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THE POWER SERIES DISTRIBUTION WITH UNKNOWN TRUNCATION PARAMETER[1]

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The minimum variance unbiased estimates are obtained when a sample is drawn from the power series distribution with unknown parameter λ and unknown truncation parameter ν. Unbiased tests of hypotheses are formulated.

1. Summary. The minimum variance unbiased estimators are obtained when a sample is drawn from the power series distribution with unknown parameter λ and unknown truncation parameter ν. Unbiased tests of hypotheses are formulated.

2. Probability model and sufficient statistics. Let $X_1, X_2, \cdots, X_n$ be a random sample from a power series distribution, see [5],

$$P[X = x] = \frac{1}{h(\nu, \lambda)} a(x)\lambda^x, \qquad x = \nu, \nu + 1, \cdots, \tag{1}$$

where $h(\nu, \lambda) = \sum_{r=\nu}^{\infty} a(r)\lambda^r$, $a(x) \geqq 0$; and the vector parameter $\boldsymbol{\theta} = (\nu, \lambda)$ is unknown. ν is the truncation parameter and $0 < \lambda < R$ where R is the radius of convergence for $h(\nu, \lambda)$. Here we shall assume that at most a finite number of $a(x)$, $x \geqq \nu$, are zero.

Let $X_{(1)} = \min [X_1, X_2, \cdots, X_n]$ and $T = \sum_{i=1}^{n} X_i$. Using (1) and the factorization theorem, it can be easily shown that $(X_{(1)}, T)$ is sufficient for $\boldsymbol{\theta}$.

3. Distribution of sufficient statistics. Let $g_n(\nu, t)$ denote the coefficient of λ^t in the expansion of $[h(\nu, \lambda)]^n$, i.e.

$$[h(\nu, \lambda)]^n = [\sum_{r=\nu}^{\infty} a(r)\lambda^r]^n = \sum_{t=\nu n}^{\infty} \lambda^t g_n(\nu, t) .$$

LEMMA 1. *The probability function of T is given by*

$$P[T = t] = \frac{\lambda^t}{[h(\nu, \lambda)]^n} g_n(\nu, t), \qquad t = n\nu, \cdots, \tag{2}$$

PROOF. Let S_j denote the number of X_i's equal to j. Then the joint probability function of $S_\nu, S_{\nu+1}, \cdots, S_T, T$ can be written

$$\begin{aligned} P[S_\nu = s_\nu, S_{\nu+1} = s_{\nu+1}, \cdots, S_t = s_t, T = t] \\ = \frac{n!\, \lambda^t}{s_\nu!\ s_{\nu+1}! \cdots s_t!} \frac{1}{[h(\nu, \lambda)]^n} \prod_{c=\nu}^{t} [a(c)]^{s_c} . \end{aligned} \tag{3}$$

The probability function of T is then

$$P[T = t] = \frac{\lambda^t}{[h(\nu, \lambda)]^n} \sum \frac{n!}{s_\nu!\ s_{\nu+1}! \cdots s_t!} \prod_{c=\nu}^{t} [a(c)]^{s_c} \tag{4}$$

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where the summation is over the nonnegative integers s_j satisfying

$$\sum_{c=\nu}^{t} s_c = n \quad \text{and} \quad \sum_{c=\nu}^{t} c s_c = t\,. \tag{5}$$

Clearly the sum is the coefficient of λ^t in the expansion of $[h(\nu, \lambda)]^n$. Thus the lemma is proved.

LEMMA 2. *The conditional probability function of $X_{(1)}$ given $T = t$ is*

$$P[X_{(1)} = x \mid T = t] = \frac{g_n(x, t) - g_n(x+1, t)}{g_n(\nu, t)}, \qquad x = \nu, \nu + 1, \cdots. \tag{6}$$

PROOF. From (3) and (4) we have

$$P[S_\nu = s_\nu, S_{\nu+1} = s_{\nu+1}, \cdots, S_t = s_t \mid T = t] = \frac{1}{g_n(\nu, t)} \frac{n!}{s_\nu!\ s_{\nu+1}! \cdots s_t!} \prod_{c=\nu}^{t} [a(c)]^{s_c}\,.$$

Hence

$$P[X_{(1)} = x \mid T = t] = \frac{1}{g_n(\nu, t)} \sum \frac{n!}{s_\nu!\ s_{\nu+1}! \cdots s_t!} \prod_{c=\nu}^{t} [a(c)]^{s_c}$$

where the summation is over the nonnegative integers s_j satisfying (5) with the restrictions

$$s_\nu = s_{\nu+1} = \cdots s_{x-1} = 0 \quad \text{and} \quad s_x \geqq 1\,.$$

Clearly the sum is $g_n(x, t) - g_n(x + 1, t)$, hence the lemma is proved.

