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NONNULL AND OPTIMALITY ROBUSTNESS OF SOME TESTS[1]

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This paper first characterizes the invariant structure of a model for which nonnull robustness holds. Applications of this result yield the nonnull robustness and optimality robustness of some tests for covariance structure including a test for sphericity. Second, we show the optimality robustness of the LBI tests in the GMANOVA(MANOVA) problem, and the problem of testing independence. In the GMANOVA problem, a robustness property of an essentially complete class of invariant tests is also shown.

1. Introduction and Summary. The robustness of a test in a hypothesis testing problem is usually studied from two viewpoints: (1) robustness under a null hypothesis and (2) robustness under an alternative hypothesis. Broadly speaking, robustness under a null hypothesis is regarded as the stability of a critical point of a test of level α, but here a test is defined to be null-robust if its null distribution remains the same for a class of distributions including the distribution under which the test is considered. Similarly, we define the nonnull robustness of a test by the invariance of its nonnull distribution for each value of the parameter under the alternative hypothesis in a class of distributions including the underlying distribution. Further, we shall call a test optimality-robust if an optimality property the test enjoys, such as UMP (uniformly most powerful), UMPI (UMP invariant), LBI (locally best invariant), etc., can be extended to a class of distributions including the distribution under which the optimality holds. Of course, optimality robustness is robustness under an alternative hypothesis. The null robustness in our sense has been considered in a class of elliptically contoured distributions or left orthogonally invariant distributions by many authors (e.g., Dempster, 1969; Kariya and Eaton, 1977; Dawid, 1977; Chmielewski, 1980; Kariya, 1981; Jensen and Good, 1981; Eaton and Kariya, 1981). On the other hand, the optimality robustness of some UMP or UMPI tests is treated in a similar set-up by Kariya and Eaton (1977), Kariya (1977, 1981), etc. Sinha (1984) treats the robustness of an LBI test.

In this paper, the nonnull robustness of invariant tests for testing certain covariance structures and the optimality robustness of some LBI and UMPI tests are considered in a class of left orthogonally invariant distributions. More specifically, in Section 2, when a problem is left invariant under a group, we characterize the structure of a model which allows the nonnull distribution of a

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maximal invariant to remain the same in a class of left orthogonally invariant distributions, where the null robustness is presupposed. Applying this result to the problem of testing a certain covariance structure in regression and the problem of testing sphericity, the robustness of the nonnull distributions of maximal invariants for these problems is established, through which the nonnull robustness of some tests and the optimality robustness of the LBI test for sphericity derived under normality by Sugiura (1972) are obtained. The null robustness of these tests follows from Kariya (1981). In Section 3, after a general discussion on the robustness of an LBI test, we treat the GMANOVA problem. There it is first shown that the essentially complete class theorem for invariant tests obtained under normality via sufficiency-invariance-sufficiency reduction (Kariya, 1978) holds for the class of all the left orthogonally invariant distributions. Second, the LBI test derived under normality by Kariya (1978) is shown to be optimality-robust for a class of left orthogonally invariant distributions under regularity conditions. As a special case, the LBI test in the MANOVA problem derived by Schwartz (1967) is shown to be optimality robust. Also in the problem of testing independence, the optimality robustness of the LBI test is obtained. The null robustness of these tests is shown in Dawid (1977), Kariya (1981) or Jensen and Good (1981).

As a technical tool, the representation theorem on the probability ratio of the distributions of a maximal invariant due to Wijsman (1967) is exploited.

Finally, throughout the paper, we denote by $\mathscr{O}(n)$ the group of $n \times n$ orthogonal matrices, by $G\ell(p)$ the group of $p \times p$ nonsingular matrices, and by R_+ the group of positive reals.

2. Nonnull robustness. In this section, when a problem is left invariant under a group, we characterize the structure of a model in which nonnull robustness holds, and show the nonnull robustness of tests for certain covariance structures. Let $\mathscr{X}$ be a nonempty open subset of R^n with the Borel σ-field $\mathscr{B}$ and let $\mathscr{G}$ be a closed subgroup of $G\ell(n)$ acting on the left of $\mathscr{X}$. Let $\mathscr{F}(\Theta)$ be a class of pdf's on $\mathscr{X}$ with respect to a relatively left invariant Borel measure τ with left multiplier δ (i.e., $\tau(gE) = \delta(g)\tau(E)$ for $g \in \mathscr{G}$ and $E \in \mathscr{B}$) such that each pdf $f(\cdot \mid \theta)$ in $\mathscr{F}(\Theta)$ is of the form

$$f(x \mid \theta) = \beta(\theta) q(\psi(x\colon \theta)), \quad \theta \in \Theta \tag{2.1}$$

where $\psi(\cdot\colon \theta)$ is a known measurable function from $\mathscr{X}$ onto $\mathscr{Y}$, $\mathscr{Y}$ is a nonempty open subset of R^m and independent of θ, q is a fixed integrable function from $\mathscr{Y}$ into $[0, \infty)$ and independent of θ, and Θ is a nonempty open subset of R^p. Suppose the group $\mathscr{G}$ leaves invariant the problem

$$H\colon \theta \in \Theta_0 \text{ vs } K\colon \theta \in \Theta_1, \tag{2.2}$$

where $\Theta_0 \cap \Theta_1 = \phi$ and $\Theta_i \subset \Theta$ $(i = 0, 1)$. Here we assume that $\mathscr{X}$ is a Cartan $\mathscr{G}$-space (see Wijsman, 1967, page 392). Then as is well known (e.g., Wijsman, 1967; or Bondar, 1976), the ratio of the densities of a maximal invariant $T = t(x)$

under $\theta_1 \in \Theta_1$ and $\theta_0 \in \Theta_0$ is given by

$$R = R(t(x)) \equiv (dP^T_{\theta_1}/dP^T_{\theta_0})(t(x)) = H(x \mid \theta_1)/H(x \mid \theta_0) \tag{2.3}$$

with

$$H(x \mid \theta) = \int_{\mathscr{G}} f(gx \mid \theta)\delta(g)\mu(dg), \tag{2.4}$$

where $f(x \mid \theta)$ is given by (2.1) and $\mu(dg)$ is a left invariant measure on $\mathscr{G}$. It is noted that $\delta(g)$ is the inverse of the Jacobian of transformation $x \to gx$, i.e., $\delta(g) = |\det(g)|$ for $g \in G\ell(n)$, and that $H(x \mid \theta_1) < \infty$ and $H(x \mid \theta_0) \neq 0$ a.e. $(P^T_{\theta_0})$. Further, it is remarked that some alternative conditions for which (2.3) holds are found in Bondar (1976) and Andersson (1982). Now to obtain a condition for (2.3) to be independent of q for all $\theta_1 \in \Theta_1$ and $\theta_0 \in \Theta_0$, we assume $\mathscr{G}$ is a product group, i.e., $\mathscr{G} = \mathscr{G}_1 \times \mathscr{G}_2$ with product invariant measure $\mu = \mu_1 \times \mu_2$ and left multiplier $\delta(g) = \delta_1(g_1)\delta_2(g_2)$ for $g = (g_1, g_2) \in \mathscr{G}$. Further we need

ASSUMPTION 2.1. The function ψ in (2.1) satisfies

$$\psi((g_1, g_2)x{:}\ \theta) = \bar{g}_1\psi((e_1, g_2)x{:}\ \theta) \tag{2.5}$$

for all $x \in \mathscr{X}$, $(g_1, g_2) \in \mathscr{G}_1 \times \mathscr{G}_2$ and $\theta \in \Theta$, and $\bar{\mathscr{G}}_1$ acts transitively on the range space $\mathscr{Y}$ of ψ, where $\bar{\mathscr{G}}_1$ is an induced group of $\mathscr{G}_1$ as a continuous homomorphic image, $\bar{g}_1 \in \bar{\mathscr{G}}_1$ and e_i is the unit element of $\mathscr{G}_i$,

