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INTERMEDIATE EFFICIENCY, THEORY AND EXAMPLES

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Comparison of tests can be made in a local way using the concept of Pitman efficiency or in a non-local way using for instance Bahadur efficiency. In the local case the level of significance is kept fixed, whereas the non-local comparison typically deals with very small levels. These points of view may be seen as the extreme points of view. Here a possibly more realistic intermediate approach is introduced: as the number of observations tends to infinity the level is sent to zero, but not so very fast, in that way filling the gap between the Pitman and Bahadur approaches. The theory is exemplified by results on likelihood ratio tests, locally most powerful tests, and simple linear rank tests.

1. Introduction. Several methods have been proposed to relate (asymptotic) performance of two different sequences of tests. In particular, the concepts of Pitman efficiency for local comparison and Bahadur efficiency for non-local comparison have obtained an especially important position in statistical literature. If one keeps the level of significance bounded away from zero, the power function of any reasonable test at a fixed alternative will tend to one as the number of observations N tends to infinity. So simple comparison of the asymptotic power does not make sense. To overcome this problem, Cochran (1952, page 323) proposes either (a) to "decrease the significance probability as N increases," or (b) to "move the alternative hypothesis steadily closer to the null hypothesis," i.e. to consider sequences of alternatives tending to the null hypothesis.

There exists a large literature using methods for comparison based on (a) *or* (b). Both principles, (a) and (b), seem to be attractive. As Hoeffding (1965, page 369) points out, it seems reasonable to let the size α_N of a test tend to zero as $N \to \infty$. On the other hand, there seems no need to use statistical methods in the case of alternatives far away from the null hypothesis. So the principle of considering points in the alternative hypothesis steadily closer to the null hypothesis is reasonable too. In Bahadur's asymptotic efficiency concept, method (a) is actually used, while fixed alternatives are under consideration, thereby ignoring principle (b). In Pitman's asymptotic efficiency concept, method (b) is used, while one deals with fixed levels, thus ignoring principle (a). From a practical and philosophical point of view, a more appealing approach is to use both attractive principles (a) and (b): considering both levels α_N tending to zero and alternatives tending to the null hypothesis. It is the purpose of this paper to provide such an approach.

Both the Pitman and Bahadur concepts can be interpreted in terms of sample sizes $N(\alpha, \beta, \theta)$, say, required to attain with a level-α test a prescribed power β at an alternative θ (cf. e.g. Serfling, 1980). In the Pitman case, α and β are kept fixed and θ is sent to the hypothesis set Θ_0; in the Bahadur case θ and β are kept fixed and α is sent to zero. Then the asymptotically required sample sizes of the tests under consideration are compared to distinguish between the two sequences of tests. Bahadur efficiency has been introduced in terms of level attained, but the above approach, which is only slightly different, seems to be more basic and stable; cf. Raghavachari (1970), Chandra and Ghosh (1978), Kallenberg (1981b).

Generally, the numbers $N(\alpha, \beta, \theta)$ are not known exactly. Therefore *some* asymptotic

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approach is justified. However, there are serious objections against above methods. As indicated above, both concepts apply one of the attractive principles (a) and (b) while ignoring the other. Moreover, as a consequence of Bahadur's approach in typical cases, the level of significance α_N required to attain a fixed power β at a fixed alternative θ tends to zero at an exponential rate as the number of observations N tends to infinity. One might say that the concepts of Pitman and Bahadur are the extreme points of view. Between these two extremes there remains a whole range of sequences of levels and it seems to be interesting to study the performance of tests with levels in this intermediate range, in that way filling the gap between the Pitman and Bahadur approaches. This is because, in a practical situation one has to decide with a fixed number of observations which test to apply. To make such a choice one can use the concept of Pitman or Bahadur efficiency. This implies that the tests are embedded in sequences of tests with either fixed levels or very small levels. In this paper the spectrum of possible levels in such an embedding is widened by adding the whole intermediate range between the two well-known extremes. This extension can also be useful in the following sense: suppose we have two sequences of tests and the Pitman efficiency of one test with respect to the other equals $e > 1$; suppose that the efficiency in the intermediate range also equals e. Then the better performance of the test remains if smaller levels of significance are under consideration. The importance of Pitman efficiency and m.m. Bahadur efficiency gathers strength by such a result.

In addition, another phenomenon has to be mentioned. It may happen that tests are asymptotically optimal when comparison is made in a non-local way, but not asymptotically optimal w.r.t. criteria based on the local performance of tests. Typical examples are likelihood ratio tests. It may be interesting to investigate whether the non-local optimality or the local non-optimality of such tests remains true, when one does not take one or another extreme point of view; cf. Section 3. (Also the reversed situation, local optimality and non-local non-optimality, may arise, e.g. with locally most powerful tests; cf. Section 4.)

By these considerations the following *i-efficiency* concept is introduced, which takes an intermediate position between the two extremes. That is, the level α_N tends to zero but not too fast, the alternative θ_N tends to the hypothesis but also not too fast and θ_N and α_N agree in such a way that the power function at θ_N stays away from 0 and 1. To be more specific: let $\mathscr{X}$ be a space of points x, $\mathscr{B}$ a σ-field of subsets of $\mathscr{X}$ and for each point θ in a set Θ let P_θ be a probability measure on $(\mathscr{X}, \mathscr{B})$. Let $S = (X_1, X_2, \cdots)$ be a sequence of i.i.d. random elements taking values in $\mathscr{X}$ according to P_θ, $\theta \in \Theta$. The distribution of S will be denoted by $\mathbb{P}_\theta$ when $\theta \in \Theta$ obtains. Suppose the hypothesis $H_0 : \theta \in \Theta_0$ has to be tested against $H_1 : \theta \in \Theta_1 \subset \Theta - \Theta_0$, where Θ_0 and Θ_1 are given subsets of Θ. A family $\{\phi_{N;\alpha};\ N \in \mathbb{N},\ 0 < \alpha < 1\}$ is a family of (randomized) tests of H_0 if for each $N \in \mathbb{N}$ and $0 < \alpha < 1$ the function $\phi_{N;\alpha}(s) = \phi_{N;\alpha}((x_1, x_2, \cdots))$ is a measurable function of $x_1, \cdots, x_N$ only, with values in [0, 1], satisfying

$$\sup_{\theta_0 \in \Theta_0} E_{\theta_0} \phi_{N;\alpha}(S) \le \alpha. \tag{1.1}$$

Suppose we have two families of tests $\{\phi_{N;\alpha}\}$ and $\{\psi_{N;\alpha}\}$. Let $\{\alpha_N\}$ be a sequence of levels with

$$\lim_{N\to\infty} \alpha_N = 0 = \lim_{N\to\infty} N^{-1} \log \alpha_N, \tag{1.2}$$

and let $\{\theta_N\}$ be a sequence of alternatives with

$$\lim_{N\to\infty} H(\theta_N, \Theta_0) = 0, \quad \lim_{N\to\infty} N H^2(\theta_N, \Theta_0) = \infty \tag{1.3}$$

and

$$0 < \liminf_{N\to\infty} E_{\theta_N} \phi_{N;\alpha_N} \le \limsup_{N\to\infty} E_{\theta_N} \phi_{N;\alpha_N} < 1. \tag{1.4}$$

Here $H(\theta, \Theta_0) = \inf_{\theta_0 \in \Theta_0} H(\theta, \theta_0)$ and $H(\theta, \theta_0)$ denotes the Hellinger distance between the probability measures P_θ and P_{θ_0}.

