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OPTIMAL STOPPING AND SEQUENTIAL TESTS WHICH MINIMIZE THE MAXIMUM EXPECTED SAMPLE SIZE[1]

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Among all sequential tests with prescribed error probabilities of the null hypothesis $H_0: \theta = -\theta_1$ versus the simple alternative $H_1: \theta = \theta_1$, where θ is the unknown mean of a normal population, we want to find the test which minimizes the maximum expected sample size. In this paper, we formulate the problem as an optimal stopping problem and find an optimal stopping rule. The analogous problem in continuous time is also studied, where we want to test whether the drift coefficient of a Wiener process is $-\theta_1$ or θ_1. By reducing the corresponding optimal stopping problem to a free boundary problem, we obtain upper and lower bounds as well as the asymptotic behavior of the stopping boundaries.

1. Introduction. Let $X_1, X_2, \cdots$ be i.i.d. $N(\theta, \sigma^2)$, where σ^2 is known and θ is an unknown parameter. The problem is to test the simple hypothesis $H_0: \theta = -\theta_1$ versus the simple alternative $H_1: \theta = \theta_1$ $(\theta_1 > 0)$ with prescribed error probabilities

$$(1) \qquad P_{-\theta_1}(H_0 \text{ is rejected}) \leqq \lambda, \quad P_{\theta_1}(H_0 \text{ is accepted}) \leqq \lambda$$

where $\lambda < \frac{1}{2}$ is a given positive constant. It is well known that the Wald sequential probability ratio test (SPRT) which stops as soon as $|S_n| \geqq b$, b being determined by (1), minimizes the expected sample size under both H_0 and H_1. Although the Wald SPRT has this optimum property, its expected sample size for $\theta \in (-\theta_1, \theta_1)$ may actually exceed the fixed sample size required by the UMP test satisfying (1), as has been pointed out by Woodroofe [14]. To remedy this, Anderson [1] has proposed a modification of the SPRT where he replaces the parallel straight lines $x = b$ and $x = -b$ by two convergent straight line boundaries which are symmetric about the line $x = 0$.

Bechhofer [2] has pointed out the desirability of constructing a test which minimizes the maximum expected sample size among all tests satisfying (1). Kiefer and Weiss [9] and later Weiss [13] have considered this problem. They have shown that such a test is a generalized sequential probability ratio test (GSPRT). Using a Bayesian approach, Weiss [13] has found the truncation

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point for such a test and has pointed out that the construction of the continuation region would be a very hard computational problem.

In Section 2, we shall formulate the problem as an optimal stopping problem. We are in the non-stationary Markov case here, and non-stationarity accounts for the difficulty in the construction of the continuation region. In Section 3, we shall study this optimal stopping problem, and it will be shown that the boundary defining the continuation region of our optimal stopping rule consists of a pair of convergent decreasing nonlinear curves, symmetric about the time axis. In Section 4, we shall consider the corresponding optimal stopping problem in continuous time, where we shall replace the partial sums of $N(0, 1)$ random variables of Section 3 by a standard Wiener process and reduce the optimal stopping problem to the solution of a free boundary problem. Throughout this paper, we shall use Φ to denote the distribution function of a standard normal random variable.

2. Formulation of the optimal stopping problem. Let $X_1, X_2, \cdots$ be i.i.d. normal random variables with known variance σ^2 and unknown mean θ. We want to test $H_0: \theta = -\theta_1$ versus $H_1: \theta = \theta_1$. A test (N, δ) consists of a stopping time N and a terminal decision rule which is represented by a critical function δ, i.e., $\delta(x_1, \cdots, x_n)$ is the probability of rejecting H_0 given that $N = n$ and $X_1 = x_1, \cdots, X_n = x_n$ are observed. Throughout this paper, we shall restrict ourselves to nonrandomized stopping rules. As in [13], we shall say that a test (N, δ) is symmetric if it satisfies the following two conditions for all n:

(A) $\delta(-x_1, \cdots, -x_n) = 1 - \delta(x_1, \cdots, x_n)$.

(B) For any Borel subset B of R^n,

$$\{N = n\} = \{(X_1, \cdots, X_n) \in B\} \Rightarrow \{N = n\} = \{(-X_1, \cdots, -X_n) \in B\}.$$

Let (N, δ) be a symmetric test such that the inequalities in (1) are actually equalities. Weiss [13] has shown that (N, δ) minimizes the maximum expected sample size among all tests satisfying (1) if there exists $p \in (0, 1)$ such that (N, δ) minimizes

$$\psi(N, \delta) = \frac{p}{2} P_{-\theta_1}(H_0 \text{ is rejected}) + (1-p)E_0 N + \frac{p}{2} P_{\theta_1}(H_0 \text{ is accepted}) \tag{2}$$

among all tests satisfying (1).

Since we assume $\lambda < \frac{1}{2}$, a test (N, δ) satisfying (1) must take at least one observation. We now assert that for any given stopping rule $N \geqq 1$, the terminal decision rule δ_0 which rejects H_0 if $S_N > 0$, accepts H_0 if $S_N < 0$ and rejects H_0 with probability $\frac{1}{2}$ if $S_N = 0$ minimizes the function $\psi(N, \delta)$ among all terminal decision rules δ. To prove this, we note that for any terminal decision rule δ,

$$
\begin{aligned}
\psi(N, \delta) = \frac{p}{2} \sum_{n=1}^{\infty} \int_{(N=n)} &(2\pi\sigma^2)^{-n/2}\delta(x_1, \cdots, x_n) \\
&\times \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i + \theta_1)^2\right\} dx_1 \cdots dx_n \\
&+ (1-p) \sum_{n=1}^{\infty} \int_{(N=n)} (2\pi\sigma^2)^{-n/2} n \\
&\times \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2\right\} dx_1 \cdots dx_n \\
&+ \frac{p}{2} \sum_{n=1}^{\infty} \int_{(N=n)} (2\pi\sigma^2)^{-n/2}(1 - \delta(x_1, \cdots, x_n)) \qquad (3) \\
&\times \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \theta_1)^2\right\} dx_1 \cdots dx_n \\
\geqq \sum_{n=1}^{\infty} (2\pi\sigma^2)^{-n/2} \int_{(N=n)} &\left\{n(1-p) + \frac{p}{2} \exp\left(-\frac{n\theta_1^2}{2\sigma^2} - \frac{\theta_1}{\sigma^2}|s_n|\right)\right\} \\
&\times \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2\right) dx_1 \cdots dx_n \\
= \psi(N, \delta_0) .
\end{aligned}
$$

It remains to find an optimal stopping rule which minimizes $\psi(N, \delta_0)$. We observe that

$$\psi(N, \delta_0) = \frac{p}{2} E_0 \left\{\alpha N + \exp\left(-\frac{N}{2}\beta^2 - \beta\left|\sum_{i=1}^{N} X_i/\sigma\right|\right)\right\}$$

where $\alpha = 2(1-p)/p$, $\beta = \theta_1/\sigma$. In the following section, we shall find an optimal stopping rule N which is symmetric, i.e., satisfies condition (B) above. Hence the test (N, δ_0) is a symmetric test. For simplicity, we shall assume that $\beta = 1$.