THEOREM 2.1. *Under Assumption* 2.1, *the probability ratio R in* (2.3) *is independent of q for all $\theta_i \in \Theta_i$ $(i = 0, 1)$. In fact, it is given by $R(t(x)) = K(x \mid \theta_1)/K(x \mid \theta_0)$ with*

$$K(x \mid \theta) = \beta(\theta) \int_{\mathscr{G}_2} \delta_1(h(x, g_2, \theta{:}\ y_0))\Delta_{1r}(h(x, g_2, \theta{:}\ y_0))\delta_2(g_2)\mu_2(dg_2), \tag{2.6}$$

where y_0 is a fixed element of $\mathscr{Y}$, $h(x, g_2, \theta{:}\ y_0)$ is an element of $\mathscr{G}_1$ such that

$$\overline{h(x, g_2, \theta{:}\ y_0)}\psi((e_1, g_2)x) = y_0, \tag{2.7}$$

and $\Delta_{1r}(\cdot)$ is a right modular function of μ_1. Here the measurability of h with respect to g_2 is assumed.

PROOF. From (2.1) and (2.4), the numerator of R is given by

$$H(x \mid \theta_1) = \int_{\mathscr{G}_2} H_1(x, g_2 \mid \theta_1)\delta_2(g_2)\mu_2(dg_2) \tag{2.8a}$$

with

$$H_1(x, g_2 \mid \theta_1) = \beta(\theta_1)\int_{\mathscr{G}_1} q[\bar{g}_1\psi((e_1, g_2)x{:}\ \theta_1)]\delta_1(g_1)\mu_1(dg_1). \tag{2.8b}$$

Since $\mathscr{G}_1$ acts transitively on the range space $\mathscr{Y}$ of $\psi(x: \theta_1)$, for any $y_0 \in \mathscr{Y}$, there exists $h(x, g_2, \theta_1 : y_0) \in \mathscr{G}$ such that (2.7) with $\theta = \theta_1$ is satisfied. Hence using the invariance of μ_1 and replacing g_1 by $g_1 h(x, g_2, \theta_1 : y_0)$ in (2.8b) yields

$$\begin{aligned} &H_1(x, g_2 \mid \theta_1) \\ &= \beta(\theta_1)\delta_1(h(x, g_2, \theta_1 : y_0))\Delta_{1r}(h(x, g_2, \theta_1 : y_0)) \int_{\mathscr{G}_1} q(\bar{g}_1 y_0)\delta_1(g_1)\mu_1(dg_1). \end{aligned} \tag{2.9}$$

Hence evaluating $H(x \mid \theta_0)$ in the same way and taking the ratio yields $R = K(x \mid \theta_1)/K(x \mid \theta_0)$, completing the proof.

In Theorem 2.1, no assumption is made on the form of q.

Now when Assumption 2.1 holds and when the null distribution $P_{\theta_0}^T$ of T does not depend on q, the nonnull distribution $P_{\theta_1}^T$ does not depend on q either, thereby establishing nonnull robustness. In such a situation, the distribution of the maximal invariant statistic is completely independent of the underlying distribution for each parameter value, and hence the whole (invariance-reduced) decision problem is also independent of the distribution. This implies that all decision theoretic properties via invariance are robust: unbiasedness, Bayes, admissibility, minimaxity, etc. We are grateful to a referee for pointing out this fact.

2.1 *Tests for covariance structure in regression.* Let us consider a regression model

$$y = X\beta + u \tag{2.10}$$

where X is an $n \times k$ fixed matrix of rank k and the error term u has a pdf of the form

$$f(u \mid \sigma^2\Sigma) = \mid \sigma^2\Sigma \mid^{-1/2} q(u'\Sigma^{-1}u/\sigma^2). \tag{2.11}$$

Here q is a function from $[0, \infty)$ into $[0, \infty)$ such that $\int_{R^n} q(u'u)\, du = 1$ and Σ is the scale matrix. Under this setup, we consider the testing problem

$$H_0\colon \Sigma \in \Lambda_0 \text{ vs } H_1\colon \Sigma \in \Lambda_1, \tag{2.12}$$

where $\Lambda_0 \cap \Lambda_1 = \phi$, $\Lambda_i \subset \Lambda (i = 0, 1)$, and Λ is the set of $p \times p$ positive definite matrices. In certain applications, Σ may be a function of an r-dimensional parameter λ ($\Sigma = \Sigma(\lambda)$) and the null and alternative hypotheses may be described in terms of λ.

This problem is left invariant under the group $\mathscr{G} = R_+ \times R^k$, where $\mathscr{G}$ acts on y by $y \to cy + Xb$ for $g = (c, b) \in \mathscr{G}$. As in Kariya (1980), choose a matrix Z: $n \times (n - k)$ such that $Z'Z = I_{n-k}$ and $ZZ' = I - X(X'X)^{-1}X'$, and define $v = (X'X)^{-1/2}X'y$, $\eta = (X'X)^{1/2}\beta$, $w = Z'y$ and $Q' = (X(X'X)^{-1/2}, Z)$. Then $w/\| w \|$ is clearly a maximal invariant under $\mathscr{G}$, $Q \in \mathscr{O}(n)$ and

$$(y - X\beta)'\Sigma^{-1}(y - X\beta) = (Qy - QX\beta)'[Q\Sigma Q']^{-1}(Qy - QX\beta),$$

where $\| w \| = (w'w)^{1/2}$. From Kelker (1970), the marginal pdf of w is of the form

$$(2.13)\quad \bar{f}(w\,|\,\theta) = |Z'\theta Z|^{-1/2}\bar{q}(\psi(w\colon \theta)) \quad \text{with} \quad \psi(w\colon \theta) = w'[Z'\theta Z]^{-1}w,$$

where $\theta = \sigma^2\Sigma$, since the pdf of $\tilde{y} = Qy$ is from (2.11)

$$|Q\theta Q'|^{-1/2}q((\tilde{y} - \tilde{\eta})'[Q\theta Q']^{-1}(\tilde{y} - \tilde{\eta})) \quad \text{with} \quad \tilde{\eta} = \begin{pmatrix}\eta\\ 0\end{pmatrix}.$$

Here $\bar{q}$ depends on q and (n, k) but not on (β, θ). In terms of w, $w/\| w \|$ is a maximal invariant under group R_+ with the action: $w \to cw$.

Now we evaluate (2.6) for the marginal pdf (2.13) of w. Take $\mathscr{G}_1 = \bar{\mathscr{G}}_1 = R_+$ and $\mathscr{G}_2 = \{e_2\}$. Then $\bar{\mathscr{G}}_1$ acts transitively on the range of ψ, and since the left invariant measure $\mu_1(dc) = dc/c$ on R_+ is also right invariant, $\Delta_{1r}(c) \equiv 1$. Further, the inverse of the Jacobian of $w \to cw$ is $\delta_1(c) = c^{n-k}$, while $h(w, \theta\colon 1) = \psi(w\colon \theta)^{-1/2}$ from $\psi(cw\colon \theta) = c^2\psi(w\colon \theta)$ and from (2.7). Therefore, from (2.6), $K(w\,|\,\theta) = |Z'\theta Z|^{-1/2}\psi(w\colon \theta)^{-(n-k)/2}$ with $\psi(w\colon \theta)$ in (2.13). This implies that for $\Sigma_i \in \Lambda_i$ $(i = 0, 1)$, the ratio R is evaluated as