From Lemmas 1 and 2 we have

$$\begin{aligned} P[X_{(1)} = x, T = t] &= \frac{\lambda^t}{[h(\nu, \lambda)]^n} [g_n(x, t) - g_n(x + 1, t)] \qquad (7) \\ &= \left\{\frac{1}{[h(\nu, \lambda)]^n} [[h(x, \lambda)]^n - [h(x + 1, \lambda)]^n]\right\} \left\{\frac{\lambda^t[g_n(x, t) - g_n(x + 1, t)]}{[h(x, \lambda)]^n - [h(x + 1, \lambda)]^n}\right\}, \end{aligned}$$

where the first and second terms are $P[X_{(1)} = x]$ and $P[T = t \mid X_{(1)} = x]$ respectively.

4. The minimum variance unbiased estimators of $\boldsymbol{\theta}$. From (7) it can be shown that $(X_{(1)}, T)$ is complete for $\boldsymbol{\theta} = (\nu, \lambda)$. Thus from the Rao–Blackwell theorem and the Lehmann–Scheffé theorem [4] the following theorems can be easily proved using (7).

THEOREM 1. *The minimum variance unbiased estimator of λ is given by*

$$\hat{\lambda} = \frac{g_n(X_{(1)}, T - 1) - g_n(X_{(1)} + 1, T - 1)}{g_n(X_{(1)}, T) - g_n(X_{(1)} + 1, T)}\,.$$

THEOREM 2. *The minimum variance unbiased estimator of ν is given by*

$$\hat{\nu} = X_{(1)} - \frac{g_n(X_{(1)} + 1, T)}{g_n(X_{(1)}, T) - g_n(X_{(1)} + 1, T)}\,.$$

Note that we have $E(X_{(1)}) = \nu + [h(\nu, \lambda)]^{-n} \sum_{x=\nu}^{\infty} [h(x+1, \lambda)]^n$. Thus to prove Theorem 2 one may show that $g_n(X_{(1)}+1, T)/[g_n(X_{(1)}, T) - g_n(X_{(1)}+1, T)]$ is the minimum variance unbiased estimator of $[h(\nu, \lambda)]^{-n} \sum_{x=\nu}^{\infty} [h(x+1, \lambda)]^n$. Now we can write

$$[h(\nu, \lambda)]^n = [\textstyle\sum_{x=\nu}^{\infty} a(x)\lambda^x]^n = \sum_{j=0}^{n} \binom{n}{j} [a(\nu)\lambda^\nu]^j [h(\nu+1, \lambda)]^{n-j} .$$

Thus we have

$$g_n(\nu, t) - g_n(\nu+1, t) = \textstyle\sum_{j=1}^{n} \binom{n}{j} [a(\nu)]^j g_{n-j}(\nu+1, t-\nu j) . \tag{8}$$

This recurrence relation can be used in the computation of the estimators.

5. Examples.

(a) *Geometric distribution.* See [2].

$$P[X = x] = pq^{x-\nu} , \qquad x = \nu, \nu+1, \cdots ,$$

where $0 < p < 1$, $q = 1 - p$, and $\nu \geqq 0$. Note that ν is the location parameter. Since $a(x) = 1$ and $h(\nu, q) = q^\nu/(1-q)$, we have

$$g_n(x, t) = \binom{n+t-nx-1}{t-nx} .$$

Hence the minimum variance unbiased estimators for q and ν are respectively given by

$$\hat{q} = \frac{\binom{n+T-nX_{(1)}-2}{T-nX_{(1)}-1} - \binom{T-nX_{(1)}-2}{T-n-nX_{(1)}-1}}{\binom{n+T-nX_{(1)}-1}{T-nX_{(1)}} - \binom{T-nX_{(1)}-1}{T-n-nX_{(1)}}} \qquad \text{and}$$

$$\hat{\nu} = X_{(1)} - \frac{\binom{T-nX_{(1)}-1}{T-n-nX_{(1)}}}{\binom{n+T-nX_{(1)}-1}{T-nX_{(1)}} - \binom{T-nX_{(1)}-1}{T-n-nX_{(1)}}} .$$

(b) *Poisson distribution.*

$$P[X = x] = \left[\textstyle\sum_{j=\nu}^{\infty} \frac{\lambda^j}{j!}\right]^{-1} \frac{\lambda^x}{x!} , \qquad x = \nu, \nu+1, \cdots ,$$

where $\lambda > 0$ and $\nu \geqq 0$.