The Hellinger distance between two probability measures P and Q on the same σ-field is defined by

$$(1.5)\qquad H(P, Q) = \left\{\int (p^{1/2} - q^{1/2})^2 d\mu\right\}^{1/2} = \left\{2 - 2\int p^{1/2}q^{1/2} d\mu\right\}^{1/2},$$

where $p = dP/d\mu$, $q = dQ/d\mu$ and μ is any σ-finite measure dominating $P + Q$. This metric is independent of the choice of μ and satisfies $0 \le H(P, Q) \le 2^{1/2}$. Contiguity of the distributions of $(X_1, \cdots, X_N)$ under θ_N and θ_0 implies $\lim\sup_{N\to\infty} NH^2(\theta_N, \theta_0) < \infty$ (cf. Oosterhoff and Van Zwet, 1979). So the sequence $\{\theta_N\}$ plays the same role as the sequence of contiguous alternatives in the Pitman case. Note that for many families of tests (1.4) and (1.2) imply (1.3). Define

$$(1.6)\qquad m_{\phi,\psi}(N) = \inf\{m;\, E_{\theta_N}\psi_{m+k;\alpha_N} \ge E_{\theta_N}\phi_{N;\alpha_N} \quad \text{for all} \quad k = 0, 1, 2, \cdots\}.$$

If the sequence of levels $\{\alpha_N\}$, apart from (1.2), satisfies

$$(1.7)\qquad -\log \alpha_N = o(N^{1/3}) \quad \text{as } N \to \infty$$

and if

$$(1.8)\qquad e_{\phi,\psi} = \lim_{N\to\infty} \frac{m_{\phi,\psi}(N)}{N}$$

exists and does not depend on the special sequences $\{\theta_N\}$, $\{\alpha_N\}$ under consideration, we say that the asymptotic i-efficiency of ϕ w.r.t. ψ equals $e_{\phi,\psi}$. If (1.7) is replaced by

$$(1.9)\qquad -\log \alpha_N = O(\log N) \quad \text{as } N \to \infty,$$

we speak of *weak* asymptotic i-efficiency of ϕ w.r.t. ψ, and use the notation $e^w_{\phi,\psi}$. Otherwise, that is if all sequences $\{\alpha_N\}$ satisfying (1.2) are under consideration, we speak of *strong* asymptotic i-efficiency of ϕ w.r.t. ψ, notation $e^s_{\phi,\psi}$. Note that $e^s_{\phi,\psi} = e \Rightarrow e_{\phi,\psi} = e \Rightarrow e^w_{\phi,\psi} = e$. So the whole intermediate range is built up with three increasing ranges. These several types of intermediate efficiency correspond with the existence of several types of moderate and Cramér type large deviation theorems. The family of tests $\{\phi_{N;\alpha}\}$ is called (weakly) (strongly) i-efficient if the (weak) (strong) asymptotic i-efficiency of ϕ w.r.t. ψ is greater or equal than 1 for all families $\{\psi_{N;\alpha}\}$ such that the efficiency exists. The above concept is related to Groeneboom's (1980, Section 3.4) definition of deficiency for moving alternatives. He studies very precisely the behavior of several tests for the multivariate linear hypothesis from an "intermediate" point of view. Rubin and Sethuraman (1965) discuss efficiency of tests from a Bayesian point of view.

For two test procedures, the ratio of the sample sizes needed to obtain equal expected risks is defined to be the Bayes Risk Efficiency. The main shortcoming of the Bayes principle is its dependence on the prior distribution. It turns out that the large sample behaviors do not depend very heavily on the prior distribution. Johnson and Truax (1974, 1978) have studied the asymptotic behavior of Bayes procedures in one-parameter and multiparameter exponential families.

In this paper it is shown that for testing simple hypotheses in multiparameter exponential families, likelihood ratio tests are strongly i-efficient (cf. Theorem 3.1). In Section 2 it is shown how to compute (weak) (strong) i-efficiency. While central limit theorems play an important role in the local case and Chernoff type large deviation theorems in the nonlocal case, here one deals with moderate and Cramér type large deviation theorems. Some results concerning the i-efficiency of likelihood ratio and locally most powerful tests are presented in Sections 3 and 4. Section 5 is devoted to intermediate efficiency in the k-sample case. In particular, the i-efficiency of simple linear rank statistics is investigated.

2. Computation of i-efficiency. Since i-efficiency as defined in Section 1 occupies an intermediate position between Pitman and Bahadur efficiency, the approaches to compute either of the two can be adapted to compute i-efficiency. Suppose that we have two sequences of test statistics, $\{T_N^{(1)}\}$ and $\{T_N^{(2)}\}$, respectively, large values of $T_N^{(i)}$ being significant, $i = 1, 2$. At first glance, the way of computing Pitman and Bahadur efficiency seems to be different. In the Pitman case one usually derives asymptotic expansions for the distribution of T_N both under the null hypothesis and under contiguous alternatives. The expansion under the null hypothesis gives an expansion for the critical value, which in combination with the expansion under contiguous alternatives results in an expression for the power of the test. Comparing the expansions of the power of $T_N^{(1)}$ and $T_N^{(2)}$ yields the Pitman efficiency of $T_N^{(1)}$ w.r.t. $T_N^{(2)}$. In the Bahadur case the situation is reversed: an expansion for the critical value of the test is obtained by considering the distribution of T_N under a fixed alternative (for efficiency purposes a weak convergence statement usually suffices). Comparing the asymptotic expansions of the levels of significance of both tests (using Chernoff type large deviation theorems) gives the Bahadur efficiency of $T_N^{(1)}$ w.r.t. $T_N^{(2)}$, cf. Theorem 7.2 in Bahadur (1971). Lemma 2.1 is more or less a modification of this theorem to our case.

As a corollary of this lemma, we obtain an adapted version of the well-known Pitman result. This shows that both approaches are not so different as it seems to be: after all, in both cases, the levels and the powers of the two tests under consideration have to be related. In the sequel it will be assumed that there always exist sequences of alternatives $\{\theta_N\}$ satisfying (1.3).

LEMMA 2.1. *Let $T_N^{(i)} = T_N^{(i)}(X_1, \cdots, X_N)$, $i = 1, 2$, $N \in \mathbb{N}$, be test statistics rejecting the hypothesis for large values of $T_N^{(i)}$. Suppose that there exist positive functions $b^{(i)}(\theta)$, $i = 1, 2$, such that for all sequences $\{\theta_N\}$ with $H(\theta_N, \Theta_0) \to 0$ and $NH^2(\theta_N, \Theta_0) \to \infty$ as $N \to \infty$*

(i) $\lim_{N\to\infty} \mathbb{P}_{\theta_N}\{1 - \varepsilon \le N^{-1/2}T_N^{(i)}/b^{(i)}(\theta_N) \le 1 + \varepsilon\} = 1$ *for each* $\varepsilon > 0$,

(ii) $\lim_{N\to\infty}\{-(Nt_N^2)^{-1}\log \sup_{\theta_0\in\Theta_0}\mathbb{P}_{\theta_0}(T_N^{(i)} \ge N^{1/2}t_N)\} = 1$ *if* $N^{1/2}t_N \to \infty$ *and* $t_N = o(N^{-1/3})$ *as* $N \to \infty$,

(iii) $\lim_{N\to\infty}\left\{\frac{b^{(1)}(\theta_N)}{b^{(2)}(\theta_N)}\right\}^2 = e$ *exists*, $0 \le e \le \infty$.