$$(2.14)\qquad R = \left(\frac{|Z'\Sigma_1 Z|}{|Z'\Sigma_0 Z|}\right)^{-1/2}\left(\frac{w'(Z'\Sigma_1 Z)^{-1}w}{w'(Z'\Sigma_0 Z)^{-1}w}\right)^{-(n-k)/2}$$

which is completely independent of $\bar{q}$ or q. Consequently, for testing $\Lambda_0 = \{\Sigma_0\}$ vs $\Lambda_1 = \{\Sigma_1\}$ in (2.12), the nonnull robustness of the test based on $S \equiv w'[Z'\Sigma_1 Z]^{-1}w/w'[Z'\Sigma_0 Z]^{-1}w < c$ holds. On the other hand, Kariya (1981) has shown that the null distribution of the maximal invariant $T = w/\| w \|$ is independent of q. Hence the null robustness of S also holds. In Kariya (1980), for testing $\rho = 0$ vs $\rho > 0$ in serially correlated errors $u_t = \rho u_{t-1} + \varepsilon_t$, in which $\Sigma(\rho)$ is approximated by $(I + \rho A)^{-1}$, the optimality robustness of the LBI test derived under normality is shown by using (2.14). On the other hand, for testing $\lambda = 0$ vs $\lambda > 0$ in the model of intra-class covariance structure $\Sigma(\lambda) = (1 - \lambda)I + \lambda ee'$ with $X = e$ where $e = (1, \cdots, 1)' \in R^n$, Kariya and Eaton (1977) observed the nonnull robustness of the UMPI test derived under normality. Of course, the expression R in (2.14) is effective for such models as a heteroscedastic model, a seemingly unrelated regression model, etc.

2.2. *Testing sphericity.* Let Z be an $n \times p$ random matrix with pdf

$$(2.15)\quad f(Z\,|\,\alpha, \sigma^2\Sigma) = |\sigma^2\Sigma|^{-n/2}q(\operatorname{tr}(Z - e\alpha')'(Z - e\alpha')(\sigma^2\Sigma)^{-1}),$$

where $e = (1, \cdots, 1)' \in R^n$ and $\alpha \in R^p$, and consider testing $\Sigma = I$ vs $\Sigma \neq I$. We may assume $\alpha = 0$ without essential loss of generality (one degree of freedom is lost). This problem is left invariant under the group $\mathscr{G} = \mathscr{G}_1 \times \mathscr{G}_2 \equiv R_+ \times \mathscr{O}(p)$ acting on the left of Z by $Z \to cZQ'$ for $g = (c, Q) \in \mathscr{G}$. Here, letting $\psi(Z\colon \theta) = \operatorname{tr} Z'Z\theta^{-1}$ with $\theta = \sigma^2\Sigma$, the subgroup R_+ acts transitively on the range of ψ, the left invariant measure dc/c is right invariant so that $\Delta_{1r} \equiv 1$, and the inverse of the Jacobian of $Z \to cZ$ is $\delta_1(c) = c^{np}$. Therefore, from $c^2\psi(ZQ'\colon \theta) = 1$, the function K corresponding to (2.6) is evaluated as

$$(2.16)\qquad K(Z\,|\,\theta) = |\theta|^{-n/2}\int_{\mathscr{O}(p)}[\operatorname{tr} QSQ'\theta^{-1}]^{-np/2}\,\mu_2(dQ),$$

where $S = Z'Z$ and $\mu_2(dQ)$ is the invariant probability measure on $\mathscr{O}(p)$ with $\delta_2(Q) \equiv 1$. Hence for testing $\theta = \sigma^2 I$ vs $\theta = \sigma^2 \Sigma$, the probability ratio R in (2.3) is $K(Z \mid \sigma^2 \Sigma)/K(Z \mid \sigma^2 I)$. It follows that

$$R = |\Sigma|^{-n/2} \int_{\mathscr{O}(p)} [1 + F]^{-np/2} \mu_2(dQ) \tag{2.17}$$
$$\text{with} \quad F = \operatorname{tr}(\Sigma^{-1} - I)QSQ'/\operatorname{tr} S,$$

which is independent of q. Since this expression is the same as the one under normality, the next theorem follows from Sugiura (1972) and the fact that the null distribution of a maximal invariant does not depend on q (Kariya, 1981).

THEOREM 2.2. *The test based on* tr $S^2/(\text{tr } S)^2 > c$ *is LBI for testing* $\sigma^2 \Sigma = \sigma^2 I$ *vs* $\sigma^2 \Sigma \neq \sigma^2 I$ *under any pdf of the form* (2.15).

In Sugiura (1972), an exact evaluation of R in (2.17) is obtained by using zonal polynomials and then it is expanded. But expanding the inside of the integrand in (2.17) up to the third order, using the boundedness of F for $\Sigma_i |\lambda_i - 1|$ small where λ_i's are the roots of Σ, and arguing as in the next section, Theorem 2.2 is also proved.

As has been remarked, in these examples not only the LBI properties but all other decision-theoretic properties of the tests are robust.

3. Optimality robustness of the LBI tests in GMANOVA and testing independence.

Except in the situation described in Section 2, nonnull robustness seldom holds in general. This is true especially when a hypothesis on a location parameter is tested. In this section, we consider the optimality robustness of the LBI tests in the GMANOVA(MANOVA) problem and the problem of testing independence. Adopting the framework given in Section 2, a basic procedure in showing the robustness of a LBI property is as follows. Assume that ψ is scalar-valued in (2.1), q and β are continuously twice differentiable, and $\Theta_0 = \{\theta_0\}$. Expand the integrand in the numerator of the ratio in (2.3) as

$$\begin{aligned} f(x \mid \theta_1) = \beta(\theta_1)\{q(\psi(x\colon \theta_0)) + q'(\psi(x\colon \theta_0))[\psi(x\colon \theta_1) - \psi(x\colon \theta_0)] \\ + q''(\psi^*(x\colon \theta_0, \theta_1))[\psi(x\colon \theta_1) - \psi(x\colon \theta_0)]^2\}, \end{aligned} \tag{3.1}$$

where $\theta_1 \in \Theta_1$, and $\psi^*(x\colon \theta_1, \theta_0) = c\psi(x\colon \theta_1) + (1 - c)\psi(x\colon \theta_0)$ for some $0 \leq c \leq 1$. Then with $D = \int_{\mathscr{G}} f(gx \mid \theta_0)\delta(g)\nu(dg)$, the ratio (2.3) is expressed as

$$\begin{aligned} R = 1 + \beta(\theta_1)\int_{\mathscr{G}} q'(\psi(gx\colon \theta_0))[\psi(gx\colon \theta_1) - \psi(gx\colon \theta_0)]\delta(g)\nu(dg)/D \\ + \kappa(\theta_1, \theta_0) + M, \end{aligned} \tag{3.2}$$

where $M \equiv M(x\colon \theta_1, \theta_0)$ is a remainder term and $\kappa(\theta_1, \theta_0) = [\beta(\theta_1)/\beta(\theta_0) - 1]$. Here if we can show (1) that the second term is expressed as $\gamma(\theta_1, \theta_0)t(x)$ with $\gamma(\theta_1, \theta_0) = O(|\theta_1 - \theta_0|)$ and $t(x)$ independent of q, (2) that

$\kappa(\theta_1, \theta_0) = O(|\theta_1 - \theta_0|)$ and (3) that for any invariant test function $\phi(x)$ and for each $q \in \mathscr{Q}$

$$\int \phi(x) M(x: \theta_1, \theta_0)\, dP_{\theta_0}^T = o(|\theta_1 - \theta_0|) \tag{3.3}$$

where $\mathscr{Q}$ is a certain class of q, then from (3.2) the power function of an invariant test ϕ is given by

$$\pi(\phi, \theta_1) = \alpha + E_{\theta_0}[\phi(x)\gamma(\theta_1, \theta_0)t(x)] + \alpha\kappa(\theta_1, \theta_2) + o(|\theta_1 - \theta_0|). \tag{3.4}$$

Hence, by the Generalized Neyman-Pearson Lemma, the test based on $t(x)$ is LBI for all $q \in \mathscr{Q}$ provided $P_{\theta_0}^T$ remains the same for all $q \in \mathscr{Q}$.

