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A GEOMETRICAL CHARACTERIZATION OF BANACH SPACES IN WHICH MARTINGALE DIFFERENCE SEQUENCES ARE UNCONDITIONAL[1]

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We study Banach-space-valued martingale transforms and, in particular, characterize those Banach spaces for which the classical theorems of the real-valued case carry over. For example, if B is a Banach space and $1 < p < \infty$, then there exists a positive real number c_p such that

$$\|\epsilon_1 d_1 + \cdots + \epsilon_n d_n\|_p \le c_p \|d_1 + \cdots + d_n\|_p$$

for all B-valued martingale difference sequences $d = (d_1, d_2, \cdots)$ and all numbers $\epsilon_1, \epsilon_2, \cdots$ in $\{-1, 1\}$ if and only if there is a symmetric biconvex function ζ on $B \times B$ satisfying $\zeta(0, 0) > 0$ and $\zeta(x, y) \le |x + y|$ if $|x| \le 1 \le |y|$.

Introduction. Let $1 < p < \infty$. For what Banach spaces B does there exist a positive real number c_p such that

$$\|\epsilon_1 d_1 + \cdots + \epsilon_n d_n\|_p \le c_p \|d_1 + \cdots + d_n\|_p$$

for all B-valued martingale difference sequences $d = (d_1, d_2, \cdots)$, all numbers $\epsilon_1, \epsilon_2, \cdots$ in $\{-1, 1\}$, and all $n \ge 1$? This and closely related questions have been of interest to Maurey [16], Pisier [17], Diestel and Uhl [12], Aldous [1], and others. Let us write $B \in UMD$ (the space B has the unconditionality property for martingale differences) if such a constant $c_p = c_p(B)$ does exist. (Maurey uses a slightly different notation.) This class of spaces appears to depend on p but, in fact, does not [16] as we shall see in another way. It was proved in [4] that $\mathbb{R} \in UMD$ and from this follows immediately that the Lebesgue spaces ℓ^r, $L^r(0, 1) \in UMD$ for $1 < r < \infty$. Any UMD-space is reflexive, in fact superreflexive [16], [1], so, for example, ℓ^1, $\ell^\infty \notin UMD$. On the other hand, Pisier [17] has constructed an example showing that a superreflexive space need not be UMD.

A function $\zeta: B \times B \to \mathbb{R}$ is *symmetric* if $\zeta(x, y) = \zeta(y, x)$ and is *biconvex* if both $\zeta(\cdot, y)$ and $\zeta(x, \cdot)$ are convex on B for all $x, y \in B$. One of our main results is that $B \in UMD$ if and only if there is a symmetric biconvex function ζ on $B \times B$ satisfying $\zeta(0, 0) > 0$ and $\zeta(x, y) \le |x + y|$ if $|x| \le 1 \le |y|$. Here $|x|$ denotes the norm of x.

It is possible to replace the ϵ_k's by suitable random coefficients and this is important for stochastic integration and many other applications. We do this now and at the same time recall some definitions.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\mathscr{A}_0, \mathscr{A}_1, \cdots$ a nondecreasing sequence of sub-σ-fields of $\mathscr{A}$. Let $f = (f_1, f_2, \cdots)$ be a B-valued martingale with difference sequence $d = (d_1, d_2, \cdots)$: $f_n = \sum_{k=1}^n d_k$ where $d_k: \Omega \to B$ is strongly measurable relative to $\mathscr{A}_k$ (the pointwise limit of a sequence of simple $\mathscr{A}_k$-measurable functions) with $\|d_k\|_1 = E|d_k|$ finite and $E(d_{k+1} \mid \mathscr{A}_k) = 0$, $k \ge 1$. (For background on B-valued martingales, see [12].) Let $v = (v_1, v_2, \cdots)$ be a real-valued predictable sequence, that is, $v_k: \Omega \to \mathbb{R}$ is $\mathscr{A}_{k-1}$-measurable, $k \ge 1$. Then $g = (g_1, g_2, \cdots)$, defined by $g_n = \sum_{k=1}^n v_k d_k$, is the transform of the martingale f by v. We write $\|f\|_p = \sup_n \|f_n\|_p$ and define the maximal function of g by $g^*(\omega) = \sup_n |g_n(\omega)|$. (There should be no confusion here with linear functionals on B.)

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Recall that if $B = \mathbb{R}$ and f and g are as above with v uniformly bounded in absolute value by 1, then [4], for $1 < p < \infty$,

$$\|f\|_1 < \infty \quad \Rightarrow g \text{ converges a.e.,} \tag{0.1}$$

$$\lambda P(g^* > \lambda) \leq c\|f\|_1, \lambda > 0, \tag{0.2}$$

$$\|g\|_p \quad \leq c_p\|f\|_p. \tag{0.3}$$

These are analogous to the classical theorems of Privalov, Kolmogorov, and M. Riesz, respectively, that show how harmonic functions control their conjugates.

For what Banach spaces B does (0.1) hold for all such B-valued f and g? (The probability space may vary as well as f and g.) Or for what Banach spaces B does there exist a positive real number c such that (0.2) holds in the same sense? A similar question can be raised with respect to (0.3). In the next section, we shall show that all three of these questions lead to the same class of Banach spaces. For convenience, we shall write $B \in MT$ (for B, martingale transforms are well-behaved) if B does enjoy at least one of these properties, hence all three.

The equivalence of (0.1) and (0.2) in a Banach space setting contrasts with the nonequivalence of two results of Doob [13]: For $B = \mathbb{R}$,

$$\|f\|_1 < \infty \Rightarrow f \text{ converges a.e.,} \tag{0.1$'$}$$

$$\lambda P(f^* > \lambda) \leq \|f\|_1, \lambda > 0. \tag{0.2$'$}$$

Chatterji [10] proved that (0.1)′ holds for all martingales with values in B if and only if B has the Radon-Nikodým property. On the other hand, (0.2)′ holds for every Banach space since $(|f_1|, |f_2|, \cdots)$ is a real-valued submartingale. Therefore, (0.1)′ and (0.2)′ are not equivalent conditions on a Banach space.

In Section 2, we shall show that $B \in \mathrm{MT}$ if and only if $B \in UMD$. Hence the ζ-condition, described above, characterizes MT-spaces also.

Section 4 contains further information about the ζ-condition. For example, if it exists, ζ determines the norm of B up to equivalence.

There are other ways that martingales f and g may be related and other kinds of transforms. Section 5 contains a brief discussion of some of the possibilities.

The ζ-condition and other results of this paper throw new light on B-valued singular integrals as we hope to show elsewhere.

The Banach space B may be either real or complex. The letter c, with or without subscripts, is used to denote a positive real number, not necessarily the same number from one use to the next. The optimum value of c in inequality (1.2), say, is denoted by $c(1.2)$ when it is necessary to be more specific.

1. Equivalent probability conditions on a Banach space: Random coefficients. In this section, g is the transform of a B-valued martingale f by a real-valued predictable sequence v uniformly bounded in absolute value by 1.

THEOREM 1.1. *Let $1 < p < \infty$. For a Banach space B, the following statements, each to hold for all such f and g, are equivalent*:

$$\|f\|_1 < \infty \Rightarrow g \textit{ converges a.e.,} \tag{1.1}$$

$$\lambda P(g^* > \lambda) \leq c\|f\|_1, \lambda > 0, \tag{1.2}$$

$$\|g\|_p \leq c_p\|f\|_p. \tag{1.3}$$

There are many other statements about martingale transforms, some presented below, that are equivalent to these, but these are basic and illustrate the possibilities. Note that (1.2), for example, is an abbreviation of the lengthier statement: There is a positive real number $c = c(B)$ such that $\cdots$. We emphasize that the underlying probability space is to vary as well as f and g. Or if a fixed probability space is preferred, it must be nonatomic and, as above, the sequence $\mathscr{A}_0, \mathscr{A}_1, \cdots$ is to vary.

