The Annals of Statistics
1984, Vol. 12, No. 3, 1100–1105

A CHARACTERIZATION THEOREM FOR EXTERNALLY BAYESIAN GROUPS

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A contribution is made to the problem of combining the subjective probability density functions $f_1, \cdots, f_n$ of n individuals for some parameter θ. More precisely, the situation is addressed which occurs when the members of a group share a common likelihood for some data and want to ensure that combining their posterior distributions for θ will yield the same result obtained by applying Bayes' rule to the aggregated prior distribution. Under certain regularity conditions to be discussed below, the logarithmic opinion pool

$$\prod_{i=1}^n f_i^{w_i} \Big/ \int \prod_{i=1}^n f_i^{w_i}\, d\mu$$

with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$ is shown to be the only pooling formula which satisfies this criterion of group rationality.

1. Introduction. Suppose that n individuals have expressed subjective opinions about some unseen quantity $\theta \in \Theta$ in terms of probability densities $f_1, \cdots, f_n$ with respect to some dominating measure μ on Θ. If a single probability density, $T(f_1, \cdots, f_n)$, is sought to typify the views of the group, it may be desirable to require that the pooling procedure, T, commute with the process of updating the f_i's upon arrival of new, jointly perceived information. That is, suppose that some objective evidence becomes available and that every member of the group agrees on the likelihood function $l\colon \Theta \to [0, \infty)$ derived from this evidence. Then it would seem reasonable to have

$$T\left(\frac{lf_1}{\int lf_1\, d\mu}, \cdots, \frac{lf_n}{\int lf_n\, d\mu}\right) = \frac{lT(f_1, \cdots, f_n)}{\int lT(f_1, \cdots, f_n)\, d\mu} \quad \mu - \text{a.e.} \tag{1.1}$$

whenever l is μ-measurable and $0 < \int lf_i\, d\mu < \infty$ for each $i = 1, \cdots, n$. Raiffa (1968) illustrates what can happen if this prescription fails.

Following Madansky (1978), any pooling formula T which satisfies (1.1) will be called *externally Bayesian*. Examples of externally Bayesian procedures are dictatorships, $T(f_1, \cdots, f_n) \equiv f_i$, and what Bacharach (1972) calls the logarithmic opinion pool:

$$T(f_1, \cdots, f_n) = \prod_{i=1}^n f_i^{w_i} \Big/ \int \prod_{i=1}^n f_i^{w_i}\, d\mu \quad \mu - \text{a.e.} \tag{1.2}$$

where $w_1, \cdots, w_n \geq 0$ are such that $\sum_{i=1}^n w_i = 1$. The former is a special case of the latter when $w_i = 1$ for some i.

Received June 1983; revised January 1984.
AMS 1980 *subject classifications*. Primary 62A99; secondary 39B40.
Key words and phrases. External Bayesianity, logarithmic opinion pool, consensus.

Let Δ denote the collection of all μ-densities f such that $f \not\equiv 0 \ \mu -$ a.e. Under a regularity condition on (Θ, μ) given below, we will show in Theorem 2.1 that (1.2) is the only externally Bayesian pooling operator $T: \Delta^n \rightarrow \Delta$ which satisfies the following "likelihood principle" for pooling operators:

$$T(f_1, \cdots, f_n)(\theta) \propto G(f_1(\theta), \cdots, f_n(\theta)) \quad \mu - \text{a.e.}$$

In the above equation, $G: [0, \infty)^n \rightarrow [0, \infty)$ is some arbitrary Lebesgue measurable function and the symbol $\propto$ is interpreted to mean "proportional up to a factor independent of θ," so that

$$(1.3) \qquad T(f_1, \cdots, f_n)(\theta) = G(f_1(\theta), \cdots, f_n(\theta)) \Big/ \int G(f_1, \cdots, f_n)\, d\mu \quad \mu - \text{a.e.}$$

Condition (1.3) can be viewed as a likelihood principle in the following sense: except for a normalization constant independent of θ, the density of the consensus distribution at θ is required to depend only on the individual densities at the actual value of the unseen quantity, and not upon the densities of the values which might have obtained but did not. It is similar in spirit to the "independence of irrelevant alternatives" hypothesis familiar to decision theorists (e.g. Bacharach, 1975) and is a natural extension of the following analogue of McConway's (1981) Strong Setwise Function Property (SSFP) for densities:

$$(1.4) \qquad T(f_1, \cdots, f_n)(\theta) = F[f_1(\theta), \cdots, f_n(\theta)] \quad \mu - \text{a.e.}$$

Condition (1.4) might be intuitively more appealing than (1.3), but it is too strong. It can be shown to characterize linear opinion pools, $\sum_{i=1}^n w_i f_i$, and thus leads to dictatorships when used in conjunction with (1.1) (cf. Genest, 1984).