Sometimes the second term in (3.2) vanishes, in which case higher order derivatives of q need to be considered. In this manner, most LBI tests derived under normality are shown to be LBI in a broader class of pdf's. But some LBI tests under normality are not robust in this sense, because $t(x)$ does depend on q, or because $P_{\theta_0}^T$ does depend on q. The readers may be referred to Giri (1968), John (1971) or Sugiura (1972) for some LBI tests.

3.1 *GMANOVA problem and MANOVA problem.* A canonical form of the GMANOVA problem is stated as follows (see Gleser and Olkin, 1970; and Kariya, 1978). Let Z be an $n \times p$ random matrix with pdf

$$f(Z \mid \Theta, \Sigma) = |\Sigma|^{-n/2} q(\operatorname{tr}(Z - \Theta)\Sigma^{-1}(Z - \Theta)'), \tag{3.5}$$

where Θ is of the structure

$$\Theta = \begin{array}{cccc} p_1 & p_2 & p_3 & \\ \Theta_{11} & \Theta_{12} & 0 & n_1 \\ \Theta_{21} & \Theta_{22} & 0 & n_2 \\ 0 & 0 & 0 & n_3 \end{array} \qquad \begin{array}{l} p_1 + p_2 + p_3 = p, \\ \\ n_1 + n_2 + n_3 = n; \end{array} \tag{3.6}$$

and q belongs to a certain class $\mathscr{Q}$, which is specified later. The problem is to test H: $\Theta_{12} = 0$ vs K: $\Theta_{12} \neq 0$. Of course, when $p_1 = p_3 = 0$, the problem is reduced to the MANOVA problem. Let $Z = (Z_{ij})$ be the decomposition of Z corresponding to the decomposition of Θ in (3.6). Then the problem is left invariant under group $\mathscr{G} = \tilde{\mathscr{O}} \times \mathscr{A} \times \mathscr{F}$ acting on the left of Z by $gZ = PZA' + F$ for $g = (P, A, F) \in \mathscr{G}$, where $\tilde{\mathscr{O}}$ is the group of $n \times n$ block diagonal matrices $\{P\}$ with diagonal blocks $P_i \in \mathscr{O}(n_i)$ $(i = 1, 2, 3)$, $\mathscr{A}$ is the group of $p \times p$ nonsingular matrices of the form $A = (A_{ij})$ with $A_{ij}: p_i \times p_j$ and $A_{ij} = 0$ for $i > j$ $(i, j = 1, 2, 3)$ and F is the group of all $n \times p$ matrices of the form $F = (F_{ij})$ with F_{ij}: $n_i \times p_j$, $F_{ij} = 0$ for $i < j$ and $F_{3k} = 0$ $(i, j = 1, 2, 3$: $k = 1, 2, 3)$. Further, let $V \equiv (V_{ij}) = (Z_{3i}'Z_{3j})$: $p \times p$ with $V_{ij}: p_i \times p_j$, $\Sigma = (\Sigma_{ij})$ with $\Sigma_{ij}: p_i \times p_j$, $V_{22.3} = V_{22} - V_{23}V_{33}^{-1}V_{32}$ and $\Sigma_{22.3} = \Sigma_{22} - \Sigma_{23}\Sigma_{33}^{-1}\Sigma_{32}$.

We shall show the LBI test derived under normality is LBI under the pdf (3.5) where q belongs to a class $\mathscr{Q}$ given in Assumption 3.1 below.

First note that the distribution of a maximal invariant under the null hypothesis remains the same for any pdf of the form (3.5) (see Kariya, 1981). Second, note that a maximal invariant under $\mathscr{G}$ is a function of $s(Z, V) = (s_1(Z, V), s_2(Z, V))$, which is a maximal invariant under the subgroup $\mathscr{H} = \mathscr{A} \times \mathscr{T}$ of $\mathscr{G}$, where

$$s_1(Z, V) = (Z_{12}, Z_{13})\begin{pmatrix} V_{22} & V_{23} \\ V_{32} & V_{33} \end{pmatrix}^{-1}(Z_{12}, Z_{13})', \tag{3.7}$$

$$s_2(Z, V) = \begin{pmatrix} Z_{13} \\ Z_{23} \end{pmatrix} V_{33}^{-1} \begin{pmatrix} Z_{13} \\ Z_{23} \end{pmatrix}';$$

hence the maximal invariant under $\mathscr{G}$ depends on Z only through

$$\bar{Z} \equiv (U_1, U_2, U_3) \equiv ((Z_{12}, Z_{13}), Z_{23}, (Z_{32}, Z_{33})). \tag{3.8}$$

Third, a maximal invariant parameter depends on (Θ, Σ) only through $\Theta_{12}\Sigma_{22.3}^{-1}\Theta_{12}' \equiv \xi\xi'$ with $\xi = \Theta_{12}\Sigma_{22.3}^{-1/2}$ (see Gleser and Olkin, 1970; or Kariya, 1978). Hence, without loss of generality, set $\Theta_{11} = 0$, $\Theta_{21} = 0$, $\Theta_{22} = 0$ and $\Sigma = I$, replace Θ_{12} by ξ, and consider the marginal pdf of $\bar{Z}$:

$$\bar{f}(\bar{Z} \mid \xi) \equiv \bar{q}(\mathrm{tr}(U_1 - \xi^*)'(U_1 - \xi^*) + \mathrm{tr}\ U_2'U_2 + \mathrm{tr} U_3'U_3) \tag{3.9}$$

where $\xi^* = (\xi, 0)$: $n_1 \times (p_2 + p_3)$. The group $\mathscr{G}$ acting on the left of Z is also reduced to the subgroup $\bar{\mathscr{G}} = \mathscr{O}(n_1) \times \bar{\mathscr{A}}$ acting on the left of $\bar{Z}$ as

$$\bar{g}\bar{Z} = \bar{g}(U_1, U_2, U_3) = (P_1U_1\bar{A}', Z_{23}A_{33}', U_3\bar{A}') \quad \text{for} \quad \bar{g} = (P_1, \bar{A}) \in \bar{\mathscr{G}}, \tag{3.10}$$

where $\bar{\mathscr{A}}$ is the group of $(p_2 + p_3) \times (p_2 + p_3)$ nonsingular matrices of the form $\bar{A} = (A_{ij})$ $(i, j = 2, 3)$ with $A_{32} = 0$. Here the subgroup $\mathscr{O}(n_2) \times \mathscr{O}(n_3)$ of $\mathscr{G}$ is ignored since it does not affect U_2 and U_3 in (3.9).

