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ON THE MEASURABILITY AND CONSISTENCY OF MAXIMUM LIKELIHOOD ESTIMATES FOR UNIMODAL DENSITIES

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This paper is concerned with maximum likelihood estimates for a large class of families of unimodal densities. The existence of measurable maximum likelihood estimates and the consistency of asymptotic maximum likelihood estimates are proved. By counterexamples it is shown that the conditions which are sufficient for consistency cannot be removed without compensation.

1. Introduction. For unimodal densities maximum likelihood estimates (m.l. estimates) have been investigated under the assumption that the mode is known (Grenander [3], Prakasa Rao [6], Robertson [7]) or that the length of the modal intervals is uniformly bounded from below (Wegman [8]).

This paper is concerned with m.l. estimates for a large class of families of unimodal densities. The existence of measurable maximum likelihood estimates and the consistency of asymptotic maximum likelihood estimates (see (1.2)) are proved.

In Section 2 we introduce a metric d_u on a special family of unimodal densities using the Lévy metric for monotone functions. In (2.6), (2.10), and (2.11) the topological structure of this metric is studied. In (2.12) we obtain that the convergence with respect to the metric d_u is equivalent to the following two conditions: Convergence at each continuity point of the limit function and convergence of the modes to the mode of the limit function. In (2.17) the convergence with respect to d_u is compared with the convergence in the mean. (2.17) also implies that the weak topology and the topology of the supremum-metric coincide on the family of probability measures with unimodal density.

The topological results of Section 2 enable us to provide a method of how to derive the existence of measurable m.l. estimates. The general method is discussed in detail (see (3.6) and (3.7)). The special cases of unimodal densities with the mode known and of unimodal densities which are uniformly bounded are used to apply these measurability statements for the m.l. estimate (see (3.8) and (3.9)).

Section 4 gives conditions which imply the consistency of asymptotic maximum likelihood estimates (a.m.l. estimates). It is shown that none of the conditions

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sufficient for consistency can be omitted without compensation. The proofs in Sections 3 and 4 are closely related to the proofs given by Landers [4] and Pfanzagl [5] for families of probability measures equipped with topologies which have a countable base, and which are either compact metric or locally compact, and admit "upper semicontinuous" densities.

We shall use the following notation. $\mathbb{N}(\mathbb{R})$ denotes the set of natural (real) numbers. Let $(X, \mathscr{A}, P)$ be a probability space. The elements of the countable Cartesian product $X^{\mathbb{N}}$ of X will be denoted by $\mathbf{x}$ or $(x_i)_{i\in\mathbb{N}}$. By $\mathscr{A}^{\mathbb{N}}$ we denote the countable product of σ-algebras $\mathscr{A}$, and by $P^{\mathbb{N}}\,|\,\mathscr{A}^{\mathbb{N}}$ the countable independent product of identical components $P\,|\,\mathscr{A}$. $\mathscr{A}^n$ will denote those sets in $\mathscr{A}^{\mathbb{N}}$ which are cylindrical with base in X^n. Given a sequence of maps $g_n: X^{\mathbb{N}} \to T$ (where T is some set) we shall always assume that $g_n(\mathbf{x})$ is depending on $x_1, \cdots, x_n$ only. $\bar{A}$ denotes the complement of a set A in X.

Let λ be the Lebesgue measure on $\mathbb{R}$. Let $d_1(f, g) \equiv \int |f - g|\, d\lambda$ for each f and $g \in \mathscr{L}_1$ where $\mathscr{L}_1$ is the family of all functions which are finitely integrable with respect to λ. Denote by $P_f(F_f)$ the appropriate measure (distribution function) with density f. Let $dP/d\lambda$ be the set of densities of P with respect to λ. $|\alpha|$ denotes the absolute value of a real number and given a function $h: \mathbb{R} \to \mathbb{R}$ let $||h|| \equiv \sup\{|h(x)|\,|\,x \in \mathbb{R}\}$.

For a pseudometric space (X, d) let $U_d(f, \varepsilon) \equiv \{g \in X\,|\,d(f, g) < \varepsilon\}$ where $f \in X$. Denote by $\mathscr{B}(d)$ the σ-algebra which is generated by the topology of d.

A function $f: \mathbb{R} \to [0, \infty]$ is unimodal iff f is non-decreasing on $]-\infty, M[$ and non-increasing on $]M, \infty[$ for some $M \in \mathbb{R}$. Then M is called the mode of f, and the set of all modes of f is called the modal interval of f. Let $\mathscr{H}$ be a family of unimodal densities equipped with the pseudometric d_1. We shall need the following definitions.

(1.1) $\varphi_n: \mathbb{R}^{\mathbb{N}} \to \mathscr{H}$, $n \in \mathbb{N}$, is called a sequence of maximum likelihood estimates (m.l. estimates) for $\mathscr{H}$ iff

$$\prod_{i=1}^n \varphi_n(\mathbf{x})(x_i) \geqq \sup\{\prod_{i=1}^n f(x_i)\,|\,f \in \mathscr{H}\} \qquad \text{for all} \quad \mathbf{x} \in \mathbb{R}^{\mathbb{N}} \quad \text{and} \quad n \in \mathbb{N}\,.$$

(1.2) $\varphi_n: \mathbb{R}^{\mathbb{N}} \to \mathscr{H}$, $n \in \mathbb{N}$, is called a sequence of asymptotic maximum likelihood estimates (a.m.l. estimates) for $\mathscr{H}$ iff

$$\lim_{n\in\mathbb{N}} \left((\prod_{i=1}^n \varphi_n(\mathbf{x})(x_i))^{-1/n} - \inf\{(\prod_{i=1}^n f(x_i))^{-1/n}\,|\,f \in \mathscr{H}\}\right) = 0 \qquad \text{for all} \quad \mathbf{x} \in \mathbb{R}^{\mathbb{N}}\,.$$

Notice that every sequence of m.l. estimates is a sequence of a.m.l. estimates.

(1.3) $\varphi_n: \mathbb{R}^{\mathbb{N}} \to \mathscr{H}$, $n \in \mathbb{N}$, is said to be strongly consistent for $f \in \mathscr{H}$ iff $\lim_{n\in\mathbb{N}} d_1(\varphi_n(\mathbf{x}), f) = 0$ for $P_f^{\mathbb{N}}$ — a.a. $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$.

We mention the following trivial statement for the sake of reference.

(1.4) Let (X, d) be a metric space. A function $f: X \to [-\infty, \infty]$ is called lower (upper) semicontinuous iff one of the following two (equivalent) assertions is valid.

(a) $\{x \in X \mid f(x) > r\}$ ($\{x \in X \mid f(x) < r\}$) is an open subset of (X, d) for each $r \in \mathbb{R}$.

(b) $\liminf_{n \in \mathrm{N}} f(x_n) \geqq f(x)$ ($\limsup_{n \in \mathrm{N}} f(x_n) \leqq f(x)$) for each $x \in X$ and each sequence $x_n \in X$, $n \in \mathbb{N}$, with $\lim_{n \in \mathrm{N}} d(x_n, x) = 0$.