REMARK 1.1 In several of the proofs, it will be helpful to assume that $E(d_1 \mid \mathscr{A}_0) = 0$ or, even more, that f starts at the origin: $f_1 = d_1 = 0$. This can be assumed, for example, in the proof of Theorem 1.1. To see this, let f, v, g, and $\mathscr{A}_0 \subset \mathscr{A}_1 \subset \cdots \subset \mathscr{A}$ be as above. We may always assume that there is a sub-σ-field $\mathscr{B}$ of $\mathscr{A}$, independent of each $\mathscr{A}_n$, on which P is nonatomic. Let r be a $\mathscr{B}$-measurable random variable taking each of the values ± 1 with probability ½ and let $d^0 = (0, rd_1, rd_2, \cdots)$, $v^0 = (1, v_1, v_2, \cdots)$, $\mathscr{A}_0^0 = \{\phi, \Omega\}$, $\mathscr{A}_1^0 = \mathscr{A}_0$, and $\mathscr{A}_n^0 = \mathscr{A}_{n-1} \vee \mathscr{B}$ for $n \geq 2$. Then, relative to this new sequence of sub-σ-fields, v^0 is predictable and f^0 is a martingale starting at the origin. Also, since $f_n^0 = rf_{n-1}$ and $g_n^0 = rg_{n-1}$ for $n \geq 2$, we see that $\|f^0\|_1 = \|f\|_1$, $(g^0)^* = g^*$, g^0 converges a.e. if and only if g does, and so forth.

PROOF. (1.1) $\Rightarrow$ (1.2). This is a straightforward application of the B-space version of Theorem 2 of [3] in the case $p = 1$. However, it will be useful for the work of Section 3 to take a slightly different approach. We show first that (1.1) implies

$$g^* > 1 \text{ a.e.} \Rightarrow \|f\|_1 \geq c. \tag{1.4}$$

Suppose, on the contrary, that no such positive number $c = c(B)$ exists. Then, for each positive integer j, there is a martingale $f_j = (f_{j1}, f_{j2}, \cdots)$, starting at the origin, with associated v_j, g_j, and $\mathscr{A}_{j0} \subset \mathscr{A}_{j1} \subset \cdots$ such that $(g_j)^* > 1$ a.e. but $\|f_j\|_1 \leq 2^{-j}$. We may assume the underlying probability space is the same for all j and the sequence $\mathscr{A}_{1\infty}$, $\mathscr{A}_{2\infty}$, $\cdots$ is independent. Here $\mathscr{A}_{j\infty}$ denotes the smallest σ-field containing every $\mathscr{A}_{jn}$, $n \geq 0$. There is a positive integer n_j such that the event

$$A_j = \{|g_{jn}| > 1 \quad \text{for some } n \leq n_j\}$$

has probability greater than ½. Let

$$\mathscr{A}_0 = \mathscr{A}_{10}, \cdots, \mathscr{A}_{n_1-1} = \mathscr{A}_{1,n_1-1},$$

$$\mathscr{A}_{n_1} = \mathscr{A}_{1n_1} \vee \mathscr{A}_{20}, \mathscr{A}_{n_1+1} = \mathscr{A}_{1n_1} \vee \mathscr{A}_{21}, \cdots,$$

$$\mathscr{A}_{n_1+n_2} = \mathscr{A}_{1n_1} \vee \mathscr{A}_{2n_2} \vee \mathscr{A}_{30}, \cdots.$$

Then, relative to this sequence of sub-σ-fields,

$$D = (d_{11}, \cdots, d_{1n_1}, d_{21}, \cdots, d_{2n_2}, \cdots)$$

is a martingale difference sequence and

$$V = (v_{11}, \cdots, v_{1n_1}, v_{21}, \cdots, v_{2n_2}, \cdots)$$

is a predictable sequence. The martingale F determined by D satisfies

$$F_{n_1+\cdots+n_k} = \textstyle\sum_{j=1}^{k} f_{jn_j}$$

with a similar formula for the transform G of F by V. Therefore,

$$\|F\|_1 \leq \textstyle\sum_{j=1}^{\infty} \|f_{jn_j}\|_1 \leq \sum_{j=1}^{\infty} 2^{-j} = 1$$

and, by the Borel-Cantelli lemma,

$$P(G \text{ diverges}) \geq P(\limsup_{m,n\to\infty} |G_n - G_m| > 1) \geq P(\limsup_{j\to\infty} A_j) = 1,$$

where we have used the independence of $A_1, A_2, \cdots$ and the divergence of the series $\sum_{j=1}^{\infty} P(A_j)$. This contradicts (1.1); therefore, (1.1) implies (1.4).

To show that (1.4) implies (1.2), we shall use a method similar to one used by Bollobás [2] in his proof of the weak-L^1 inequality for the martingale square function. Fix (f, v, g) where f starts at the origin. If n is a positive integer and $g_n^*(\omega) = \sup_{1\leq k\leq n} |g_k(\omega)|$, then, as we shall show,

$$cP(g_n^* > 2) \le \|f\|_1 \tag{1.5}$$

where $c = c(1.4)$, the optimum constant of (1.4). This implies, by taking limits and scaling, that (1.2) holds with $c(1.2) \le 2/c(1.4)$. To prove (1.5), we may assume that the probability on the left-hand side is positive. For each positive integer j, let $g_j = (g_{j1}, g_{j2}, \cdots)$ be the transform of a martingale f_j by a predictable sequence v_j such that (f_j, v_j, g_j) has the same distribution as (f, v, g). We may assume, as above, that the underlying probability space is the same for all j and that $\{\mathscr{A}_{j\infty}, j \ge 1\}$ is independent. Let

$$D = (d_{11}, \cdots, d_{1n}, u_1 d_{21}, \cdots, u_2 d_{2n}, u_1 u_2 d_{31}, \cdots)$$

and

$$V = (v_{11}, \cdots, v_{1n}, v_{21}, \cdots, v_{2n}, v_{31}, \cdots)$$

where u_j is the indicator function of the set $\{g_{jn}^* \le 2\}$. The corresponding sequence of sub-σ-fields is as above with $n_j = n$. Then the martingale F determined by D satisfies

$$F_{kn} = f_{1n} + u_1 f_{2n} + \cdots + u_1 u_2 \cdots u_{k-1} f_{kn}$$

with a similar formula for the transform G of F by V. Therefore,

$$\|F\|_1 \le \|f\|_1 E(1 + u_1 + u_1 u_2 + \cdots) = \|f\|_1 / P(g_n^* > 2)$$

and $G^* > 1$ a.e. so that, by (1.4), $c \le \|F\|_1$. This gives (1.5) and completes the proof of (1.1) $\Rightarrow$ (1.2).