3. The continuation region and other properties of the optimal stopping rule τ. Let $X_1, X_2, \cdots$ be i.i.d. $N(0, 1)$ random variables and $S_n = X_1 + \cdots + X_n$ $(S_0 = 0)$. Let $\alpha > 0$ and define $Z_n = \alpha n + \exp(-\frac{1}{2}n - |S_n|)$. In this section, we shall find an optimal stopping rule τ for the loss sequence Z_n, i.e., $EZ_\tau = \inf_T EZ_T$, where the infimum is taken over all stopping rules T. From the theory of optimal stopping, we are in the non-stationary Markov case here (see [4], Chapter 5). In this connection, we remark that if we replace Z_n by $\tilde{Z}_n = \alpha n + \exp(-\frac{1}{2}n - S_n)$, then the loss sequence $\tilde{Z}_n$ is a submartingale and the optimal rule is not to take any observation. For the sequence Z_n, it is obvious that one should not continue sampling after stage n if $\exp(-\frac{1}{2}n - |S_n|) \leqq \alpha$, or equivalently if $|S_n| + \frac{1}{2}n \geqq -\log \alpha$. In particular, τ is bounded by M, where M is the smallest nonnegative integer $\geqq -2 \log \alpha$. This agrees with a result of Weiss [13]. From now on, we shall assume that $\alpha < 1$ so that $M \geqq 1$. Let Y be a standard normal random variable and define

$$
\begin{aligned}
v(x, n) &= \exp\left(-\frac{n}{2} - |x|\right); \\
u(x, n) &= v(x, n)\,, && \text{if } n \geqq M\,, \\
(4) \qquad &= \min\{v(x, n), \alpha + Eu\{x + Y, n + 1)\}\,, \\
& && \text{if } n = 0, 1, \cdots, M - 1\,; \\
d(x, n) &= v(x, n) - u(x, n)\,; \\
\tau &= \inf\{n \geqq 0 : d(S_n, n) = 0\}\,.
\end{aligned}
$$

It is easy to see ([4], Chapters 3 and 5) that τ is an optimal stopping rule for Z_n. As first indicated by Kiefer and Weiss [9] for the corresponding sequential test, we now prove directly that the continuation region of our optimal rule τ is defined by a non-increasing sequence (b_n) with $b_n = 0$ for $n \geqq M$, i.e., τ is the first $n \geqq 0$ such that $|S_n| \geqq b_n$.

THEOREM 1. *There exists a nonnegative sequence* $(b_n, 0 \leqq n \leqq M)$ *with* $b_M = 0$ *such that the optimal stopping rule* τ *is given by* $\tau = \inf\{n \geqq 0 : |S_n| \geqq b_n\}$. *Furthermore, for each n fixed, the functions d, u, v are continuous even functions in x and* $d(x, n)$, $u(x, n)$ *are both non-increasing in* $x \geqq 0$.

PROOF. We shall prove the theorem by backward induction. We note that $\tau \leqq M$ and $d(x, n) = 0$ if $|x| + \frac{1}{2}n \geqq -\log\alpha$. Suppose that for $i = n + 1, \cdots, M$, $d(x, i)$ and $u(x, i)$ are both continuous even functions and are both non-increasing in $x \geqq 0$. Then we define $b_i = \inf\{x \geqq 0; d(x, i) = 0\}$ and note that

$$
\begin{aligned}
u(x, i) &= \alpha + Eu(x + Y, i + 1) && \text{if } |x| < b_i\,, \\
&= v(x, i) && \text{if } |x| \geqq b_i\,.
\end{aligned}
$$

Obviously $u(x, n)$ is a continuous even function in x. We now construct b_n. At stage n, if we observe $S_n = x \geqq 0$, then the optimal rule τ will stop sampling if

$$
\begin{aligned}
v(x, n) \leqq \alpha &+ EI_{[|x+Y| \geqq b_{n+1}]} \exp\left(-\frac{n+1}{2} - |x + Y|\right) \\
&+ EI_{[|x+Y| < b_{n+1}]} u(x + Y, n + 1)
\end{aligned}
$$

or equivalently if

$$
(5) \qquad 0 \leqq \alpha + E\exp\left(-\frac{n+1}{2} - |x + Y|\right) - v(x, n) - Ed(x + Y, n + 1)\,.
$$

Now $E\exp(-\frac{1}{2}(n + 1) - |x + Y|) - v(x, n) = e^{-n/2}\{e^x\Phi(-1 - x) - e^{-x}\Phi(1 - x)\}$ is a continuous increasing function in x. Also the family of densities of the random variables $|x + Y|$, $x \geqq 0$, has monotone likelihood ratio. By induction assumption, $-d(z, n + 1)$ is a continuous non-decreasing function in $z \geqq 0$. Hence $-Ed(|x + Y|, n + 1)$ is a continuous non-decreasing function in $x \geqq 0$ ([11], page 74). Therefore the set of all $x \geqq 0$ satisfying (5) is an interval, say $[b_n, \infty)$, with $b_n \leqq (-\log\alpha - \frac{1}{2}n)^+$.

