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RATE OF CONVERGENCE OF ONE- AND TWO-STEP *M*-ESTIMATORS WITH APPLICATIONS TO MAXIMUM LIKELIHOOD AND PITMAN ESTIMATORS

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A one-step version $M_n^{(1)}$ and a two-step version $M_n^{(2)}$ of a general M-estimator M_n are suggested such that $M_n - M_n^{(1)} = O_P(n^{-1})$ and $M_n - M_n^{(2)} = O_P(n^{-3/2})$ for every $n^{1/2}$-consistent initial estimator and under some regularity conditions. In the special case of maximum likelihood estimation, this among other yields that the second-order efficiency properties of $M_n^{(2)}$ coincide with those of M_n. An application to the Pitman estimator of location is considered.

1. Introduction. Let $X_1, X_2, \cdots$ be independent identically distributed (i.i.d.) random variables with common density $f(x, \theta)$ (with respect to Lebesgue measure), where $\theta \in \Theta \subset \mathbb{R}$. Moreover, we assume that the parameter space Θ is an open interval. The true value of θ will be denoted by θ_0.

Let $\psi: \mathbb{R} \times \Theta \to \mathbb{R}$ be a function such that $E_{\theta_0}\psi(X_1, \theta)$ exists for all $\theta \in \Theta$ and has a unique zero at $\theta = \theta_0$.

A consistent estimator M_n of θ_0 which is a solution with respect to t of the equation

$$\sum_{i=1}^n \psi(X_i, t) = 0 \tag{1.1}$$

will be called an M-estimator corresponding to ψ. Regularity conditions on ψ and f under which M-estimators exist, can be found, e.g., in Serfling (1980), Chapter 7 and Huber (1981), Chapter 6.

It is often difficult to find a consistent solution of Equation (1.1) in an explicit way. Therefore a standard technique is to look at an iterative solution of (1.1) (see, e.g., Dzhaparidze (1983) for an excellent account). We shall consider the iterative solution $M_n^{(k)}$ of (1.1) in the form

$$M_n^{(k)} = \begin{cases} M_n^{(k-1)} - (n\hat{\gamma}_n^{(k-1)})^{-1} \sum_{i=1}^n \psi(X_i, M_n^{(k-1)}) & \text{if } \hat{\gamma}_n^{(k-1)} \neq 0 \\ M_n^{(k-1)} & \text{if } \hat{\gamma}_n^{(k-1)} = 0 \end{cases} \tag{1.2}$$

$k = 1, 2, \cdots$, where $M_n^{(0)}$ is a consistent initial estimator of θ_0 and $\hat{\gamma}_n^{(k)}$ $(k = 0, 1, \cdots)$ is a consistent estimator of $\gamma(\theta_0)$, with

$$\gamma(\theta) = \int \dot{\psi}(x, \theta) f(x, \theta)\, dx, \tag{1.3}$$

$\dot{\psi}(x, \theta)$ denoting the partial derivative with respect to θ.

Before we proceed, we introduce $E(\cdot)$ and $P(\cdot)$ as a shorthand notation for $E_{\theta_0}(\cdot)$ and $P_{\theta_0}(\cdot)$.

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Theorems showing that a one-step version (say T_n) of M_n is asymptotically equivalent to M_n, i.e.,

$$M_n - T_n = o_P(n^{-1/2}),$$

can be established, assuming that the initial estimator $M_n^{(0)}$ is $n^{1/2}$-consistent (i.e., $n^{1/2}(M_n^{(0)} - \theta_0) = O_P(1)$) and that ψ and f satisfy some regularity conditions. A discussion on this subject can be found in Bickel (1975), Huber (1981), Chapter 6 and in Lehmann (1983), Chapter 6 for the case that M_n is a maximum likelihood estimator (MLE).

Jurečková (1983) studied the regression case and showed that $M_n - M_n^{(1)} = O_P(n^{-1})$; in (1985) she derived the second-order asymptotic distribution of M_n in the location case and concluded that $M_n - M_n^{(1)} = o_P(n^{-1})$, provided $M_n^{(0)}$ is equally asymptotically efficient as M_n.