(1.2) $\Rightarrow$ (1.3): Suppose that Φ is a convex function from $[0, \infty)$ onto $[0, \infty)$ satisfying the growth condition

$$\Phi(2\lambda) \le c\Phi(\lambda), \quad \lambda > 0, \tag{1.6}$$

and set $\Phi(\infty) = \infty$. Then (1.2) implies

$$E\Phi(g^*) \le cE\Phi(f^*) \tag{1.7}$$

where $c(1.7)$ depends only on $c(1.2)$ and $c(1.6)$. In particular, $\Phi(\lambda) = \lambda^p$ gives

$$\|g^*\|_p \le c_p \|f^*\|_p.$$

In view of $\|g\|_p \le \|g^*\|_p$ and $\|f^*\|_p \le q\|f\|_p$ (Doob's inequality with $1/p + 1/q = 1$), we see that (1.3) follows.

In the real case, (1.7) is an immediate consequence of the two-sided Φ-inequality between the square function $S(f)$ and the maximal function f^* obtained in [9] and the fact that $S(g) \le S(f)$. Here the square function inequality is no longer generally valid; however, a direct approach to g^* is possible. One may use the Banach-space version of Theorem 2.1 of [9] or proceed more concretely as follows.

As above, we may assume that f starts at the origin. Let w_k be an $\mathscr{A}_{k-1}$ measurable majorant of $|d_k|$. Let $\delta > 0$, $\beta > \delta + 1$,

$$\mu(\omega) = \inf\{n: |g_n(\omega)| > \lambda\},$$

$$\nu(\omega) = \inf\{n: |g_n(\omega)| > \beta\lambda\},$$

$$\sigma(\omega) = \inf\{n: |f_n(\omega)| > \delta\lambda \text{ or } w_{n+1}(\omega) > \delta\lambda\},$$

and u_k be the indicator function of $\{\mu < k \le \nu \wedge \sigma\}$. Then $u = (u_1, u_2, \cdots)$ is a predictable sequence and F, the transform of f by u, is a martingale satisfying $F^* \le 3\delta\lambda$ on $\{\mu < \infty\} = \{g^* > \lambda\}$ and $F^* = 0$ elsewhere. It follows that $\|F\|_1 \le 3\delta\lambda\, P(g^* > \lambda)$. If G is the transform of F by v, hence also the transform of g by u, then

$$P(g^* > \beta\lambda, f^* \vee w^* \le \delta\lambda) \le P(G^* > (\beta - \delta - 1)\lambda) \le c(1.2)\, \|F\|_1 / (\beta - \delta - 1)\lambda.$$

Therefore, (1.2) implies that

$$P(g^* > \beta\lambda, f^* \vee w^* \leq \delta\lambda) \leq \epsilon P(g^* > \lambda) \tag{1.8}$$

where $\epsilon = 3c(1.2)\delta/(\beta - \delta - 1)$. This gives, by Lemma 7.1 of [5], that

$$E\Phi(g^*) \leq cE\Phi(f^* \vee w^*). \tag{1.9}$$

This inequality can now be applied to the good part of Davis's decomposition [11] of f. The bad part can be handled separately. Inequality (1.7) then follows. For an illustration of the details of the method, see page 34 of [5]. This completes the proof of (1.2) $\Rightarrow$ (1.3).

REMARK 1.2. With a slight change, the above proof of (1.2) $\Rightarrow$ (1.7) also gives (1.3) $\Rightarrow$ (1.7): Use the fact that $\{|G_n|, n \geq 1\}$ is a nonnegative submartingale to obtain

$$P(G^* > (\beta - \delta - 1)\lambda) \leq \|G\|_p^p/(\beta - \delta - 1)^p\lambda^p$$

and notice that (1.8) must then hold with $\epsilon = [3c(1.3)\delta/(\beta - \delta - 1)]^p$. Similar reasoning shows that (1.7) holding for one Φ implies that it holds for all Φ satisfying the required conditions.

(1.3) $\Rightarrow$ (1.1): The special case of (1.3) in which the coefficients are constant and of modulus one (the UMD property) implies that B is superreflexive [16], [1]. Therefore, B has the Radon-Nikodým property (for example, see [12]). Accordingly, if G is a B-valued martingale with $\|G\|_1$ finite, then G converges a.e. (Chatterji [10]).

Consider f, v, g with $\|f\|_1$ finite and let h be the nonnegative submartingale defined by $h_n = |f_n|$. Let $\mu = \inf\{n: h_n > \lambda\}$ and F be the transform of f by the predictable sequence u where u_k is the indicator function of $\{\mu \geq k\} = \{f_{k-1}^* \leq \lambda\}$. Then $F_n^* \leq \lambda + h_{\mu\wedge n}$ where

$$Eh_{\mu\wedge n} \leq Eh_n \leq \|h\|_1 = \|f\|_1$$

so $EF^* \leq \lambda + \|f\|_1$ is finite. By Remark 1.2 and (1.7), with $\Phi(\lambda) = \lambda$, it follows that G, the transform of g by u hence also the transform of F by v, satisfies $\|G\|_1 \leq EG^* \leq cEF^* < \infty$. Therefore, by Chatterji's result mentioned above, G converges a.e. Since $g = G$ on $\{\mu = \infty\} = \{f^* \leq \lambda\}$, it follows that g converges a.e. on $\{f^* \leq \lambda\}$. In view of the inequality $\lambda P(f^* > \lambda) \leq \|f\|_1$, we conclude that g converges a.e. This completes the proof of Theorem 1.1.

2. Equivalent probability conditions on a Banach space: Nonrandom coefficients. In the following theorem, g is the transform of a B-valued martingale f by a sequence of numbers in $\{-1, 1\}$.

THEOREM 2.1. *Let $1 < p < \infty$. For a Banach space B, the following statements, each to hold for all such f and g, are equivalent*:

$$\|f\|_1 < \infty \Rightarrow g \text{ converges a.e.}, \tag{2.1}$$

$$\lambda P(g^* > \lambda) \leq c\|f\|_1, \qquad \lambda > 0, \tag{2.2}$$

$$\|g\|_p \leq c_p\|f\|_p. \tag{2.3}$$

The $\pm$ 1-version of (1.7) is also equivalent to each of the above statements as is

$$g^* > 1 \text{ a.e.} \Rightarrow \|f\|_1 \geq c. \tag{2.4}$$

The proof of Theorem 1.1 carries over to the present setting.

How are the two theorems related? First note that if $B \in UMD$, then (2.3) holds and, since $\|f_1\|_p \leq \cdots \leq \|f_n\|_p$, the converse is also true.

THEOREM 2.2. *If B is a Banach space, then*

$$B \in MT \Leftrightarrow B \in UMD. \tag{2.5}$$

PROOF. We shall show a little more: $c_p(1.3) = c_p(2.3)$. Let f be a martingale with difference sequence d and g a transform of f by v as in Theorem 1.1. It is enough to show that if $\|f\|_p$ is finite, then

$$\|g_n\|_p \le c_p(2.3)\,\|f_n\|_p \tag{2.6}$$

and further reductions are possible:

(i) It is enough to show that (2.6) holds in the special case

$$v_k = H_k(f_0, \cdots, f_{k-1}),\ 1 \le k \le n, \tag{2.7}$$

where $f_0 = 0$ and $H_k: B \times \cdots \times B \to [-1, 1]$ is continuous. This may be seen by slightly modifying the proof of the comparable step in [6].