We remark that a unimodal function f is upper semicontinuous iff f is right (left) continuous where it is non-decreasing (non-increasing). Each unimodal function is continuous λ-almost everywhere.

2. The topological background. First we assume that the mode is zero. We define

(2.1) $$\mathscr{F}^0 \equiv \{f: \mathbb{R} \to [0, \infty] \mid f \text{ unimodal, upper semicontinuous, } \textstyle\int f\, d\lambda \leqq 1, f(0) = \infty\}$$

and

(2.2) $$\mathscr{F}_1^0 \equiv \{f \in \mathscr{F}^0 \mid \textstyle\int f\, d\lambda = 1\}\,.$$

Now we introduce a metric d_u on $\mathscr{F}^0$ using the Lévy metric L for monotone functions. For each $f \in \mathscr{F}^0$ define

$$\begin{aligned} f^*(x) &\equiv f(x) && x < 0 \\ &\equiv \infty && x \geqq 0 \end{aligned}$$

and

$$\begin{aligned} f^{**}(x) &\equiv f(-x) && x < 0 \\ &\equiv \infty && x \geqq 0\,. \end{aligned}$$

Let $\mathscr{L} \equiv \{f^* \mid f \in \mathscr{F}^0\}$ and let L be the Lévy metric for right-continuous, non-decreasing functions. Thus

$$L(f, g) \equiv \inf\{\varepsilon > 0 \mid f(x - \varepsilon) - \varepsilon \leqq g(x) \leqq f(x + \varepsilon) + \varepsilon \ \text{ for all } \ x \in \mathbb{R}\}$$

where f and $g \in \mathscr{L}$.

In essentially the same way as in [2] (Theorem 1, (III) $\to$ (I) on page 33 and Theorem 3 on page 38) we obtain the following two results ((2.3) and (2.4)):

(2.3) Let h_n and $h \in \mathscr{L}$ where $n \in \mathbb{N}$. Then $\lim_{n \in \mathrm{N}} L(h_n, h) = 0$ is equivalent to $\lim_{n \in \mathrm{N}} h_n(x) = h(x)$ at all continuity points x of h.

(2.4) $(\mathscr{L}, L)$ is a compact metric space.

Now define for each f and $g \in \mathscr{F}^0$:

(2.5) $$d_u(f, g) \equiv \max\{L(f^*, g^*), L(f^{**}, g^{**})\}\,.$$

It is easy to see that for each f and $g \in \mathscr{F}^0$:

(2.5′) $$d_u(f, g) = \inf\{\varepsilon > 0 \mid f(x + \operatorname{sgn}(x)\varepsilon) - \varepsilon \leqq g(x) \ \text{ and } \ g(x + \operatorname{sgn}(x)\varepsilon) - \varepsilon \leqq f(x) \ \text{ for all } \ x \neq 0\}\,.$$

(2.6) THEOREM. *$(\mathscr{F}^0, d_u)$ is a compact metric space.*

PROOF. Because of (2.4), $\mathscr{L} \times \mathscr{L}$ equipped with the maximum-metric is a compact metric space. Obviously, the map $\tau : \mathscr{F}^0 \to \mathscr{L} \times \mathscr{L}$ (where $\tau(f) \equiv (f^*, f^{**})$) is injective. In view of (2.5) the proof will be concluded by showing that $\tau(\mathscr{F}^0)$ is a closed subset of $\mathscr{L} \times \mathscr{L}$. It suffices to show that $\tau(\mathscr{F}^0)$ contains all its limit points. Let $f_n \in \mathscr{F}^0$, $n \in \mathbb{N}$, and $(h, g) \in \mathscr{L} \times \mathscr{L}$ such that $\lim_{n\in\mathbb{N}} \max\{L(f_n^*, h), L(f_n^{**}, g)\} = 0$. Let $f(x) \equiv h(x)I_{]-\infty,0]}(x) + g(-x)I_{[0,\infty[}(x)$. We derive from (2.3) that $\lim_{n\in\mathbb{N}} f_n(x) = f(x)$ λ-a.e. According to Fatou's lemma, this implies $\int f\,d\lambda \leqq 1$. Hence $f \in \mathscr{F}^0$ and $(h, g) = \tau(f)$ is an element of $\tau(\mathscr{F}^0)$.

For each unimodal function g denote by M_g the center of the modal interval I_g of g. Denote by L_g (R_g) the left (right) corner of the modal interval of g.

Furthermore, we define

(2.7) $$\mathscr{F} \equiv \{f: \mathbb{R} \to [0, \infty] \mid f \text{ unimodal, upper semicontinuous, } \int f\,d\lambda \leqq 1, f(M_f) = \infty\}$$

and

(2.8) $$\mathscr{F}_1 \equiv \{f \in \mathscr{F} \mid \int f\,d\lambda = 1\}\,.$$

Now we define a metric d_u on $\mathscr{F}$ which is an extension of $d_u \mid \mathscr{F}^0$ to $\mathscr{F}$. Hence the use of the same symbol is justified. Define

(2.9) $$d_u(f, g) \equiv \max\{(d_u \mid \mathscr{F}^0)(f(x + M_f), g(x + M_g)), |M_f - M_g|\}$$

where f and $g \in \mathscr{F}$.

(2.10) THEOREM. *$(\mathscr{F}, d_u)$ is a locally compact, σ-compact metric space (with countable base).*

PROOF. (2.6) implies that the metric space $\mathscr{F}^0 \times \mathbb{R}$ (equipped with the maximum-metric) has the desired properties. And $(\mathscr{F}, d_u)$ can be identified with $\mathscr{F}^0 \times \mathbb{R}$ using the map $\vartheta : \mathscr{F} \to \mathscr{F}^0 \times \mathbb{R}$ where $\vartheta(f) \equiv (f(x + M_f), M_f)$ for each $f \in \mathscr{F}$.

(2.11) THEOREM. *A subset $\mathscr{K}$ of $\mathscr{F}$ is relatively compact* iff *$\{M_f \mid f \in \mathscr{K}\}$ is bounded.*

PROOF. Let $\vartheta : \mathscr{F} \to \mathscr{F}^0 \times \mathbb{R}$ be as in the proof of (2.10). Then $\mathscr{K}$ is relatively compact iff $\vartheta(\mathscr{K})$ has this property in $\mathscr{F}^0 \times \mathbb{R}$. This is equivalent to $\{M_f \mid f \in \mathscr{K}\}$ bounded because $\mathscr{F}^0$ is a compact space.

Now we characterize convergence with respect to the metric d_u by pointwise convergence.

(2.12) THEOREM. *Let f_n and $f \in \mathscr{F}$, $n \in \mathbb{N}$. Then the following two assertions are equivalent:*

(i) $\lim_{n\in\mathbb{N}} d_u(f_n, f) = 0$.