The aim of the present paper is to show that $M_n^{(1)}$ and $M_n^{(2)}$, with properly defined $\hat{\gamma}_n^{(0)}$ and $\hat{\gamma}_n^{(1)}$, are not only asymptotically equivalent to M_n for every $n^{1/2}$-consistent initial estimator $M_n^{(0)}$, but even that

$$M_n - M_n^{(1)} = O_P(n^{-1}) \quad \text{and} \quad M_n - M_n^{(2)} = O_P(n^{-3/2}).$$

These asymptotic relations among others yield that the second-order efficiency properties of $M_n^{(2)}$ (in the sense of Akahira and Takeuchi, 1981) coincide with those of M_n. This is of special interest in the maximum likelihood case.

The basic technical tool is the second-order asymptotic linearity in t and u of

$$\textstyle\sum_{i=1}^n \psi(X_i, \theta_0 + n^{-1/2}t + n^{-\tau}u), \quad \text{where} \quad \tau \geq ½.$$

This is established in Section 2. The main result is proved in Section 3. Section 4 specifies the main result for maximum likelihood estimation. In Section 5 we consider the application concerning the Pitman estimator.

2. Second-order asymptotic linearity. Let $X_1, X_2, \cdots$ be i.i.d. random variables with common density $f(x, \theta_0)$ and let Θ be an open interval in $\mathbb{R}$. Let ψ be a function from $\mathbb{R} \times \Theta$ into $\mathbb{R}$ and denote the first and second derivative of ψ with respect to the second argument as $\dot{\psi}$ and $\ddot{\psi}$. Further assume that ψ satisfies the following conditions:

(A1) ψ and $\dot{\psi}$ are absolutely continuous in the second argument.

(A2) There exists a $\delta > 0$ and a positive constant K_1 such that

$$E[(\dot{\psi}(X_1, \theta_0 + t))^2] \leq K_1 \quad \text{for} \quad |t| \leq \delta.$$

Moreover we will assume either condition (A3) or the somewhat stronger condition (A3)′:

(A3) There exists a $\delta > 0$ and a positive constant K_2 such that

$$E|\ddot{\psi}(X_1, \theta_0 + t)| \leq K_2 \quad \text{for} \quad |t| \leq \delta.$$

(A3)′ There exists a $\delta > 0$ and a positive constant K_3 such that

$$E[(\ddot{\psi}(X_1, \theta_0 + t))^2] \leq K_3 \quad \text{for} \quad |t| \leq \delta.$$

The first theorem states, under the conditions given above, the asymptotic linearity in t and u of $\sum_{i=1}^{n} \psi(X_i, \theta_0 + n^{-1/2}t + n^{-\tau}u)$.

THEOREM 2.1. *Let $X_1, X_2, \cdots$ be i.i.d. random variables with common density $f(x, \theta_0)$ and let ψ be a function from $\mathbb{R} \times \Theta$ into $\mathbb{R}$ satisfying* (A1) *and* (A2). *Then,*

(i) *if ψ satisfies* (A3), *we have for every $C > 0$*

$$\sup_{|t| \le C} | n^{-1/2} \textstyle\sum_{i=1}^{n} [\psi(X_i, \theta_0 + n^{-1/2}t) - \psi(X_i, \theta_0)] - \gamma(\theta_0)t | = O_P(n^{-1/2}). \tag{2.1}$$

(ii) *if ψ satisfies* (A3)$'$, *we have for every $\tau \ge 1/2$, $C_1 > 0$, $C_2 > 0$*

$$\begin{aligned} \sup_{|t| \le C_1, |u| \le C_2} | n^{-1/2} \textstyle\sum_{i=1}^{n} [\psi(X_i, \theta_0 + n^{-1/2}t + n^{-\tau}u) - \psi(X_i, \theta_0 + n^{-1/2}t)] \\ - \gamma(\theta_0)n^{1/2-\tau}u | = O_p(n^{-\tau}). \end{aligned} \tag{2.2}$$

PROOF. We only give a proof of (ii) since the proof of (i) is completely analogous and simpler. Note that $\gamma(\theta_0)$ exists by condition (A2) and that $\theta_0 + n^{-1/2}t + n^{-\tau}u \in \Theta$ for n sufficiently large (say $n > n_0$). Denote

$$Z_n(t, u) = n^{-1/2} \textstyle\sum_{i=1}^{n} [\psi(X_i, \theta_0 + n^{-1/2}t + n^{-\tau}u) - \psi(X_i, \theta_0 + n^{-1/2}t)]$$

and

$$Z_n^0(t, u) = Z_n(t, u) - EZ_n(t, u), \quad n = 1, 2, \cdots.$$

Then we have to show that

$$\sup_{|t| \le C_1, |u| \le C_2} | Z_n^0(t, u) | = O_P(n^{-\tau}) \tag{2.3}$$

and

$$\sup_{|t| \le C_1, |u| \le C_2} | EZ_n(t, u) - \gamma(\theta_0)n^{1/2-\tau}u | = O(n^{-\tau}). \tag{2.4}$$