Now to derive an LBI test under f in (3.5) or $\bar{f}$ in (3.9), we evaluate the probability ratio R in (2.3) for the marginal pdf $\bar{f}$. The numerator of R is given by

$$\begin{aligned}\int_{\bar{\mathscr{G}}} \bar{q}(\mathrm{tr}(P_1U_1\bar{A}' - \xi^*)'(P_1U_1\bar{A}' - \xi^*)& \\ + \mathrm{tr}\ A_{33}Z_{23}'Z_{23}A_{33}' + \mathrm{tr}\ \bar{A}U_3'U_3\bar{A}')\mu(d\bar{A})\nu(dP_1)&\end{aligned} \tag{3.11}$$

where

$$\mu(d\bar{A}) = (|A_{22}A_{22}'|^{(M-p_2)/2}dA_{22})(|A_{33}A_{33}'|^{(n-p_3)/2}dA_{33})dA_{23}$$

with $M = n_1 + n_3 - p_3$ and ν is the invariant probability measure on $\mathscr{O}(n_1)$. It is noted that a left invariant measure on $\bar{A}$ is

$$(|A_{22}A_{22}'|^{-(p_2+p_3)/2}dA_{22})(|A_{33}A_{33}'|^{-p_3/2}dA_{33})dA_{23}$$

and that the inverse of the Jacobian of transformation (3.10) is $|A_{22}A_{22}'|^{(n_1+n_3)/2}|A_{33}A_{33}'|^{n/2}$. Here, transforming $\bar{A}$ into $\bar{A}C$ in (3.11) where $C =$

$(C_{ij}) \in \bar{A}$ and $C(U_1'U_1 + U_3'U_3)C' = I_{p_2+p_3}$, the inside of $\bar{q}$ in (3.11) becomes

$$\text{tr}\,\bar{A}\bar{A}' - 2\,\text{tr}\,\xi' P_1(W_2A_{22}' + W_3A_{23}') + \text{tr}\,A_{33}C_{33}Z_{23}'Z_{23}C_{33}'A_{33}' + \delta, \tag{3.12}$$

where $\delta = \text{tr}\,\xi\xi' = \text{tr}\,\Theta_{12}\Sigma_{22.3}^{-1}\Theta_{12}'$ and

$$(W_2, W_3) \equiv U_1C' = (Z_{12}C_{22}' + Z_{13}C_{23}', Z_{13}C_{33}'). \tag{3.13}$$

(We always ignore multiplicative constants coming out by transformations when they are cancelled out with those of the denominator.) Further, transforming A_{33} into $A_{33}(I + C_{33}Z_{23}'Z_{23}C_{33}')^{-1/2}$, the inside of $\bar{q}$ or (3.12) becomes

$$\text{tr}\,\bar{A}\bar{A}' - 2\,\text{tr}\,\xi' P_1(W_2A_{22}' + W_3A_{23}') + \delta \equiv \text{tr}\,\bar{A}\bar{A}' - 2\eta + \delta, \quad \text{say}. \tag{3.14}$$

Under these transformations, the ratio becomes

$$\begin{aligned} R &\equiv dP_\xi^T/dP_0^T \\ &= \int_{\mathscr{G}} \bar{q}(\text{tr}\,\bar{A}\bar{A}' - 2\eta + \delta)\nu(dP_1)\mu(d\bar{A}) \Big/ \int_{\mathscr{G}} \bar{q}(\text{tr}\,\bar{A}\bar{A}')\nu(dP_1)\mu(d\bar{A}), \end{aligned} \tag{3.15}$$

where P_ξ^T denotes the distribution of a maximal invariant T under f with $\xi = \Theta_{12}\Sigma_{22.3}^{-1/2}$. Since R is a pdf of T with respect to P_0^T, and since C in (3.13) does not depend on Z_{23} so that R does not depend on Z_{23}, it follows that the density of T evaluated at $T(U_1, U_2, U_3)$ does not involve U_2. This implies that the density is a function of T_0, where T_0 is a maximal invariant function of (U_1, U_3) under the action (3.10) with the U_2 part ignored. Therefore, since P_0^T is independent of ξ, T_0 is sufficient for T by Neyman's Factorization Theorem (symbolically $dP_\xi^T = R_\xi(T_0)dP_0^T$). (This result also follows from the Stein Theorem (Hall, Wijsman and Ghosh, 1965) as in the normal case (see Kariya, 1978, Theorem 3.1).) Thus we obtain

THEOREM 3.1. *The class of invariant tests based on* $s_1(Z, V)$ *in* (4.2) *and* $T_2 \equiv Z_{13}V_{33}^{-1}Z_{13}'$ *only forms an essentially complete class.*

Note that in this theorem no assumption is made on q in (3.5). Second, this theorem implies that the statistics $T_3 \equiv Z_{13}V_{33}^{-1}Z_{23}'$ and $T_4 \equiv Z_{23}V_{33}^{-1}Z_{23}'$ in the maximal invariant (3.7) under $\mathscr{H} = \mathscr{A} \times \mathscr{F}$ can be discarded when an invariant test under $\mathscr{G}$ is considered. Third, the above argument gives a simpler proof even for the normal case. In fact, Kariya (1978) derived the result via the distributional properties of $s_1(Z, V)$ and $s_2(Z, V)$ under normality. But the above proof makes it clear that this result is more related to the invariant structure of the problem than the distributional structure. This point of interest has been emphasized by a referee. Of course, when $p_3 = 0$, or especially in the MANOVA problem, the result in Theorem 3.1 becomes trivial.

Next, to establish a robustness property of the LBI test derived under normality we assume

ASSUMPTION 3.1. The function q in the pdf (3.5) belongs to $\mathscr{Q}$, where $\mathscr{Q}$ is the class of continuously three times differentiable functions from $[0, \infty)$ into $[0, \infty)$ such that

$$\int_{R^{np}} q(\operatorname{tr} Z'Z)\, dZ = 1, \tag{3.16}$$

$$\int_{\bar{\mathscr{A}}} (\operatorname{tr} \bar{A}\bar{A}')^{i/2} \,|\, \bar{q}^{(i)}(\operatorname{tr} \bar{A}\bar{A}') \,|\, \mu(d\bar{A}) < \infty \quad (i = 1, 2, 3), \tag{3.17}$$

$$\bar{q}^{(3)}(x) \le 0, \quad \text{and} \quad \bar{q}^{(3)} \text{ is nondecreasing,} \tag{3.18}$$

where $\bar{q}^{(i)}(x) = d^i\bar{q}(x)/dx^i$.

Under this assumption, we prove the following result:

THEOREM 3.2. (GMANOVA). *Under Assumption* 3.1, *the test based on the critical region*

$$a_0 \operatorname{tr} X(X'X + V_{22.3})^{-1}X'(I + T_2)^{-1} - \operatorname{tr}(I + T_2)^{-1} > c \quad \textit{with} \tag{3.19}$$

$$\begin{gathered} a_0 = (n_1 + n_3 - p_3)/2, \\ X = (I + T_2)^{-1/2}(Z_{12} - Z_{13}V_{33}^{-1}V_{32}) \quad \textit{and} \quad T_2 = Z_{13}V_{33}^{-1}Z_{13}' \end{gathered} \tag{3.20}$$

is LBI for testing $H\colon \Theta_{12} = 0$ *vs* $K\colon \Theta_{12} \neq 0$ *under the pdf* (3.5).

COROLLARY 3.1. (MANOVA). *Under Assumption* 3.1 *and when* $p_1 = p_3 = 0$, *the test based on*

$$\operatorname{tr} Z_{12}(Z_{12}'Z_{12} + V_{22})^{-1}Z_{12}' > c \tag{3.21}$$

is LBI for testing $H\colon \Theta_{12} = 0$ *vs* $K\colon \Theta_{12} \neq 0$ *under the pdf* (3.5).

Corollary 3.1 follows directly from Theorem 3.2 by setting $p_1 = p_3 = 0$.

Two remarks follow. First, the LBI test in (3.19) under the pdf (3.5) is the same as the LBI test derived under normality. This implies that the LBI property is robust at least within the class of pdf's specified by Assumption 3.1. Similarly the LBI test in (3.21) coincides with the Pillai test in the MANOVA, the LBI property of which is shown under normality by Schwartz (1967). Hence Corollary 3.1 also shows the robustness of the LBI property of the test up to the class $\mathscr{Q}$. Second, the conditions in Assumption 3.1 are satisfied for a large class of pdf's, especially in the case of normal mixture: $q(x) = \int_0^\infty e^{-ax}\, dF(a)$ provided the conditions on the moments hold. On the other hand, in the MANOVA problem, it is shown in Kariya (1981) that when $\min(n_1, p_2) = 1$, without any condition on q except the convexity of q, the UMPI (uniformly most powerful invariant) property of the test (3.21) is guaranteed.