This paper relates to the recent literature on the problem of pooling opinions. In particular, its analysis pertains to what French (1983) calls the "textbook problem" in which the group takes upon itself to summarize its beliefs for others to use at some time in the future in as yet undefined circumstances. The result presented here, therefore, does not presuppose the existence of a decision maker who might elect to use the experts' opinions as data in order to update his/her own distribution. For a survey of other approaches to the consensus problem, the reader is referred to the review papers by Hogarth (1975), Weerahandi and Zidek (1981) and French (1983).

2. A characterization theorem. We will now derive the following characterization of the logarithmic opinion pool.

THEOREM 2.1. *Let $T: \Delta^n \rightarrow \Delta$ be an externally Bayesian pooling operator of the form* (1.3). *Also assume that for any $\varepsilon > 0$, there exists $A_\varepsilon \subset \Theta$ with $0 < \mu(A_\varepsilon) < \varepsilon$. Then T is a logarithmic opinion pool with some arbitrary constants $w_1, \cdots, w_n \geq 0$ satisfying $\sum_{i=1}^n w_i = 1$.*

For clarity, the proof is broken up into two lemmas. First note that the assumption on Θ implies the existence of any finite number $k > 1$ of disjoint subsets $A_j \subset \Theta$ with $0 < \mu(A_j) < \varepsilon$, $1 \leq j \leq k$.

LEMMA 2.2. *The function G in* (1.3) *is homogeneous* i.e. $G(cx_1, \cdots, cx_n) = cG(x_1, \cdots, x_n)$ *for all* $c > 0$ *and* $x_i > 0$, $i = 1, \cdots, n$.

PROOF. Let $\delta = (c + 1)^{-1}\min\{1/x_i: 1 \le i \le n\}$ and choose A_j, $1 \le j \le 5$, five disjoint subsets of Θ with $0 < \mu(A_j) < \delta$. Write $K = 2/\mu(A_1 \cup A_2)$, let $\gamma > 0$ be such that

$$\gamma < K \min\{1/x_i - \mu(A_1) - c\mu(A_2): 1 \le i \le n\}$$

and pick $0 < \lambda, \xi < \infty$ so that

$$\lambda < \min\{- \gamma + K[1/x_i - \mu(A_1) - c\mu(A_2)]: 1 \le i \le n\}$$
$$\le \max\{- \gamma + K[1/x_i - \mu(A_1) - c\mu(A_2)]: 1 \le i \le n\} < \xi.$$

Now for each $i = 1, \cdots, n$, there exists $m_i \in (0, 1)$ such that

$$\lambda m_i + \xi(1 - m_i) = - \gamma + K[1/x_i - \mu(A_1) - c\mu(A_2)].$$

Define

$$f_i = [K/4]I(A_1 \cup A_2) + [m_i/4\mu(A_3)]I(A_3)$$
$$+ [(1 - m_i)/4\mu(A_4)]I(A_4) + [h/4R]I(N)$$

where $N = \Theta\backslash(\cup_{j=1}^{4} A_j)$, $R = \int_N h d\mu$, $h \in \Delta$ is arbitrary and, in general, $I(A)$ denotes the characteristic function of the set A. Note that $R > 0$, for otherwise h would vanish on $A_5 \subset N$, thus contradicting the fact that $h \not\equiv 0$ μ − a.e. Thus f_i is well-defined and belongs to Δ.

Now consider the likelihood

$$l = I(A_1) + cI(A_2) + \lambda I(A_3) + \xi I(A_4) + \gamma I(N)$$

for which $\int l f_i \, d\mu = K/4x_i$, $i = 1, \cdots, n$.

Since T is externally Bayesian,

$$G\left(\frac{l(\theta)f_1(\theta)}{\int lf_1 \, d\mu}, \cdots, \frac{l(\theta)f_n(\theta)}{\int lf_n \, d\mu}\right) \Big/ l(\theta)G(f_1(\theta), \cdots, f_n(\theta))$$
$$(2.1) \qquad = \int G\left(\frac{lf_1}{\int lf_1 \, d\mu}, \cdots, \frac{lf_n}{\int lf_n \, d\mu}\right) d\mu \Big/ \int lG(f_1, \cdots, f_n) \, d\mu$$

μ − a.e. and since the right-hand side of this identity does not vary with θ, the left-hand side assumes the same value whether θ belongs to A_1 or A_2.