We only deal with (2.3) and (2.4) in the case where the supremum is taken over the region $(t, u) \in [0, C_1] \times [0, C_2]$; the other quadrants are treated similarly. For $0 \le t_1 \le t_2 \le C_1$ and $0 \le u_1 \le u_2 \le C_2$, we have

$$\begin{aligned} &E[(Z_n^0(t_2, u_2) - Z_n^0(t_1, u_1))^2] \\ &\quad \le E[(\psi(X_1, \theta_0 + n^{-1/2}t_2 + n^{-\tau}u_2) - \psi(X_1, \theta_0 + n^{-1/2}t_2) \\ &\qquad\qquad - \psi(X_1, \theta_0 + n^{-1/2}t_1 + n^{-\tau}u_1) + \psi(X_1, \theta_0 + n^{-1/2}t_1))^2] \\ &\quad = E\left[\left(\int_{n^{-\tau}u_1}^{n^{-\tau}u_2} \dot\psi(X_1, \theta_0 + n^{-1/2}t_2 + v)\, dv \right.\right. \\ &\qquad\qquad \left.\left. + \int_0^{n^{-\tau}u_1} \int_{n^{-1/2}t_1}^{n^{-1/2}t_2} \ddot\psi(X_1, \theta_0 + w + v)\, dw\, dv\right)^2\right] \\ &\quad \equiv E[(A_{n1}(X_1) + A_{n2}(X_1))^2]. \end{aligned}$$

Now

$$E[A_{n1}^2(X_1)]$$
$$\le \int_{n^{-\tau}u_1}^{n^{-\tau}u_2} \int_{n^{-\tau}u_1}^{n^{-\tau}u_2} \{E[(\dot\psi(X_1, \theta_0 + n^{-1/2}t_2 + v))^2]E[(\dot\psi(X_1, \theta_0 + n^{-1/2}t_2 + v'))^2]\}\, dv\, dv'$$
$$\le K_1 n^{-2\tau}(u_2 - u_1)^2, \quad \text{using (A2)},$$

and similarly, by using (A3)$'$ we obtain that

$$E[A_{n2}^2(X_1)] \le K_3 n^{-2\tau}u_1^2 n^{-1}(t_2 - t_1)^2.$$

Hence, for $n > n_0$ there is a constant K such that

$$E[(Z_n^0(t_2, u_2) - Z_n^0(t_1, u_1))^2] \le Kn^{-2\tau}[(t_2 - t_1)^2 + (u_2 - u_1)^2].$$

From this it follows by the multivariate version of Theorem 12.1 of Billingsley (1968) (see Jurečková and Sen, 1984, Section 3) that, for every positive integer m

$$P(\max_{0\le k\le m, 0\le k'\le m} | n^\tau Z_n^0(kC_1/m, k'C_2/m) | \ge M) \le K'M^{-2},$$

where K' is a positive constant, independent of m. Since $Z_n^0(t, u)$ is continuous, we conclude by letting $m \to \infty$ that $\sup_{0\le t\le C_1, 0\le u\le C_2} | Z_n^0(t, u) | = O_P(n^{-\tau})$. For the proof of (2.4), we have for $0 \le t \le C_1$ and $0 \le u \le C_2$ and $n > n_0$

$$| EZ_n(t, u) - \gamma(\theta_0)n^{1/2-\tau}u |$$
$$= n^{1/2} | E[\psi(X_1, \theta_0 + n^{-1/2}t + n^{-\tau}u) - \psi(X_1, \theta_0 + n^{-1/2}t) - n^{-\tau}u\dot\psi(X_1, \theta_0)] |$$
$$= n^{1/2} \left| E\left[\int_0^{n^{-\tau}u} \int_0^v \ddot\psi(X_1, \theta_0 + n^{-1/2}t + w)\, dw\, dv - \int_0^{n^{-\tau}u} \int_0^{n^{-1/2}t} \ddot\psi(X_1, \theta_0 + w)\, dw\, dv\right]\right|$$
$$\le K_2 n^{1/2}[n^{-2\tau}(u^2/2) + n^{-1/2-\tau}tu] = O(n^{-\tau}).$$

Hence the proof is complete. □

Theorem 2.1 has an easy corollary, which will be useful in the sequel.