We now show that $d(x, n)$ is a non-increasing function in $x \geqq 0$. For $x \geqq b_n$, $d(x, n) = 0$. Now let $x < b_n$. Then $d(x, n) > 0$ and

$$\begin{aligned} d(x, n) &= v(x, n) - u(x, n) \\ &= v(x, n) - E \exp\left(-\frac{n+1}{2} - |x + Y|\right) \\ &\qquad - \alpha + Ed(x + Y, n + 1) \\ &= e^{-n/2}\{e^{-x}\Phi(1 - x) - e^{x}\Phi(-1 - x)\} - \alpha + Ed(x + Y, n + 1)\,. \end{aligned} \tag{6}$$

Hence $d(x, n)$ is non-increasing in $x \geqq 0$. Similarly, since $u(x, n) = \alpha + Eu(x + Y, n + 1)$ if $|x| < b_n$, and $= v(x, n)$ if $|x| \geqq b_n$, we can prove that $u(x, n)$ is non-increasing in $x \geqq 0$. □

THEOREM 2. *For each x, the functions $d(x, n)$ and $u(x, n)$ defined by* (4) *are both non-increasing in n. Consequently the sequence $(b_n, 0 \leqq n \leqq M)$ constructed in Theorem* 1 *is non-increasing.*

PROOF. We first show that for $x \geqq 0$,

$$d(x, n) \geqq d(x, n + 1) \qquad \text{with strict inequality if} \quad x < b_n\,. \tag{7}$$

We shall prove (7) by backward induction. Obviously (7) is true for $n \geqq M$. Now assume that (7) holds for $n = m + 1$. In the case where $x \geqq 0$ is a continuation point at stages m and $m + 1$, i.e., $x < b_m$ and $x < b_{m+1}$, then it follows from (6) that $d(x, m) > d(x, m + 1)$, since using the induction assumption, we have $Ed(x + Y, m + 1) \geqq Ed(x + Y, m + 2)$. We now show that $b_m \geqq b_{m+1}$, so that the only remaining cases are $b_{m+1} \leqq x < b_m$ (and therefore $d(x, m + 1) = 0 < d(x, m)$) together with $x \geqq b_m$ (and therefore $d(x, m) = d(x, m + 1) = 0$). If $b_m = 0$, then obviously $b_{m+1} = 0$. Now assume $b_m > 0$. To prove $b_m > b_{m+1}$, we recall that $b_n = \inf\{x \geqq 0 : x \text{ satisfies } (5)\}$, and so it suffices to show that

$$\alpha + E \exp\left(-\frac{m+2}{2} - |b_m + Y|\right) - v(b_m, m + 1) - Ed(b_m + Y, m + 2) > 0$$

or equivalently,

$$\begin{aligned} \alpha + e^{-(m+1)/2}\{e^{b_m}\Phi(-1 - b_m) - e^{-b_m}\Phi(1 - b_m)\} \\ - Ed(b_m + Y, m + 2) > 0\,. \end{aligned} \tag{8}$$

Since $b_m > 0$,

$$\alpha + e^{-m/2}\{e^{b_m}\Phi(-1 - b_m) - e^{-b_m}\Phi(1 - b_m)\} - Ed(b_m + Y, m + 1) = 0\,. \tag{9}$$

Using induction assumption, $Ed(b_m + Y, m + 1) \geqq Ed(b_m + Y, m + 2)$. Also $e^{b_m}\Phi(-1 - b_m) - e^{-b_m}\Phi(1 - b_m) < 0$. Therefore (8) follows from (9).

To prove that $u(x, n)$ is non-increasing in n, we note that this is obviously true for $n \geqq M$. Defining $b_n = 0$ for $n \geqq M$, we can write for all $n = 1, 2, \cdots$,

$$\begin{aligned} u(x, n) &= \alpha + Eu(x + Y, n + 1) && \text{if} \quad |x| < b_n\,, \\ &= v(x, n) && \text{if} \quad |x| \geqq b_n\,. \end{aligned}$$

Using backward induction, we find that if $0 \leqq x < b_{n+1}$, then $Eu(x+Y, n+1) \geqq Eu(x+Y, n+2)$ and therefore $u(x,n) \geqq u(x, n+1)$. If $x \geqq b_n$, then $u(x,n) = v(x,n) \geqq v(x, n+1) = u(x, n+1)$. Finally, if $b_{n+1} \leqq x < b_n$, then

$$u(x, n+1) = v(x, n+1) \leqq \alpha + Eu(x+Y, n+2) \leqq \alpha + Eu(x+Y, n+1) = u(x,n)\,. \quad \square$$

The following theorem gives upper and lower bounds and the asymptotic behavior of b_n as $\alpha \downarrow 0$.

THEOREM 3. *The sequence b_n, $0 \leqq n \leqq M$, constructed in Theorem* 1 *satisfies the following inequalities*:

$$\beta_n \leqq b_n \leqq \left(-\frac{n}{2} - \log \alpha\right)^+ \tag{10}$$

where $x = \beta_n$ is the positive solution of

$$\alpha e^{n/2} = e^{-x}\Phi(1-x) - e^{x}\Phi(-1-x) \tag{11}$$

if (11) *has a positive solution and β_n is defined as* 0 *if otherwise. Furthermore, as $\alpha \to 0$, we have $M \to \infty$ and for each fixed $n = 0, 1, 2, \cdots$,*

$$b_n \sim -\log \alpha\,. \tag{12}$$

PROOF. We have already shown that $b_n \leqq (-\frac{1}{2}n - \log \alpha)^+$. Since b_n satisfies (5) and $d \geqq 0$, it follows that

$$\alpha + e^{-n/2}\{e^{b_n}\Phi(-1-b_n) - e^{b_n}\Phi(1-b_n)\} \geqq 0\,.$$

Therefore $b_n \geqq \beta_n = \inf\{x \geqq 0 : \alpha e^{n/2} \geqq e^{-x}\Phi(1-x) - e^{x}\Phi(-1-x)\}$, noting that the function $e^{-x}\Phi(1-x) - e^{x}\Phi(-1-x)$ is a continuous positive decreasing function for $x \geqq 0$.