Then the i-efficiency of $T_N^{(1)}$ w.r.t. *$T_N^{(2)}$ exists and is given by*

$$e_{T^{(1)},T^{(2)}} = e. \tag{2.1}$$

If in (ii) *$t_N = o(N^{-1/3})$ is replaced by $t_N = O(N^{-1/2}(\log N)^{1/2})$ or $t_N = o(1)$, respectively, then the weak or strong i-efficiency, respectively, of $T_N^{(1)}$* w.r.t. *$T_N^{(2)}$ exists and is equal to e.*

PROOF. Let $\{\alpha_N\}$ be a sequence of levels with $\lim_{N\to\infty}\alpha_N = 0$ and $-\log \alpha_N = o(N^{1/3})$ as $N \to \infty$, and let $\{\theta_N\}$ be a sequence of alternatives with $\lim_{N\to\infty}H(\theta_N, \Theta_0) = 0$, $\lim_{N\to\infty}NH^2(\theta_N, \Theta_0) = \infty$ and

$$0 < \liminf_{N\to\infty} E_{\theta_N}\phi_{N;\alpha_N}^{(1)} \le \limsup_{N\to\infty} E_{\theta_N}\phi_{N;\alpha_N}^{(1)} < 1, \tag{2.2}$$

where $\phi_{N;\alpha}^{(1)}$ denotes the level-α test based on $T_N^{(1)}$. Let $N^{1/2}t_N^{(1)}$ be the critical value of $\phi_{N;\alpha_N}^{(1)}$, i.e.

$$N^{1/2}t_N^{(1)} = \inf\{d; \sup_{\theta_0\in\Theta_0}\mathbb{P}_{\theta_0}(T_N^{(1)} > d) \le \alpha_N\}.$$

Then for all $d < N^{1/2}t_N^{(1)}$

$$\sup_{\theta_0\in\Theta_0}\mathbb{P}_{\theta_0}(T_N^{(1)} > N^{1/2}t_N^{(1)}) \le \alpha_N < \sup_{\theta_0\in\Theta_0}\mathbb{P}_{\theta_0}(T_N^{(1)} \ge d). \tag{2.3}$$

Condition (ii) holds for *all* sequences $\{t_N\}$ such that $N^{1/2}t_N \to \infty$ and $N^{1/3}t_N \to 0$ as $N \to \infty$. Therefore, (ii) implies that for all $a \in \mathbb{R}$ and $i = 1, 2$

$$\liminf_{N\to\infty}\log \sup_{\theta_0\in\Theta_0}\mathbb{P}_{\theta_0}(T_N^{(i)} \ge a) > -\infty.$$

Hence $N^{1/2}t_N^{(1)}, \to \infty$, because $\alpha_N \to 0$.

Let $\delta_N = (-2N^{-1/3}\log \alpha_N)^{1/2}$, then $\delta_N = o(1)$, because $-\log \alpha_N = o(N^{1/3})$. Application of (ii) with $t_N = \delta_N N^{-1/3}$ yields

$$1 = \lim_{N\to\infty}(2\log \alpha_N)^{-1}\log \sup_{\theta_0\in\Theta_0} \mathbb{P}_{\theta_0}(T_N^{(1)} \geq N^{1/2}\delta_N N^{-1/3})$$

and hence by (2.3) $t_N^{(1)} \leq \delta_N N^{-1/3}$ for sufficiently large N, and thus $t_N^{(1)} = o(N^{-1/3})$. So, (ii) can be applied with $t_N = t_N^{(1)}$. By (i) and (2.2) we have

$$t_N^{(1)} = b^{(1)}(\theta_N)\{1 + o(1)\} \quad \text{as } N \to \infty. \tag{2.4}$$

In combination with (ii) and (2.3), the result

$$\log \alpha_N = -N\{b^{(1)}(\theta_N)\}^2\{1 + o(1)\} \quad \text{as } N \to \infty, \tag{2.5}$$

is now obtained.

Let $\phi_{N;\alpha}^{(2)}$ denote the level-α test based on $T_N^{(2)}$ and let

$$m = m_{\phi^{(1)},\phi^{(2)}}(N).$$

Since there is no information about the behavior of $m^{-1/2}T_m^{(2)}/b^{(2)}(\theta_N)$, we cannot proceed in exactly the same way as above for the sequence $\{\phi_{N;\alpha}^{(2)}\}$. Let $\{\gamma_j\}$ be any sequence of real numbers satisfying $e < \lim_{j\to\infty}\gamma_j \leq \infty$ and let $\{N_j\}$ be any sequence of natural numbers with $\lim_{j\to\infty}N_j = \infty$ and such that $\gamma_1 N_1 < \gamma_2 N_2 < \cdots$ and $\gamma_j N_j \in \mathbb{N}, j = 1, 2, \cdots$. It will be shown that

$$\lim_{j\to\infty} E_{\theta_{N_j}}\phi_{\gamma_j N_j;\alpha_{N_j}}^{(2)} = 1. \tag{2.6}$$

Since $\lim\sup_{N\to\infty} E_{\theta_N}\phi_{m-1;\alpha_N}^{(2)} < 1$ it then follows that $\lim\sup_{N\to\infty} m/N = \lim\sup_{N\to\infty}(m-1)/N \leq e$. For short we write $n = \gamma_j N_j$.

To apply (i) to $n^{-1/2}T_n^{(2)}/b^{(2)}(\theta_{N_j})$, a new sequence of alternatives is introduced as follows: $\theta_n^* = \theta_{N_j}$ and $\theta_k^* = \theta_k$ if $k \neq \gamma_j N_j$ for all j. It is easily seen that $H(\theta_N^*, \Theta_0) \to 0$ and $NH^2(\theta_N^*, \Theta_0) \to \infty$ as $N \to \infty$. Writing $n^{1/2}t_{N_j}^{(2)}$ for the critical value of $\phi_{n;\alpha_{N_j}}^{(2)}$ we obtain, by the same line of argument as above, $n^{1/2}t_{N_j}^{(2)} \to \infty, t_{N_j}^{(2)} = o(n^{-1/3})$ and hence by (ii)

$$\log \alpha_{N_j} = -n(t_{N_j}^{(2)})^2\{1 + o(1)\} \quad \text{as} \quad j \to \infty. \tag{2.7}$$

(Note that here we have used $-\log \alpha_{N_j} = o(N_j^{1/3}) = o(n^{1/3})$ as $j \to \infty$.) Combination of (2.5), (2.7) and (iii) yields

$$\begin{aligned}\lim_{j\to\infty}\left\{\frac{t_{N_j}^{(2)}}{b^{(2)}(\theta_{N_j})}\right\}^2 &= \lim_{j\to\infty}\frac{n(t_{N_j}^{(2)})^2}{N_j\{b^{(1)}(\theta_{N_j})\}^2}\left\{\frac{b^{(1)}(\theta_{N_j})}{b^{(2)}(\theta_{N_j})}\right\}^2\frac{N_j}{n}\\ &= e/\lim_{j\to\infty}\gamma_j < 1.\end{aligned} \tag{2.8}$$

Because $\theta_n^* = \theta_{N_j}$, we have

$$\begin{aligned}E_{\theta_{N_j}}\phi_{n;\alpha_{N_j}}^{(2)} &\geq \mathbb{P}_{\theta_{N_j}}(T_n^{(2)} > n^{1/2}t_{N_j}^{(2)})\\ &\geq \mathbb{P}_{\theta_n^*}\{n^{-1/2}T_n^{(2)}/b^{(2)}(\theta_n^*) > t_{N_j}^{(2)}/b^{(2)}(\theta_{N_j})\},\end{aligned}$$

which tends to 1 as $j \to \infty$ in view of (i) and (2.8), thus establishing (2.6). This completes the proof of $\lim\sup_{N\to\infty} m/N \leq e$.

Similarly one shows $\lim\inf_{N\to\infty} m/N \geq e$.

Following the same line of proof the results concerning weak and strong i-efficiency are obtained. □

REMARK 2.1. If in condition (ii) of Lemma 2.1, $(Nt_N^2)^{-1}$ is replaced by $\{Nf^{(i)}(t_N)\}^{-1}$, where $f^{(i)}(x) = c^{(i)}x^2 + o(x^2)$ as $x \to 0$ for some $c^{(i)} > 0$, $i = 1, 2$, the lemma remains true if in (2.1) e is replaced by $ec^{(1)}(c^{(2)})^{-1}$. Note that in typical cases the function f in Bahadur's

(1971) Theorem 7.2 satisfies $f(x) = cx^2 + o(x^2)$ as $x \to 0$, thus affirming the correspondence between Lemma 2.1 and Theorem 7.2 of Bahadur (1971).