(ii) (a) $\lim_{n\in\mathbb{N}} |M_{f_n} - M_f| = 0$ *and*

(b) $\lim_{n\in\mathbb{N}} f_n(x) = f(x)$ *at each continuity point $x \neq M_f$ of f.*

PROOF. (2.3) together with the definitions (2.5) and (2.9) imply that $\lim_{n\in\mathbb{N}} d_u(f_n, f) = 0$ is equivalent to (a) and (b′) $\lim_{n\in\mathbb{N}} f_n(x + M_{f_n}) = f(x + M_f)$ for all continuity points $(x + M_f)$ of f whenever $x \neq 0$. By the monotony of f and f_n we easily derive the equivalence between (a), (b′) and (a), (b).

Without proof we mention the following corollary.

(2.13) COROLLARY. *Let h_n and $h \in \mathscr{F}$, $n \in \mathbb{N}$, be such that* $\lim_{n\in\mathbb{N}} d_u(h_n, h) = 0$. *Then*

$$h(x^-) \leqq \liminf_{n\in\mathbb{N}} h_n(x) \leqq \limsup_{n\in\mathbb{N}} h_n(x) \leqq h(x) \qquad (x < M_f)$$

and

$$h(x) \geqq \limsup_{n\in\mathbb{N}} h_n(x) \geqq \liminf_{n\in\mathbb{N}} h_n(x) \geqq h(x^+) \qquad (x < M_f)\,.$$

Hereafter we shall always assume in this section that each unimodal density is right or left continuous at the mode except for the case that we explicitly state that it is an element of $\mathscr{F}$.

(2.14) PROPOSITION. *Let f_n and $f \in \mathscr{L}_1$, $n \in \mathbb{N}$, be unimodal functions such that $f_n \geqq 0$, $n \in \mathbb{N}$, $f \geqq 0$, and* $\lim_{n\in\mathbb{N}} \|F_{f_n} - F_f\| = 0$. *Then* $\liminf_{n\in\mathbb{N}} L_{f_n} \geqq L_f$ *and* $\limsup_{n\in\mathbb{N}} R_{f_n} \leqq R_f$.

PROOF. Assume, on the contrary, that $\liminf_{n\in\mathbb{N}} L_{f_n} < L_f$. Then there exists $\varepsilon > 0$ and a subsequence $\mathbb{N}_0 \subset \mathbb{N}$ such that $L_{f_n} \leqq L_f - \varepsilon$ for all $n \in \mathbb{N}_0$. We can find some $\delta > 0$ and a number $L_f - \varepsilon < a < L_f$ and an interval J with $\lambda(J) > 0$ containing L_f such that $f(a) + 2\delta \leqq \inf\{f(x) \mid x \in J\}$. Let first $f_n(a) \geqq f(a) + \delta$, $n \in \mathbb{N}_0$, then $f_n - f \geqq \delta$ on $[L_f - \varepsilon, a]$. Otherwise, if $f_n(a) < f(a) + \delta$, $n \in \mathbb{N}_0$, then $f - f_n \geqq \delta$ on J. These statements imply that $\limsup_{n\in\mathbb{N}} \|F_{f_n} - F_f\| \geqq \delta \min\{\lambda(J), a - L_f + \varepsilon\}/2 > 0$ in contradiction to the assumption. Hence we have $\liminf_{n\in\mathbb{N}} L_{f_n} \geqq L_f$. In a similar way we can show that $\limsup_{n\in\mathbb{N}} R_{f_n} \leqq R_f$.

We remark that (2.14) implies $\lim_{n\in\mathbb{N}} M_{f_n} = M_f$ if the mode of f is uniquely determined.

(2.15) PROPOSITION. *Let f_n, $n \in \mathbb{N}$, and f be unimodal functions such that $f_n \geqq 0$, $n \in \mathbb{N}$, $f \geqq 0$, and $\int f_n \, d\lambda \leqq 1$, $n \in \mathbb{N}$, $\int f \, d\lambda \leqq 1$. Suppose that* $\lim_{n\in\mathbb{N}} \|F_{f_n} - F_f\| = 0$. *Then there are versions h_n and g_n of f_n and f respectively which are elements of $\mathscr{F}$ such that* $\lim_{n\in\mathbb{N}} d_u(h_n, g_n) = 0$. *And this is valid for all versions h_n and g_n with* $\lim_{n\in\mathbb{N}} |M_{h_n} - M_{g_n}| = 0$.

PROOF. It is easy to check that the assertion is equivalent to the following:

(2.16) There exist $\eta_n > 0$, $n \in \mathbb{N}$, and $(a_n, b_n) \in I_{f_n} \times I_f$, $n \in \mathbb{N}$, such that $\lim_{n\in\mathbb{N}} \eta_n = 0$ and $\lim_{n\in\mathbb{N}} \gamma_n = 0$ (where $\gamma_n \equiv b_n - a_n$) and the following conditions are fulfilled. For each $n \in \mathbb{N}$

$$f(x + \gamma_n - \eta_n)\{\begin{smallmatrix}-\\+\end{smallmatrix}\}\eta_n\{\begin{smallmatrix}\leqq\\\geqq\end{smallmatrix}\}f_n(x)\{\begin{smallmatrix}\leqq\\\geqq\end{smallmatrix}\}f(x + \gamma_n + \eta_n)\{\begin{smallmatrix}+\\-\end{smallmatrix}\}\eta_n \qquad \text{for each} \quad x\{\begin{smallmatrix}<\\>\end{smallmatrix}\}a_n\{\begin{smallmatrix}-\\+\end{smallmatrix}\}\eta_n\,.$$

We shall now verify (2.16). As a consequence of (2.14) we know that there exist $(a_n, b_n) \in I_{f_n} \times I_f$, $n \in \mathbb{N}$, such that $\lim_{n\in\mathbb{N}} \gamma_n = 0$ (where $\gamma_n \equiv b_n - a_n$). Since $\lim_{n\in\mathbb{N}} f(x + \gamma_n) = f(x)$ λ-a.e. Scheffé's lemma implies $\lim_{n\in\mathbb{N}} \|F_f - F_{f(x+\gamma_n)}\| = 0$. Furthermore, $\|F_{f_n} - F_{f(x+\gamma_n)}\| \leqq \|F_{f_n} - F_f\| + \|F_f - F_{f(x+\gamma_n)}\|$.

