k+p-2)!}$$

and corresponds to using a linear combination of the seminorms employed by Ajne's and by Giné's tests for uniformity. See Giné (1975, pages 1261–1262) and Prentice (1978, page 172). Note that, as all tests in this section are conditional on $(\varepsilon_1, \varepsilon_2)$, there is no need to evaluate the last term on the right of (2.7) or (2.8).

3. Uniform scores tests. If both $\mathbb{X}$ and $\mathbb{Y}$ are the circle, S^1, then we can use a probability integral transform to define uniform scores. Recall that, once an origin and an orientation of S^1 have been chosen, each λ in $\mathscr{P}(S^1)$ determines the corresponding probability integral transform H_λ: $S^1 \to S^1$ by $H_\lambda(\theta) = 2\pi\lambda([0, \theta])$, where S^1 has unit radius. Similarly, ν in $\mathscr{P}(S^1 \times S^1)$ with marginals ν_1, ν_2 in $\mathscr{P}(S^1)$ determines H_ν: $S^1 \times S^1 \to S^1 \times S^1$ by $H_\nu(\theta, \phi) = (H_{\nu_1}(\theta), H_{\nu_2}(\phi))$. In particular, the probability integral transform H_ε of the empirical distribution ε on $S^1 \times S^1$ transforms ε into the uniform-scores distribution η which has "discrete uniform" marginals η_1 and η_2. Following the use of uniform scores in two-sample tests by Wheeler and Watson (1964), Mardia (1967), and Beran (1969) and in tests of symmetry by Jupp and Spurr (1983), we propose tests of independence which reject this hypothesis for large values of

$$T_n^* = n \| \eta - \eta_1 \otimes \eta_2 \|^2. \tag{3.1}$$

In the absence of ties, T_n^* can be computed using (2.7) with

$$(x_i, y_i) = (i2\pi/n, r_i(2\pi/n)), \quad \text{for} \quad i = 1, \cdots, n,$$

where the r_i are the ranks of the second variable. An alternative to T_n^* requiring less computation is

$$T_n^{**} = n \| \eta \|^2, \tag{3.2}$$

which is equivalent to taking only the first term in (2.7). Using condition C it can be shown that $n \| \eta_i \|^2 \to 0$ as $n \to \infty$ for $i = 1, 2$. It follows that T_n^* and T_n^{**} are asymptotically equal.

Both T_n^* and T_n^{**} are quadratic forms in linear rank statistics. They are invariant under separate continuous invertible transformations of the two angles and so are well-defined independently of the origins and orientations of the two circles. Most importantly, they are distribution-free under sampling from continuous distributions with independent marginals. The asymptotic null distribution is given in the following theorem.

THEOREM 3.1. *Under random sampling from a continuous distribution on* $S^1 \times S^1$ *with independent marginals*

$$T_n^* \to_d \textstyle\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \alpha_k^2 \beta_\ell^2 H_{k,\ell} \quad \textit{and} \quad T_n^* - T_n^{**} \to_d 0$$

where $\{H_{k,\ell}\}_{k,\ell=1}^{\infty}$ *is a collection of independent chi-squared random variables with four degrees of freedom.*

Approximations to the tail areas of the asymptotic distribution of T_n^* can be derived from work of Hoeffding (1964).

The population version of η is $\nu \circ H_\nu$, which may be regarded as "ν with marginals made uniform." Then the population version of $n^{-1}T_n^*$ is $\| \nu \circ H_\nu \|^2$, which measures how far $\nu \circ H_\nu$ is from the uniform distribution on $S^1 \times S^1$. Note that $\nu \circ H_\nu$ is uniform if and only if $\nu = \nu_1 \otimes \nu_2$. These considerations lead naturally to the following consistency properties of our tests, analogous to those of the corresponding two-sample tests (Beran, 1969) and tests of symmetry (Jupp and Spurr, 1983).

THEOREM 3.2. (Consistency). *Let* ν *be a continuous distribution on* $S^1 \times S^1$. *Then, if* $\| \nu \circ H_\nu \|^2 > 0$, *the sequence of tests based on* T_n^* *or on* T_n^{**} *is consistent against* ν. *In particular, a sequence of tests is consistent against all alternatives if* $\alpha_k \neq 0$ *and* $\beta_k \neq 0$ *for all* k.

Finally we consider some examples. Note that, as both $\mathbb{X}$ and $\mathbb{Y}$ are the unit circle, the functions h_A and h_B used in (2.7) take the form

$$h_A(x, y) = 2 \textstyle\sum_{k=1}^{\infty} \alpha_k^2 \cos k(x - y)$$

$$h_B(x, y) = 2 \textstyle\sum_{k=1}^{\infty} \beta_k^2 \cos k(x - y).$$

(Compare formula (6.1) of Giné (1975).) Note also that a sequence $A = \{\alpha_k\}$ satisfying $\sup_k | k^s \alpha_k | < \infty$ for some $s > 1/2$ gives rise to both a Sobolev test for uniformity on S^1 based on $\| \cdot \|_A$ and (taking $A = B$) a uniform scores test of independence using T_n^* based on $\| \cdot \|_{A \otimes A}$.

EXAMPLE 1. The simplest Sobolev test of uniformity on S^1 is Rayleigh's test. Giné (1975) shows that this is obtained by taking $\alpha_1 = 1$ and $\alpha_k = 0$ for $k \geq 2$.