REMARK 2.2. Lemma 2.1 is related to Wieand's (1976) Theorem. In his investigations α and θ are sent separately to 0 and Θ_0, respectively. Then he presents a theorem, which states that under some conditions the order in which the two limiting processes are executed does not matter. This result suggests that for intermediate α_N and θ_N corresponding to it, a good approximation of m_N/N is obtained by this common limit. Our investigations refer directly to intermediate levels and corresponding alternatives; in other words, α_N and θ_N are not sent separately, the one after the other, but *simultaneously* to 0 and Θ_0, respectively. It seems to be more appropriate in approximating the finite case to apply one limiting process than to apply two limiting processes in one or another order. If the intermediate efficiency equals Wieand's common limit, such a result may be seen as a proof of the above mentioned suggestion. Technically, the difference lies in Conditions I and II of Wieand (1976) and Condition (ii) of Lemma 2.1.

As a corollary of Lemma 2.1 the following adapted version of the well-known Pitman result is obtained.

COROLLARY 2.2. *Suppose that* $\Theta = [0, \infty)$, $\Theta_0 = \{0\}$ *and* $\Theta_1 = (0, \infty)$. *Let* $U_N^{(i)} = U_N^{(i)}(X_1, \ldots, X_N)$, $i = 1, 2$, $N \in \mathbb{N}$, *be test statistics rejecting the hypothesis for large values of* $U_N^{(i)}$. *Suppose there exist functions* $\mu^{(i)}$ *on* Θ *with a righthand derivative* $\mu^{(i)\prime}(0) > 0$ *at* $\theta = 0$ *and functions* $\sigma^{(i)}$ *on* Θ *satisfying* $\lim_{\theta \downarrow 0} \sigma^{(i)}(\theta) = \sigma^{(i)}(0) > 0$, $i = 1, 2$, *such that*

(i) $\mathbb{P}_0[N^{1/2}\{U_N^{(i)} - \mu^{(i)}(0)\}/\sigma^{(i)}(0) \geq x_N] = \Phi(-x_N)\{1 + o(1)\}$ *if* $0 < x_N < o(N^{1/6})$ *as* $N \to \infty$,

(ii) $\mathbb{P}_{\theta_N}[N^{1/2}\{U_N^{(i)} - \mu^{(i)}(\theta_N)\}/\sigma^{(i)}(\theta_N) \geq x] \to \Phi(x)$ *if* $\theta_N \to 0$ *and* $N^{1/2}\theta_N \to \infty$ *as* $N \to \infty$, *where* Φ *denotes the standard normal distribution function, and suppose that*

(iii) $H(\theta, 0) \to 0 \Leftrightarrow \theta \to 0$, *and* $H(\theta, 0) = a\theta^2 + o(\theta^2)$ *as* $\theta \to 0$ *for some* $a > 0$.

Then the i-efficiency of $U_N^{(1)}$ w.r.t. $U_N^{(2)}$ *exists and is given by*

$$e_{U^{(1)},U^{(2)}} = \left\{\frac{\mu^{(1)\prime}(0)\sigma^{(2)}(0)}{\mu^{(2)\prime}(0)\sigma^{(1)}(0)}\right\}^2. \tag{2.9}$$

PROOF. Apply Lemma 2.1 with $T_N^{(i)} = 2^{1/2}N^{1/2}\{U_N^{(i)} - \mu^{(i)}(0)\}/\sigma^{(i)}(0)$ and $b^{(i)}(\theta) = 2^{1/2}\{\mu^{(i)}(\theta) - \mu^{(i)}(0)\}/\sigma^{(i)}(0)$, $i = 1, 2$, where it has to be noted that sequences $\{\theta_N\}$ with $H(\theta_N, \Theta_0) \to 0$ and $NH^2(\theta_N, \Theta_0) \to \infty$ correspond with sequences $\{\theta_N\}$ with $\theta_N \to 0$ and $N^{1/2}\theta_N \to \infty$. □

REMARK 2.3. If in (i) $o(N^{1/6})$ is replaced by $O((\log N)^{1/2})$ or $o(N^{1/2})$, respectively (in which case it seems to be more appropriate to add a factor $\exp[c^{(i)}x_N^3 N^{-1/2}\{1 + o(1)\}]$ with some constants $c^{(i)} > 0$, $i = 1, 2$, on the right-hand side of (i)), then the weak or strong i-efficiency, respectively, of $U_N^{(1)}$ w.r.t. $U_N^{(2)}$ exists and is given by the right-hand side of (2.9).

EXAMPLE 2.1. Let $X_1, X_2, \cdots$ be i.i.d. random variables with a normal $N(\theta, 1)$ distribution and suppose $H_0: \theta = 0$ is to be tested against $H_1: \theta > 0$. Consider $U_N^{(1)}(S) = U_N^{(1)}(X_1, X_2, \cdots) = N^{-1}\sum_{i=1}^N X_i$ (the Gauss test) and $U_N^{(2)}(S) = N^{-1}\sum_{i=1}^N 1_{(X_i>0)}$ (the sign test). Application of Corollary 2.2 and Remark 2.3 yields $e^s_{U^{(1)},U^{(2)}} = \frac{1}{2}\pi$, agreeing with the Pitman efficiency of $U^{(1)}$ w.r.t. $U^{(2)}$. Next consider $T_N^{(1)} = N^{-1/2}\sum_{i=1}^N X_i$ (the one-sided Gauss test) and $T_N^{(2)} = |T_N^{(1)}|$ (the two-sided Gauss test). Application of Lemma 2.1 yields $e^s_{T^{(1)},T^{(2)}} = 1$, agreeing with the Bahadur efficiency of $T^{(1)}$ w.r.t. $T^{(2)}$.

EXAMPLE 2.2. Let $X_1, X_2, \cdots$ be i.i.d. random variables with distribution function

$$F_\theta(x) = F(x - \theta), \quad x, \theta \in \mathbb{R},$$

where F is assumed to be symmetric about zero and to have density f. Consider the testing problem $H_0:\theta = 0$ against $H_1:\theta > 0$. We will apply Corollary 2.2 with $U_N^{(1)}$ equal to the signed-rank Wilcoxon statistic and $U_N^{(2)}$ equal to the one-sample t-statistic. To be more precise:

$$U_N^{(1)} = U_N^{(1)}(X_1, \cdots, X_N) = \{N(N+1)\}^{-1}\sum_{X_i>0} R_i^+,$$

where $R_1^+, \cdots, R_N^+$ are the ranks of $|X_1|, \cdots, |X_N|$, and $U_N^{(2)} = \bar{X}_N S_N^{-1}$ with $S_N^2 = (N-1)^{-1}\sum_{i=1}^N (X_i - \bar{X}_N)^2$. It will be assumed that the density f is such that (iii) holds, that $\int \exp(tx^2)f(x)\,dx < \infty$ for some $t > 0$ and that $\int f^2(x)\,dx < \infty$. Note that $2(N+1)(N-1)^{-1}U_N^{(1)} = V_N + 2(N-1)^{-1}W_N$, where V_N is a U-statistic defined by

$$V_N = \binom{N}{2}^{-1} \sum_{1\le i<j\le N} h(X_i, X_j)$$

with

$$h(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 + x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

and W_N is the sign-test statistic $N^{-1}\sum_{X_i>0} 1$. Using Vandemaele's (1980) Cramér type large deviation result concerning U-statistics it is seen that (i) of Corollary 2.2 holds for $U_N^{(1)}$, cf. also Vandemaele (1981, page 47–49). (The functions $\mu^{(i)}$ and $\sigma^{(i)}$ are the same as in the Pitman case.) Similarly as in Vandemaele and Veraverbeke (1983) it can be shown that (i) also holds for $U_N^{(2)}$ (cf. Callaert, 1981). By Theorem 3 of Hušková (1970), (ii) is established for $U_N^{(1)}$, while (ii) is easily proved for the t-statistic. So the i-efficiency of the Wilcoxon signed-rank test w.r.t. the one-sample t-test is the same as the Pitman efficiency and equals $12\sigma^2\{\int_{-\infty}^{\infty} f^2(x)\,dx\}^2$, where $\sigma^2 = \mathrm{var}_0 X_1$.