COROLLARY 2.1. *Let T_n be an $n^{1/2}$-consistent estimator of θ_0. Then,*

(i) *under the conditions of Theorem* 2.1 (i), *we have*

$$n^{-1/2} \textstyle\sum_{i=1}^n [\psi(X_i, T_n) - \psi(X_i, \theta_0)] - \gamma(\theta_0)n^{1/2}(T_n - \theta_0) = O_P(n^{-1/2}). \tag{2.5}$$

(ii) *under the conditions of Theorem* 2.1 (ii) *and with U_n a statistic satisfying $U_n = O_P(n^{-\tau})$, we have*

$$n^{-1/2} \textstyle\sum_{i=1}^n [\psi(X_i, T_n + U_n) - \psi(X_i, T_n)] - \gamma(\theta_0)n^{1/2}U_n = O_P(n^{-\tau}). \tag{2.6}$$

3. Application to one- and two-step estimators. Assume that there exists an $n^{1/2}$-consistent solution M_n of (1.1) and that

(A4) $$\gamma(\theta_0) \neq 0.$$

Then, it follows from (2.5) that

(3.1) $$n^{1/2}(M_n - \theta_0) = -n^{-1/2}(\gamma(\theta_0))^{-1} \textstyle\sum_{i=1}^n \psi(X_i, \theta_0) + O_P(n^{-1/2}).$$

Let $M_n^{(0)}$ be an arbitrary $n^{1/2}$-consistent initial estimator of θ_0. Consider the one-step version $M_n^{(1)}$ and the two-step version $M_n^{(2)}$ of M_n defined in (1.2) with $\hat{\gamma}_n^{(k)}$ $(k = 0, 1)$ defined as follows

(3.2) $$\hat{\gamma}_n^{(k)} = n^{-1/2}(t_2 - t_1)^{-1} \textstyle\sum_{i=1}^n [\psi(X_i, M_n^{(k)} + n^{-1/2}t_2) - \psi(X_i, M_n^{(k)} + n^{-1/2}t_1)]$$

where t_1 and t_2 $(t_1 < t_2)$ are arbitrary fixed real numbers. It easily follows from Corollary 2.1 that $\hat{\gamma}_n^{(0)}$ is an $n^{1/2}$-consistent estimator of $\gamma(\theta_0)$. The following theorem shows that, under the assumptions of Section 2,

$$M_n - M_n^{(1)} = O_P(n^{-1}) \quad \text{and} \quad M_n - M_n^{(2)} = O_P(n^{-3/2}),$$

where we remark that the $n^{1/2}$-consistency of $\hat{\gamma}_n^{(1)}$ follows from $M_n - M_n^{(1)} = O_P(n^{-1})$.

THEOREM 3.1. *Let $X_1, X_2, \cdots$ be i.i.d. random variables with common density $f(x, \theta_0)$. Assume that*

(i) *M_n is an $n^{1/2}$-consistent solution of the equation* (1.1),
(ii) *$M_n^{(0)}$ is an $n^{1/2}$-consistent initial estimator of θ_0,*
(iii) *ψ satisfies* (A1), (A2), *and* (A4).

Let $M_n^{(1)}$ and $M_n^{(2)}$ be the one- and two-step estimators defined in (1.2) *with $\hat{\gamma}_n^{(k)}$ given in* (3.2) *for $k = 0, 1$. Then, under condition* (A3), *we have*

(3.3) $$M_n - M_n^{(1)} = O_P(n^{-1})$$

and under condition (A3)$'$, *we have*

(3.4) $$M_n - M_n^{(2)} = O_P(n^{-3/2})$$

PROOF. Since $\gamma(\theta_0) \neq 0$ and $\hat{\gamma}_n^{(0)} - \gamma(\theta_0) = O_P(n^{-1/2})$, we have

(3.5) $$\gamma(\theta_0)/\hat{\gamma}_n^{(0)} - 1 = O_P(n^{-1/2}).$$

Moreover, since $P(|\hat{\gamma}_n^{(0)}| < \delta)$ can be made arbitrarily small for $\delta < |\gamma(\theta_0)|$ and