For the corresponding test of independence we have (for $n \geq 2$)

$$T_n^* = T_n^{**} = 2n(\bar{R}_+^2 + \bar{R}_-^2)$$

where $\bar{R}_\pm$ is the mean resultant length of $2\pi n^{-1}(i \mp r_i)$, $i = 1, \cdots, n$. This follows from calculations of Mardia (1975, page 360). Mardia considered the circular rank correlation coefficient $r_0 = \max(\bar{R}_+^2, \bar{R}_-^2)$ and Fisher and Lee (1982) suggested $\hat{\Pi}_n = \bar{R}_+^2 - \bar{R}_-^2$ as a circular analogue of Spearman's rho. Under independence T_n^* is distributed asymptotically as χ_4^2.

EXAMPLE 2. A test which is consistent against a wider class of alternatives is obtained by taking $A = B = \{\alpha_k\}$ with $\alpha_{2k} = 0$, $\alpha_{2k+1} = (2k+1)^{-1}$, so that, as shown by Giné (1975), the test of uniformity on S^1 given by A is Ajne's (1968) A_n-test. The Fourier expansion $|\theta| = \pi/2 - \sum_{k=0}^{\infty} (2k+1)^{-2}\cos(2k+1)\theta$ for $|\theta| < \pi$ shows that $h_A(x, y) = (\pi/2)(\pi/2 - \widehat{xy})$ where $\widehat{xy}$ denotes the angle between x and y. It follows that

$$T_n^{**} = \pi^4(16n)^{-1} \sum_{i=1}^n \sum_{j=1}^n \{1 - 4n^{-1}d(i, j)\}\{1 - 4n^{-1}d(r_i, r_j)\}$$

where $d(i, j) = \min(|i - j|, n - |i - j|)$. Note that if n is even we have $\int \mathbf{t}\, d\eta_1 = \mathbf{0} = \int \mathbf{u}\, d\eta_2$ and so $T_n^* = T_n^{**}$.

Consider the antipodal action on $S^1 \times S^1$ which takes (θ, ϕ) to $(\theta + \pi, \phi + \pi)$ and let $\bar{\varepsilon}$ denote the corresponding symmetrization of the empirical distribution. Puri and Rao (1977) introduced $n \| \bar{\varepsilon} \|_{A\otimes A}^2$ as a test of circular independence which is distribution-free for continuous distributions with antipodal symmetry. In contrast, $T_n^{**} = n \| \eta \|_{A\otimes A}^2$ does not require symmetry in order to be distribution-free.

EXAMPLE 3. Let m_1 and m_2 be integers with $m_i \geq 2$, for $i = 1, 2$, and take $A = \{\alpha_k\}$, $B = \{\beta_k\}$ where $\alpha_k = m_1(k\pi)^{-1}\sin(k\pi/m_1)$, $\beta_k = m_2(k\pi)^{-1}\sin(k\pi/m_2)$, so that the corresponding test of uniformity is that of Rao (1972) and Rothman (1972). For $(\theta, \phi) \in S^1 \times S^1$ let $\chi^2_{(\theta,\phi)}$ denote the usual χ^2-statistic for testing independence in the contingency table with cells.

$$(\theta + 2\pi m_1^{-1}(j-1), \theta + 2\pi m_1^{-1}j) \times (\phi + 2\pi m_2^{-1}(q-1), \phi + 2\pi m_2^{-1}q),$$

for $j = 1, \cdots, m_1$ and $q = 1, \cdots, m_2$, and with observations $(2\pi n^{-1}i, 2\pi n^{-1}r_i)$, for $i = 1, \cdots, n$, given by η. If both m_1 and m_2 divide n, then $\eta_1 \otimes \eta_2$ gives probability $(m_1m_2)^{-1}$ to each cell. Let $O_{ij}(\theta, \phi)$ denote the number of observations in the (i, j)th cell. Then by calculations similar to those of Rao (1972) and Puri and Rao (1977), we find that the double Fourier expansion of $O_{ij}(\theta, \phi)$ yields

$$\begin{aligned}\{O_{ij}(\theta, \phi) &- n(m_1m_2)^{-1}\}^2 \\ &= 16n^2(m_1m_2)^{-1} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \alpha_k^2\beta_\ell^2\{(T_{cc}^{k,\ell})^2 + (T_{cs}^{k,\ell})^2 + (T_{sc}^{k,\ell})^2 + (T_{ss}^{k,\ell})^2\}\end{aligned}$$

where, e.g.,

$$T_{cc}^{k,\ell} = n^{-1} \sum_{i=1}^n \cos(2\pi n^{-1}ki)\cos(2\pi n^{-1}\ell r_i).$$

It follows that, if m_1 and m_2 divide n,

$$T_n^* = T_n^{**} = (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \chi^2_{(\theta,\phi)}\, d\theta\, d\phi.$$

Thus the uniform scores test based on T_n^* is an averaged χ^2 test of independence analogous to Rao's (1972) averaged χ^2 goodness-of-fit test.