To prove the asymptotic formula (12), we now derive another lower bound of b_n. For any $k = 1, 2, \cdots$, we note that

$$\begin{aligned} u(x,n) &\leqq \alpha k + e^{-(k+n)/2} E\exp(-|S_k + x|) \\ &= \alpha k + e^{-n/2}\{e^{-x}(1 - \Phi(k^{-\frac{1}{2}}(k-x))) + e^{x}\Phi(k^{-\frac{1}{2}}(-k-x))\}\,. \end{aligned}$$

Therefore $d(x,n) \geqq e^{-n/2}\{e^{-x}\Phi(k^{-\frac{1}{2}}(k-x)) - e^{x}\Phi(k^{-\frac{1}{2}}(-k-x))\} - \alpha k$. Since $d(b_n, n) = 0$, it then follows that

$$e^{-n/2}\{e^{-b_n}\Phi(k^{-\frac{1}{2}}(k-b_n)) - e^{b_n}\Phi(k^{-\frac{1}{2}}(-k-b_n))\} \leqq \alpha k\,. \tag{13}$$

It is easy to see from (10) that for n fixed, $b_n \to \infty$ as $\alpha \downarrow 0$. Setting $k = [b_n]$ in (13), we obtain that as $\alpha \downarrow 0$,

$$(\tfrac{1}{2} + o(1))\exp\left(-\frac{n}{2} - b_n\right) \leqq \alpha b_n$$

and therefore $b_n(1 + o(1)) \geqq -\log \alpha$. This inequality, together with the upper bound in (10), gives the asymptotic formula (12). $\square$

Let $\alpha = e^{-M/2}$ where M is a positive integer. Define $\pi_0(x) = g_0(x) = e^{-x}(1 - \Phi(1 - x)) + e^x\Phi(-1 - x)$ and let $x = c_1$ be the positive solution of

$$\pi_0(x) + e^{-\frac{1}{2}} = e^{-x} . \tag{14}$$

Then define

$$g_1(x) = e^{-x}(1 - \Phi(1 + c_1 - x)) + e^x\Phi(-1 - c_1 - x) ,$$
$$\pi_1(x) = (2\pi e)^{-\frac{1}{2}} \int_{-c_1}^{c_1} \pi_0(z) \exp(-(z - x)^2/2)\, dz + g_1(x) ,$$
$$\lambda_1(x) = \Phi(c_1 - x) - \Phi(-c_1 - x)$$

and let $x = c_2$ be the positive solution of

$$\pi_1(x) + e^{-1}(1 + \lambda_1(x)) = e^{-x} . \tag{15}$$

In general, having defined $c_1, \cdots, c_i$, we let $x = c_{i+1}$ be the positive solution of

$$\pi_i(x) + e^{-(i+1)/2}(1 + \lambda_i(x) + \lambda_{i-1,i}(x) + \cdots + \lambda_{1,\cdots,i}(x)) = e^{-x} \tag{16}$$

where

$$\pi_i(x) = (2\pi e)^{-\frac{1}{2}} \int_{-c_i}^{c_i} \pi_{i-1}(z) \exp(-(z-x)^2/2)\, dz + g_i(x) ,$$
$$g_i(x) = e^{-x}(1 - \Phi(1 + c_i - x)) + e^x\Phi(-1 - c_i - x) ,$$
$$\lambda_i(x) = \Phi(c_i - x) - \Phi(-c_i - x) ,$$
$$\lambda_{j,j+1}(x) = (2\pi)^{-\frac{1}{2}} \int_{-c_{j+1}}^{c_{j+1}} \lambda_j(z) \exp(-(z - x)^2/2)\, dz ,$$
$$\lambda_{j,j+1,j+2}(x) = (2\pi)^{-\frac{1}{2}} \int_{-c_{j+2}}^{c_{j+2}} \lambda_{j,j+1}(z) \exp(-(z - x)^2/2)\, dz , \qquad \text{etc.}$$

Note that the functions g_i, π_i, λ_i, $\lambda_{j,j+1}$, etc., defined above are all even functions. Numerical solution gives us $c_1 = .08$, $c_2 = .4$, $c_3 = .73$, $c_4 = 1.1$, $c_5 = 1.5$, etc., and the points (i, c_i) are not collinear.

We assert that $b_M = 0$ and $b_{M-i} = c_i$ for $i = 1, \cdots, M$. To prove this, we note that if $b_n > 0$, then $x = b_n$ is the positive solution of

$$e^{-M/2} + EI_{[|x+Y|\geqq b_{n+1}]} \exp\left(-\frac{n+1}{2} - |x + Y|\right) + EI_{[|x+Y|<b_{n+1}]}u(x + Y, n + 1) = v(x, n) \tag{17}$$

where Y is $N(0, 1)$. Since $b_M = 0$ and $u(x, M) = v(x, M)$, equation (17) with $n = M - 1$ reduces to equation (14), and so $b_{M-1} = c_1$. Then for $x \geqq 0$,

$$\begin{aligned} u(x, M - 1) &= e^{-M/2} + e^{-(M-1)/2}g_0(x) && \text{if} \quad x < c_1 , \\ &= v(x, M - 1) && \text{if} \quad x \geqq c_1 . \end{aligned}$$

Therefore equation (17) with $n = M - 2$ reduces to equation (15) upon multiplication of both sides of (17) by $e^{(M-2)/2}$, and so $b_{M-2} = c_2$. By induction, we see that in general, for $x \geqq 0$ and $i = 2, \cdots, M$,

$$\begin{aligned} u(x, M - i) &= e^{-M/2}(1 + \lambda_{i-1}(x) + \lambda_{i-2,i-1}(x) + \cdots + \lambda_{1,\cdots,i-1}(x)) \\ &\quad + e^{-(M-i)/2}\pi_{i-1}(x) && \text{if} \quad x < c_i , \\ &= v(x, M - i) && \text{if} \quad x \geqq c_i , \end{aligned}$$

and equation (17) with $n = M - (i + 1)$ reduces to equation (16). Therefore $b_{M-(i+1)} = c_{i+1}$.

We now consider the case where $\alpha = e^{-(M-\delta)/2}$, M being a positive integer and $\delta \in (0, 1)$. First suppose that $e^{-(1-\delta)/2} < \Phi(1) - \Phi(-1)$. Then the equation

$$\pi_0(x) + e^{-(1-\delta)/2} = e^{-x} \tag{18}$$

has a unique positive solution $x = c_1^*$. Define g_1^*, π_1^*, λ_1^* as before, only replacing c_1 by c_1^*. In general, we shall replace equation (16) by

$$\pi_i^*(x) + e^{-(i+1-\delta)/2}(1 + \lambda_i^*(x) + \cdots + \lambda_{1,\cdots,i}^*(x)) = e^{-x} \tag{19}$$

and let $x = c_{i+1}^*$ be the positive solution of (19), where π_i^*, λ_i^*, etc., are defined as before with c_i^* taking the place of c_i. In this case, we have $b_{M-i} = c_i^*$ $(i = 1, \cdots, M)$.