Thus, there exists $\alpha_n > 0$, $n \in \mathbb{N}$, such that $\lim_{n \in \mathbb{N}} \alpha_n = \infty$ and $\alpha_n \|F_{f_n} - F_{f(x+\gamma_n)}\| \leqq 1$ for each $n \in \mathbb{N}$. Let us take $\eta_n \equiv \alpha_n^{-\frac{1}{2}}$. We prove that $f(x + \gamma_n - \eta_n) - \eta_n \leqq f_n(x)$ for all $x < a_n - \eta_n$. The other inequalities of (2.16) follow similarly. Let $x < a_n - \eta_n$ be fixed. According to the definition of a_n and b_n,

$$f(x + \gamma_n - \eta_n) - f_n(x) \leqq f(y + \gamma_n) - f_n(y) \qquad \text{for all} \quad y \in [x - \eta_n, x] .$$

Hence

$$\begin{aligned} f(x + \gamma_n - \eta_n) - f_n(x) &\leqq \eta_n^{-1} \textstyle\int_{x-\eta_n}^{x} (f(y + \gamma_n) - f_n(y))\, dy \\ &\leqq \alpha_n^{\frac{1}{2}} \|F_{f_n} - F_{f(y+\gamma_n)}\| \leqq \eta_n . \end{aligned}$$

(2.17) THEOREM. *Let f_n, $n \in \mathbb{N}$, and f be unimodal probability densities. Then the following three assertions are equivalent*:

(i) *There are versions h_n and g_n of f_n and f respectively which are elements of $\mathscr{F}_1$ for which* $\lim_{n \in \mathbb{N}} d_u(h_n, g_n) = 0$.
(ii) $\lim_{n \in \mathbb{N}} d_1(f_n, f) = 0$.
(iii) $\lim_{n \in \mathbb{N}} \|F_{f_n} - F_f\| = 0$.

PROOF. (i) $\rightarrow$ (ii). It suffices to show that each subsequence $(f_n)_{n \in \mathbb{N}_0}$ of $(f_n)_{n \in \mathbb{N}}$ contains a subsequence which converges to f with respect to d_1. Since $J \equiv \{M_g \mid g$ version of f and $g \in \mathscr{F}_1\}$ is a compact interval there exists a subsequence $(M_{g_n})_{n \in \mathbb{N}_1}$ of $(M_{g_n})_{n \in \mathbb{N}_0}$ which converges to some $a \in J$. Denote by g_a the version of f which is an element of $\mathscr{F}_1$ and has the mode at a. By (2.12) we obtain that $\lim_{n \in \mathbb{N}_1} d_u(g_n, g_a) = 0$ and this implies that $\lim_{n \in \mathbb{N}_1} d_u(h_n, g_a) = 0$. Hence by (2.12) and Scheffé's lemma: $\lim_{n \in \mathbb{N}_1} d_1(f_n, f) = 0$.
(ii) $\rightarrow$ (iii) is well known.
(iii) $\rightarrow$ (i). See (2.15).
As a consequence of (2.15) and (2.17) we obtain

(2.18) COROLLARY. *Let h_n and $h \in \mathscr{F}_1$, $n \in \mathbb{N}$, have the same mode. Then the following three assertions are equivalent*:

(i) $\lim_{n \in \mathbb{N}} d_u(h_n, h) = 0$.
(ii) $\lim_{n \in \mathbb{N}} d_1(h_n, h) = 0$.
(iii) $\lim_{n \in \mathbb{N}} \|F_{h_n} - F_h\| = 0$.

3. Measurability. First we shall give some auxiliary lemmas which will also be used for the consistency arguments in Section 4. Let $\mathscr{F}$ be as in (2.7).

(3.1) PROPOSITION. $A(x, r) \equiv \{g \in \mathscr{F} \mid g(x) \geqq r\}$ *is a compact subset of* $(\mathscr{F}, d_u)$ *for each $x \in \mathbb{R}$ and $r > 0$. And the function $g \rightarrow g(x)$ is upper semicontinuous on $\mathscr{F}$ for each $x \in \mathbb{R}$.*

PROOF. Let $x \in \mathbb{R}$ and $r > 0$ be fixed. According to (2.11), we have to show that (a) $A(x, r)$ is a closed subset of $\mathscr{F}$ and (b) $\{M_g \mid g \in A(x, r)\}$ is bounded. We show that $A(x, r)$ contains all its limit points. Let $g_n \in A(x, r)$, $n \in \mathbb{N}$, and $g \in \mathscr{F}$ be such that $\lim_{n \in \mathbb{N}} d_u(g_n, g) = 0$. We show that $g(x) \geqq r$ and hence $g \in A(x, r)$.

First suppose that $x = M_g$. Then $g(x) = \infty \geqq r$. Otherwise $x \neq M_g$. Then Corollary (2.13) implies $g(x) \geqq r$. Next we verify (b). Let $g \in A(x, r)$ then $g(x) \geqq r > 0$. Because of $\int g \, d\lambda \leqq 1$ we have $|x - M_g| \leqq r^{-1}$. Thus (b) is proved.

The second part of the assertion follows from (1.4).

(3.2) PROPOSITION. *The function* $x \to \sup_{g \in \mathscr{K}} g(x)$ *is* $\mathscr{B}$*-measurable for all compact subsets* $\mathscr{K}$ *of* $\mathscr{F}$.

PROOF. By (3.1) and [5], Corollary (3.6), it suffices to show that $x \to \sup_{g \in \mathscr{U}} g(x)$ is $\mathscr{B}$-measurable for each open set $\mathscr{U} \subset \mathscr{F}$. We even show that $\{x \in \mathbb{R} \mid \sup_{g \in \mathscr{U}} g(x) > r\}$ is open for each $r > 0$. Let $r > 0$ be fixed and $x \in \mathbb{R}$ such that $\sup_{g \in \mathscr{U}} g(x) > r$. Then there exists $f \in \mathscr{U}$ such that $f(x) > r$ and $x \neq M_f$. Choose a $\delta > 0$ such that $U_{d_u}(f, \delta) \subset \mathscr{U}$. Let $0 < \gamma < \delta$ be such that $M_f + \gamma < x$ $(M_f - \gamma > x)$. Then we have $f(y + \gamma/2) \in U_{d_u}(f, \delta)$ $(f(y - \gamma/2) \in U_{d_u}(f, \delta))$ and $f(y + \gamma/2) > \gamma$ $(f(y - \gamma/2) > r)$ on $]x - \gamma/2, x + \gamma/2[$. Thus $\sup_{g \in \mathscr{U}} g(y) > r$ for each $y \in \,]x - \gamma/2, x + \gamma/2[$ and the assertion follows.

Let $\mathscr{H}$ be a closed subset $\mathscr{F}$ and let $\psi : \mathbb{R} \to \mathbb{R}$ be $\mathscr{B}$-measurable. Suppose that $g(x) \leqq \psi(x)$ for all $x \neq M_g$ and $g \in \mathscr{H}$. Define

$$\begin{aligned} f_g(x) &\equiv g(x) & x &\neq M_g \\ &\equiv \psi(x) & x &= M_g \end{aligned}$$

and $\sup f_{\mathscr{K}}(x) \equiv \sup_{g \in \mathscr{K}} f_g(x)$ for $\mathscr{K} \subset \mathscr{H}$. From (3.1) and (3.2) we derive two corollaries for $(f_g)_{g \in \mathscr{H}}$ and $\sup f_{\mathscr{K}}$. The proofs are trivial.

(3.3) COROLLARY. $\{g \in \mathscr{H} \mid f_g(x) \geqq r\}$ *is a compact subset of* $\mathscr{F}$ *for all* $x \in \mathbb{R}$ *and* $r > 0$.

(3.4) COROLLARY. $x \to \sup f_{\mathscr{K}}(x)$ *is* $\mathscr{B}$*-measurable for each compact set* $\mathscr{K} \subset \mathscr{H}$.