Let $\mathscr{F}$ denote the class of all continuous distribution functions symmetric around 0. Consider the testing problem $H_0:F \in \mathscr{F}$ against $H_1:F_\theta(x) = F_0(x-\theta)$, $\theta > 0$, where $F_0 \in \mathscr{F}$ has a density f_0 with $\int f_0^2(x)\,dx < \infty$ and finite variance σ^2. We shall further assume that $H(F_\theta, \mathscr{F}) = H(F_\theta, F_0) \to 0 \Leftrightarrow \theta \to 0$ and $H(F_\theta, F_0) = a\theta^2 + o(\theta^2)$ as $\theta \to 0$ for some $a > 0$. Now we apply Lemma 2.1 with $T_N^{(i)} = 2^{1/2}N^{1/2}\{U_N^{(i)} - \mu^{(i)}(0)\}/\sigma^{(i)}(0)$ and $b^{(i)}(\theta) = 2^{1/2}\{\mu^{(i)}(\theta) - \mu^{(i)}(0)\}/\sigma^{(i)}(0)$, where $U_N^{(i)}$, $\mu^{(i)}$ and $\sigma^{(i)}$ are as above, $i = 1, 2$. The proof of (i), (iii), and (ii) w.r.t. $T_N^{(1)}$ is easily obtained by the above mentioned results. (Note that $T_N^{(1)}$ is distribution free under H_0.) To prove (ii) w.r.t. $T_N^{(2)}$ we use (2.7) and (2.8) of Jones and Sethuraman (1978) with a replaced by $N^{1/2}\{2(N-1)\}^{-1/2}t_N$. (Note that $(1+x)\log(1+x) + (1-x)\log(1-x) = x^2\{1 + o(1)\}$ as $x \to 0$.) So again the i-efficiency equals $12\sigma^2\{\int_{-\infty}^{\infty} f_0^2(x)\,dx\}^2$.

3. Likelihood ratio tests in exponential families. It is well-known that in many testing problems, likelihood ratio (LR) tests are asymptotically optimal when comparison of tests is made in a non-local way; cf. Bahadur (1965), Brown (1971), Kallenberg (1981b). On the other hand LR tests usually are not asymptotically optimal w.r.t. criteria based on the local performance of tests. Therefore it may be interesting to investigate the i-efficiency of LR tests to find out whether the non-local optimality or the local non-optimality of LR tests remains true in the intermediate case.

In this section it will be assumed that $X_i (i = 1, 2, \cdots)$ is distributed according to an exponential family

$$dP_\theta(x) = \exp\{\theta' x - \psi(\theta)\}\,d\mu(x), \quad \theta \in \Theta \subset \mathbb{R}^k, x \in \mathbb{R}^k, \tag{3.1}$$

where μ is a σ-finite non-degenerate measure, Θ denotes the natural parameter space, i.e. $\Theta = \{\theta \in \mathbb{R}^k; \int \exp(\theta' x)\,d\mu(x) < \infty\}$, and

$$\psi(\theta) = \log \int \exp(\theta' x)\,d\mu(x), \quad \theta \in \Theta. \tag{3.2}$$

Here $\theta'x$ denotes the inner product of θ and x. Without loss of generality assume that Θ has a non-empty interior and that μ is not supported on a flat. Let $\Theta^* = \{\theta \in \Theta; E_\theta \|X_1\| < \infty\}$, where $\|\cdot\|$ denotes the Euclidean norm. Note that $\text{int}\,\Theta \subset \Theta^* \subset \Theta$. For $\theta \in \Theta^*$ define

$$\lambda(\theta) = E_\theta X_1. \tag{3.3}$$

The mapping λ is $1-1$ on Θ^* (cf. Lemma 2.2 in Berk, 1972). Defining

$$\Lambda = \lambda(\Theta^*) = \{\lambda(\theta); \theta \in \Theta^*\},$$

the inverse mapping λ^{-1} exists on Λ. Note that $\lambda(\theta) = \text{grad}\,\psi(\theta)$ if $\theta \in \text{int}\,\Theta$. Moreover, for $\theta \in \text{int}\,\Theta$, the covariance matrix Σ_θ of X_1 is the Hessian of ψ.

The Hellinger distance is now given by

$$H^2(\theta_1, \theta_0) = 2 - 2\exp\{\psi(\tfrac{1}{2}(\theta_0 + \theta_1)) - \tfrac{1}{2}\psi(\theta_0) - \tfrac{1}{2}\psi(\theta_1)\}, \quad \theta_0, \theta_1 \in \Theta. \tag{3.4}$$

Since with N observations $X_1, \cdots, X_N$ the sample mean $\bar{X}_N = N^{-1}\sum_{i=1}^N X_i$ is sufficient, LR tests and most powerful (MP) tests only depend on $\bar{X}_N$. The distribution of $\bar{X}_N$ is denoted by $\bar{P}_\theta^N$.

For the testing problem $H_0: \theta \in \Theta_0$ against $H_1: \theta \in \Theta_1 = \Theta - \Theta_0$ we define

$$L(x) = \begin{cases} \infty & \text{if } \sup_{\theta_0 \in \Theta_0}\{\theta_0'x - \psi(\theta_0)\} = \infty \\ \sup_{\theta \in \Theta}\{\theta'x - \psi(\theta)\} - \sup_{\theta_0 \in \Theta_0}\{\theta_0'x - \psi(\theta_0)\} & \text{otherwise.} \end{cases} \tag{3.5}$$

With this notation, the size-α LR test of H_0 based on N observations is given by

$$\varphi_{N;\alpha}^{\text{LR}}(\bar{X}_N) = \begin{cases} 1 & > \\ \delta_{N;\alpha} \quad \text{if} \quad L(\bar{x}_N) & = d_{N;\alpha}, \\ 0 & < \end{cases}$$

where $d_{N;\alpha} = \inf\{d;\ \sup_{\theta_0 \in \Theta_0} \mathbb{P}_{\theta_0}(L(\bar{X}_N) > d) \le \alpha\}$ and $\delta_{N;\alpha} = \sup\{\delta \in [0, 1]; \sup_{\theta_0 \in \Theta_0} E_{\theta_0}\varphi_{N;\alpha}^{\text{LR}}(\bar{X}_N) \le \alpha\}$. Then we have for all $\delta < d_{N;\alpha}$

$$\sup_{\theta_0 \in \Theta_0} \mathbb{P}_{\theta_0}(L(\bar{X}_N) > d_{N;\alpha}) \le \alpha < \sup_{\theta_0 \in \Theta_0} \mathbb{P}_{\theta_0}(L(\bar{X}_N) \ge d).$$

If $\sup_{\theta_0 \in \Theta_0} \mathbb{P}_{\theta_0}(L(\bar{X}_N) \ge t)$ is a left-continuous function of t, then

$$\sup_{\theta_0 \in \Theta_0} E_{\theta_0}\varphi_{N;\alpha}^{\text{LR}}(\bar{X}_N) = \alpha.$$

Now we have the following.

THEOREM 3.1. *If $\Theta_0 \subset A \subset \text{int}\,\Theta$ for some compact subset A, the* LR *test is strongly i-efficient.*

Note that the case of a simple hypothesis is covered by this theorem.