(ii) To prove (2.6), we may assume that $f_1, \cdots, f_n$ are simple functions: Let d_{jk} be a simple $\mathscr{A}_k$-measurable function such that $\|d_{jk} - d_k\|_p \le 2^{-j-1}$. Let $D_{j1} = d_{j1}$ and $D_{jk} = d_{jk} - E(d_{jk} \mid \mathscr{A}_{j,k-1})$, $k > 1$, where $\mathscr{A}_{jk}$ is the field generated by $d_{j1}, \cdots, d_{jk}$. Then, for $k > 1$,

$$\|D_{jk} - d_k\|_p = \|d_{jk} - d_k - E(d_{jk} - d_k \mid \mathscr{A}_{j,k-1})\|_p \le 2^{-j}.$$

Therefore, the martingale F_j corresponding to $D_j = (D_{j1}, D_{j2}, \cdots)$ satisfies $\|F_{jk} - f_k\|_p \le k2^{-j}$ and $F_{jk} \to f_k$ a.e. as $j \to \infty$. Accordingly, by (i), $V_{jk} = H_k(F_{j0}, \cdots, F_{j,k-1}) \to v_j$ a.e. Also, by Lebesgue's dominated convergence theorem,

$$\|v_k d_k - V_{jk}D_{jk}\|_p \le \|(v_k - V_{jk})d_k\|_p + \|d_k - D_{jk}\|_p$$

converges to 0, which implies that $\|G_{jn} - g_n\|_p$ converges to 0, as $j \to \infty$. So (2.6) must hold if

$$\|G_{jn}\|_p \le c_p(2.3)\,\|F_{jn}\|_p$$

and this gives the desired reduction to simple functions.

(iii) To prove (2.6), we may assume that g is the transform of f by a sequence of numbers in $[-1, 1]$: If the map $\omega \to (f_1(\omega), \cdots, f_n(\omega))$ has finite range, as in (ii), and (2.7) holds, then there is a martingale difference sequence D, numbers $a_1, a_2, \cdots$ in $[-1, 1]$, and a positive integer N such that

$$f_n = \textstyle\sum_{k=1}^N D_k, \tag{2.8}$$

$$g_n = \textstyle\sum_{k=1}^N a_k D_k. \tag{2.9}$$

(In addition, $f_n^* = F_N^*$ and $g_n^* = G_N^*$.) First let $D_1 = d_1$ and $a_1 = v_1 = H_1(0)$. At the second stage, let m_1 denote the number of points in the range of f_1 and define the next m_1 differences D_k and coefficients a_k in any order by

$$D_k = I(f_1 = x_1)d_2 \quad \text{and} \quad a_k = H_2(0, x_1)$$

where $I(A)$ is the indicator function of the set A and x_1 varies over the range of f_1. At the third stage, m_2 is the number of points in the range of $\omega \to (f_1(\omega), f_2(\omega))$ and the formulas become

$$D_k = I(f_1 = x_1, f_2 = x_2)d_3,$$

$$a_k = H_3(0, x_1, x_2),$$

and so forth. Let $N = 1 + m_1 + \cdots + m_{n-1}$; set $D_k = 0$ and $a_k = 0$ for $k > N$. Relative to a suitable sequence of sub-σ-fields (e.g., the smallest possible), $D = (D_1, D_2, \cdots)$ is a martingale difference sequence and both (2.8) and (2.9) hold.

So we can complete the proof of Theorem 2.2 by showing that if d is a martingale difference sequence and $a_1, a_2, \cdots$ are numbers in $[-1, 1]$, then

$$\|\textstyle\sum_{k=1}^n a_k d_k\|_p \le c_p(2.3)\,\|\sum_{k=1}^n d_k\|_p.$$

(Since this holds for $a_k = \pm 1$, this is an example of the contraction principle; cf. Kahane [14].) Write $a_k = \sum_{j=1}^\infty \epsilon_{jk} 2^{-j}$ where $\epsilon_{jk} = \pm 1$. Then

$$\|\sum_{k=1}^{n} a_k d_k\|_p = \|\sum_{j=1}^{\infty} 2^{-j} \sum_{k=1}^{n} \epsilon_{jk} d_k\|_p \leq \sum_{j=1}^{\infty} 2^{-j} \|\sum_{k=1}^{n} \epsilon_{jk} d_k\|_p \leq c_p(2.3) \|\sum_{k=1}^{n} d_k\|_p.$$

3. A geometrical characterization.

THEOREM 3.1. *A Banach space B is an MT-space (equivalently, $B \in UMD$) if and only if there is a symmetric biconvex function ζ: $B \times B \to \mathbb{R}$ such that $\zeta(0, 0) > 0$ and*

$$\zeta(x, y) \leq |x + y| \quad \text{if} \quad |x| \leq 1 \leq |y|. \tag{3.1}$$

To gain some feeling for this condition, note that $\zeta(x, y) = |x + y|$ is symmetric, biconvex, and satisfies (3.1) but does not satisfy $\zeta(0, 0) > 0$. In fact, any function ζ satisfying (3.1) must also satisfy $\zeta(x, -x) \leq |x - x| = 0$ for all x on the boundary of the unit ball, so if $\zeta(0, 0)$ is to be positive, ζ cannot be convex.

Here is an example in the positive direction: If B is a space with an inner product, let

$$\zeta(x, y) = 1 + (x, y) \tag{3.2}$$

($1 + \mathrm{Re}(x, y)$ in the complex case). Then ζ is symmetric, biconvex, $\zeta(0, 0) = 1$, and (3.1) follows from

$$[1 + (x, y)]^2 \leq 1 + 2(x, y) + |x|^2|y|^2 = |x + y|^2 + (1 - |x|^2)(1 - |y|^2). \tag{3.3}$$

Actually, the converse holds, although we shall not prove it here: If the ζ-condition is satisfied with $\zeta(0, 0) = 1$ (the extreme case), then B is an inner product space. See [7].

Sometimes it is convenient to replace (3.1) by the condition

$$\zeta(x, y) \leq |x + y| \quad \text{if} \quad |y| \geq 1, \tag{3.4}$$

and we shall do this without loss of generality in the proof of Theorem 3.1: Suppose that ζ satisfies the conditions of Theorem 3.1 and let

$$\begin{aligned}
\zeta_1(x, y) &= \zeta(x, y) \vee |x + y| && \text{if} \quad |y| < 1,\\
&= |x + y| && \text{if} \quad |y| \geq 1,\\
\zeta_2(x, y) &= \zeta(x, y) \vee |x + y| && \text{if} \quad |x| \vee |y| < 1,\\
&= |x + y| && \text{if} \quad |x| \vee |y| \geq 1.
\end{aligned}$$

Then, $\zeta_1 = \zeta_2$: By definition, there is equality on the set where $|x| \vee |y| < 1$ or $|y| \geq 1$. Equality also holds on the complementary set, where $|y| < 1 \leq |x|$, because $\zeta(x, y) \leq |x + y|$ by (3.1) and the symmetry of ζ. Now notice that $\zeta_1(x, y) = \zeta_2(x, y) = \zeta_2(y, x) = \zeta_1(y, x)$, and $\zeta_1(\cdot, y)$ is convex. Therefore, ζ_1 is symmetric, biconvex, $\zeta_1(0, 0) = \zeta(0, 0) > 0$, and ζ_1 satisfies (3.4).

PROOF OF THEOREM 3.1. Let $M(x, y)$ be the class of all B-valued martingales f starting at x such that, for some sequence $(1, \epsilon_2, \epsilon_3, \cdots)$ in $\{-1, 1\}$, the transform g of f by this sequence satisfies

$$P(|g_n - y| \geq 1 \quad \text{for some} \quad n \geq 1) = 1. \tag{3.5}$$

Let $\psi(x, y) = \inf\{\|f\|_1 : f \in M(x, y)\}$. Then, as we shall show,

$$\psi(x, y) = \psi(x, 2x - y), \tag{3.6}$$

$$\psi(\cdot, y) \text{ is convex,} \tag{3.7}$$

$$\psi(x, y) \leq |x| \quad \text{if} \quad |y| \geq 1, \tag{3.8}$$

and $B \in UMD$ if and only if

$$\psi(0, 0) > 0. \tag{3.9}$$

These properties of ψ imply that ζ, defined by

$$\zeta(x, y) = 2\psi\left(\frac{x+y}{2}, y\right), \tag{3.10}$$

is symmetric, biconvex, and satisfies (3.4), hence (3.1). Furthermore, if $B \in UMD$, then $\zeta(0, 0) > 0$.