(3.5) PROPOSITION. $\mathscr{F}_1$ *is the countable intersection of open sets* (*with respect to* $(\mathscr{F}, d_u)$).

PROOF. First we show that $h \to \int h \, d\lambda$ is a lower semicontinuous function on $\mathscr{F}$ (see (1.4)). Let h_n and $h \in$ [illegible] $\in \mathbb{N}$, and let $\lim_{n \in \mathbb{N}} d_u(h_n, h) = 0$. By (2.12) we have $\lim_{n \in \mathbb{N}} h_n(x) = h(x)$ λ-a.e., and according to Fatou's lemma, $\liminf_{n \in \mathbb{N}} \int h_n(x) \, dx \geqq \int h(x) \, dx$. By (1.4) we know that $\{h \in \mathscr{F} \mid \int h \, d\lambda > r\}$ is an open subset of $\mathscr{F}$ for each $r \in \mathbb{R}$. Since $\mathscr{F}_1 = \bigcap_{n \in \mathbb{N}} \{h \in \mathscr{F} \mid \int h \, d\lambda > 1 - 1/n\}$ the proposition follows.

Next we explain a method for proving the existence of a sequence of measurable m.l. estimates for families of unimodal probability densities. Let $\psi : \mathbb{R} \to [0, \infty]$ be upper semicontinuous, and let $\mathscr{H}$ be a measurable subset of $(\mathscr{F}_1, \mathscr{B}(d_u))$ such that $g(x) \leqq \psi(x)$ for each $x \neq M_g$ and $g \in \mathscr{H}$. Denote by $\mathscr{H}^c$ the closure of $\mathscr{H}$ in $(\mathscr{F}, d_u)$.

(3.6) LEMMA. *For each* $n \in \mathbb{N}$ *there exists a map* $\varphi_n' : \mathbb{R}^N \to \mathscr{H}^c$ *which is* $\mathscr{B}^n$, $\mathscr{B}(d_u \mid \mathscr{H}^c)$*-measurable and fulfills the condition*

$$\prod_{i=1}^n f_{\varphi_n'(\mathbf{x})}(x_i) = \sup \{\prod_{i=1}^n f_g(x_i) \mid g \in \mathscr{H}^c\} \qquad \textit{for each} \quad \mathbf{x} \in \mathbb{R}^N .$$

PROOF. Using the upper semicontinuity of ψ we easily derive from (2.13) that $g(x) \leqq \psi(x)$ for each $x \neq M_g$ and $g \in \mathscr{H}^c$. According to (2.10), $(\mathscr{H}^c, d_u)$ is a locally compact metric space with countable base. We derive from (3.3) that $\{g \in \mathscr{H}^c \mid -\log f_g(x) \leqq r\}$ is compact for all x and $r \in \mathbb{R}$. (3.4) implies that $x \to \inf_{g \in \mathscr{K}} (-\log f_g(x))$ is $\mathscr{B}$-measurable for each compact set $\mathscr{K} \subset \mathscr{H}^c$. Because of these statements [4], (Theorem (1.9) and Corollary (1.11)) are applicable and imply the assertion.

Now we use the map $\varphi_n' \colon \mathbb{R}^N \to \mathscr{H}^c$ for constructing a $\mathscr{B}^n$, $\mathscr{B}(d_u \mid \mathscr{H})$-measurable m.l. estimate for $\mathscr{H}$.

Let φ_n', $n \in \mathbb{N}$, be given as in (3.6). Let B_n be a $\mathscr{B}^n$-measurable subset of $\mathbb{R}^N$ such that for each $\mathbf{x} \in B_n$ and $g \in \mathscr{H}^c$ the relation

$$\prod_{i=1}^n f_g(x_i) = \sup \{\prod_{i=1}^n f_h(x_i) \mid h \in \mathscr{H}^c\}$$

implies $g \in \mathscr{H}$. Furthermore, let there be a $\mathscr{B}^n \mid \bar{B}_n$, $\mathscr{B}(d_u \mid \mathscr{H})$-measurable map $\tau_n \colon \bar{B}_n \to \mathscr{H}$ such that for each $\mathbf{x} \in \bar{B}_n$: $\prod_{i=1}^n f_{\tau_n(\mathbf{x})}(x_i) = \sup \{\prod_{i=1}^n f_h(x_i) \mid h \in \mathscr{H}\}$.

Define $\varphi_n \colon \mathbb{R}^N \to \mathscr{H}$ by

$$\begin{aligned} \varphi_n(\mathbf{x}) &\equiv \varphi_n'(\mathbf{x}) \qquad \mathbf{x} \in B_n \\ &\equiv \tau_n(\mathbf{x}) \qquad \mathbf{x} \in \bar{B}_n . \end{aligned}$$

(3.7) THEOREM. *$\varphi_n \colon \mathbb{R}^N \to \mathscr{H}$, $n \in \mathbb{N}$, is a sequence of $\mathscr{B}^n$, $\mathscr{B}(d_u \mid \mathscr{H})$-measurable (and hence $\mathscr{B}^n$, $\mathscr{B}(d_1 \mid \mathscr{H})$-measurable)* m.l. *estimates for $\mathscr{H}$.*

PROOF. (3.6) combined with (3.5) implies the first part of the assertion. Since id: $(\mathscr{H}, d_u) \to (\mathscr{H}, d_1)$ is continuous (see (2.17)) the second part follows immediately.

Let us now consider the case when the mode is known to be zero. Prakasa Rao [6] describes the m.l. estimate as the slope of the greatest convex minorant on $]-\infty, 0]$ and the smallest concave majorant on $[0, \infty[$ of the empirical distribution function. (As the referee pointed out this characterization is due to Reid [1].) The m.l. estimate $\varphi_n(\mathbf{x})$ is uniquely determined (except for the value at the mode) and upper semicontinuous for each $\mathbf{x} \in \mathbb{R}^N$ with $x_i \neq 0$ where $i = 1, \cdots, n$. If $x_i = 0$ for some $i \in \{1, \cdots, n\}$ define $\varphi_n(\mathbf{x}) = g_0$ where g_0 is an arbitrary but then fixed unimodal probability density with $g_0(0) = \infty$.

(3.8) COROLLARY. *The* m.l. *estimate φ_n for the sample size n for the family of all unimodal probability densities with mode at zero is $\mathscr{B}^n$, $\mathscr{B}(d_1)$-measurable. Furthermore, for $x \neq 0$ fixed $\mathbf{x} \to \varphi_n(\mathbf{x})(x)$ is $\mathscr{B}^n$-measurable.*

PROOF. Let $\mathscr{H} = \mathscr{F}_1^0$ in (3.7) (where $\mathscr{F}_1^0$ is defined in (2.2)). $\mathscr{F}_1^0$ is a closed subset of $\mathscr{F}_1$ with respect to d_u. Let $\psi(x) \equiv 1/|x|$. $B_n \equiv \{x \in \mathbb{R}^N \mid x_i \neq 0,\ i = 1, \cdots, n\}$ and $\varphi_n(\mathbf{x}) = g_0$ for $\mathbf{x} \in \bar{B}_n$ where $g_0 \in \mathscr{F}_1^0$ is fixed. (3.7) implies that $\varphi_n \colon \mathbb{R}^N \to \mathscr{F}_1^0$ is $\mathscr{B}^n$, $\mathscr{B}(d_1 \mid \mathscr{F}_1^0)$-measurable. In view of the remarks made above we easily see that an "arbitrary" m.l. estimate (for the family of all unimodal probability densities with mode equal to zero) is $\mathscr{B}^n$, $\mathscr{B}(d_1)$-measurable. This proves the first part of the assertion.