(3.6) $$\begin{aligned} &P(n|M_n - M_n^{(1)}| > C) \\ &\qquad \le P(n|M_n - M_n^{(1)}| > C, \hat{\gamma}_n^{(0)} \neq 0) + P(|\hat{\gamma}_n^{(0)}| < \delta), \end{aligned}$$

it suffices to find a bound for $M_n - M_n^{(1)}$ in the case that $\hat{\gamma}_n^{(0)} \neq 0$. Then, using (3.1), we have

$$n^{1/2}(M_n - M_n^{(1)})$$

$$(3.7)\qquad = \frac{\gamma(\theta_0)}{\hat{\gamma}_n^{(0)}} \left\{ n^{-1/2} \frac{1}{\gamma(\theta_0)} \sum_{i=1}^n [\psi(X_i, M_n^{(0)}) - \psi(X_i, \theta_0)] - n^{1/2}(M_n^{(0)} - \theta_0) \right\}$$

$$+ \left(\frac{\gamma(\theta_0)}{\hat{\gamma}_n^{(0)}} - 1 \right) \left\{ n^{-1/2} \frac{1}{\gamma(\theta_0)} \sum_{i=1}^n \psi(X_i, \theta_0) + n^{1/2}(M_n^{(0)} - \theta_0) \right\} + O_P(n^{-1/2}).$$

The statement (3.3) then easily follows from (3.5)–(3.7). If we now assume (A3)′ and put $T_n = M_n$ and $U_n = M_n^{(1)} - M_n$ in (2.6), we get, for $\hat{\gamma}_n^{(1)} \neq 0$,

$$n^{1/2}(M_n - M_n^{(2)}) = n^{1/2}(M_n - M_n^{(1)}) + n^{-1/2}(1/\hat{\gamma}_n^{(1)}) \sum_{i=1}^n \psi(X_i, M_n^{(1)})$$

$$= ((\gamma(\theta_0)/\hat{\gamma}_n^{(1)}) - 1) n^{1/2}(M_n^{(1)} - M_n) + O_P(n^{-1})$$

and this implies (3.4). □

4. Application to maximum likelihood estimators. The maximum likelihood estimator is an M-estimator generated by the function

$$(4.1)\qquad \psi(x, t) = (\partial/\partial t)\log f(x, t)$$

There exists a host of literature about the asymptotic efficiency of MLE of the first and second orders. A good review of such results may be found in Akahira and Takeuchi (1981).

One-step versions of the MLE $\hat{\theta}_n$ were discussed in Lehmann (1983), Chapter 6, where also other references can be found; however, only the relation $\hat{\theta}_n - T_n^{(1)} = o_P(n^{-1/2})$ was proved for such one-step version $T_n^{(1)}$.

The results of Section 3 yield, under some regularity conditions, the existence of one- and two-step versions $T_n^{(1)}$ and $T_n^{(2)}$ of $\hat{\theta}_n$ such that

$$\hat{\theta}_n - T_n^{(1)} = O_P(n^{-1}) \quad \text{and} \quad \hat{\theta}_n - T_n^{(2)} = O_P(n^{-3/2}).$$

Being combined with the results of Akahira and Takeuchi (1981), this means that the second-order efficiency properties of $T_n^{(2)}$ coincide with those of $\hat{\theta}_n$.

The regularity conditions under which the likelihood equation provides us with an $n^{1/2}$-consistent estimator $\hat{\theta}_n$ of θ_0 are well known (see, e.g., Theorem 6.2.3 of Lehmann, 1983). One of these regularity conditions is typically that the Fisher information $I(\theta)$ satisfies

$$(4.2)\qquad 0 < I(\theta) = E_\theta[((\partial/\partial\theta)\log f(X_1, \theta))^2] = E_\theta[-(\partial^2/\partial\theta^2)\log f(X_1, \theta)] < \infty$$

for all $\theta \in \Theta$. If we also assume that the function ψ in (4.1) satisfies the conditions (A1)–(A2) of Section 2, then $\gamma(\theta_0) = -I(\theta_0)$, so that (A4) holds because of (4.2).

If there exists an $n^{1/2}$-consistent initial estimator of θ_0, then Theorem 3.1 applies, so that we could summarize the results in the following theorem.