By considering the transforms of f in pairs, relative to both $(1, \epsilon_2, \epsilon_3, \cdots)$ and $(1, -\epsilon_2, -\epsilon_3, \cdots)$, we see that $M(x, y) = M(x, 2x - y)$. This gives (3.6).

To prove (3.7), let $x, x_1, x_2 \in B$ and $\alpha_1 > 0$, $\alpha_2 > 0$ satisfy $\alpha_1 + \alpha_2 = 1$ and $x = \alpha_1 x_1 + \alpha_2 x_2$. Let $\delta > 0$ and, for $j \in \{1, 2\}$, let $f_j = (f_{j1}, f_{j2}, \cdots)$ be a martingale in $M(x_j, y)$, relative to $\mathscr{A}_{j1} \subset \mathscr{A}_{j2} \subset \cdots$, such that $\| f_j \|_1 \le \psi(x_j, y) + \delta$. Let $(1, \epsilon_{j2}, \epsilon_{j3}, \cdots)$ be a sequence of coefficients giving (3.5). We may assume that the underlying probability space is the same for both values of j and that $\mathscr{A}_{1\infty} = \bigvee_n \mathscr{A}_{1n}$, $\mathscr{A}_{2\infty}$, and $\mathscr{B}$ are independent where $\mathscr{B}$ is a sub-σ-field on which P is nonatomic. Let $u_1 = 1 - u_2$ be a $\mathscr{B}$-measurable random variable with $P(u_1 = 1) = \alpha_1$, $P(u_1 = 0) = \alpha_2$. Then, with $s_j = x_j - x$,

$$D = (x, u_1 s_1 + u_2 s_2, u_1\, d_{12}, u_2\, d_{22}, u_1\, d_{13}, u_2\, d_{23}, \cdots)$$

is a martingale difference sequence relative to

$$(\{\phi, \Omega\}, \mathscr{B}, \mathscr{A}_{12} \vee \mathscr{B}, \mathscr{A}_{12} \vee \mathscr{A}_{22} \vee \mathscr{B}, \cdots).$$

Consider F and G determined by D and the coefficient sequence

$$(1, 1, \epsilon_{12}, \epsilon_{22}, \epsilon_{13}, \epsilon_{23}, \cdots).$$

Then, $F_1 = x$, $F_{2n} = u_1 f_{1n} + u_2 f_{2n}$, and $G_{2n} = u_1 g_{1n} + u_2 g_{2n}$. Thus, $F \in M(x, y)$ and

$$\| F_{2n} \|_1 \le \alpha_1 \| f_{1n} \|_1 + \alpha_2 \| f_{2n} \|_1 \le \alpha_1 \psi(x_1, y) + \alpha_2 \psi(x_2, y) + \delta$$

giving $\psi(x, y) \le \alpha_1 \psi(x_1, y) + \alpha_2 \psi(x_2, y)$ and the convexity of $\psi(\cdot, y)$.

To prove (3.8), let $|y| \ge 1$ and consider first the special case $|x - y| \ge 1$. If $f_n = g_n = x$ for all n, then $f \in M(x, y)$ and $\| f \|_1 = |x|$. So (3.8) holds in this case. In particular, $\psi(0, y) = 0$. Now let $x \ne 0$. Then, for $\lambda > 1$ large enough, we have that $|\lambda x - y| \ge 1$ so, by convexity and the above special case,

$$\psi(x, y) \le (1 - \lambda^{-1})\psi(0, y) + \lambda^{-1}\psi(\lambda x, y) = \lambda^{-1}\psi(\lambda x, y) \le \lambda^{-1} |\lambda x| = |x|.$$

Now suppose that $B \in UMD$. Let $f \in M(0, 0)$ and g be as in the definition of $M(0, 0)$ so that g satisfies (3.5) with $y = 0$. Then, $g^* \ge 1$ a.e. so $(\lambda g)^* > 1$ a.e. and $\| f \|_1 \ge c(2.4)/\lambda$, for all $\lambda > 1$, giving $\psi(0, 0) \ge c(2.4)$ and (3.9).

To go in the other direction, suppose that f and g are as in Theorem 2.1. Then, by Remark 1.1 and the definition of ψ,

$$g^* > 1 \text{ a.e.} \Rightarrow \| f \|_1 \ge \psi(0, 0). \tag{3.11}$$

Therefore, $B \in UMD$ if $\psi(0, 0) > 0$.

We shall complete the proof of the theorem by showing that if there is a symmetric biconvex function ζ satisfying (3.4) and $\zeta(0, 0) > 0$, then $\psi(0, 0) > 0$. This will be accomplished by showing that

$$\psi(x, y) \ge \varphi(x, y) \tag{3.12}$$

where $\varphi(x, y) = \frac{1}{2}\zeta(2x - y, y)$. Note that

$$\varphi(x, y) = \varphi(x, 2x - y), \tag{3.13}$$

$$\varphi(\cdot, y) \text{ is convex}, \tag{3.14}$$

$$\varphi(x, y) \le |x| \quad \text{if} \quad |y| \ge 1. \tag{3.15}$$

(Since these are the only properties of φ that will be used in the proof of (3.12), ψ is the greatest function φ with these three properties.)

Let $M_n^+(x, y)$ be the class of all $f \in M(x, y)$ such that, for some sequence $(1, 1, \epsilon_3, \epsilon_4, \cdots)$ in $\{-1, 1\}$, the transform g of f by this sequence satisfies

$$P(|g_k - y| \geq 1 \quad \text{for some} \quad k \leq n) = 1.$$

If $n \geq 2$, as we shall assume, this class is nonempty. Let $\psi_n^+(x, y) = \inf\{\| f_n \|_1 : f \in M_n^+(x, y)\}$ and define $M_n^-(x, y)$ and $\psi_n^-(x, y)$ similarly using coefficient sequences of the form $(1, -1, \epsilon_3, \epsilon_4, \cdots)$. If there is no restriction on $\epsilon_2 \in \{-1, 1\}$, write $M_n(x, y)$ and $\psi_n(x, y)$. Note that $\psi_n(x, y) = \psi_n^+(x, y) \wedge \psi_n^-(x, y)$ and $M_n^-(x, y) = M_n^+(x, 2x - y)$ so

$$\psi_n(x, y) = \psi_n^+(x, y) \wedge \psi_n^+(x, 2x - y). \tag{3.16}$$

The main step in proving (3.12) is to show that

$$\psi_n(x, y) \geq \varphi(x, y). \tag{3.17}$$

But this follows, by (3.13) and (3.16), from the simpler inequality

$$\psi_n^+(x, y) \geq \varphi(x, y). \tag{3.18}$$

To prove this inequality for $n = 2$, let $f \in M_2^+(x, y)$. Then $g_2 = f_2$ and either $|x - y| \geq 1$ or $|f_2 - y| \geq 1$ a.e. We may assume the latter. An immediate consequence of (3.13) and (3.15) is that $\varphi(x, y) \leq |x|$ if $|x - y| \geq 1$. Therefore, $\varphi(f_2, y) \leq |f_2|$ a.e. and, by Jensen's inequality,

$$\varphi(x, y) \leq E\varphi(f_2, y) \leq \| f_2 \|_1.$$

Accordingly, both (3.18) and (3.17) hold for $n = 2$.