EXAMPLE 4. Watson's (1961) U^2-test of uniformity on the circle is consistent against all alternatives. Giné (1975) showed that the U^2-test is a Sobolev test using $\alpha_k = k^{-1}$ for $k \geq 1$. We now show that the corresponding uniform scores test is Rothman's (1971) C_n-test of independence. Given origins and orientations on $S^1 \times S^1$, the distribution functions F_μ, F_{μ_1}, F_{μ_2} of $\mu \in \mathscr{P}(S^1 \times S^1)$ are defined by $F_\mu(\theta, \phi) = \mu([0, \theta] \times [0, \phi])$, $F_{\mu_1}(\theta) = F_\mu(\theta, 2\pi)$, $F_{\mu_2}(\phi) = F(2\pi, \phi)$. Then one measure of dependence between μ_1 and μ_2 is the function T_μ defined by

$$T_\mu(\theta, \phi) = F_\mu(\theta, \phi) - F_{\mu_1}(\theta)F_{\mu_2}(\phi).$$

If the function Z_μ is defined by

$$Z_\mu(\theta, \phi) = T_\mu(\theta, \phi) - \int T_\mu(\theta, \phi)\, d\mu_1(\theta) - \int T_\mu(\theta, \phi)\, d\mu_2(\phi) + \int\int T_\mu(\theta, \phi)\, d\mu(\theta, \phi),$$

then

$$C_\mu = \int\int \{Z_\mu(\theta, \phi)\}^2\, d\mu(\theta, \phi)$$

is well-defined. Rothman's statistic takes μ to be the empirical distribution ε and is

$$C_n = nC_\varepsilon.$$

To calculate C_n for a sample $(\theta_1, \phi_1), \cdots, (\theta_n, \phi_n)$, it is convenient to use the following alternative expression given by Rothman's equation (9):

$$C_n = n^{-2} \textstyle\sum_{j=1}^n (\sum_{k=1}^n \{T_\varepsilon(\theta_j, \phi_j) - T_\varepsilon(\theta_k, \phi_j) - T_\varepsilon(\theta_j, \phi_k) + T_\varepsilon(\theta_k, \phi_k)\})^2.$$

Thus it can be seen that C_n is rank-invariant and so $C_\varepsilon = C_\eta$. Note that $T_\eta(\theta, \phi)$ and so $Z_\eta(\theta, \phi)$ are constant on each of the regions $(i/n, (i+1)/n) \times (j/n, (j+1)/n)$ for $i, j = 0, \cdots, n-1$. It follows that

$$\int\int [Z_\eta(\theta, \phi)]^2\, d\eta(\theta, \phi) = \int\int [Z_\eta(\theta, \phi)]^2 (2\pi)^{-2}\, d\theta\, d\phi.$$

Double Fourier expansion of $Z_\eta(\theta, \phi)$ together with Parseval's formula yields

$$C_\eta = (4\pi^2)^{-2} \| \eta - \eta_1 \otimes \eta_2 \|^2_{A \otimes A},$$

where $A = \{\alpha_k\}$ with $\alpha_k = k^{-1}$ for $k \geq 1$. Thus

$$T_n^* = 16\pi^4 C_n.$$

By Theorem 3.2, Rothman's test is consistent against all continuous alternatives. From Hoeffding's (1964) results on the tail probabilities of weighted sums of χ^2 distributions, we obtain

$$\begin{aligned} \lim_{n \to \infty} P(T_n^* > x) &\simeq 2.932[P(\chi_4^2 > x) - 1.110P(\chi_2^2 > x)] \\ &= (-0.322 + 1.466x)e^{-x/2} \end{aligned}$$

and so approximate critical values of T_n^* are 9.91, 11.62, and 15.42 at significance levels 0.1, 0.05, and 0.01, respectively.

4. An example. As an illustration of the use of the tests of Section 2, we consider Stephens' (1979) data on directions of magnetization of rock before and after heat treatment. Here $\mathbb{X} = \mathbb{Y} = S^2$ and $n = 6$. As we know of no distribution-free tests for independence on the sphere and as the small sample size renders the use of asymptotic results inappropriate, it seems that we must use a randomization test. The quick Rayleigh-type test of (2.8) yields $T_n = 0.007$, which is within the lower 15% of the randomization distribution. Similarly, the Ajne-Giné-type test mentioned after (2.8) with $c = d = 1$ yields $T_n = 0.162$, which is within the lower 22% of the appropriate distribution. Thus neither test detects any evidence of dependence.

APPENDIX

The proofs of our theorems indicated here parallel the proofs given by Giné (1975) and Wellner (1979) of their results, with the simplification that we work entirely in the Hilbert space $L^2(\mathbb{X} \times \mathbb{Y})$ without passing to Banach spaces analogous to their $C(B_s)$. Giné (1975, pages 1252–1253) remarks that there are simple Hilbert space proofs of his results. This approach works also for the tests of Wellner (1979) and Jupp and Spurr (1983).

The following auxiliary result is proved using a multivariate form of Hájek's (1961) permutational central limit theorem.