Proof of Theorem 3.2. Expand the integrand of $\bar{q}$ in the numerator of (3.15) as

$$\begin{aligned}\bar{q}(\operatorname{tr}\bar{A}\bar{A}') &+ \bar{q}^{(1)}(\operatorname{tr}\bar{A}\bar{A}')(-2\eta+\delta) \\ &+ \tfrac{1}{2}\bar{q}^{(2)}(\operatorname{tr}\bar{A}\bar{A}')(-2\eta+\delta)^2 + \tfrac{1}{6}\bar{q}^{(3)}(z)(-2\eta+\delta)^3,\end{aligned} \tag{3.22}$$

where $z = \operatorname{tr}\bar{A}\bar{A}' + (1-\alpha)(-2\eta+\delta)$ with $0 \le \alpha \le 1$. We evaluate the integrals of each term in (3.22). First, since the integral of $(\operatorname{tr} P_1 Q)^k$ over $\mathscr{O}(n_1)$ with respect to $\nu(dP_1)$ is zero for k odd, from (3.14) and (3.16) the integration of the second term of (3.22) over $\bar{\mathscr{G}}$ is simply $\delta \int_{\bar{\mathscr{A}}} \bar{q}^{(1)}(\operatorname{tr}\bar{A}\bar{A}')\mu(d\bar{A})$. Second, the integration of the third term of (3.22) becomes

$$\begin{aligned}\frac{2}{n_1}\int_{\bar{\mathscr{A}}} [\operatorname{tr}(W_2A_{22}' + W_3A_{23}')\xi'\xi(W_2A_{22}' + W_3A_{23}')']\bar{q}^{(2)}(\operatorname{tr}\bar{A}\bar{A}')\mu(d\bar{A}) \\ + \frac{\delta^2}{2}\int_{\bar{\mathscr{A}}} \bar{q}^{(2)}(\operatorname{tr}\bar{A}\bar{A}')\mu(d\bar{A}),\end{aligned} \tag{3.23}$$

since $\int_{\mathscr{O}(n_1)} (\operatorname{tr} P_1Q)^2\, \nu(dP_1) = \operatorname{tr} Q'Q/n_1$. In the first term of (3.23), the measure $\bar{q}^{(2)}(\operatorname{tr}\bar{A}\bar{A}')\mu(d\bar{A})$ is invariant under the sign change $A_{22} \longrightarrow -A_{22}$ and so the integration of $\operatorname{tr} W_2A_{22}'\xi'\xi A_{23}W_3'$ is zero. To evaluate the integral of $\operatorname{tr} W_2A_{22}'\xi'\xi A_{22}W_2'$ in (3.23), let $\bar{\bar{q}}^{(2)}(\operatorname{tr} A_{22}A_{22}')\mu_2(dA_{22})$ be the marginal measure of A_{22}, where $\mu_2(dA_{22}) = |A_{22}A_{22}'|^{(M-p_2)/2}dA_{22}$, and decompose $G\ell(p_2) = G_T(p_2) \times \mathscr{O}(p_2)$ and $\mu(dA_{22}) = \lambda_2(dB_2)\tau_2(dQ_2)$. Here $G_T(p_2)$ is the group of $p_2 \times p_2$ lower triangular matrices with positive diagonal elements, $\tau_2(dQ_2)$ is the invariant probability measure on $\mathscr{O}(p_2)$, and $\lambda_2(dB_2) = |B_2B_2'|^{M/2}(\Pi b_{ii}^{-i})\, dB_2$ with $B_2 = (b_{ij})$ (see, e.g., Wijsman, 1967, page 398; or Eaton, 1983, page 213). Note that $(\Pi b_{ii}^{-i})\, dB_1$ is a left invariant measure on $G_T(p_2)$. Under this decomposition, with $A_{22} = B_2Q_2$, the integration of $\operatorname{tr} W_2A_{22}'\xi'\xi A_{22}W_2'$, after integration over $\mathscr{O}(p_2)$, is

$$\frac{1}{p_2}(\operatorname{tr} W_2'W_2)\int_{G_T(p_2)} (\operatorname{tr} B_2B_2'\xi'\xi)\bar{\bar{q}}^{(2)}(\operatorname{tr} B_2B_2')\lambda_2(dB_2) = \frac{\delta\beta_2}{p_2}\operatorname{tr} W_2'W_2 \tag{3.24}$$

with

$$\beta_2p_2 = \int_{G_T(p_2)} (\operatorname{tr} B_2B_2')\bar{\bar{q}}^{(2)}(\operatorname{tr} B_2B_2')\lambda_2(dB_2), \tag{3.25}$$

where we used the facts that $\int_{\mathscr{O}(p_2')} \operatorname{tr} AQ_2BQ_2'\tau_2(dQ_2) = \operatorname{tr} AB/p_2$ and $\int_{G_T(p_2)} B_2B_2'\bar{\bar{q}}^{(2)}(\operatorname{tr} B_2B_2')\lambda_2(dB_2) = \beta_2 I$. Further to evaluate the integral of $\operatorname{tr} W_3A_{23}'\xi'\xi A_{23}W_3'$ in (3.23), let $\tilde{\bar{\bar{q}}}^{(2)}(\operatorname{tr} A_{23}A_{23}')dA_{23}$ be the marginal measure of A_{23}. Then a direct evaluation of the integration of $\operatorname{tr} W_3A_{23}'\xi'\xi A_{23}W_3'$ with respect to this measure yields $\delta\beta_3 \operatorname{tr} W_3'W_3$, with

$$\beta_3p_2p_3 = \int_{R^{p_2p_3}} (\operatorname{tr} A_{23}A_{23}')\tilde{\bar{\bar{q}}}^{(2)}(\operatorname{tr} A_{23}A_{23}')\, dA_{23}. \tag{3.26}$$

Therefore (3.23) is finally evaluated as

$$(2/n_1)\delta[(\beta_2/p_2)\operatorname{tr} W_2'W_2 + \beta_3 \operatorname{tr} W_3'W_3] + o(\delta). \tag{3.27}$$

Third, we show that the integral of the fourth term in (3.22) is $o(\delta)$. Since

$$|\eta| \leq (\mathrm{tr}\,\bar{A}\bar{A}')^{1/2}(\mathrm{tr}\,\xi' HH'\xi)^{1/2} \leq (\mathrm{tr}\,\bar{A}\bar{A}')^{1/2}\delta^{1/2}$$

from $W_2W_2' + W_3W_3' \leq I$ where $H = (P_1W_2, P_1W_3)$, we have $z \geq \mathrm{tr}\,\bar{A}\bar{A}' - 2(\mathrm{tr}\,\bar{A}\bar{A}')^{1/2}\delta^{1/2} + \delta$. Hence from (3.18),

$$\begin{aligned} &\left|\int_{\bar{\mathscr{G}}} \bar{q}^{(3)}(z)(-2\eta + \delta)^3\nu(dP_1)\mu(d\bar{A})\right| \\ (3.28)\qquad &\leq \int_{\bar{\mathscr{G}}} -\bar{q}^{(3)}(z)\,|-2\eta + \delta\,|^3\nu(dP_1)\mu(d\bar{A}) \\ &\leq \int_{\bar{\mathscr{G}}} -\bar{q}^{(3)}(\mathrm{tr}\,\bar{A}\bar{A}' - 2(\mathrm{tr}\,\bar{A}\bar{A}')^{1/2}\delta^{1/2} + \delta)\,|-2\eta + \delta\,|^3\nu(dP_1)\mu(d\bar{A}). \end{aligned}$$