We now use induction (cf. [2] where induction is used to obtain a new proof of the weak-L^1 inequality for the martingale square function in the real case). Suppose (3.17) holds for some $n \geq 2$ and $f \in M_{n+1}^+(x, y)$. We may assume that f_2 has finite range $\{x_1, \cdots, x_m\}$ and that $\alpha_j = P(f_2 = x_j) > 0$ for $j = 1, \cdots, m$. (See step (ii) in the proof of Theorem 2.2.) We may also assume that the g associated with f in the definition of $M_{n+1}^+(x, y)$ satisfies

$$P(|g_k - y| \geq 1 \quad \text{for some} \quad k = 2, \cdots, n + 1) = 1. \tag{3.19}$$

Consider

$$\| f_{n+1} \|_1 = \textstyle\sum_{j=1}^m \displaystyle\int_{\{f_2 = x_j\}} | x_j + d_3 + \cdots + d_{n+1} | \, dP$$

and the martingale difference sequence $D_j = (x_j, d_3, d_4, \cdots)$ relative to $(\Omega, \mathscr{A}, P_j)$ and $(\mathscr{A}_2, \mathscr{A}_3, \cdots)$ where P_j is the conditional probability $P(\cdot | f_2 = x_j)$. By (3.19), the corresponding martingale F_j belongs to $M_n(x_j, y)$ and, by the induction hypothesis,

$$\| f_{n+1} \|_1 = \textstyle\sum_{j=1}^m \alpha_j \displaystyle\int_\Omega | F_{jn} | \, dP_j \geq \textstyle\sum_{j=1}^m \alpha_j \psi_n(x_j, y) \geq \sum_{j=1}^m \alpha_j \varphi(x_j, y) \geq \varphi(x, y).$$

Therefore, (3.18) and (3.17) hold for $n + 1$.

The final step in the proof is to show that (3.17) implies (3.12). Let f and g be as in the definition of $M(x, y)$ and let $\delta > 0$. Then there is a positive integer n such that the event

$$A = \{|g_k - y| \geq 1 \quad \text{for some} \quad k \leq n\}$$

has probability greater than $1 - \delta$. Let u be the indicator function of the complement of A. We may assume there is a sub-σ-field $\mathscr{B}$ independent of $\mathscr{A}_n$ on which P is nonatomic. Let r be $\mathscr{B}$-measurable with $P(r = \pm 1) = ½$ and let $z \in B$ satisfy $|z| = 2$. Then

$$D = (d_1, \cdots, d_n, urz, 0, 0, \cdots)$$

is a martingale difference sequence relative to

$$(\mathscr{A}_1, \cdots, \mathscr{A}_n, \mathscr{A}_n \vee \mathscr{B}, \cdots).$$

The corresponding martingale F belongs to $M_{n+1}(x, y)$: If G, the transform of F by the same sequence giving g from f, has not visited the complement of the open ball with center y and radius 1 by n, then G will visit it at $n + 1$. Therefore, by (3.17),

$$\varphi(x, y) \le \psi_{n+1}(x, y) \le \| F_{n+1} \|_1 \le \| f_n \|_1 + | z | Eu \le \| f \|_1 + 2\delta.$$

This implies (3.12) and completes the proof of the theorem.

Additional information can be obtained by modifying the above proof. The next two theorems illustrate some of the possibilities.

THEOREM 3.2. *If B is a Banach space, $\Lambda: B \times B \times [0, 1] \to \mathbb{R}$ satisfies*

$$\Lambda(x, y, t) = \Lambda(y, x, t), \tag{3.20}$$

(3.21) *the mapping $(s, t) \to \Lambda(x + \alpha s, y + \beta s, t)$ is convex on $B \times [0, 1]$ for all $\alpha, \beta \ge 0$,*

$$\Lambda(x, y, 1) \le | x + y | \quad \text{if} \quad | x | \le 1 \le | y |, \tag{3.22}$$

$$\Lambda(x, y, 0) \le | x + y | \quad \text{if} \quad | x | \le | y | \le 1, \tag{3.23}$$

and g is the transform of a B-valued martingale f by a real predictable sequence v uniformly bounded in absolute value by 1, then

$$\lambda\Lambda(0, 0, P(g^* > \lambda)) \le 2 \| f \|_1, \lambda > 0. \tag{3.24}$$

For example, if B is an inner product space, then

$$\Lambda(x, y, t) = t + (x, y) \tag{3.25}$$

$(t + Re(x, y)$ in the complex case) satisfies (3.20), the convexity property (3.21) follows from

$$\Lambda(x + \alpha s, y + \beta s, t) = t + (x, y) + \alpha(s, y) + \beta(x, s) + \alpha\beta | s |^2,$$

inequality (3.22) is implied by (3.3), and (3.23) follows from

$$2 | x + y | - 2(x, y) \ge 2 | x + y | - | x + y |^2 = 1 - (1 - | x + y |)^2.$$

Here $\Lambda(0, 0, t) = t$ so (3.24) becomes

$$\lambda P(g^* > \lambda) \le 2 \| f \|_1, \qquad \lambda > 0, \tag{3.26}$$

and 2 is the best constant as in the real case [6]. Conversely, if B is a Banach space such that 2 is the best constant in (1.2) or (2.2), then the ζ-condition is satisfied with $\zeta(0, 0) = 1$ (in the proof of Theorem 3.1, the function ψ must satisfy $\psi(0, 0) = ½$) and, consequently [7], B is an inner product space.

PROOF. We may replace (3.22) and (3.23) by

$$\Lambda(x, y, 1) \le | x + y | \quad \text{if} \quad | y | \ge 1, \tag{3.27}$$

$$\Lambda(x, y, 0) \le | x + y |. \tag{3.28}$$

To show this, consider Λ_0 defined by

$$\begin{aligned} \Lambda_0(x, y, t) &= \Lambda(x, y, t) \vee | x + y | && \text{if} \quad | x | \vee | y | < 1, \\ &= | x + y | && \text{if} \quad | x | \vee | y | \ge 1. \end{aligned}$$

Clearly, $\Lambda_0(0, 0, t) \ge \Lambda(0, 0, t)$ and Λ_0 satisfies (3.20), (3.22), and (3.23). To show that Λ_0 satisfies (3.21), we shall use, among other things, that

$$\Lambda(x, y, t) \le | x + y | \quad \text{if} \quad | x | \vee | y | = 1, \tag{3.29}$$

which follows from the convexity, symmetry, and other properties of Λ. Let

$$H_1(s, t) = \Lambda(x + \alpha s, y + \beta s, t),$$

$$H_2(s, t) = | x + \alpha s + y + \beta s |.$$

Then both H_1 and H_2 are convex on the line segment joining any two points (s_1, t_1), (s_2, t_2) in $B \times [0, 1]$. Let $0 \le \delta < 1$,

$$K_j(t) = H_j((1-t)s_1 + ts_2, (1-t)t_1 + tt_2),$$

and

$$\begin{aligned} K_\delta(t) &= (K_1(t) - \delta) \vee K_2(t) \qquad &\text{if} \quad t \in T^-, \\ &= K_2(t) &\text{if} \quad t \in T^0 \cup T^+, \end{aligned}$$

where

$$T^- = \{t \in [0, 1] : | x + \alpha[(1-t)s_1 + ts_2] | \vee | y + \beta[(1-t)s_1 + ts_2]| < 1\},$$

T^0 is the set where equality holds, and T^+ is the set where the inequality sign is reversed. To show that Λ_0 satisfies (3.21), it is enough to show that K_0 is convex on [0, 1] and this follows from the fact that K_δ is convex on [0, 1] for $0 < \delta < 1$: Clearly, K_δ is convex on each component of T^- and on each component of T^+. Suppose $t_0 \in T^0$. Then $K_1(t_0) \le K_2(t_0)$ by (3.29). Therefore, there is a relatively open subinterval I of [0, 1] containing t_0 such that

$$K_1(t) - \delta < K_1(t_0) - \delta/2 < K_2(t), \qquad t \in I,$$

implying that $K_\delta(t) = K_2(t)$, $t \in I$. Thus, K_δ is locally convex on [0, 1] and, hence, is convex.