In the case where $e^{-(1-\delta)/2} \geqq \Phi(1) - \Phi(-1)$, we define $\tilde{c}_0 = 0$ and let $x = \tilde{c}_1$ be the positive solution of

$$\pi_0(x) + e^{-(2-\delta)/2} = e^{-x} . \tag{20}$$

In general we shall replace equation (16) by

$$\tilde{\pi}_i(x) + e^{-(i+2-\delta)/2}(1 + \tilde{\lambda}_1(x) + \cdots + \tilde{\lambda}_{1,\cdots,i}(x)) = e^{-x} \tag{21}$$

and let $x = \tilde{c}_{i+1}$ be the positive solution of (21), where $\tilde{\pi}_i$, $\tilde{\lambda}_i$, etc., are defined as before with $\tilde{c}_i$ taking the place of c_i. In this case, we have $b_{M-i} = \tilde{c}_{i-1}$ for $i = 1, \cdots, M$.

In practice, although we can solve numerically the system of equations (14), (16) (or the system (18), (19); or the system (20), (21)) to obtain the sequence b_n defining the optimal stopping rule τ, the operating characteristics of the test using such a stopping boundary cannot be obtained analytically since the stopping boundary is given only numerically and does not have a nice functional form. In practical situations, the preassigned upper bound λ in (1) for the error probabilities of the test is usually small, and in this case, it is easy to see that the number α defining the corresponding loss sequence Z_n is also small. The following theorem shows that as $\alpha \downarrow 0$, the (upper) stopping boundary b_n is asymptotically linear with slope $-\frac{1}{2}$ and intercept $-\log \alpha$, and let us approximate it by the line $x = (-\frac{1}{2}n - \log \alpha)^+$, $n = 0, 1, \cdots$. The corresponding approximation to our test is then to stop as soon as $|S_n| + \frac{1}{2}n \geqq -\log \alpha$, and is therefore a special case of Anderson's test whose operating characteristics have been derived in [1].

THEOREM 4. *As $\alpha \downarrow 0$, the asymptotic shape of the boundary b_n is linear; more precisely,*

$$b_{[-t \log \alpha]} \sim |\log \alpha| \left(1 - \frac{t}{2}\right) \qquad \text{if} \quad 0 \leqq t < 2 ,$$

$$b_{[-2 \log \alpha]} = O(1) ,$$

and $b_n = 0$ if $n > -2 \log \alpha$.

PROOF. We have already seen that $b_n = 0$ if $n > -2\log\alpha$ and $b_{[-2\log\alpha]} = 0$ or c_1^*, where c_1^* is defined by (18). Now let $0 < t < 2$ and in relation (13), set $n = [-t\log\alpha]$ and $k = [b_n]$. With these values of n and k, since (10) implies that $b_n \to \infty$ as $\alpha \downarrow 0$, (13) then reduces to

$$(\tfrac{1}{2} + o(1))\exp\left(-\frac{n}{2} - b_n\right) \leqq \alpha b_n$$

and therefore

$$\frac{t}{2}\log\alpha - b_{[-t\log\alpha]}(1 + o(1)) \leqq \log\alpha\,.$$

This inequality, together with the upper bound in (10), gives the desired conclusion. □

4. Extension to the continuous-time analogue. Let $W(t)$ be a standard Wiener process and let $X(t) = W(t) + \theta t$. We want to test $H: \theta = -1$ versus $K: \theta = 1$ with prescribed error probabilities

$$P_{-1}(H \text{ is rejected}) \leqq \lambda\,, \qquad P_1(H \text{ is accepted}) \leqq \lambda \tag{22}$$

where $\lambda < \frac{1}{2}$ is a given positive constant. As in the discrete-time case, a symmetric test for which the inequalities in (22) become equalities minimizes the maximum expected sample size among all tests satisfying (22) if we stop as soon as $|X(t)| \geqq b(t)$, where for some $\alpha > 0$, $b(t)$ and $-b(t)$ are the boundary curves of the continuation region of the optimal stopping rule σ which we shall construct below for the loss process $Z(t) = \alpha t + \exp(-\frac{1}{2}t - |W(t)|)$, and if the terminal decision rule rejects H when $X(t) > 0$ and accepts H when $X(t) < 0$.

We now proceed to find an optimal stopping rule σ for the loss process $Z(t)$, i.e., $EZ(\sigma) = \inf_T EZ(T)$, where the infimum is taken over all stopping times T. Let $Y(t)$ be the space-time Brownian motion, i.e., $Y(t)$ is a continuous stationary Markov process whose transition function $P(t, y, \Lambda)$ with $y = (x, s)$ and Λ being any measurable subset of $(-\infty, \infty) \times [0, \infty)$ is determined by

$$\begin{aligned} P(t, (x, s), \Gamma \times C) &= P[W(s+t) \in \Gamma \mid W(s) = x] \qquad \text{if} \quad s + t \in C\,,\\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

Properties of space-time processes and their harmonic functions have been studied by Doob [5], Itô and McKean [8] and Lai [10]. If G is an open subset of $(-\infty, \infty) \times [0, \infty)$ and $h: G \to (-\infty, \infty)$, then it is known ([10], Theorem 3) that h is harmonic for $Y(t)$ iff h is a continuous (and therefore by the maximum principle C^∞) solution of

$$\frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial t} = 0$$

on G. (We refer the reader to Chapter 12 of [6] for the definition and properties of harmonic functions of a continuous Markov process.) Define

$$g(x, t) = \alpha t + \exp\left(-\frac{t}{2} - |x|\right) = \alpha t + v(x, t)\,, \tag{23}$$

$$f(x, t) = \inf_T E[g(Y(T)) \mid Y(0) = (x, t)] = \alpha t + \varphi(x, t)$$

where $\varphi(x, t) = \inf_T \{\alpha ET + e^{-t/2} E \exp(-|x + W(T)| - \frac{1}{2}T)\}$.
It is known (see [7]) that the function $-f$ is the least excessive majorant of the function $-g$ and that f is harmonic for the space-time process $Y(t)$ on the set $\{(x, t) : f(x, t) < g(x, t)\}$ which is an open subset of $(-\infty, \infty) \times [0, \infty)$.