The LR test can be expressed in terms of Kullback-Leibler information numbers defined by

$$\begin{aligned} K(\theta_1, \theta_0) &= E_{\theta_1}\{\log dP_{\theta_1}/dP_{\theta_0}(X_1)\} \\ &= \psi(\theta_0) - \psi(\theta_1) + (\theta_1 - \theta_0)'\lambda(\theta_1),\ \theta_1 \in \Theta^*,\ \theta_0 \in \Theta. \end{aligned} \tag{3.6}$$

If $\bar{x}_N \in \Lambda$, $\lambda^{-1}(\bar{x}_N)$ is the maximum likelihood estimate of θ, since $\sup_{\theta \in \Theta}\theta'\bar{x}_N - \psi(\theta) = \lambda^{-1}(\bar{x}_N)'\bar{x}_N - \psi(\lambda^{-1}(\bar{x}_N))$, and thus

$$\varphi_{N;\alpha}^{\text{LR}}(\bar{x}_N) = \begin{cases} 1 & > \\ \delta_{N;\alpha} \quad \text{if} \quad K(\lambda^{-1}(\bar{x}_N), \Theta_0) & = d_{N;\alpha}, \\ 0 & < \end{cases}$$

where $K(\theta, \Theta_0) = \inf\{K(\theta, \theta_0); \theta_0 \in \Theta_0\}$.

Before proving Theorem 3.1, we present the following technical lemma.

LEMMA 3.2.

(a) *Let $B \subset \operatorname{int} \Theta$ be compact. Then* $\| \lambda(\theta) - \lambda(\xi)\| / \| \theta - \xi \|$, $\| \theta - \xi \|^{-2} K(\theta, \xi)$, $K(\theta, \xi)/[(\theta - \xi)'\{\lambda(\theta) - \lambda(\xi)\}]$ *and* $H^2(\theta, \xi)/K(\theta, \xi)$ *are uniformly bounded away from* 0 *and* ∞ *for* $\theta, \xi \in B$, $\theta \neq \xi$.

(b) *Let* $\Theta_0 \subset A \subset \operatorname{int} \Theta$ *for some compact subset A. Let* $\{\theta_N\}$ *be a sequence with* $K(\theta_N, \Theta_0) \to 0$ *and* $NK(\theta_N, \Theta_0) \to \infty$. *For all positive constants c and* γ *we have for all sufficiently large N*

$$\| x - \lambda(\theta_N) \| \le cN^{-1/2} \Rightarrow | K(\lambda^{-1}(x), \Theta_0) - K(\theta_N, \Theta_0)| < \gamma K(\theta_N, \Theta_0).$$

PROOF. For a proof of (a) cf. Kallenberg (1981b, Lemma 3.1a). (b) Let $x_N \in \Lambda$ satisfy $\| x_N - \lambda(\theta_N) \| \le cN^{-1/2}$. Without loss of generality Θ_0 may be assumed to be a closed and hence compact subset of int Θ. Define θ_{0N} by $K(\theta_N, \Theta_0) = K(\theta_N, \theta_{0N})$ and θ_N^* by $K(\lambda^{-1}(x_N), \Theta_0) = K(\lambda^{-1}(x_N), \theta_N^*)$. In view of (a) there exist positive constants c_1 and c_2 such that for all sufficiently large N

$$\begin{aligned} &K(\lambda^{-1}(x_N), \Theta_0) - K(\theta_N, \Theta_0) \\ &\qquad \le K(\lambda^{-1}(x_N), \theta_{0N}) - K(\theta_N, \theta_{0N}) \\ &\qquad = K(\lambda^{-1}(x_N), \theta_N) + (\theta_N - \theta_{0N})' \{x_N - \lambda(\theta_N)\} \\ &\qquad \le K(\theta_N, \Theta_0)\{c_1 N^{-1} K(\theta_N, \Theta_0)^{-1} + c_2 K(\theta_N, \Theta_0)^{-1/2} N^{-1/2}\}, \end{aligned} \tag{3.7}$$

implying $K(\lambda^{-1}(x_N), \theta_N^*) = K(\lambda^{-1}(x_N), \Theta_0) \le 2K(\theta_N, \Theta_0)$ for all sufficiently large N. Therefore there exist positive constants c_3 and c_4 such that $\{\lambda^{-1}(x_N) - \theta_N^*\}'\{\lambda(\theta_N) - x_N\} \le c_3 K(\lambda^{-1}(x_N), \theta_N^*)^{1/2} N^{-1/2} \le 2^{1/2} c_3 N^{-1/2} K(\theta_N, \Theta_0)^{1/2}$ and hence

$$\begin{aligned} &K(\theta_N, \Theta_0) - K(\lambda^{-1}(x_N), \Theta_0) \\ &\qquad \le K(\theta_N, \theta_N^*) - K(\lambda^{-1}(x_N), \theta_N^*) \\ &\qquad = K(\theta_N, \lambda^{-1}(x_N)) + \{\lambda^{-1}(x_N) - \theta_N^*\}'\{\lambda(\theta_N) - x_N\} \\ &\qquad \le K(\theta_N, \Theta_0)\{c_4 N^{-1} K(\theta_N, \Theta_0)^{-1} + 2^{1/2} c_3 N^{-1/2} K(\theta_N, \Theta_0)^{-1/2}\} \end{aligned} \tag{3.8}$$

for all sufficiently large N. Combination of (3.7) and (3.8) completes the proof of the lemma. □

PROOF OF THEOREM 3.1. Let $\{\psi_{N;\alpha}\}$ be a family of tests for which $e^s_{\varphi^{\mathrm{LR}},\psi}$ exists, let $\{\alpha_N\}$ be a sequence of levels with $\lim_{N\to\infty} \alpha_N = 0 = \lim_{N\to\infty} N^{-1} \log \alpha_N$, and let $\{\theta_N\}$ be a sequence of alternatives with $\lim_{N\to\infty} H(\theta_N, \Theta_0) = 0$, $\lim_{N\to\infty} NH^2(\theta_N, \Theta_0) = \infty$ and

$$0 < \liminf_{N\to\infty} E_{\theta_N} \varphi^{\mathrm{LR}}_{N;\alpha_N} \le \limsup_{N\to\infty} E_{\theta_N} \varphi^{\mathrm{LR}}_{N;\alpha_N} < 1. \tag{3.9}$$

By Lemma 3.2 (a) it follows that $\lim_{N\to\infty} K(\theta_N, \Theta_0) = 0$ and $\lim_{N\to\infty} NK(\Theta_N, \Theta_0) = \infty$. By definition $e^s_{\varphi^{\mathrm{LR}},\psi} = \lim_{N\to\infty} m_{\varphi^{\mathrm{LR}},\psi}(N)/N$. Since power functions are continuous on int Θ, it is no restriction to assume that Θ_0 is closed. Moreover $K(\theta_N, .)$ is a continuous function on int Θ and hence there exists $\theta_{0N} \in \Theta_0$ such that $K(\theta_N, \theta_{0N}) = K(\theta_N, \Theta_0)$. The MP level-$\alpha_N$ test of θ_{0N} against θ_N based on j observations is given by

$$\varphi_j^{+,N}(\bar{x}_j) = \begin{cases} 1 & > \\ \gamma_{j,N} \quad \text{if} \quad (\theta_N - \theta_{0N})'\bar{x}_j & = c_{j,N}, \\ 0 & < \end{cases}$$

where $\gamma_{j,N}$ and $c_{j,N}$ satisfy $E_{\theta_{0N}} \varphi_j^{+,N}(\bar{X}_j) = \alpha_N$, $j = 1, 2, \cdots$, $N = 1, 2, \cdots$. Defining

$$m_N^+ = \inf\{m; E_{\theta_N} \varphi_m^{+,N} \ge E_{\theta N} \varphi^{\mathrm{LR}}_{N;\alpha_N}\}$$

it follows that

$$m_N^+ \le m_{\varphi^{\mathrm{LR}},\psi}(N), \quad N = 1, 2, \cdots,$$

cf. (1.6).