We may also assume in the proof that $\lambda = 1$, that v is a sequence $(a_1, a_2, \cdots)$ of real numbers (see the proof of Theorem 2.2), and that $\Lambda(x, y, t)$ is nondecreasing in t (otherwise, replace Λ by $\sup_{u \le t}\Lambda(x, y, u)$).

Let $M(x, y, t)$ be the class of all B-valued martingales f starting at x such that, for some sequence $(1, a_2, a_3, \cdots)$ in $[-1, 1]$, the transform g of f by this sequence satisfies

$$P(| g_n - y | \ge 1 \quad \text{for some} \quad n \ge 1) \ge t. \tag{3.30}$$

Let $\psi(x, y, t) = \inf\{\| f \|_1 : f \in M(x, y, t)\}$. Then, the desired inequality (3.24) follows, for $\lambda = 1$, from Remark 1.1 and

$$\psi(x, y, t) \ge \varphi(x, y, t) \tag{3.31}$$

where $\varphi(x, y, t) = ½\Lambda(2x - y, y, t)$. The proof of (3.31) is nearly the same as the proof of (3.12) and we shall only sketch it. Here φ satisfies

$$\varphi(x, y, t) = \varphi(x, 2x - y, t), \tag{3.32}$$

(3.33) the mapping $(s, t) \to \varphi(x + s, y + \theta s, t)$ is convex on $B \times [0, 1]$ for all $\theta \in [0, 2]$,

$$\varphi(x, y, 1) \le | x | \qquad \text{if} \quad | y | \ge 1, \tag{3.34}$$

$$\varphi(x, y, 0) \le | x |, \tag{3.35}$$

$$\varphi(x, y, t) \text{ is nondecreasing in } t. \tag{3.36}$$

(The function ψ is the greatest function φ with these five properties.)

Consider the analogue of (3.17):

$$\psi_n(x, y, t) \ge \varphi(x, y, t). \tag{3.37}$$

Let $f \in M_2(x, y, t)$ so that, for some $a_2 \in [-1, 1]$, either $| x - y | \ge 1$ or

$$P(| x + a_2\, d_2 - y | \ge 1) \ge t.$$

We may suppose the latter inequality holds. Let u be the indicator function of the event on the left-hand side and $\theta = 1 - a_2$. By (3.32) and (3.34),

$$\varphi(x, y, 1) \le | x | \qquad \text{if} \quad | x - y | \ge 1$$

implying, by (3.34) and (3.35), that

$$\varphi(x + d_2, y + \theta\, d_2, u) \le | x + d_2 |.$$

Therefore, by (3.33),

$$\varphi(x, y, t) \le \varphi(x, y, Eu) \le E\varphi(x + d_2, y + \theta\, d_2, u) \le \| f_2 \|_1,$$

which implies that (3.37) holds for $n = 2$.

In the induction step, suppose that (3.37) holds for some $n \ge 2$ and $f \in M_{n+1}(x, y, t)$. Let f_2 have finite range $\{x_1, \cdots, x_m\}$ as in the proof of Theorem 3.1 and write $x_j = x + s_j$. Let

$$P_j(| g_k - y | \ge 1 \quad \text{for some} \quad k = 2, \cdots, n + 1) = t_j.$$

Then $F_j \in M_n(x + s_j, y + \theta s_j, t_j)$ so that

$$\| f_{n+1} \|_1 = \textstyle\sum_{j=1}^m \alpha_j \displaystyle\int_\Omega | F_{jn} | \, dP_j \ge \textstyle\sum_{j=1}^m \alpha_j \varphi(x + s_j, y + \theta s_j, t_j)$$
$$\ge \varphi(x, y, \textstyle\sum_{j=1}^m \alpha_j t_j) \ge \varphi(x, y, t).$$

THEOREM 3.3 *If B is a Banach space, $1 < p < \infty$, $\Gamma: B \times B \times [0, \infty) \to \mathbb{R}$ satisfies*

(3.38) $$\Gamma(x, y, t) = \Gamma(y, x, t),$$

(3.39) *the mapping $(x, t) \to \Gamma(x, y, t)$ is convex on $B \times [0, \infty)$,*

(3.40) $$\Gamma(x, y, t) \le \left| \frac{x + y}{2} \right|^p \quad \textit{if} \quad \left| \frac{x - y}{2} \right|^p \ge t,$$

and g is the transform of a B-valued martingale f by a real predictable sequence v uniformly bounded in absolute value by 1, *then*

(3.41) $$\Gamma(0, 0, 1) \| g \|_p^p \le \| f \|_p^p.$$

So, if $\Gamma(0, 0, 1) > 0$, then $B \in MT$. Furthermore, if $B \in MT$ and Γ is the greatest function satisfying (3.38), (3.39), and (3.40), then $\Gamma(0, 0, 1) > 0$ and the best constant in (1.3) is given by

$$c_p(1.3) = [1/\Gamma(0, 0, 1)]^{1/p}.$$

For example, if B is an inner product space and $p = 2$, consider

$$\Gamma(x, y, t) = t + (x, y)$$

(or the real part of this expression in the complex case). Clearly, (3.38) and (3.39) are satisfied and (3.40) follows from

$$t + (x, y) = t + \left| \frac{x + y}{2} \right|^2 - \left| \frac{x - y}{2} \right|^2.$$

PROOF. We may assume that $\Gamma(x, y, t)$ is nondecreasing in t and, by the proof of Theorem 2.2, that v is a sequence of numbers in $\{-1, 1\}$.

Here let $M(x, y, t)$ be the class of all B-valued martingales f starting at x such that, for some sequence $(1, \epsilon_2, \epsilon_3, \cdots)$ in $\{-1, 1\}$, the transform g of f by this sequence satisfies

$$\sup_n \| g_n - y \|_p^p \ge t.$$

Let $\psi(x, y, t) = \inf\{\| f \|_p^p : f \in M(x, y, t)\}$. The inequality (3.41) then follows, with the aid of a scaling argument and Remark 1.1, from

(3.42) $$\psi(x, y, t) \ge \varphi(x, y, t)$$

where $\varphi(x, y, t) = \Gamma(2x - y, y, t)$. This function φ satisfies

(3.43) $$\varphi(x, y, t) = \varphi(x, 2x - y, t),$$

(3.44) the mapping $(x, t) \to \varphi(x, y, t)$ is convex on $B \times [0, \infty)$, (3.45) $$\varphi(x, y, t) \le |x|^p \quad \text{if} \quad |x - y|^p \ge t,$$

(3.46) $\varphi(x, y, t)$ is nondecreasing in t.