THEOREM A. *Let* $\{(x_{n,i}, y_{n,i}): 1 \le i \le n, n = 1, 2, \cdots\}$ *be a triangular array in* $\mathbb{X} \times \mathbb{Y}$ *and let* $\nu^{(n)}$ *denote the probability measure giving mass* n^{-1} *to* $(x_{n,i}, y_{n,i})$ *for* $1 \le i \le n$. *Define the random element* X_n *of* $L^2(\mathbb{X} \times \mathbb{Y})$ *by*

$$X_n = n^{-1/2} \textstyle\sum_{i=1}^n (\mathbf{t}(x_{n,i}) - \overline{\mathbf{t}}) \otimes (\mathbf{u}(y_{n,\sigma(i)}) - \overline{\mathbf{u}}),$$

where σ *is distributed uniformly over the permutations of* $\{1, \cdots, n\}$,

$$\overline{\mathbf{t}} = n^{-1} \textstyle\sum_{i=1}^n \mathbf{t}(x_{n,i}), \quad \overline{\mathbf{u}} = n^{-1} \sum_{i=1}^n \mathbf{u}(y_{n,i}) \quad \text{and} \quad L^2(\mathbb{X}) \otimes L^2(\mathbb{Y})$$

has been identified with its image in $L^2(\mathbb{X} \times \mathbb{Y})$. *Suppose that* $w^*\text{-lim}\ \nu^{(n)} = \nu$ *exists and has marginal distributions* ν_1 *and* ν_2 *on* $\mathbb{X}$ *and* $\mathbb{Y}$. *Then*

$$X_n \to_d W^{(\nu)}$$

where $W^{(\nu)}$ *has the Gaussian distribution on* $L^2(\mathbb{X} \times \mathbb{Y})$ *with mean zero and covariance structure specified by*

$$\mathrm{Var}(\langle W^{(\nu)}, \mathbf{f} \otimes \mathbf{g}\rangle) = \mathrm{Var}_{\nu_1}(\langle \mathbf{t}(X), \mathbf{f}\rangle)\mathrm{Var}_{\nu_2}(\langle \mathbf{u}(Y), \mathbf{g}\rangle)$$

for $\mathbf{f} \in L^2(\mathbb{X})$, $\mathbf{g} \in L^2(\mathbb{Y})$.

To establish Theorem A we shall use several lemmas.

First we investigate some properties of the function t; $\mathbb{X} - L^2(\mathbb{X})$ defined by

$$t(x) = \textstyle\sum_{k=1}^\infty \alpha_k \sum_{f_i \in E_k} f_i(x) f_i.$$

LEMMA 1. *If* $\{\alpha_k\}$ *satisfies condition* C *then*

$$\sum_{k=1}^{\infty} \alpha_k \sum_{f_i \in E_k} f_i(x) f_i$$

in uniformly convergent.

PROOF. Condition C states that $\sup_k |\alpha_k \sigma_k^{s/2}| < \infty$ for some $s > (\dim \mathbb{X})/2$. Choose $r \in ((\dim \mathbb{X})/2, s)$. Then there exists $K > 0$ such that

$$1 + \sum_{k=1}^{\infty} \sigma_k^{-r} \sum_{f_i \in E_k} (f_i(x))^2 = \| \delta_x \|_{-r}^2 \leq K \quad \text{for all } x \text{ in } \mathbb{X}.$$

(cf. Lemma 3.1 of Giné (1975).) Thus for $N \geq 1$

$$\sum_{k=1}^{N} \sigma_k^{-r} \sum_{f_i \in E_k} (f_i(x))^2 < K \quad \text{for all } x \text{ in } \mathbb{X}.$$

Also, because $\sigma_k \to \infty$ as $k \to \infty$ (see, e.g., page 254 of Warner, 1971), σ_k^{r-s} tends monotonically to zero. Then by the normed space version of Dirichlet's test

$$\sum_{k=1}^{\infty} \sigma_k^{r-s} \sigma_k^{-r} \sum_{f_i \in E_k} f_i(x) f_i$$

is uniformly convergent.

For $n > m$ we have

$$\begin{aligned} \| \textstyle\sum_{k=m}^{n} \alpha_k \sum_{f_i \in E_k} f_i(x) f_i \|^2 &= \textstyle\sum_{k=m}^{n} \alpha_k^2 \sum_{f_i \in E_k} [f_i(x)]^2 \\ &\leq (\sup_k |\alpha_k \sigma_k^{s/2}|)^2 \textstyle\sum_{k=m}^{n} \sigma_k^{-s} \sum_{f_i \in E_k} [f_i(x)]^2 \end{aligned}$$

which tends to zero uniformly in x as $m, n \to \infty$. Thus the Cauchy condition is satisfied and uniform convergence follows. □

COROLLARY. *If* $\{\alpha_k\}$ *satisfies condition* C *then* $t\colon \mathbb{X} \to L^2(\mathbb{X})$ *is uniformly continuous.*

PROOF. Continuity follows from the uniform convergence above. Uniform continuity follows from compactness of $\mathbb{X}$. □

LEMMA 2. *If* X_n *is as defined in Theorem* A *then* $\{\mathscr{L}(X_n)\}_{n=1}^{\infty}$ *is relatively compact in* $\mathscr{P}(L^2(\mathbb{X} \times \mathbb{Y}))$.

PROOF. If $\{f_i\}_{i=1}^{\infty}$, $\{g_j\}_{j=1}^{\infty}$ are orthonormal bases of $L^2(\mathbb{X})$, $L^2(\mathbb{Y})$ then $\{f_i \otimes g_j\}_{i,j=1}^{\infty}$ is an orthonormal basis of $L^2(\mathbb{X} \times \mathbb{Y})$.