Therefore

$$\frac{G(x_1, \cdots, x_n)}{G(K/4, \cdots, K/4)} = \frac{G(cx_1, \cdots, cx_n)}{cG(K/4, \cdots, K/4)}$$

from which the asserted conclusion follows. □

Not all homogeneous G's generate an externally Bayesian pooling procedure of the form (1.3). Consider, for instance, $G(x_1, \cdots, x_n) = \max\{x_i: 1 \le i \le n\}$ which gives rise to the operator

$$(2.2) \qquad T(f_1, \cdots, f_n) = \max\{f_i\} \Big/ \int \max\{f_i\}\, d\mu.$$

The following lemma will show that procedures such as (2.2) do not obey (1.1).

LEMMA 2.3. *The function G in* (1.3) *satisfies the functional equation*

$$G(x_1y_1, \cdots, x_ny_n)G(1, \cdots, 1) = G(x_1, \cdots, x_n)G(y_1, \cdots, y_n)$$

$$\textit{for all} \quad x_1, \cdots, x_n, \quad y_1, \cdots, y_n \quad \textit{in} \quad (0, \infty).$$

PROOF. Let $0 < \gamma < \min\{1, 1/x_i: 1 \le i \le n\}$, $0 < \delta < \min\{(1 - \gamma)/y_i, (1/x_i - \gamma)/y_i: 1 \le i \le n\}$, and, as before, let A_j, $1 \le j \le 5$, be five disjoint subsets of Θ with $0 < \mu(A_j) < \delta$. If we write $t_i = 1 - \gamma - y_i\mu(A_2)$, then $t_i > 0$ and $1/x_i > \gamma + y_i\mu(A_2)$ for all $1 \le i \le n$.

Next, choose $0 < \lambda, \xi < \infty$ so that

$$\lambda < \min\{(1/t_i)[1/x_i - \gamma - y_i\mu(A_2)]: 1 \le i \le n\}$$
$$\le \max\{(1/t_i)[1/x_i - \gamma - y_i\mu(A_2)]: 1 \le i \le n\} < \xi.$$

Then for each $i = 1, \cdots, n$ there exists $m_i \in (0, 1)$ such that $\lambda m_i + \xi(1 - m_i) = [1/x_i - \gamma - y_i\mu(A_2)]/t_i$.

Define

$$f_i = [\gamma/2\mu(A_1)]I(A_1) + y_iI(A_2) + [t_im_i/\mu(A_3)]I(A_3)$$
$$+ [t_i(1 - m_i)/\mu(A_4)]I(A_4) + [\gamma h/2R]I(N)$$

where, as before, $A_5 \subset N = \Theta\backslash(\cup_{j=1}^4 A_j)$, $R = \int_N h d\mu >$ and $h \in \Delta$ is arbitrary.

Letting $= I(A_1 \cup A_2) + \lambda I(A_3) + \xi I(A_4) + I(N)$, we see that $\int lf_i\, d\mu = 1/x_i$ for all $i = 1, \cdots, n$. Since the left-hand side of (2.1) assumes the same value whether θ belongs to A_1 or A_2, it follows that

$$\frac{G(Kx_1, \cdots, Kx_n)}{G(K, \cdots, K)} = \frac{G(x_1y_1, \cdots, x_ny_n)}{G(y_1, \cdots, y_n)}$$

where $K = \gamma/2\mu(A_1)$. But by Lemma 2.2, the left-hand side of this identity equals $G(x_1, \cdots, x_n)/G(1, \cdots, 1)$, whence the conclusion. □

To complete the proof of Theorem 2.1, it suffices to observe that the function $H(x_1, \cdots, x_n) \equiv_\Delta G(x_1, \cdots, x_n)/G(1, \cdots, 1)$ is Lebesgue measurable and satisfies Cauchy's functional equation, $H(x_1y_1, \cdots, x_ny_n) = H(x_1, \cdots, x_n)H(y_1, \cdots, y_n)$, on $(0, \infty)^n$. Therefore, $H(x_1, \cdots, x_n) = \prod_{i=1}^n x_i^{w_i}$ for some $w_1, \cdots, w_n \in \mathbb{R}$, and since $H(cx_1, \cdots, cx_n) = cH(x_1, \cdots, x_n)$ for all $c > 0$ and $x_1, \cdots, x_n > 0$ by Lemma 2.2 it is clear that $\sum_{i=1}^n w_i = 1$. The nonnegativity of the weights derives from the fact that $\int \prod_{i=1}^n f_i^{w_i}\, d\mu$ must be finite for all choices of $f_1, \cdots, f_n \in \Delta$.