We now approximate this continuous-time optimal stopping problem by a sequence of discrete-time problems. For fixed x real, $t \geqq 0$ and $N = 0, 1, 2, \cdots$, let $Z^N_{x,t}$ denote the sequence $(Z^N_{x,t}(n),\, n = 0, 1, 2, \cdots)$ where

$$Z^N_{x,t}(n) = \alpha n 2^{-N} + e^{-t/2} \exp(-|x + W(n2^{-N})| - n2^{-N-1})\,.$$

For $x = 0$, $t = 0$ and $N = 0$, this reduces to the sequence considered in the previous section. The value of the loss sequence $Z^N_{x,t}$ is given by $u_N(x, t)$, where we define

$$\begin{aligned}
M_N(t) &= \text{smallest nonnegative integer } n \text{ such that } t + n2^{-N} \geqq \\
&\qquad -2 \log 2^{-N}\alpha\,; \\
u_N(x, n; t) &= v(x, t + n2^{-N}) \qquad\qquad \text{if } n \geqq M_N(t)\,, \\
&= \min\{v(x, t + n2^{-N}), \alpha 2^{-N} + E u_N(x + W(2^{-N}), n + 1; t)\} \\
&\qquad\qquad \text{if } n < M_N(t)\,; \qquad (24) \\
u_N(x, t) &= u_N(x, 0; t)\,, \\
d_N(x, t) &= v(x, t) - u_N(x, t)\,, \\
b_N(t) &= \inf\{x \geqq 0 : d_N(x, t) = 0\}\,.
\end{aligned}$$

Since the process $\alpha s + \exp(-|x + W(s)| - \frac{1}{2}(t + s))$, $s \geqq 0$, is continuous and bounded by the function $\alpha s + 1$, it follows from [12] that

$$u_N(x, t) \downarrow \varphi(x, t) \qquad \text{as} \quad N \uparrow \infty\,. \tag{25}$$

For $t \geqq 0$, since $u_N(x, t)$ is an even function in x, we obtain from (25) that $\varphi(x, t)$ is also an even function in x. Defining $\rho(x, t) = v(x, t) - \varphi(x, t) = g(x, t) - f(x, t)$ and $b(t) = \inf\{x \geqq 0 : \rho(x, t) = 0\}$, we find that as $N \uparrow \infty$, $d_N(x, t) \uparrow \rho(x, t)$ and $b_N(t) \uparrow b(t)$, and it can be shown that the function b is everywhere positive and $b_N(t)$ is strictly increasing in N for all sufficiently large N.

Using a similar argument as in the proof of Theorem 1, we can prove that for each t fixed, $d_N(x, t)$ and $u_N(x, t)$ are both non-increasing in $x \geqq 0$. Therefore $\rho(x, t)$ and $\varphi(x, t)$ are both non-increasing in $x \geqq 0$. Hence $f(x, t) < g(x, t)$ iff $x < b(t)$, and since f is harmonic for $Y(t)$ on $G = \{(x, t) : f(x, t) < g(x, t)\}$, it follows that $f \in C^\infty(G)$ and

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t} = 0\,, \qquad |x| < b(t) \tag{26}$$

$$f(x, t) = g(x, t)\,, \qquad |x| \geqq b(t)\,. \tag{27}$$

Since for fixed $t \geqq 0$, $f(x, t)$ is an even function in x, we have

$$\frac{\partial f}{\partial x}(0, t) = 0\,. \tag{28}$$

In fact $(\partial^n/\partial x^n)f(0, t) = 0$ if n is odd. From now on, we need only restrict ourselves to the quadrant $\{(x, t): x \geqq 0, t \geqq 0\}$ and the function g is a C^∞ function on this set. We remark that the restriction of $-f$ to this quadrant is the least excessive majorant of the restriction of $-g$ to the same set with respect to the space-time process of reflecting Brownian motion and equation (28) represents the boundary condition at the reflecting barrier $x = 0$.

We now make use of the discrete-time approximation of $\varphi(x, t)$ together with equations (26), (27), (28) to obtain further properties of the function f.

LEMMA 1. *If* $0 \leqq x < b(t)$, *then*

$$-\exp\left(-\frac{t}{2} - x\right) \leqq \frac{\partial f}{\partial x}(x, t) \leqq 0\,, \tag{29}$$

$$\alpha - \tfrac{1}{2}\exp\left(-\frac{t}{2} - x\right) \leqq \frac{\partial f}{\partial t}(x, t) \leqq \alpha\,. \tag{30}$$

PROOF. Obviously $\partial f/\partial x = \partial\varphi/\partial x$ and $\partial f/\partial t = \alpha + \partial\varphi/\partial t$, and so we need only consider the function φ. Since φ and ρ are non-increasing in $x \geqq 0$, $\partial\varphi/\partial x \leqq 0$ and $\partial\rho/\partial x \leqq 0$, i.e., $\partial v/\partial x \leqq \partial\varphi/\partial x \leqq 0$. To prove (30), let $q_1 = k_1 2^{-m}$, $q_2 = k_2 2^{-m}$ be two dyadic rationals with $k_2 > k_1 \geqq 0$. Then by a similar argument as the proof of Theorem 2, we can show that $d_N(x, q_1) \geqq d_N(x, q_2)$ for all $N > m$. Therefore for fixed $x \geqq 0$, $\rho(x, q)$ is a non-increasing function on the set of dyadic rationals. Hence if $0 \leqq x < b(t)$, then $(\partial/\partial t)\rho(x, t) \leqq 0$ and therefore $(\partial/\partial t)v(x, t) \leqq (\partial/\partial t)\varphi(x, t)$. Similarly we can show that $(\partial/\partial t)\varphi(x, t) \leqq 0$. □

LEMMA 2. *For each* $t \geqq 0$, $b(t) < \infty$. *Furthermore*, $\lim_{t\to\infty} b(t) = 0$ *and the function* b *is continuous and strictly decreasing on* $[0, \infty)$.