For every i and j, we have by the usual finite-sampling calculation

$$\begin{aligned} E[\langle X_n, f_i \otimes g_j \rangle^2] &= \operatorname{Var}(\langle X_n, f_i \otimes g_j \rangle) \\ &= (n/(n-1)) \operatorname{Var}_{\nu_1^{(n)}}(\langle t(X), f_i \rangle) \operatorname{Var}_{\nu_2^{(n)}}(\langle u(Y), g_j \rangle) \\ &\leq (n/(n-1)) E_{\nu_1^{(n)}}[\langle t(X), f_i \rangle^2] E_{\nu_2^{(n)}}[\langle u(Y), g_j \rangle^2] \\ &\leq 2 E_{\nu_1^{(n)}}[\langle t(X), f_i \rangle^2] E_{\nu_2^{(n)}}[\langle u(Y), g_j \rangle^2], \end{aligned}$$

where $\nu_1^{(n)}$ and $\nu_2^{(n)}$ are the marginals of $\nu^{(n)}$ on $\mathbb{X}$ and $\mathbb{Y}$. Thus

$$\begin{aligned}E(\textstyle\sum_{\max(i,j)>r}\langle X_n, f_i \otimes g_j\rangle^2) &\le 2 \textstyle\sum_{\max(i,j)>r} E_{\nu_1^{(n)}}[\langle t(X), f_i\rangle^2]E_{\nu_2^{(n)}}[\langle u(Y), g_j\rangle^2] \\ &\le 2 \textstyle\sum_{i>r} E_{\nu_1^{(n)}}[\langle t(X), f_i\rangle^2] \sum_{j=1}^{\infty} E_{\nu_2^{(n)}}[\langle u(Y), g_j\rangle^2] \\ &\quad + 2 \textstyle\sum_{i=1}^{\infty} E_{\nu_1^{(n)}}[\langle t(X), f_i\rangle^2] \sum_{j>r} E_{\nu_2^{(n)}}[\langle u(Y), g_j\rangle^2] \\ &\le 2 \sup_{y\in\mathbb{Y}} \| u(y) \|^2 \textstyle\sum_{i>r} E_{\nu_1^{(n)}}[\langle t(X), f_i\rangle^2] \\ &\quad + 2 \sup_{x\in\mathbb{X}} \| t(x) \|^2 \textstyle\sum_{j>r} E_{\nu_2^{(n)}}[\langle u(Y), g_j\rangle^2],\end{aligned}$$

and by Lemma 1 this tends to zero uniformly in n as $r \to \infty$. Theorem 1.13 of Prohorov (1956) now shows that $\{\mathscr{L}(X_n)\}_{n=1}^{\infty}$ is contained in a compact subset of $\mathscr{P}(L^2(\mathbb{X} \times \mathbb{Y}))$. □

The next lemma is a multivariate permutational central limit theorem.

LEMMA 3. *Let $\mathbb{E}$ and $\mathbb{F}$ be finite-dimensional inner product spaces and let $\{(c_{n,i}, d_{n,i}): 1 \le i \le n, n = 1, 2, \cdots\}$ be a triangular array in $\mathbb{E} \times \mathbb{F}$. Define the random element S_n of $\mathbb{E} \otimes \mathbb{F}$ by*

$$S_n = \textstyle\sum_{i=1}^{n} c_{n,i} \otimes d_{n,\sigma(i)},$$

where σ is distributed uniformly over the permutations of $\{1, \cdots, n\}$. Suppose that

(i) $\sum_{i=1}^{n} c_{n,i} = 0, \quad \sum_{i=1}^{n} d_{n,i} = 0, \quad n = 1, 2, \cdots,$

(ii) $n^{-1} \sum_{i=1}^{n} c_{n,i} \otimes c_{n,i} \to \Sigma_c, \quad \sum_{i=1}^{n} d_{n,i} \otimes d_{n,i} \to \Sigma_d$

for some $\Sigma_c \in \mathbb{E} \otimes \mathbb{E}$, $\Sigma_d \in \mathbb{F} \otimes \mathbb{F}$,

(iii) $n^{-1/2}\max_{1\le i\le n} \| c_{n,i} \| \to 0, \quad \max_{1\le i\le n} \| d_{n,i} \| \to 0,$

(iv) *for every $\varepsilon > 0$, $n^{-1} \sum_{\{\delta_{n_{ij}}>\varepsilon\}}\delta^2_{n_{ij}} \to 0$ where $\delta_{n_{ij}} = \| c_{n,i} \| \| d_{n,j} \|$.*

Then $S_n \to_d N(0, \Sigma)$ where the variance Σ is the Kronecker product $\Sigma_c \otimes \Sigma_d$.

PROOF. The proof is a generalization of that of Hájek's (1961) version of the Wald-Wolfowitz-Noether permutational central limit theorem. For each n define $a = a_n: (0, 1] \to \{1, \cdots, n\}$ by

$$a_n(t) = j \quad \text{for} \quad (j-1)/n < t \le j/n, \quad j = 1, \cdots, n.$$

For each n, let $U_1, \cdots, U_n$ be independent random variables uniformly distributed on $(0, 1]$. Let R_i be the rank of U_i in $\{U_1, \cdots, U_n\}$. Then we may take

$$S_n = \textstyle\sum_{i=1}^{n} c_{n,i} \otimes d_{n,R_i}.$$

Define

$$T_n = \textstyle\sum_{i=1}^{n} c_{n,i} \otimes d_{a(U_i)}.$$

That S_n and T_n are asymptotically equivalent in the mean, i.e.,

$$\lim_{n\to\infty} E[\| S_n - T_n \|^2]/E[\| T_n - E[T_n] \|^2] = 0,$$

follows from a straightforward generalization of Hájek's Theorem 3.1. The only nontrivial part of this generalization is the following version of his Lemma 2.1:

If dim $\mathbb{E} = p$, then

$$E[\| c_{n,a(U_1)} - c_{n,R_1} \|^2] \leq 2(2p)^{1/2}/n \max_{1\leq i\leq n} \| c_{n,i} \| (\textstyle\sum_{i=1}^n \| c_{n,i} \|^2)^{1/2}.$$

This is obtained by summing the corresponding inequalities for the components of $c_{n,a(U_1)} - c_{n,R_1}$ given by Hájek's inequality (2.5) and then using Jensen's inequality and concavity of the function $t \mapsto t^{1/2}$.