The second part follows immediately from (2.13) and (3.7).

Notice that the assertion of Corollary (3.8) may also be derived from the description of the m.l. estimate given above.

(3.9) COROLLARY. *There exists a sequence of* $\mathscr{B}^n$, $\mathscr{B}(d_1)$*-measurable* m.l. *estimates for the family of all unimodal probability densities which are uniformly bounded by some* $S > 0$.

PROOF. Let $\mathscr{H} \equiv \{g \in \mathscr{F}_1 \mid g(x) \leqq S$ for all $x \neq M_g\}$ and $\psi(x) \equiv S$ in (3.7). Let $B_n \equiv \{x \in \mathbb{R}^{\mathrm{N}} \mid \max\{x_1, \cdots, x_n\} - \min\{x_1, \cdots, x_n\} \geqq 1/S\}$, and $\tau_n : \bar{B}_n \to \mathscr{H}$ is defined in the following way

$$\begin{aligned}\tau_n(\mathbf{x})(x) &\equiv \infty \qquad x = \min\{x_1, \cdots, x_n\}\\ &\equiv S \qquad x \in]\min\{x_1, \cdots, x_n\}, \min\{x_1, \cdots, x_n\} + 1/S]\\ &\equiv 0 \qquad \text{otherwise}.\end{aligned}$$

It is easy to see that τ_n is continuous. By (3.7) there exists a $\mathscr{B}^n$, $\mathscr{B}(d_1 \mid \mathscr{H})$-measurable estimate φ_n for $\mathscr{H}$, and then f_{φ_n} is a $\mathscr{B}^n$, $\mathscr{B}(d_1)$-measurable m.l. estimate for the family of all unimodal probability densities which are uniformly bounded by S.

4. Consistency. Concerning consistency it makes no sense to define a sequence of m.l. estimates for the family of all unimodal probability densities since every density with mode at one of the observations x_i, $i \in \{1, \cdots, n\}$ and value $+\infty$ at x_i may be taken as image of $(x_i)_{i \in \mathrm{N}}$ under the m.l. estimate. Therefore we shall always assume that the given family $\mathscr{H}$ of unimodal densities is bounded by a function ψ. Let all $f \in \mathscr{H}$ have a mode in the closed set $A \subset \mathbb{R}$. If the mode is known to be $a \in \mathbb{R}$ we take $A = \{a\}$, and if the mode is unknown we take $A = \mathbb{R}$.

(4.1) Hereafter we shall always assume in this section that for each $f \in \mathscr{H}$ all versions f' of f, where f' is upper semicontinuous, $M_{f'} \in A$, and $f'(M_{f'}) = \psi(M_{f'})$, are elements of $\mathscr{H}$. Then each m.l. estimate (if existent) for the sample $(x_1, \cdots, x_n)$ has mode at one of the observations x_i, $i \in \{1, \cdots, n\}$.

(4.2) THEOREM. *Let* $\mathscr{H}$ *be a family of unimodal probability densities which are bounded by* ψ *(equipped with the pseudometric* d_1*). Suppose that* ψ *is upper semi-continuous. Then any sequence of* a.m.l. *estimates is strongly consistent for each* $f \in \mathscr{H}$ *fulfilling*

(a) $\int f \log \psi \, d\lambda < \infty$ *and*
(b) $\int f \log f \, d\lambda > -\infty$.

We prove Theorem (4.2) with the help of Lemma (4.3) which is a modification of results of Pfanzagl ([5], Theorem (1.12) and Corollary (1.16)).

(4.3) LEMMA. *Let* (T, d) *be a locally compact metric space with countable base, denote by* $\mathscr{M}$ *the power set of* T, *let* $\mathscr{P} \mid \mathscr{A}$ *be a family of probability measures on the measurable space* $(X, \mathscr{A})$ *and let* $\vartheta : \mathscr{P} \to \mathscr{M}$ *be a map such that* $\vartheta(P_1) \cap \vartheta(P_2) = \emptyset$ *for all* $P_1, P_2 \in \mathscr{P}$ *with* $P_1 \neq P_2$. *Let* $f_t : X \to [-\infty, \infty]$, $t \in T$, *be such that*

(a) $\{t \in T \mid f_t(x) \leqq r\}$ *is a compact set in* T *for all* x *and* $r \in \mathbb{R}$,

(b) $x \to \inf f_C(x)$ *is* $\mathscr{A}$*-measurable for all compact sets* $C \subset T$.

Let $P_0 \in \mathscr{P}$ *be such that, in addition,*

(c) $P_0(\inf f_H) > -\infty$ *for any closed set* $H \subset T$ *with* $H \cap \vartheta(P_0) = \emptyset$,

(d) $P_0(f_t) > P_0(f_l)$ *for all* $t \in T \backslash \vartheta(P_0)$ *and* $l \in \vartheta(P_0)$,

(e) $P_0(f_t) = P_0(f_{t'})$ *for all* $t, t' \in \vartheta(P_0)$.

Let $\varphi_n : X^{\mathbb{N}} \to T$, $n \in \mathbb{N}$, *be such that,*

(f) $$\lim_{n \in \mathbb{N}} (\exp(n^{-1} \sum_{i=1}^n f_{\varphi_n(\mathbf{x})}(x_i)) - \inf\{\exp(n^{-1} \sum_{i=1}^n f_t(x_i)) \mid t \in \bigcup_{P \in \mathscr{P}} \vartheta(P)\}) = 0 .$$

Then there exists a sequence $t_n(\mathbf{x}) \in \vartheta(P_0)$, $n \in \mathbb{N}$, *such that* $\lim_{n \in \mathbb{N}} d(\varphi_n(\mathbf{x}), t_n(\mathbf{x})) = 0$ *for* $P_0^{\mathbb{N}}$-a.a. $\mathbf{x} \in X^{\mathbb{N}}$.

PROOF. We shall only sketch the proof as it is closely related to that of Pfanzagl's theorems cited above.

Let $P_0 \in \mathscr{P}$ be such that (c), (d), and (e) are fulfilled. Let $(T^*, \mathscr{T}^*)$ be the one-point-compactification of (T, d) and t^* the point at infinity. Let $f_{t^*} \equiv \infty$. The conditions (c), (d), and (e) still hold for P_0 if we put T^* and $(f_t)_{t \in T^*}$ in place of T and $(f_t)_{t \in T}$ respectively.