It suffices to show that $\liminf_{N\to\infty} m_N^+/N \geq 1$. The proof of this is along the same lines as the proof of Lemma 2.1: (Lemma 2.1 can not be directly applied, because φ^+ is a test of θ_{0N} against θ_N and not a test of H_0 against θ_N)

By Lemma 3.2 (b) it follows that

$$\lim_{N\to\infty} \mathbb{P}_{\theta_N}[1 - \varepsilon \leq \{K(\lambda^{-1}(\bar{X}_N), \Theta_0)\}^{1/2}\{K(\theta_N, \Theta_0)\}^{-1/2} \leq 1 + \varepsilon] = 1$$

for each $\varepsilon > 0$. Since $K(\lambda^{-1}(x), \Theta_0) \geq d$ implies $K(\lambda^{-1}(x), \theta_0) \geq d$, $\theta_0 \in \Theta_0$, it follows by Lemma 3.2 in Kallenberg (1981b) that

$$\liminf_{N\to\infty}\{NK(\theta_N, \Theta_0)\}^{-1}|\log \alpha_N| \geq 1. \tag{3.10}$$

Let $\{\zeta_j\}$ be any sequence of real numbers satisfying $\lim_{j\to\infty} \zeta_j < 1$ and let $\{N_j\}$ be any sequence of natural numbers with $\lim_{j\to\infty} N_j = \infty$ and such that $\zeta_1 N_1 < \zeta_2 N_2 < \cdots$ and $\zeta_j N_j \in \mathbb{N}, j = 1, 2, \cdots$. It suffices to show that

$$\lim_{j\to\infty} E_{\theta N_j}\varphi^{+,N_j}_{\zeta_j N_j;\alpha N_j} = 0. \tag{3.11}$$

Let

$$c_j^* = \max[0, \{c_{\zeta_j N_j, N_j} - (\theta_{N_j} - \theta_{0N_j})'\lambda(\theta_{N_j})\}\|\theta_{N_j} - \theta_{0N_j}\|^{-1}(\zeta_j N_j)^{1/2}]$$

and

$$A_j = \{x; (\theta_{N_j} - \theta_{0N_j})'\{x - \lambda(\theta_{N_j})\}\|\theta_{N_j} - \theta_{0N_j}\|^{-1}(\zeta_j N_j)^{1/2} \in (c_j^*, c_j^* + 1)\}.$$

Then

$$\alpha_{N_j} \geq \exp\{-\zeta_j N_j K(\theta_{N_j}, \theta_{0N_j})\} \int_{A_j} \exp[-\zeta_j N_j(\theta_{N_j} - \theta_{0N_j})'\{x - \lambda(\theta_{N_j})\}]\, d\bar{P}^{\zeta_j N_j}_{\theta N_j}(x)$$

$$\geq \exp\{-\zeta_j N_j K(\theta_{N_j}, \theta_{0N_j}) - (\zeta_j N_j)^{1/2}\|\theta_{N_j} - \theta_{0N_j}\|(c_j^* + 1)\}\, \bar{P}^{\zeta_j N_j}_{\theta N_j}(A_j).$$

By (3.10) it follows that $c_j^* \to \infty$, implying (3.11). □

4. Locally most powerful tests in curved exponential families. It is well-known that under weak conditions LMP tests are Pitman efficient. On the other hand LMP tests are far from optimal from a non-local point of view. In this section the performance of LMP tests is considered in the intermediate case.

Suppose that the density of X_i equals

$$\exp\{\gamma_\theta' x - \psi(\gamma_\theta)\}, \quad \theta \in \Theta, \tag{4.1}$$

with respect to a σ-finite measure μ on $\mathbb{R}^k$. Here Θ is an interval in $\mathbb{R}^1$, γ_θ a differentiable bijection from Θ onto $\gamma(\Theta) \subset \Gamma = \{\gamma \in \mathbb{R}^k; \int \exp(\gamma' x)\, d\mu(x) < \infty\}$ and $\psi(\gamma) = \log \int \exp(\gamma' x)\, d\mu(x)$. So the distribution of X_i belongs to a curved exponential family in the terminology of Efron (1975). This means that our one-parameter family is smooth in the sense that it can be embedded in an exponential family in a suitable way. We consider the testing problem $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$, where $\theta_0 \in \Theta$ is given. Indicating differentiation with respect to θ by a dot, the size-α LMP test of H_0 against H_1 based on N observations is given by

$$\varphi^L_{N;\alpha}(\bar{x}_N) = \begin{cases} 1 & > \\ \delta_{N;\alpha} & \dot{\gamma}'_{\theta_0}\bar{x}_n = d_{N;\alpha}, \\ 0 & < \end{cases} \tag{4.2}$$

where $\bar{x}_N = N^{-1}\sum_{i=1}^N x_i$ and the constants $d_{N;\alpha}$ and $\delta_{N;\alpha}$ satisfy $E_{\theta_0}\varphi^L_{N;\alpha}(\bar{X}_N) = \alpha$; cf. Efron (1975).

Now we have the following result.

THEOREM 4.1. *Assume that $\gamma_{\theta_0} \in \operatorname{int} \Gamma$, that $H(\theta_N, \theta_0) \to 0 \Leftrightarrow \theta_N \to \theta_0$, and that the Fisher information of X_i at θ_0 is positive. Then the* LMP *test of H_0 against H_1 is strongly i-efficient.*

Note that a positive Fisher information at θ_0 implies $\dot{\gamma}_{\theta_0} \neq 0$. The proof of Theorem 4.1 is similar to the proof of Theorem 3.1 and is therefore omitted.

Related results concerning the behavior of LMP tests in curved exponential families can be found in Kallenberg (1981a).

5. The *k*-sample case. The concept of *i*-efficiency has been defined so far for testing problems in the one-sample case. Here we will extend the definition to the *k*-sample case.

Let $S = (X_\infty^{(1)}; X_\infty^{(2)}; \cdots; X_\infty^{(k)})$ be k independent sequences $X_\infty^{(j)} = (X_1^{(j)}, X_2^{(j)}, \cdots)$ of independent random elements $X_m^{(j)}$ taking values in some measurable space $(\mathscr{X}, \mathscr{B})$ according to a probability measure $P_{\theta^{(j)}}$ $\theta^{(j)} \in \Theta^{(j)}$, $(m = 1, 2, \cdots; j = 1, 2, \cdots, k)$. The distribution of S will be denoted by $\mathbb{P}_\theta$ where $\theta = (\theta^{(1)}, \cdots, \theta^{(k)}) \in \Theta = \Theta^{(1)} \times \Theta^{(2)} \times \cdots \times \Theta^{(k)}$ obtains. For each $N = k, k+1, \cdots$ let $N^{(1)}(N), \cdots, N^{(k)}(N)$ be positive integers such that $N = \sum_{j=1}^k N^{(j)}(N)$ and

$$\lim_{N\to\infty} \lambda_N^{(j)} = \lim_{N\to\infty} N^{-1} N^{(j)}(N) = p^{(j)} \in (0, 1), \quad j = 1, \cdots, k. \tag{5.1}$$

With N observations, the random elements $X_1^{(1)}, \cdots, X_{N^{(1)}}^{(1)}; \cdots; X_1^{(k)}, \cdots; X_{N^{(k)}}^{(k)}$ are available. Consider the testing problem $H_0: \theta \in \Theta_0$ against $H_1: \Theta_1$, where Θ_0 and Θ_1 are given subsets of Θ, satisfying $\Theta_0 \cap \Theta_1 = \emptyset$. Let $\{\theta_N\} = \{(\theta_N^{(1)}, \cdots, \theta_N^{(k)})\}$ be a sequence of alternatives. Replacing (1.3) by

$$\lim_{N\to\infty} \inf_{\theta_0\in\Theta_0} \max_{1\le j\le k} H(\theta_N^{(j)}, \theta_0^{(j)}) = 0, \quad \lim_{N\to\infty} \inf_{\theta_0\in\Theta_0} \sum_{j=1}^k N^{(j)} H^2(\theta_N^{(j)}, \theta_0^{(j)}) = \infty, \tag{5.2}$$

the concept of *i*-efficiency is defined (with some obvious modifications) as in Section 1. The following modified version of Lemma 2.1 will be used later.