PROOF. Integrating equation (26) from $x = 0$ to $x = \xi$ and using (28), we obtain

$$\frac{1}{2}\frac{\partial f}{\partial x}(\xi, t) + \int_0^\xi \frac{\partial f}{\partial t}(x, t)\,dx = 0\,, \qquad 0 \leqq \xi < b(t)\,. \tag{31}$$

Suppose $b(t) = \infty$. Then it is easy to see from (30) that $\int_0^\infty (\partial/\partial t)f(x, t)\,dx = \infty$. Therefore (31) implies that $\lim_{\xi\to\infty}(\partial/\partial x)f(\xi, t) = -\infty$, contradicting (29). Hence we must have $b(t) < \infty$. A similar application of (29), (30) and (31) proves that $\lim_{t\to\infty} b(t) = 0$. It is also easy to check that $b(t)$ is continuous and strictly decreasing on $[0, \infty)$. □

LEMMA 3. *The function φ is continuous on $[0, \infty) \times [0, \infty)$.*

PROOF. We know that $\varphi \in C^\infty(G)$ and $\varphi(x, t) = v(x, t)$ if $x \geqq b(t)$. Therefore it remains to prove that φ is continuous at the points $(b(t), t)$, $t \geqq 0$. Let (x_n, t_n) be any sequence in G converging to $(b(t), t)$. By the mean value theorem and Lemma 1, it follows that $\varphi(x_n, t_n) - \varphi(x_m, t_m) \to 0$ as $n \to \infty$, $m \to \infty$. Therefore by the Cauchy criterion, $\lim_{n\to\infty} \varphi(x_n, t_n)$ exists. Since $\varphi \leqq v$ and $\varphi(\xi, t)$ is non-increasing in $\xi \geqq 0$, it is easy to see that $\lim_{\xi \uparrow b(t)} \varphi(\xi, t) = \varphi(b(t), t)$. □

THEOREM 5. *Let $\sigma = \inf\{t \geqq 0 : |W(t)| \geqq b(t)\}$. Then σ is optimal for the process $Z(t)$, i.e., $EZ(\sigma) = \inf_T EZ(T)$. Furthermore, defining $\sigma(x, s) = \inf\{t \geqq 0 : |x + W(t)| \geqq b(t + s)\}$, we have*

$$\varphi(x, s) = \alpha E\sigma(x, s) + e^{-s/2}Ev(x + W(\sigma(x, s)), \sigma(x, s)) .$$

PROOF. See Theorem 7.3 and Section 10 of [12].

LEMMA 4. *For fixed $t \geqq 0$, $\partial f/\partial x$ is a continuous function in x, and the smooth fit property*

$$\frac{\partial f}{\partial x}(b(t), t) = \frac{\partial g}{\partial x}(b(t), t) \tag{32}$$

holds for all $t \geqq 0$.

PROOF. Let $\xi = b(t)$. Then by (26) and (30), $(\partial^2/\partial x^2)f = -2(\partial/\partial t)f$ is bounded on G. Therefore using the mean value theorem as in the proof of Lemma 3, it is easy to see that $\lim_{x\uparrow\xi} f_x(x, t)$ exists and is finite. Since $f(x, t)$ is continuous at $x = \xi$, an easy application of the mean value theorem shows that the left-hand derivative $f_x^-(\xi, t)$ exists and is equal to $\lim_{x\uparrow\xi} f_x(x, t)$.

To show the smooth fit property (32), we follow Chernoff ([3], page 233). Obviously the right-hand derivative $\varphi_x^+(\xi, t) = v_x(\xi, t)$. For $0 \leqq x < \xi$, $\varphi(x, t) \leqq v(x, t)$ and therefore $\varphi_x^-(\xi, t) \geqq v_x(\xi, t)$. To prove the reverse inequality, we note that since $\varphi_{xx} = f_{xx}$ is bounded on G and $\varphi_{xx} = v_{xx}$ is bounded on $\{(x, t) : x \geqq b(t)\}$, we have

$$\begin{aligned} \varphi(\xi + x\delta^{\frac12}, t) &= \varphi(\xi, t) + x\delta^{\frac12}\varphi_x^+(\xi, t) + x^2O(\delta) , && 0 < x \leqq \delta^{-\frac12} \\ &= \varphi(\xi, t) + x\delta^{\frac12}\varphi_x^-(\xi, t) + x^2O(\delta) , && -\xi\delta^{-\frac12} \leqq x < 0 . \end{aligned} \tag{33}$$

Since φ is bounded, it then follows from (33) that

$$E\varphi(\xi + W(\delta), t) = \varphi(\xi, t) + (\delta/2\pi)^{\frac12}\{v_x(\xi, t) - \varphi_x^-(\xi, t)\} + O(\delta) . \tag{34}$$

Now $\varphi(\xi, t) \leqq \alpha\delta + E\varphi(\xi + W(\delta), t)$ and so (34) implies that $v_x(\xi, t) \geqq \varphi_x^-(\xi, t)$. □

The equations (26), (27), (28) and (32) restricted to the quadrant $\{(x, t) : x \geqq 0, t \geqq 0\}$ constitute a generalized Stefan problem (or free boundary problem). We now integrate these equations to obtain upper and lower bounds and the asymptotic behavior of the stopping boundary $b(t)$.

THEOREM 6. *Define*

$$\beta(t) = -\frac{t}{2} - \log 2\alpha - \log\left(-\frac{t}{2} - \log 2\alpha\right), \qquad \text{if} \quad -\frac{t}{2} - \log 2\alpha \geqq 1 ,$$
$$= \left(-\frac{t}{2} - \log 2\alpha\right)^+ \qquad \text{otherwise.}$$

Then for $t \geqq 0$,

$$\beta(t) \leqq b(t) \leqq \frac{1}{2\alpha} e^{-t/2} . \tag{35}$$

Furthermore,

$$b(t) \sim \frac{1}{2\alpha} e^{-t/2} \qquad \text{as} \quad t \to \infty . \tag{36}$$

For fixed $t \geqq 0$,

$$b(t) \sim -\log \alpha \qquad \text{as} \quad \alpha \downarrow 0 . \tag{37}$$

PROOF. Letting $\xi \uparrow b(t)$ in (31) and using Lemma 4, we obtain

$$\tfrac{1}{2} \exp\left(-\frac{t}{2} - b(t)\right) = \int_0^{b(t)} \frac{\partial f}{\partial t}(x, t)\, dx . \tag{38}$$

It then follows from (30) and (38) that

$$\alpha b(t) + \tfrac{1}{2} e^{-t/2}(e^{-b(t)} - 1) \leqq \tfrac{1}{2} \exp\left(-\frac{t}{2} - b(t)\right) \leqq \alpha b(t) . \tag{39}$$

From the above inequalities, we easily obtain (35) and (36).