As in [5], Theorem (1.12), (1.13), the strong law of large numbers together with (f) implies that

$$\limsup_{n \in \mathbb{N}} n^{-1} \sum_{i=1}^n f_{\varphi_n(\mathbf{x})}(x_i) \leqq P_0(f_l) \qquad (P_0^{\mathbb{N}}\text{-a.e.}) \quad \text{for all} \quad l \in \vartheta(P_0) . \qquad (^+)$$

Let $m \in \mathbb{N}$ be fixed. Let $U_m \equiv \bigcup_{t \in \vartheta(P_0)} U_d(t, 1/m)$. U_m is an open set in T^* and $\vartheta(P_0) \subset U_m$.

As in [5], Theorem (1.12), (1.14), the conditions (a) through (e) (with T^* and $(f_t)_{t \in T^*}$ instead of T and $(f_t)_{t \in T}$ respectively) imply

$$P_0(f_l) < \inf_{t \in \bar{U}_m} P_0(f_t) \leqq \liminf_{n \in \mathbb{N}} \inf_{t \in \bar{U}_m} n^{-1} \sum_{i=1}^n f_t(x_i) \qquad (P_0^{\mathbb{N}}\text{-a.e.}) \quad \text{for all} \quad l \in \vartheta(P_0) . \qquad (^{++})$$

By $(^+)$ and $(^{++})$

$$\limsup_{n \in \mathbb{N}} n^{-1} \sum_{i=1}^n f_{\varphi_n(\mathbf{x})}(x_i) < \liminf_{n \in \mathbb{N}} \inf_{t \in \bar{U}_m} n^{-1} \sum_{i=1}^n f_t(x_i) \qquad (P_0^{\mathbb{N}}\text{-a.e.}) . \qquad (^{+++})$$

Let $Z_m \subset X^{\mathbb{N}}$ with $P_0^{\mathbb{N}}(Z_m) = 0$ such that $(^{+++})$ holds for all $\mathbf{x} \in \bar{Z}_m$. Let $Z = \bigcup_{m \in \mathbb{N}} Z_m$. Let $\mathbf{x} \in \bar{Z}$ and $m \in \mathbb{N}$ be fixed. Then $(^{+++})$ implies that $\varphi_n(\mathbf{x}) \in U_m$ for sufficiently large $n \in \mathbb{N}$. Define $(n_m)_{m \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $\varphi_n(\mathbf{x}) \in U_m$ for all $n \geqq n_m$. Without loss of generality we may assume that $n_1 < n_2 < n_3 < \cdots$. Choose $t_n^m(\mathbf{x}) \in \vartheta(P_0)$ such that $d(\varphi_n(\mathbf{x}), t_n^m(\mathbf{x})) < 1/m$ for each $n \geqq n_m$. Define $t_n(\mathbf{x}) \equiv t_n^m(\mathbf{x})$ for $n \in \{n_m, \cdots, n_{m+1} - 1\}$. It follows immediately that $\lim_{n \in \mathbb{N}} d(\varphi_n(\mathbf{x}), t_n(\mathbf{x})) = 0$ where $t_n(\mathbf{x}) \in \vartheta(P_0)$ for each $n \in \mathbb{N}$.

Please note that (4.3) implies essentially the result of [5], Corollary (1.16) if $\vartheta(P)$ contains only one element for each $P \in \mathscr{P}$. Later we shall indicate why this corollary is not applicable for our purpose.

PROOF OF THEOREM (4.2). Let $\mathscr{K} \equiv \{g \in \mathscr{F}_1 \mid g(x) \leqq \psi(x)$ for $x \neq M_g$, $\min\{g(x), \psi(x)\} \in \mathscr{H}\}$. Denote by $\mathscr{K}^c$ the closure of $\mathscr{K}$ in $\mathscr{F}$. Let φ_n, $n \in \mathbb{N}$, be a sequence of a.m.l. estimates for $\mathscr{H}$. Define a map $\varphi_n' : \mathbb{R}^{\mathbb{N}} \to \mathscr{K}^c$ such that $\varphi_n'(\mathbf{x}) = \varphi_n(\mathbf{x})$ λ-a.e. for each $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Let $f_g \equiv -\log(\min\{g, \psi\})$, $\mathscr{P} \equiv \{P_g \mid g \in \mathscr{K}\}$ and $\vartheta(P) \equiv (dP/d\lambda) \cap \mathscr{K}$ for $P \in \mathscr{P}$. Next we show that the assumptions of Lemma (4.3) are fulfilled for $(\mathscr{K}^c, d_u)$ (in place of (T, d)).

As a consequence of (2.10) we have that $(\mathscr{K}^c, d_u)$ is a locally compact metric space with countable base. It trivially holds that $\vartheta(P_1) \cap \vartheta(P_2) = \varnothing$ for $P_1 \neq P_2$. Now we verify the conditions (a) through (b) of (4.3).

(a) and (b) follow immediately from (3.3) and (3.4). Let $g \in \mathscr{K}$ be such that conditions (a) and (b) of the theorem are fulfilled and put P_g in place of P_0 in (4.3).

For (c): We derive that $h(x) \leqq \psi(x)$ for each $x \neq M_h$ and $h \in \mathscr{K}^c$. Hence $P_g(\inf f_{\mathscr{K}^c}) > -\infty$.

(d) follows immediately as an application of [5], Example (1.3).

(e) is trivial.

For (f): In view of the general assumption (4.1) and since φ_n is an a.m.l. estimate for $\mathscr{H}$, we easily conclude that φ_n' fulfills (f).

Lemma (4.3) implies that there exists $g_n(\mathbf{x}) \in (dP_g/d\lambda) \cap \mathscr{K}$ such that

$$\lim_{n \in \mathbb{N}} d_u(\varphi_n'(\mathbf{x}), g_n(\mathbf{x})) = 0 \quad P_g^{\mathbb{N}}\text{-a.e.}$$

By (2.17) this is equivalent to $\lim_{n \in \mathbb{N}} d_1(\varphi_n(\mathbf{x}), g) = 0$ $P_g^{\mathbb{N}}$-a.e.

We have proved the assertion of the theorem for each $g \in \mathscr{K}$, and hence the assertion is valid for each $f \in \mathscr{H}$, as we can find a version $g \in \mathscr{K}$ for each $f \in \mathscr{H}$.

Now we discuss the method of the proof. As a consequence of Theorem (4.2) we know already that we have to consider d_1-convergence or convergence in some stronger sense. If the mode is known we may apply Pfanzagl's results (that is Lemma (4.3) with the one-to-one parametrization $\vartheta : \mathscr{P} \to (\mathscr{F}_1^0, d_1)$) to obtain the consistency statement. In other words, we have the following interdependence: $\mathscr{P}$ is a family of probability measures equipped with the supremum-metric. Choose a version $f_P \in dP/d\lambda$ for each $P \in \mathscr{P}$ such that the function $P \to f_P(x)$ is upper semicontinuous for each $x \in \mathbb{R}$. $(^+)$

If the mode is not known the following example shows that it is not generally possible to choose versions $f_P \in dP/d\lambda$, $P \in \mathscr{P}$, such that $(^+)$ is fulfilled.