Assumptions (ii) and (iii) ensure that

$$\lim_{n\to\infty}(\max_{1\leq i\leq n} \| c_{n,i} \|^2/\textstyle\sum_{i=1}^n \| c_{n,i} \|^2) = 0$$

and so S_n and T_n are asymptotically equivalent in the mean.

Every subsequence of $\{\mathscr{L}(S_n)\}_{n=1}^\infty$ has a convergent subsequence. By asymptotic equivalence, the same is true for $\{\mathscr{L}(T_n)\}_{n=1}^\infty$. Using assumptions (ii), (iii) and (iv), it can be seen (cf. Hájek's Theorem 4.1) that for any $v \in \mathbb{E} \otimes \mathbb{F}$ any convergent subsequence of $\{\mathscr{L}(\langle T_n, v\rangle)\}_{n=1}^\infty$ converges to $N(0, \langle \Sigma, v \otimes v\rangle)$. Thus $\mathscr{L}(T_n)$ and $\mathscr{L}(S_n)$ converge to $N(0, \Sigma)$. □

Proof of Theorem A. For any finite subset $\{(h_\alpha, k_\alpha): \alpha = 1, \cdots, p\}$ of $L^2(\mathbb{X}) \times L^2(\mathbb{Y})$, define the triangular array $\{(c_{n,i}, d_{n,i}): 1 \leq i \leq n, n = 1, 2, \cdots\}$ in $\mathbb{R}^p \times \mathbb{R}^p$ by

$$c_{n,i,\alpha} = \langle t(x_{n,i}) - \bar{t}, h_\alpha\rangle \quad d_{n,i,\alpha} = n^{-1/2}\langle u(y_{n,i}) - \bar{u}, k_\alpha\rangle,$$

where $c_{n,i,\alpha}$ and $d_{n,i,\alpha}$ denote the αth components of $c_{n,i}$ and $d_{n,i}$, $\alpha = 1, \cdots, p$. Conditions (i) and (ii) of Lemma 3 are clearly satisfied. As $t(\mathbb{X})$ and $u(\mathbb{Y})$ are bounded, conditions (iii) and (iv) are also satisfied. It follows that the asymptotic distribution of $\{\langle X_n, h_\alpha \otimes k_\beta\rangle: \alpha, \beta = 1, \cdots, p\}$ is multivariate Normal. In particular, if $v = \sum_{\alpha=1}^p h_\alpha \otimes k_\alpha$ then

$$\langle X_n, v\rangle \to_d N(0, \langle \Sigma, v \otimes v\rangle)$$

where Σ is the element of $L^2(\mathbb{X} \times \mathbb{Y}) \otimes L^2(\mathbb{X} \times \mathbb{Y})$ determined by

$$\langle \Sigma, (f \otimes g) \otimes (f \otimes g)\rangle = \mathrm{Var}_{\nu_1}(\langle t(X), f\rangle)\mathrm{Var}_{\nu_2}(\langle u(Y), g\rangle)$$

for $f \in L^2(\mathbb{X})$, $g \in L^2(\mathbb{Y})$. Thus for v in $L^2(\mathbb{X}) \otimes L^2(\mathbb{Y})$, we have $\langle X_n, v\rangle \to_d \langle W^{(\nu)}, v\rangle$ and so $\phi_n(v) \to \phi_W(v)$ where ϕ_n and ϕ_W denote the characteristic functions of $\mathbb{X}_n$ and $W^{(\nu)}$. By Lemma 2, every subsequence of $\{\mathscr{L}(X_n)\}_{n=1}^\infty$ has a convergent subsequence, say $\mathscr{L}(X_{n_i}) - \mathscr{L}(Y)$. Then, for all v in $L^2(\mathbb{X}) \otimes L^2(\mathbb{Y})$, $\phi_{n_i}(v) \to \phi_Y(v) = \phi_W(v)$. As $L^2(\mathbb{X}) \otimes L^2(\mathbb{Y})$ is dense in its completion $L^2(\mathbb{X} \times \mathbb{Y})$ and as characteristic functions are continuous, we have $\mathscr{L}(Y) = \mathscr{L}(W^{(\nu)})$. Thus every subsequence of $\{\mathscr{L}(X_n)\}_{n=1}^\infty$converges to $\mathscr{L}(W^{(\nu)})$ and the result lows. □

Proof of Theorem 2.1. Taking $\nu^{(n)} = \varepsilon$ in Theorem A, we have $\| X_n \|^2 = T_n$. For random sampling from some ν with independent marginals, the distribution of $\| X_n \|^2$ is that of T_n conditional on $(\varepsilon_1, \varepsilon_2)$. Thus the asymptotic null conditional distribution on T_n exists. Now put

$$S_{12} = E_\varepsilon[t \otimes u] - E_{\varepsilon_1}[t] \otimes E_{\varepsilon_2}[u].$$