Since

$$|-2\eta + \delta\,|^3 \leq 8(\mathrm{tr}\,\bar{A}\bar{A}')^{3/2}\delta^{3/2} + 12(\mathrm{tr}\,\bar{A}\bar{A}')\delta^2 + 6(\mathrm{tr}\,\bar{A}\bar{A}')^{1/2}\delta^{5/2} + \delta^3,$$

and since $\delta \geq 0$, the right side of (3.28) is further bounded above by

$$(3.29)\qquad \textstyle\sum_{i=1}^4 c_i \displaystyle\int_{\bar{\mathscr{A}}} -\bar{q}^{(3)}(\mathrm{tr}\,\bar{A}\bar{A}' - 2(\mathrm{tr}\,\bar{A}\bar{A}')^{1/2}\delta^{1/2})(\mathrm{tr}\,\bar{A}\bar{A}')^{(4-i)/2}\delta^{(1+i)/2}\mu(d\bar{A}),$$

where $c_1 = 8$, $c_2 = 12$, $c_3 = 6$ and $c_4 = 1$. We split the domain of this integral into $E_0 = \{\mathrm{tr}\,\bar{A}\bar{A}' \leq 1\}$ and $E_1 = \{\mathrm{tr}\,\bar{A}\bar{A}' > 1\}$. On E_1, by (3.18) replacing $(\mathrm{tr}\,\bar{A}\bar{A}')^{1/2}$ in the inside of $\bar{q}^{(3)}$ by $\mathrm{tr}\,\bar{A}\bar{A}'$, and changing $\bar{A}$ into $(1 - 2\delta^{1/2})^{1/2}\bar{A}$ for $\delta < 1/8$, (3.29) is shown to be $o(\delta)$, while on E_0, it is $o(\delta)$ from the boundedness of $\bar{q}^{(3)}$.

Consequently, noticing that the denominator of the ratio R is simply $D = \int_{\bar{\mathscr{A}}} \bar{Q}(\mathrm{tr}\,\bar{A}\bar{A}')\mu(d\bar{A})$, we obtain

LEMMA 3.1. *The ratio* R *in* (3.15) *is evaluated as*

$$R = 1 + \frac{\delta}{D}\left[\beta_1 + \frac{2}{n_1}\left(\frac{\beta_2}{p_2}\,\mathrm{tr}\,W_2'W_2 + \beta_3\mathrm{tr}\,W_3'W_3\right)\right] + o(\delta)$$

where $\beta_1 = \int_{\bar{\mathscr{A}}} \bar{q}^{(1)}(\mathrm{tr}\,\bar{A}\bar{A}')\mu(d\bar{A})$, W_i*'s are defined in* (3.13), β_2 *and* β_3 *are given by* (3.25) *and* (3.26), *respectively, and* $o(\delta)$ *is uniform in* Z.

LEMMA 3.2.

(1) $\mathrm{tr}\,W_2'W_2 = \mathrm{tr}\,X(X'X + V_{22.3})^{-1}X'(I + T_2)^{-1}$

(2) $\mathrm{tr}\,W_3'W_3 = -\mathrm{tr}(I + T_2)^{-1} + n_1$

(3) $\beta_2/\beta_3 = M \equiv n_1 + n_3 - p_3$

PROOF. The proofs of (1) and (2) are straightforward but tedious (see the

Appendix). To prove (3), consider the marginal density of A_{22} and A_{23} with respect to $|A_{22}A_{22}'|^{(M-p_2)/2}dA_{22}dA_{23}$:

$$h(\text{tr } A_{22}A_{22}' + \text{tr } A_{23}A_{23}')$$
$$= \int_{G\ell(p_3)} \bar{q}^{(2)}(\text{tr } A_{22}A_{22}' + \text{tr } A_{23}A_{23}' + \text{tr } A_{33}A_{33}') \times |A_{33}A_{33}'|^{(n-p_3)/2}\, dA_{33}.$$

As before, decompose A_{22} as $A_{22} = B_2Q_2$, with $Q_2 \in \mathscr{O}(p_2)$ and $B_2 \in G_T(p_2)$. Then the marginal measure of B_2 and A_{23} is given by

$$(3.30) \quad \psi(dB_2, dA_{23}) \equiv h(\text{tr } B_2B_2' + \text{tr } A_{23}A_{23}')|B_2B_2'|^{M/2}(\Pi(b_{ii}^2)^{-i/2})dB_2dA_{23}.$$

Note that

$$\beta_2p_2 = \int \text{tr } B_2B_2'\psi(dB_2, dA_{23}) \quad \text{and} \quad \beta_2p_2p_3 = \int \text{tr } A_{23}A_{23}'\psi(dB_2, dA_{23}).$$

Define $B_2 = (b_{ij})$, $A_{23} = (a_{ij})$, $L = \text{tr } B_2B_2' + \text{tr } A_{23}A_{23}'$, $e_0 = \text{tr } A_{23}A_{23}'/L = \sum a_{ij}^2/L$, $e_i = b_{ii}^2/L$ $(i = 1, \cdots, p_2)$, $e_{i+p_2} = b_{i+1,i}^2/L$ $(i = 2, \cdots, p_2-1)$, $e_{i+p_2+(p_2-1)} = b_{i+2,i}^2/L$ $(i = 3, \cdots, p_2 - 2)$, $\cdots$, and $e_{p_2(p_2+1)/2} = b_{p_2,1}^2/L$. Further extend the domain $[0, \infty)$ of b_{ii} into $(-\infty, \infty)$, which simply gives a multiplicative constant, say c_0, to the right side of (3.30), and let

$$K = \int c_0h(\text{tr } B_2B_2' + \text{tr } A_{23}A_{23}')\, dB_2\, dA_{23}.$$

Then since $c_0h(\text{tr } B_2B_2' + \text{tr } A_{23}A_{23}')/K$ is a spherical density of b_{ij}'s and a_{ij}'s, L and $e \equiv (e_0, e_1, \cdots, e_{p_2(P_2+1)/2})$ are independent and e obeys a Dirichlet distribution $D(p_1p_2/2, ½, \cdots, ½)$ (see, e.g., Kariya and Eaton, 1977). Hence, letting $N = \int L^{\Sigma(m-i)/2}c_0h(\text{tr } B_2B_2' + \text{tr } A_{23}A_{23}')\, dB_2dA_{23}$, which is finite since $\beta_2 < \infty$ and $\beta_3 < \infty$, we obtain

$$\beta_3p_2p_3/KN = E(e_0 \textstyle\prod_{i=1}^{p_2} e_i^{(M-i)/2}).$$

This is directly evaluated from the formula (77.9) of Wilks (1962, page 179). Similarly, evaluating

$$\beta_2p_2/KN = E[\textstyle\sum_{j=1}^{p_2(p_2+1)} e_j \prod_{i=1}^{p_2} e_i^{(M-i)/2}]$$
$$= \textstyle\sum_{i=1}^{p_2} E(\prod_{i=1}^{p_2} e_i^{(M-i)/2}) + \sum_{j=p_2+1}^{p_2(p_2+1)/2} E(e_j \prod_{i=1}^{p_2} e_i^{(M-i)/2})$$

and computing the ratio $\beta_2p_2/\beta_3p_2p_3$ yields

$$\textstyle\sum_{i=1}^{p_2} (M - i + 1)/p_2p_3 + \sum_{i>j} 1/p_2p_3 = M/p_3.$$

Hence $\beta_2/\beta_3 = M$, completing the proof.

Now from these lemmas and the general argument given in the first part of this section, Theorem 3.2 is obtained.