(4.4) EXAMPLE. Let $f(x) \equiv I_{]0,1[}(x)$ and let J_n, $n \in \mathbb{N}$, be subintervals of $[0, 1]$ such that $\limsup_{n \in \mathbb{N}} J_n =]0, 1[$ and $\lim_{n \in \mathbb{N}} \lambda(J_n) = 0$. Define $f_n(x) \equiv I_{[\lambda(J_n),1]}(x) + I_{J_n}(x)$. Then we have $d_1(f_n, f) \leqq 2\lambda(J_n)$ and hence $\lim_{n \in \mathbb{N}} d_1(f_n, f) = 0$. But $\limsup_{n \in \mathbb{N}} f_n(x) = 2f(x)$. In order to obtain $(^+)$ it is easy to see that $\lim_{n \in \mathbb{N}} d_1(f_n, f) = 0$ has to imply $\limsup_{n \in \mathbb{N}} f_n(x) \leqq f(x)$ λ-a.e. (see also (1.4)).

If the mode is not known d_1-convergence implies only convergence at continuity points outside of the modal interval of the limit function (see (2.17), (2.12), and also (4.4)). Given the family of all unimodal probability densities with mode at zero, we can prove that for any $\alpha > 0$ the m.l. estimate exceeds α near the mode for infinitely many $n \in \mathbb{N}$ (with probability one). This "peaking" near the mode indicates that we have to consider d_1-convergence and not convergence in some stronger sense.

(4.5) COROLLARY. *Let $\mathscr{H}^a$ be the family of all unimodal probability densities with mode at $a \in \mathbb{R}$. Any sequence of* a.m.l. *estimates for $\mathscr{H}^a$ is strongly consistent for each $f \in \mathscr{H}^a$ fulfilling*

(a) $\int f(x) \log |x - a| \, dx > -\infty$ *and*
(b) $\int f(x) \log f(x) \, dx > -\infty$.

PROOF. We have $f(x) \leqq 1/|x - a|$ for each $f \in \mathscr{H}^a$. Hence the assertion follows from (4.2) with $\psi(x) \equiv 1/|x - a|$.

It is well known that the m.l. estimate is strongly consistent for each $f \in \mathscr{H}^a$ (without assumptions (a) and (b)). Robertson [7] uses the representation of the m.l. estimate as a conditional expectation given a σ-lattice for proving consistency.

In examples (4.7) and (4.8) we shall find special families of unimodal probability densities such that the assumptions of Theorem (4.2) are fulfilled except for condition (4.2)(a) or (4.2)(b) and such that any sequence of m.l. estimates may fail to be consistent.

(4.6) COROLLARY. *Let $\mathscr{H}_S$ be the family of all unimodal probability densities which are uniformly bounded by $S > 0$. Then any sequence of* a.m.l. *estimates for $\mathscr{H}_S$ is strongly consistent for each $f \in \mathscr{H}_S$ fulfilling $\int f \log f \, d\lambda > -\infty$.*

PROOF. Apply (4.2) for $\psi(x) \equiv S$. (4.2)(a) is trivially fulfilled.

We do not know whether the condition $\int f \log f \, d\lambda > -\infty$ can be removed in Corollary (4.6).

(4.7) EXAMPLE. Let

$$\begin{aligned} g_0(x) &\equiv (-x \log x (\log \log x^{-1})^2)^{-1} && (0 < x \leqq \exp(-e)) \\ &\equiv \infty && (x = 0) \\ &\equiv 0 && (\text{otherwise}) . \end{aligned}$$

For $0 < \alpha \leqq \exp(-e)$

$$\begin{aligned} g_\alpha(x) &\equiv g_0(x)/2 && (\alpha < x \leqq \exp(-e)) \\ &\equiv -\log \alpha((\log \log \alpha^{-1})^2 + \log \log \alpha^{-1}) g_0(\alpha)/2 && (0 < x \leqq \alpha) \\ &\equiv \infty && (x = 0) \\ &\equiv 0 && (\text{otherwise}) . \end{aligned}$$

g_α, $\alpha \in [0, \exp(-e)]$ are unimodal probability densities with mode at zero. Let $\psi(x) \equiv 1/|x|$. By (3.7) we know that a sequence of m.l. estimates exists. We show that any sequence of m.l. estimates is not consistent for g_0 (for which (4.2)(a) is not fulfilled).

Let $B_n \equiv \{\mathbf{x} \in \mathbb{R}^{\mathrm{N}} \mid x_n \leqq \exp(-\exp n)\}$. We obtain

$$\sum_{n \in \mathrm{N}} P_{g_0}^{\mathrm{N}}(B_n) = \sum_{n \in \mathrm{N}} 1/n = \infty .$$

According to the 2nd Lemma of Borel-Cantelli, this implies

$$P_{g_0}^{\mathbb{N}}(\limsup_{n \in \mathbb{N}} B_n) = 1 .$$

Let $\mathbf{x} \in \limsup_{n \in \mathbb{N}} B_n$ be such that $x_i \neq 0$ for all $i \in \mathbb{N}$. There exists a subsequence $(i(n))_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $0 < \min\{x_1, \cdots, x_{i(n)}\} \leqq \exp(-\exp i(n))$.

It can be easily checked that

$$\prod_{j=1}^{i(n)} g_{\min\{x_1, \cdots x_{i(n)}\}}(x_j) > \prod_{j=1}^{i(n)} g_0(x_j) .$$

This implies that every m.l. estimate for the sample $(x_1, \cdots, x_{i(n)})$ is an element of $\mathscr{G} \equiv \{g_\alpha \mid \alpha \in]0, \exp(-e)]\}$. Since g_0 is no d_1-limit point of $\mathscr{G}$ any sequence of m.l. estimates cannot be consistent for g_0.

(4.8) EXAMPLE. Let

$$\begin{aligned} g_0(x) &\equiv (x \log x (\log\log x)^2)^{-1} && (x > \exp(e)) \\ &\equiv \infty && (x = \exp(e)) \\ &\equiv 0 && (\text{otherwise}) . \end{aligned}$$

For $\exp(e) < \alpha < \infty$ let

$$\begin{aligned} g_\alpha(x) &\equiv (g_0(x) + g(\alpha)\alpha(\log \alpha) \\ &\qquad \times ((\log\log \alpha)^2 + \log\log \alpha)/(\alpha - \exp(e)))/2 && (x > \exp(e)) \\ &\equiv \infty && (x = \exp(e)) \\ &\equiv 0 && (\text{otherwise}) . \end{aligned}$$

g_α, $\alpha \in [\exp(e), \infty[\cup \{0\}$ are unimodal probability densities with mode $\exp(e)$. The assumptions of (4.2) are fulfilled except for condition (b) for g_0. In a similar way as in (4.7) we can show that every sequence of m.l. estimates is not consistent for g_0.

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