Then

$$n^{1/2}S_{12} = n^{1/2}E_\varepsilon[(t - E_{\nu_1}[t]) \otimes (u - E_{\nu_2}[u])]$$
$$- n^{-1/2}E_\varepsilon[n^{1/2}(t - E_{\nu_1}[t])] \otimes E_\varepsilon[n^{1/2}(u - E_{\nu_2}[u])]$$

and the first term on the right-hand side tends to a Gaussian random vector with mean $\Sigma_{12} = E_\nu[t \otimes u] - E_{\nu_1}[t] \otimes E_{\nu_2}[u]$, while the second term tends in probability to zero. Thus, if $\| \nu - \nu_1 \otimes \nu_2 \|^2_{A\otimes B} > 0$, we have

$$T_n/n = \| S_{12} \|^2 \to \| \Sigma_{12} \|^2 = \| \nu - \nu_1 \otimes \nu_2 \|^2 > 0,$$

so the sequence of tests is consistent. If $\| \nu - \nu_1 \otimes \nu_2 \|^2 = 0$, then T_n has a limiting distribution with support containing zero and so the sequence of tests is not consistent. □

Proof of Theorem 2.2. In the above notation, we have

$$n^{1/2}S_{12} = n^{1/2}(S_{12} - \Sigma_{12}^{(n)}) + n^{1/2}(\Sigma_{12}^{(n)} - \Sigma_{12})$$

where $\Sigma_{12}^{(n)} = E_{\nu^{(n)}}[t \otimes u] - E_{\nu_1^{(n)}}[t] \otimes E_{\nu_2^{(n)}}[u]$ and $\Sigma_{12} = 0$ as $\nu = \nu_1 \otimes \nu_2$. Now

$$E_{\nu^{(n)}}[\langle n^{1/2}(S_{12} - \Sigma_{12}^{(n)}), f_i \otimes g_j\rangle^2] \le E_{\nu_1^{(n)}}[\langle t(X), f_i\rangle^2]E_{\nu_2^{(n)}}[\langle u(Y), g_j\rangle^2]$$

and so tightness of $\{\mathscr{L}(n^{1/2}(S_{12} - \Sigma_{12}^{(n)}))\}_{n=1}^{\infty}$ follows as in the proof of Lemma 2. Then it is routine to verify that $n^{1/2}(S_{12} - \Sigma_{12}^{(n)}) \to_d W^{(\nu)}$. Finally, because $\nu = \nu_1 \otimes \nu_2$ we have

$$\langle W^{(\nu)}, f_i \otimes g_j\rangle = \alpha_k\beta_\ell Z^{(\nu)}(f_i g_j) \quad \text{for} \quad f_i \in E_k \quad \text{and} \quad g_j \in F_\ell. \quad \square$$

Proof of Theorem 2.3. As in the proof of Theorem 2.1, $n^{1/2}(S_{12} - \Sigma_{12})$ tends to a Gaussian random vector in $L^2(\mathbb{X} \times \mathbb{Y})$ with mean zero and with variance that of $\mathbf{v}(X, Y)$. Now apply the Hilbert space version of (ii) on page 321 of Rao (1965) to the function $x \mapsto \| x \|^2$ from $L^2(\mathbb{X} \times \mathbb{Y})$ to $\mathbb{R}$. □

Proof of Theorem 3.1. Take $x_{n,i} = y_{n,i} = i(2\pi/n)$ in Theorem A. If ν has independent marginals then

$$\mathscr{L}(T_n^*) = \mathscr{L}(\| X_n \|^2) \to \mathscr{L}(\| W^{(\mu)} \|^2)$$

where μ is the uniform distribution on $S^1 \times S^1$.

To show asymptotic equivalence of T_n^* and T_n^{**}, first note that by Condition C there exists $M > 0$ such that $\sup_k | \alpha_k k^s | < M$ for some $s > 1/2$. Then

$$\| E_{\eta_1}[t] \|^2 = \| \eta_1 \|_A^2 = \textstyle\sum_{n|k} 2\alpha_k^2 = 2 \sum_{i=1}^{\infty} \alpha_{in}^2 < 2M^2 \sum_{i=1}^{\infty} (in)^{-2s}.$$

Thus $n \| E_{\eta_1}[t] \|^2 < 2n^{1-2s} \sum_{i=1}^{\infty} i^{-2s} \to 0$ because $2s > 1$. Similarly $n \| E_{\eta_2}[u] \|^2 \to 0$.

Now

$$\begin{aligned} |T_n^{**} - T_n^*| &= |n \| E_\eta[t \otimes u] \|^2 - n \| E_\eta[t \otimes u] - E_{\eta_1}[t] \otimes E_{\eta_2}[u] \|^2 | \\ &= n | \langle 2E_\eta[t \otimes u] - E_{\eta_1}[t] \otimes E_{\eta_2}[u], E_{\eta_1}[t] \otimes E_{\eta_2}[u] \rangle | \\ &\le \| 2E_\eta[t \otimes u] - E_{\eta_1}[t] \otimes E_{\eta_2}[u] \| (n \| E_{\eta_1}[t] \| n \| E_{\eta_2}[u] \|)^{1/2} \\ &\to 0 \end{aligned}$$

as $t(S^1)$ and $u(S^1)$ are bounded. □

PROOF OF THEOREM 3.2. This follows from

$$n^{-1}T_n^* = \| \eta - \eta_1 \otimes \eta_2 \|^2 \to \| \nu_\circ H_\nu - \mu \|^2 = \| \nu_\circ H_\nu \|^2. \quad \square$$

Acknowledgement. We are very grateful to two referees for their constructive comments and for pointing out an error in an earlier "proof" of Theorem 2.1. This has led to substantial improvements in the paper.