3. Discussion. Theorem 2.1 provides a characterization of the logarithmic opinion pool in a large variety of settings, including the important case of parameter estimation where Θ is an open subset of $\mathbb{R}^m$ and μ is Lebesgue measure. The equally important case where Θ is finite and μ is a counting-type measure is not covered by our analysis, however. This is because of the assumed existence of subsets of Θ with arbitrarily small measure. In that case, a similar result has been derived by McConway (1978). The formula he obtains is of the form

$$T(f_1, \cdots, f_n) = g \prod_{i=1}^n f_i^{w_i} \Big/ \int g \prod_{i=1}^n f_i^{w_i} \, d\mu \tag{3.1}$$

where $g\colon \Theta \longrightarrow (0, \infty)$ is a likelihood-type function and $w_1, \cdots, w_n$ are nonnegative numbers such that $\sum_{i=1}^n w_i = 1$; it applies whenever Θ is countable and contains at least three distinct points. We conjecture that formulas of this type would be the only ones to qualify as externally Bayesian if the function G in (1.3) were allowed to depend on θ. Note that (3.1) reduces to a logarithmic opinion pool if T is required to preserve unanimity, i.e. $T(f, \cdots, f) = f$ for all $f \in \Delta$.

The problem of choosing the weights w_i in (1.2) remains and is not addressed here. This difficulty is common to most axiomatic approaches, however (cf. McConway, 1981; Bordley, 1982; Wagner, 1982); Winkler (1968) discusses some pragmatic solutions to the problem. Another potential criticism of Theorem 2.1 might arise from its failure to address the "deadlock" situation in which the distributions of two assessors have different, possibly disjoint support sets. This is because our analysis was restricted to nonzero densities, an assumption which could be justified using "Cromwell's Rule" (Lindley, 1982). In our opinion, it makes no sense to try to combine distributions which may be as much as mutually singular, since such distributions appear to indicate a logical, rather than probabilistic, disagreement between the assessors.

Finally, something should be said about Lindley's (1983) recent contention that the only available normative approach to the consensus problem requires the introduction of a decision maker who treats the group's opinions as data which he/she incorporates into his/her own, possibly vague prior in a conventional Bayesian manner. In the normal model that Lindley presents, the result is a logarithmic-like pooling formula with variable weights which are derived from the decision maker's assessment of the experts' opinions. The formula is not externally Bayesian, except in special circumstances which Lindley describes. This leads him to reject the externally Bayesian criterion as being fallacious and essentially ad hoc.

As French (1983) points out, however, there are situations in which a group of individuals cannot, or is unwilling, to produce the "superBayesian" required by Lindley. This is the situation in the domain of our analysis, and French describes the difficulties which may arise, in this case, if an intermediary is arbitrarily invested with the task of collating the group's opinions. Within the context of the textbook problem, the one we addressed, the decision maker represents the "synthetic personality" of the group (Hogarth, 1975, page 282), and he most certainly does *not* have an opinion.

We agree with Lindley that, after some discussion, the members of a panel of experts should revise their opinions using Bayesian methods. Even if this procedure were applied iteratively, however, the group as a whole might still fall short of the required unanimity; this possibility has already been demonstrated by French (1981). If the experts attribute their differences of opinion to the lack of empirical evidence, they might elect to resolve this difficulty by summarizing their diverse beliefs using an externally Bayesian prescription. In this way, they would ensure that once new, objective information becomes available, a potential user could update this summary opinion with the same effect as if the experts themselves had observed the data jointly. In the circumstances described by our theorem, a logarithmic opinion pool would then be the inevitable consequence.

4. Acknowledgments. This work constitutes part of the author's doctoral dissertation which was completed in 1983 at the University of British Columbia, under the supervision of Professor James V. Zidek. It is a pleasure to thank him for many enlightening discussions, as well as the Natural Sciences and Engineering Research Council of Canada for generous financial support.

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