Testing independence. Again let Z be an $n \times p$ random matrix with pdf (3.5). In the present problem, let $Z = (Z_1, Z_2)$ with Z_i: $n \times p_i$ $(i = 1, 2)$, $\Theta = (\Theta_1, \Theta_2)$ with Θ_i: $n \times p_i$ $(i = 1, 2)$ and $\Sigma = (\Sigma_{ij})$, Σ_{ij}: $p_i \times p_j$ $(i, j = 1, 2)$ and $p_1 + p_2 = p$.

We consider the problem of testing $\Sigma_{12} = 0$ where $n \ge p$. For simplicity, we assume $\Theta = 0$. Write $Z'Z \equiv S = (S_{ij})$ with $S_{ij} = Z_i'Z_j$. Under the pdf (3.5), the problem remains invariant under the group $\mathscr{A} = G\ell(p_1) \times G\ell(p_2)$ acting on the left of Z by $Z \rightarrow (Z_1A_1', Z_2A_2')$. With Θ assumed to be zero, S is sufficient and a maximal invariant statistic and a maximal invariant parameter are, respectively, the latent roots $\{d_1 \ge \cdots \ge d_{\min(p_1,p_2)}\}$ of $S_{12}S_{22}^{-1}S_{21}S_{11}^{-1}$ and the latent roots $\{\rho_1 \ge \cdots \ge \rho_{\min(p_1,p_2)}\}$ of $\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}$. Hence without loss of generality, let $p_1 \le p_2$ and

$$\Sigma = \begin{pmatrix} I, & \Gamma \\ \Gamma', & I \end{pmatrix} \quad \text{with} \quad \Gamma = (\Delta, 0) \tag{3.31}$$

where $\Delta = \text{diag}\{\rho_1, \cdots, \rho_{p_1}\}$. With this Σ^{-1}, the ratio R in (2.3) is $R = N/D$ where

$$N = N(\Delta) = \int_{\mathscr{A}} |\Sigma^{-1}|^{n/2} q(\text{tr } \Sigma^{-1}ASA')\mu(dA), \tag{3.32}$$

where $D = N(0)$, $A = \left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right)$ and $\mu(dA) = \Pi_{i=1}^{2} |A_iA_i'|^{(n-p_i)/2} dA_i$ (see Schwartz, 1967, for the normal case).

THEOREM 3.3. *Under Assumption* 3.1 *with* $\bar{q} = q$ *and* $\bar{A} = A$, *the test based on the critical region*

$$\text{tr } S_{12}S_{22}^{-1}S_{21}S_{11}^{-1} > c \tag{3.33}$$

is LBI for testing $\Sigma_{12} = 0$ *vs* $\Sigma_{12} \ne 0$ *under the pdf* (3.5).

The proof is similar to the GMANOVA case and omitted. The constant c in (3.33) does not depend on q because the null robustness of the test in (3.33) holds (see Kariya, 1981).

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APPENDIX

Proof of (1) *and* (2) *in Lemma* 3.2. (1) Note that $U_1 = (Z_{12}, Z_{13})$, $U_3 = (Z_{32}, Z_{33})$, $(W_2, W_3) = U_1C' = (Z_{12}C_{22}' + Z_{13}C_{23}', Z_{13}C_{33}')$ and $C(U_1'U_1 + U_3'U_3)C' = I$, where $C = (C_{ij}) \in \mathscr{A}$. Write $U_1'U_1 + U_3'U_3 = C^{-1}C'^{-1} \equiv (H_{ij})$ $(i, j = 2, 3)$.

Then

$$C_{22} = (H_{22} - H_{23}H_{33}^{-1}H_{32})^{-1/2} = H_{22.3}^{-1/2} \quad \text{and} \quad C_{23}' = -H_{33}^{-1}H_{32}C_{22}',$$

while

$$H_{22} = Z_{12}'Z_{12} + V_{22}, \quad H_{23} = H_{32}' = Z_{12}'Z_{13} + V_{23} \quad \text{and} \quad H_{33} = Z_{13}'Z_{13} + V_{33}.$$

Hence

$$\begin{aligned} W_2 &= Z_{12}C'_{22} + Z_{13}C'_{23} = (Z_{12} - Z_{13}H_{33}^{-1}H_{32})H_{22.3}^{-1/2} \\ &= [Z_{12} - Z_{13}(Z'_{13}Z_{13} + V_{33})^{-1}(Z'_{13}Z_{12} + V_{32})]H_{22.3}^{-1/2} \\ &= [Z_{12} - \tilde{Z}_{13}(\tilde{Z}'_{13}\tilde{Z}_{13} + I)^{-1}(\tilde{Z}'_{13}Z_{12} + V_{33}^{-1/2}V_{32})]H_{22.3}^{-1/2} \end{aligned}$$

where $\tilde{Z}_{13} = Z_{13}V_{33}^{-1/2}$. Here using $(\tilde{Z}'_{13}\tilde{Z}_{13} + I)^{-1} = I - \tilde{Z}'_{13}(I + \tilde{Z}_{13}\tilde{Z}'_{13})^{-1}\tilde{Z}_{13}$ and $(I + \tilde{Z}_{13}\tilde{Z}'_{13})^{-1} = I - (I + \tilde{Z}_{13}\tilde{Z}'_{13})^{-1}\tilde{Z}_{13}\tilde{Z}'_{13}$,

$$\begin{aligned} W_2 &= [Z_{12} - (I + \tilde{Z}_{13}\tilde{Z}'_{13})^{-1}(\tilde{Z}_{13}\tilde{Z}'_{13}Z_{12} + \tilde{Z}_{13}V_{33}^{-1/2}V_{32})]H_{22.3}^{-1/2} \\ &= [(I + T_2)^{-1}(Z_{12} - Z_{13}V_{33}^{-1}V_{32})]H_{22.3}^{-1/2}, \end{aligned}$$

where $T_2 = Z_{13}V_{33}^{-1}Z'_{13}$. On the other hand, using the same relations

$$\begin{aligned} H_{22.3} &= Z'_{12}Z_{12} + V_{22} - (Z'_{12}Z_{13} + V_{23})(Z'_{13}Z_{13} + V_{33})^{-1}(Z'_{13}Z_{12} + V_{32}) \\ &= Z'_{12}Z_{12} + V_{22} - (Z'_{12}\tilde{Z}_{13} + V_{23}V_{33}^{-1/2})(\tilde{Z}'_{13}\tilde{Z}_{13} + I)^{-1}(\tilde{Z}'_{13}Z_{12} + V_{33}^{-1/2}V_{32}) \\ &= V_{22.3} + X'X, \end{aligned}$$

with $X = (I + T_2)^{-1/2}(Z_{12} - Z_{13}V_{33}^{-1}V_{32})$. Therefore,

$$\operatorname{tr} W'_2W_2 = \operatorname{tr}(I + T_2)^{-1}X(X'X + V_{22.3})^{-1}X',$$

as is to be proved.

(2)
$$\begin{aligned} \operatorname{tr} W'_3W_3 &= \operatorname{tr} Z_{13}(Z'_{13}Z_{13} + V_{33})^{-1}Z'_{13} \\ &= \operatorname{tr} \tilde{Z}_{13}(\tilde{Z}'_{13}\tilde{Z}_{13} + I)^{-1}\tilde{Z}'_{13} \\ &= \operatorname{tr} \tilde{Z}_{13}\tilde{Z}'_{13} - \operatorname{tr} \tilde{Z}_{13}\tilde{Z}'_{13}(I + \tilde{Z}_{13}\tilde{Z}'_{13})^{-1}\tilde{Z}_{13}\tilde{Z}'_{13} \\ &= \operatorname{tr} T_2(I + T_2)^{-1} = n_1 - \operatorname{tr}(I + T_2)^{-1}. \end{aligned}$$

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