REFERENCES

AJNE, B. (1968). A simple test for uniformity of a circular distribution. *Biometrika* **55** 343–354.

BERAN, R. J. (1969). The derivation of nonparametric two-sample tests from tests for uniformity of a circular distribution. *Biometrika* **56** 561–570.

BLUM, J. R., KIEFER, J., and ROSENBLATT, M. (1961). Distribution free tests of independence based on the sample distribution function. *Ann. Math. Statist.* **32** 485–498.

DOWNS, T. D. (1974). Rotational angular correlations. In *Biorhythms and Human Reproduction*, (M. Ferin, F. Halberg, R. M. Richart and L. van der Wiele, eds.), 97–104. Wiley, New York.

EPP, R. J., TUKEY, J. W. and WATSON, G. S. (1971). Testing unit vectors for correlation. *J. Geophys. Res.* **76** 8480–8483.

FISHER, N. I. and LEE, A. J. (1982). Nonparametric measures of angular-angular association. *Biometrika* **69** 315–321.

FISHER, N. I. and LEE, A. J. (1983). A correlation coefficient for circular data. *Biometrika* **70** 327–332.

GINÉ, M. E. (1975). Invariant tests for uniformity on compact Riemannian manifolds based on Sobolev norms. *Ann. Statist.* **3** 1243–1266.

HÁJEK, J. (1961). Some extensions of the Wald-Wolfowitz-Noether theorem. *Ann. Math. Statist.* **32** 506–523.

HOEFFDING, W. (1948). A non-parametric test of independence. *Ann. Math. Statist.* **19** 546–557.

HOEFFDING, W. (1964). On a theorem of V. M. Zolotarev. *Theory Probab. Appl.* **9** 89–91.

JOHNSON, R. A. and WEHRLY, T. (1977). Measures and models for angular correlation and angular-linear correlation. *J. Roy. Statist. Soc. Ser. B* **39** 222–229.

JUPP, P. E. and MARDIA, K. V. (1980). A general correlation coefficient for directional data and related regression problems. *Biometrika* **67** 163–173.

JUPP, P. E. and SPURR, B. D. (1983). Sobolev tests for symmetry of directional data. *Ann. Statist.* **11** 1225–1231.

MACKENZIE, J. K. (1957). The estimation of an orientation relationship. *Acta Cryst.* **10** 61–62.

MARDIA, K. V. (1967). A non-parametric test for the bivariate two-sample location problem. *J. Roy. Statist. Soc. Ser. B* **29** 340–342.

MARDIA, K. V. (1972). *Statistics of Directional Data.* Academic, London.

MARDIA, K. V. (1975). Statistics of directional data. *J. Roy. Statist. Soc. Ser. B* **37** 349–393.

MARDIA, K. V. and PURI, M. L. (1978). A spherical correlation coefficient robust against scale. *Biometrika* **65** 391–395.

PRENTICE, M. J. (1978). On invariant tests of uniformity for directions and orientations. *Ann. Statist.* **6** 169–176.

PROHOROV, YU. V. (1956). Convergence of random processes and limit theorems in probability theory. *Theory Probab. Appl.* **1** 157–214.

PURI, M. L. and RAO, J. S. (1977). Problems of association for bivariate circular data and a new test of independence. In *Multivariate Analysis IV*, (P. R. Krishnaiah, ed.) 513–522. North-Holland, Amsterdam.

RAO, C. R. (1965). *Linear Statistical Inference and Its Applications*. Wiley, New York.

RAO, J. S. (1972). Some variants of chi-square for testing uniformity on the circle. *Z. Wahrsch. verw. Gebiete* **22** 33–44.

RIVEST, L.-P. (1982). Some statistical methods for bivariate circular data. *J. Roy. Statist. Soc. Ser. B* **44** 81–90.

ROTHMAN, E. D. (1971). Tests of coordinate independence for a bivariate sample on a torus. *Ann. Math. Statist.* **42** 1962–1969.

ROTHMAN, E. D. (1972). Tests for uniformity of a circular distribution. *Sankhyā Ser. A* **34** 23–32.

STEPHENS, M. A. (1979). Vector correlation. *Biometrika* **66** 41–48.

WARNER, F. W. (1971). *Foundations of Differentiable Manifolds and Lie Groups*. Scott, Foresman & Co., Glenville, Ill.

WATSON, G. S. (1961). Goodness-of-fit tests on a circle. *Biometrika* **48** 109–114.

WATSON, G. S. and BERAN, R. J. (1967). Testing a sequence of unit vectors for serial correlation. *J. Geophys. Res.* **72** 5655–5659.

WELLNER, J. A. (1979). Permutation tests for directional data. *Ann. Statist.* **7** 929–943.

WHEELER, S. and WATSON, G. S. (1964). A distribution-free two-sample test on the circle. *Biometrika* **51** 256–257.

MATHEMATICAL INSTITUTE
UNIVERSITY OF ST. ANDREWS
NORTH HAUGH
ST. ANDREWS KY16 9SS
SCOTLAND