For this distribution we do not have a simple closed form for $g_n(x, t)$. However we have $g_n(0, t) = n^t/t!$, and $t!\, g_n(1, t)/n!$ is a Stirling number of the second kind (see [1], pages 177–179 or [6], page 33). One can easily compute the estimator for λ and ν for the moderate value of n, using (8) and

$$t!\, g_n(\nu, t) = \sum \frac{n!}{s_\nu!\, s_{\nu+1}! \cdots s_t!} \frac{t!}{(\nu!)^{s_\nu} ((\nu+1)!)^{s_{\nu+1}} \cdots (t!)^{s_t}}$$

where the summation is over the nonnegative integers satisfying (5).

When ν is known, $\nu = \nu_0$ say, the minimum variance unbiased estimator of λ is obtained in [7], and using our notations it can be written $g_n(\nu_0, T-1)/g_n(\nu_0, T)$.

6. Tests of hypothesis. For simplicity, we shall restrict our testing to one-sided hypotheses although a two-sided hypothesis can be easily formulated ([3], Chapter 4).

THEOREM 3. *For testing the hypothesis* $H_\nu: \nu \leqq \nu_0$ *against the alternative* K_ν: $\nu > \nu_0$, *the uniformly most powerful* (u.m.p.) *unbiased test is given by*

$$\begin{aligned} \phi(X_{(1)}) &= 1 && \textit{if} \quad X_{(1)} > C(T) \\ &= \gamma(T) && \textit{if} \quad X_{(1)} = C(T) \\ &= 0 && \textit{if} \quad X_{(1)} < C(T) \end{aligned} \tag{9}$$

where $C(T)$ *and* $\gamma(T)$ *are uniquely determined from*

$$\sum_{x=\nu_0}^{\infty} \phi(x)[g_n(x, t) - g_n(x+1, t)]/g_n(\nu_0, t) = \alpha .$$

PROOF. From (2) it follows that the statistic T is sufficient and complete on the boundary $\nu = \nu_0$, $0 < \lambda < \infty$. From (6) it can be easily shown that the conditional probability function of $X_{(1)}$ given $T = t$ has the monotone likelihood ratio property. Thus the test (9) is the u.m.p. similar test of size α on the boundary $\nu = \nu_0$, $0 < \lambda < \infty$. This completes the proof of the theorem.

THEOREM 4. *For testing the hypothesis*

$$H_\lambda: \lambda \leqq \lambda_0 \quad \textit{against the alternative} \quad K_\lambda: \lambda > \lambda_0 ,$$

the u.m.p. *unbiased test is given by*

$$\begin{aligned} \phi(T) &= 1 && \textit{if} \quad T > C(X_{(1)}) , \\ &= \gamma(X_{(1)}) && \textit{if} \quad T = C(X_{(1)}) , \\ &= 0 && \textit{if} \quad T < C(X_{(1)}) , \end{aligned} \tag{10}$$

where $C(X_{(1)})$ *and* $\gamma(X_{(1)})$ *are uniquely determined so that*

$$\sum_{t=0}^{\infty} \phi(t) \frac{[g_n(x, t) - g_n(x+1, t)]\lambda_0^t}{[h(x, \lambda_0)]^n - [h(x+1, \lambda_0)]^n} = \alpha .$$

PROOF. From (7) it follows that the statistic $X_{(1)}$ is sufficient and complete on the boundary $\lambda = \lambda_0$, $0 < \nu < \infty$, and that the conditional probability function of T given $X_{(1)} = x$ is in the exponential family. Hence the u.m.p. similar test of size α on the boundary $\lambda = \lambda_0$, $0 < \nu < \infty$, is given by (10). Thus the theorem follows.

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