THEOREM 4.1. *Let $X_1, X_2, \cdots$ be i.i.d. random variables with common density $f(x, \theta_0)$ satisfying* (4.2). *Assume that*

(i) $\hat{\theta}_n$ *is an $n^{1/2}$-consistent solution of the maximum likelihood equation* $\sum_{i=1}^n (\partial/\partial t)\log f(X_i, t) = 0$,
(ii) $T_n^{(0)}$ *is an $n^{1/2}$-consistent initial estimator of θ_0*,
(iii) $\psi(x, t) = (\partial/\partial t)\log f(x, t)$ *satisfies* (A1)–(A2).

Let

$$T_n^{(k)} = \begin{cases} T_n^{(k-1)} - (n\hat{\gamma}_n^{(k-1)})^{-1} \sum_{i=1}^n \psi(X_i, T_n^{(k-1)}) & \text{if } \hat{\gamma}_n^{(k-1)} \neq 0 \\ T_n^{(k-1)} & \text{if } \hat{\gamma}_n^{(k-1)} = 0 \end{cases}$$

$k = 1, 2$, *be the one- and two-step MLE with $\hat{\gamma}_n^{(k)}$ given in* (3.2) *for $k = 0, 1$. Then, under condition* (A3) *we have*

$$\hat{\theta}_n - T_n^{(1)} = O_P(n^{-1})$$

and under condition (A3)′ *we have*

$$\hat{\theta}_n - T_n^{(2)} = O_P(n^{-3/2}).$$

5. Application to Pitman's estimator in the location case. Let us now consider the location model in which $f(x, \theta) = f(x - \theta)$ so that the function ψ in (4.1) now takes the form $\psi(x, t) = \psi(x - t)$, where

$$\psi(x) = -(f'(x)/f(x)). \tag{5.1}$$

It is then natural to restrict our considerations to the translation-equivariant estimators, i.e., to estimators T_n which satisfy $T_n(X_1 + a, \cdots, X_n + a) = T_n(X_1, \cdots, X_n) + a$ for all $a \in \mathbb{R}$. Since the maximum likelihood equation now reduces to $\sum_{i=1}^n \psi(X_i - t) = 0$, it is immediately clear that a MLE will be translation-equivariant.

We shall consider the situation where the loss incurred by estimating θ_0 by t is the square deviation $(t - \theta_0)^2$. The minimum risk estimator is then the Pitman estimator T_n^* which could be written in the form

$$T_n^* = \tilde{\theta}_n - E_0(\tilde{\theta}_n \mid \mathbf{Y}) \tag{5.2}$$

where $\tilde{\theta}_n$ is an arbitrary initial translation-equivariant estimator with finite risk and

$$\mathbf{Y} = (Y_2, \cdots, Y_n) = (X_2 - X_1, \cdots, X_n - X_1). \tag{5.3}$$

Let $\hat{\theta}_n$ be the $n^{1/2}$-consistent solution of the likelihood equation. It follows from Theorem 5.3.1 of Akahira and Takeuchi (1981) that if $0 < I(f) < \infty$ and under the conditions (A1), (A2) and (A3)″ with

(A3)″ There exists a $\delta > 0$ and a positive constant K such that

$$E \mid \ddot{\psi}(X_1, \theta_0 + t) \mid^3 \le K \quad \text{for} \quad \mid t \mid \le \delta,$$

the Pitman estimator satisfies

$$T_n^* - \hat{\theta}_n = O_P(n^{-1}). \tag{5.4}$$

Let us select an $n^{1/2}$-consistent translation-equivariant initial estimator $T_n^{(0)}$ and put

$$T_n^{(1)} = T_n^{(0)} - (1/nI(f)) \sum_{i=1}^n f'(X_i - T_n^{(0)})/f(X_i - T_n^{(0)}). \tag{5.5}$$

Then $T_n^{(1)}$ is translation-equivariant and (5.4) combined with Theorem 4.1 implies

$$T_n^* - T_n^{(1)} = O_P(n^{-1}). \tag{5.6}$$

REMARK. The choice $T_n^{(0)} = \bar{X}_n = (1/n) \sum_{i=1}^n x_i$ is of special interest; then we should suppose that the underlying distribution has finite variance. Another possibility is to use robust estimators of location which are $n^{1/2}$-consistent and translation-equivariant under general conditions.

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