LEMMA 5.1. *Let $T_N^{(i)} = T_N^{(i)}(X_1^{(1)}, \cdots, X_{N^{(k)}}^{(k)})$, $i = 1, 2$, $N \in \mathbb{N}$, be test statistics rejecting the hypothesis for large values of $T_N^{(i)}$. Suppose that there exist positive functions $b^{(i)}(p_1, \cdots, p_k; \theta)$, $i = 1, 2$, such that for all sequences $\{\theta_N\}$ satisfying* (5.2)

(i) $\lim_{N\to\infty} \mathbb{P}_{\theta_N}\{1 - \varepsilon \le N^{-1/2} T_N^{(i)} / b^{(i)}(\lambda_N^{(1)}, \cdots, \lambda_N^{(k)}; \theta_N) \le 1 + \varepsilon\} = 1$ *for each* $\varepsilon > 0$,

(ii) $\lim_{N\to\infty}\{-(N t_N^2)^{-1} \log \sup_{\theta_0\in\Theta_0} \mathbb{P}_{\theta_0}(T_N^{(i)} \ge N^{1/2} t_N)\} = 1$ *if* $N^{1/2} t_N \to \infty$ *and* $t_N = o(N^{-1/3})$ *as* $N \to \infty$,

(iii) *if* $n = n(N) \to \infty$ *as* $N \to \infty$, *then* $\lim_{N\to\infty} \left\{ \dfrac{b^{(1)}(\lambda_N^{(1)}, \cdots, \lambda_N^{(k)}; \theta_N)}{b^{(2)}(\lambda_n^{(1)}, \cdots, \lambda_n^{(k)}; \theta_N)} \right\}^2 = e$ *exists*, $0 \le e \le \infty$.

Then the i-efficiency of $T_N^{(1)}$ w.r.t *$T_N^{(2)}$ exists and is given by*

$$e_{T^{(1)}, T^{(2)}} = e. \tag{5.3}$$

If in (ii) *$t_N = o(N^{-1/3})$ is replaced by $t_N = O(N^{-1/2}(\log N)^{1/2})$ or $t_N = o(1)$, respectively, then the weak or strong i-efficiency, respectively, of $T_N^{(1)}$* w.r.t. *$T_N^{(2)}$ exists and is equal to e.*

The proof of Lemma 5.1 is analogous to the proof of Lemma 2.1. Note that in (2.8) we now use

$$\lim_{j\to\infty} \left\{ \frac{b^{(1)}(\lambda_{N_j}^{(1)}, \cdots, \lambda_{N_j}^{(k)}; \theta_{N_j})}{b^{(2)}(\lambda_n^{(1)}, \cdots, \lambda_n^{(k)}; \theta_{N_j})} \right\}^2 = e,$$

according to (iii).

Similarly, Corollary 2.2 can be adapted to the *k*-sample case. As an application the intermediate efficiency of simple linear rank statistics in the two-sample case will be discussed. So let $k = 2$, $\Theta = \{(F, G); F, G \in \mathscr{F}\}$, where $\mathscr{F}$ denotes the set of continuous distribution functions on $\mathbb{R}^1$. The null hypothesis is given by $H_0: \theta \in \{\theta \in \Theta; \theta^{(1)} = \theta^{(2)}\} = \{(F, F); F \in \mathscr{F}\}$ and the alternative hypothesis by $H_1 = H_1(F) = \{(F, F_\Delta); \Delta \in (0, \infty)\}$, where $F \in \mathscr{F}$ satisfies $H(F_\Delta, F) \to 0 \Leftrightarrow \Delta \to 0$ and $NH^2(F_{\Delta_N}, F) \to \infty \Leftrightarrow N^{1/2}\Delta_N \to \infty$. We consider simple linear rank statistics of the form

$$S_N = S_n(\varphi) = \sum_{i=1}^{N^{(1)}} a_N(R_{iN}), \tag{5.3}$$

where R_{iN} is the rank of $X_i^{(1)}$ among $X_1^{(1)}, \cdots, X_{N^{(1)}}^{(1)}, X_1^{(2)}, \cdots, X_{N^{(2)}}^{(2)}$ and where the scores $a_N(1), \cdots, a_N(N)$ are given in either of the following ways:

$$a_N(i) = \varphi(i/(N+1)), \quad 1 \le i \le N, \tag{5.4}$$

$$a_N(i) = E\varphi(U_N^{(i)}), \quad 1 \le i \le N. \tag{5.5}$$

Here $U_N^{(i)}$ denotes the ith order statistic in a sample of size N from the uniform distribution on (0, 1) and

(5.6) $\varphi(t)$ is a non-constant function on (0, 1) with bounded, continuous first derivative φ' on (0, 1).

Define $\sigma^2(\varphi) = \int \{\varphi(t) - \int \varphi(u)\, du\}^2\, dt$. Let $\{\Delta_N\}$ be a sequence of positive, real numbers with $\lim_{N\to\infty}\Delta_N = 0$, $\lim_{N\to\infty} N^{1/2}\Delta_N = \infty$, and let $H_N(x) = \lambda_N^{(1)} F(x) + \lambda_N^{(2)} F_{\Delta_N}(x)$. The asymptotic normality of

$$N^{1/2}\left\{N^{-1}S_N - \lambda_N^{(1)} \int \varphi(H_N(x))\, dF(x)\right\}\{\sigma^2(\varphi)p^{(1)}p^{(2)}\}^{-1/2}$$

is well-known.

A Cramér type large deviation theorem for simple linear rank statistics under the null hypothesis (cf. Lemma 5.1 condition (ii)) can be found in Kallenberg (1982). Suppose that the derivative $f_\Delta = \partial F_\Delta/\partial\Delta$ exists in $(0, \bar{\Delta})$ for some $\bar{\Delta} > 0$ and that there exists a function f such that $\lim_{\Delta\to 0} f_\Delta = f$, F-a.e. To compute the i-efficiency of $S_{N,1} = S_N(\varphi_1)$ w.r.t. $S_{N,2} = S_N(\varphi_2)$ with φ_1, φ_2 satisfying (5.6), it is further supposed that there are F-integrable functions h_i such that $|\varphi_i'(H_N)f_{\Delta_N}| \le h_i$, F-a.e., $i = 1, 2$, $N = 2, 3, \cdots$ and $\int \varphi_i'(F) f\, dF > 0$, $i = 1, 2$. Then the i-efficiency of $S_{N,1}$ w.r.t. $S_{N,2}$ exists and is given by

$$\left\{\frac{\int \varphi_1'(F) f\, dF}{\int \varphi_2'(F) f\, dF}\right\}^2 \frac{\sigma^2(\varphi_2)}{\sigma^2(\varphi_1)}. \tag{5.7}$$

To prove this result, Lemma 5.1 is applied with

$$T_N^{(i)} = 2^{1/2}N^{1/2}\{\sigma^2(\varphi_i)p^{(1)}p^{(2)}\}^{-1/2}\left\{N^{-1}S_{N,i} - \lambda_N^{(1)} \int \varphi_i(F)\, dF\right\},$$

and

$$b^{(i)}(\lambda_N^{(1)}, \lambda_N^{(2)}; \Delta_N) = 2^{1/2}\lambda_N^{(1)}\{\sigma^2(\varphi_i)p^{(1)}p^{(2)}\}^{-1/2}\left\{\int \varphi_i(H_N(x))\, dF(x) - \int \varphi_i(F)\, dF\right\}, \quad i = 1, 2.$$

So, again we have that the i-efficiency coincides with the Pitman efficiency; cf. Chernoff and Savage (1958). This result is related to Corollary 5 of Kremer (1979). In Kremer's, and Wieand's (1976), approach, first the alternative is fixed, then the limit for $N \to \infty$ is taken and afterwards the alternative is sent to the hypothesis. This suggests that with small levels and local alternatives, (5.7) is a good approximation. Here the case of small levels and local alternatives is directly attacked, proving Kremer's suggestion.

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