To prove (37), we shall write $b(t) = b(t; \alpha)$ and $\varphi(x, t) = \varphi(x, t; \alpha)$ for clarity. In view of the lower bound in (35), we need only show that

$$\limsup_{\alpha \downarrow 0} b(t; \alpha)/|\log \alpha| \leqq 1 .$$

First consider a fixed $t > 0$. Suppose there exist $\varepsilon > 0$ and a sequence $\alpha_n \downarrow 0$ such that $b(t; \alpha_n) > -(1 + \varepsilon) \log \alpha_n > 0$ for all n. Let $x_n = -(1 + \frac{1}{2}\varepsilon) \log \alpha_n$ and take a fixed $t_1 \in (0, t)$. Then clearly $x_n < b(t_1; \alpha_n)$ and $\varphi(x_n, t_1; \alpha_n) \geqq \alpha_n E\sigma_n$, where $\sigma_n = \inf\{s \geqq 0 \colon |x_n + W(s)| \geqq b(t_1 + s; \alpha_n)\}$. Since $b(t; \alpha_n) - x_n > \frac{1}{2}\varepsilon |\log \alpha_n|$, we have $\liminf_{n\to\infty} E\sigma_n \geqq (t - t_1)$. Therefore

$$\alpha_n(t - t_1)(1 + o(1)) \leqq \alpha_n E\sigma_n < v(x_n, t_1) = e^{-t_1/2} \alpha_n^{1+(\varepsilon/2)}$$

which is a contradiction.

We now observe that

$$\varphi(x, t; \alpha) = \inf_T \left\{\alpha ET + e^{-t/2} E \exp\left(-|x + W(T)| - \frac{T}{2}\right)\right\}$$
$$= e^{-t/2} \varphi(x, 0; \alpha e^{t/2}) ,$$

and so $b(0; \alpha) = b(t; \alpha e^{-t/2})$. Hence the asymptotic formula (37) for $t = 0$ follows from the asymptotic result which we proved for $t > 0$. □

For fixed x and t, let us write $h(\alpha) = h(\alpha; x, t) = \varphi(x, t; \alpha)$ and define $\alpha(x, t) = \inf\{\alpha > 0 : \varphi(x, t; \alpha) = v(x, t)\}$. The function h has some interesting properties. It is continuous on $[0, \infty)$, strictly increasing on $[0, \alpha(x, t)]$, $h(0) = 0$ and $h \in C^\infty(0, \alpha(x, t))$. We note that

$$f(x, t; \alpha) = \alpha t + \varphi(x, t; \alpha) = \alpha t + e^{-t/2}\varphi(x, 0; \alpha e^{t/2})$$
$$= \alpha t + e^{-t/2}h(\alpha e^{t/2}) .$$

Therefore $(\partial/\partial t)f(x, t; \alpha) = \alpha - \frac{1}{2}e^{-t/2}h(\alpha e^{t/2}) + \frac{1}{2}\alpha h'(\alpha e^{t/2})$. Hence (30) implies

$$(40) \qquad 0 \leqq \alpha h'(\alpha) \leqq h(\alpha) , \qquad 0 < \alpha < \alpha(x, t) .$$

Obviously h is concave, and this in turn implies that if $|x| < b(t)$, then

$$\frac{\partial^2 f}{\partial t^2} \leqq \tfrac{1}{4}e^{-t/2}h(\alpha e^{t/2}) - \tfrac{1}{4}\alpha h'(\alpha e^{t/2})$$
$$= \tfrac{1}{4}e^{-t/2}h(\alpha e^{t/2}) \leqq \tfrac{1}{4}e^{-t/2-|x|} ,$$

and therefore $(\partial^2/\partial t^2)f \leqq (\partial^2/\partial t^2)g$.

The following theorem gives the asymptotic shape of the boundary $b(t)$ as $\alpha \downarrow 0$.

Theorem 7. *As* $\alpha \downarrow 0$

$$b(-s\log\alpha) \sim |\log\alpha|\left(1 - \frac{s}{2}\right) \qquad \text{if} \quad 0 \leqq s < 2 ,$$
$$b(-s\log\alpha) \sim \tfrac{1}{2}\alpha^{(s/2)-1} \qquad \text{if} \quad s > 2 ,$$

and $b(-2\log\alpha)$ *is a positive constant independent of* α.

Proof. Let $0 < s < 2$. In view of the lower bound in (35), we need only show that

$$(41) \qquad \limsup_{\alpha\downarrow 0} b(-s\log\alpha; \alpha)/|\log\alpha| \leqq 1 - \frac{s}{2}$$

where we write $b(t) = b(t; \alpha)$ and $\varphi(x, t) = \varphi(x, t; \alpha)$ as in the proof of Theorem 6. Suppose there exist $\varepsilon > 0$ and a sequence $\alpha_n \downarrow 0$ such that $b(-s\log\alpha_n; \alpha_n) > -(1 + \varepsilon - \frac{1}{2}s)\log\alpha_n$ and $\alpha_n < e^{-1}$ for all n. Let $x_n = -(1 + \frac{1}{2}\varepsilon - \frac{1}{2}s)\log\alpha_n$. Clearly $x_n < b(-s\log\alpha_n - 1; \alpha_n)$. Using a similar argument as in Theorem 6, we obtain that

$$\alpha_n(1 + o(1)) \leqq \varphi(x_n, -s\log\alpha_n - 1; \alpha_n)$$
$$< v(x_n, -s\log\alpha_n - 1) = e^{\frac{1}{2}}\alpha_n^{1+(\varepsilon/2)} ,$$

which is a contradiction. Hence (41) must hold.

We now consider the case $s > 2$. It follows from the upper bound in (35) that $\lim_{\alpha\downarrow 0} b(-s\log\alpha; \alpha) = 0$. Therefore setting $t = -s\log\alpha$ in (39), we obtain $b(-s\log\alpha; \alpha) \sim \frac{1}{2}\alpha^{s/2-1}$. Finally, since $b(0; \alpha) = b(t; \alpha e^{-t/2})$, it is easy to see that $b(-2\log\alpha; \alpha) = b(0; 1)$. □

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