(As before, ψ is the greatest function with these properties so the mapping $(x, y, t) \to \psi\left(\frac{x+y}{2}, y, t\right)$ is the greatest function satisfying (3.38), (3.39), and (3.40). This implies the remark, following the statement of Theorem 3.3, about best constants.)

Now define $M_n^+(x, y, t)$, $\psi_n^+(x, y, t)$, and so forth, in analogy with the corresponding objects of the proof of Theorem 3.1. Let $f_2 \in M_2^+(x, y, t)$. Then $g_2 = f_2$ and $\| f_2 - y \|_p^p \ge t$. By (3.45),

$$\varphi(f_2, y, |f_2 - y|^p) \le |f_2|^p.$$

Therefore, by (3.44) and (3.46),

$$\varphi(x, y, t) \le \varphi(x, y, \| f_2 - y \|_p^p) \le \| f_2 \|_p^p.$$

Accordingly, $\varphi(x, y, t) \le \psi_2^+(x, y, t)$ and, as in the proof of Theorem 3.1, this implies $\varphi(x, y, t) \le \psi_2(x, y, t)$. The remainder of the proof is a straightforward modification of the corresponding parts of the proofs of Theorems 3.1 and 3.2.

4. Some further properties of ζ. Any function ζ satisfying the conditions of Theorem 3.1 determines the norm of B up to equivalence.

THEOREM 4.1. *Suppose that $\zeta: B \times B \to \mathbb{R}$, with $\zeta(0, 0) > 0$, is a symmetric biconvex function satisfying* (3.1). *If $\| \cdot \|$ is a norm on B such that ζ also satisfies* (3.1) *with respect to $\| \cdot \|$, then*

(4.1) $$\zeta(0, 0)|x| \le \|x\| \le |x|/\zeta(0, 0).$$

In particular, $\zeta(0, 0) = 1$ implies uniqueness. To prove the theorem, we shall need the following lemma.

LEMMA 4.1. *Under the conditions of the above theorem,*

(4.2) $$|\zeta(x, y) - \zeta(x', y')| \le |x - x'| + |y - y'|$$

if $|x'| \vee |y| \le 1$ *or* $|x| \vee |y'| \le 1$.

PROOF. Suppose that $|x'| \vee |y| \le 1$ and $x \ne x'$. Then, for all large $\lambda > 1$, $|x + \lambda(x' - x)| > 1$ so, by the convexity and other assumptions,

$$\begin{aligned}\zeta(x', y) - \zeta(x, y) &\le \lambda^{-1}[\zeta(x + \lambda(x' - x), y) - \zeta(x, y)] \\ &\le \lambda^{-1}[|x| + |y| + \lambda |x' - x| - \zeta(x, y)]\end{aligned}$$

and the last expression converges to $|x' - x|$ as $\lambda \to \infty$. The desired inequality (4.2) easily follows.

Note that if ζ satisfies (3.4), then (4.2) holds for all choices of the arguments.

PROOF OF THEOREM 4.1. It is enough to prove the left-hand side of (4.1). Furthermore, we may assume that

(4.3) $$\zeta(x, y) = \zeta(-x, -y)$$

(otherwise, replace ζ by the average of these two expressions). Then, by convexity,

(4.4) $$\zeta(0, 0) \le \zeta(\pm x, 0).$$

If $|x| = 1$, as we may suppose, and $\|x\| \le 1$, then $\zeta(x, -x) \le |x - x| = 0$, by (3.1), so Lemma 4.1 applied to $\|\cdot\|$ gives

$$\zeta(0, 0) \le \zeta(x, 0) \le \zeta(x, 0) - \zeta(x, -x) \le \|x\|,$$

which is the left-hand side of (4.1). If $|x| = 1 < \|x\|$, the left-hand side is trivial because $\zeta(0, 0) \le \zeta(x, 0) \le |x| = 1$.

REMARK 4.1. Suppose that ζ satisfies the conditions of Theorem 3.1 relative to some unknown norm $|\cdot|$ on B. How can ζ be used to construct an equivalent norm? Simply let V be the smallest convex set containing all x in B satisfying $\zeta(\alpha x, -\alpha x) > 0$ for all scalars α such that $|\alpha| \le 1$. Clearly, $\alpha V = V$ if $|\alpha| = 1$. Also, it is easy to check, using Lemma 4.1, that $\zeta(0, 0)U \subset V \subset U$ where $U = \{x \in B : |x| < 1\}$. Therefore, $\|x\| = \inf\{\lambda > 0 : x \in \lambda V\}$ defines a norm on B satisfying $|x| \le \|x\| \le |x|/\zeta(0, 0)$.

REMARK 4.2. Suppose that $|\cdot|$ and $\|\cdot\|$ are norms on B satisfying $\alpha|x| \le \|x\| \le \beta|x|$ where α and β are positive numbers. Suppose that ζ is symmetric, biconvex, $\zeta(0, 0) > 0$, and (3.4) is satisfied by ζ and $|\cdot|$. What is an appropriate ζ-function for $(B, \|\cdot\|)$? If

$$\zeta_0(x, y) = \frac{\alpha}{\beta}\zeta(\beta x, \beta y),$$

then ζ_0 has the desired properties. In particular, if $\|y\| \ge 1$, then $|\beta y| \ge 1$ so that

$$\zeta_0(x, y) \le \frac{\alpha}{\beta}|\beta x + \beta y| = \alpha|x + y| \le \|x + y\|.$$

5. Concluding remarks.

(i) The equivalent probability conditions of Sections 1 and 2 determine a rather large class of Banach spaces. For example, the Lebesgue spaces l^r and $L^r(0, 1)$ belong to this class for all $r \in (1, \infty)$. (See the Introduction.) Here we describe a condition that looks similar but which, in fact, gives rise to a much smaller class of spaces, albeit an important one.

Let f and g be B-valued martingales, relative to the same sequence of sub-σ-fields, such that their respective difference sequences d and e satisfy $|e_k(\omega)| \le |d_k(\omega)|$. Let $1 < p < \infty$. For what Banach spaces B does there exist a positive real number c_p such that

$$\|g\|_p \le c_p\|f\|_p \tag{5.1}$$

for all such f and g?

Before answering this question, we note that the analogue of Theorem 1.1 holds and, except for obvious minor changes, has the same proof. One simple consequence is that if (5.1) holds for some $p \in (1, \infty)$ then it holds for all such p. Therefore, in the following, we may assume that $p = 2$.

Let $x_1, x_2, \cdots$ and z be elements of B, with $|z| = 1$, and define d and e by $d_k = r_k x_k$ and $e_k = r_k|x_k|z$ where $r_1, r_2 \cdots$ is an independent sequence of random variables satisfying $P(r_k = \pm 1) = ½$. Then d and e are B-valued martingale difference sequences satisfying $|d_k(\omega)| = |e_k(\omega)|$. If (5.1) holds for $p = 2$, then

$$\|\textstyle\sum_{k=1}^n r_k x_k\|_2 \approx \|\sum_{k=1}^n r_k|x_k|z\|_2 = \|\sum_{k=1}^n r_k|x_k|\|_2 = (\sum_{k=1}^n |x_k|^2)^{1/2}.$$

Therefore, by a result of Kwapień [15], B is isomorphic to an inner product space.

Now suppose that B is isomorphic to an inner product space. In fact, to prove (5.1), we may suppose that $B = l^2$. Then, it is elementary that $\|f\|_2^2 = E(\sum_{k=1}^\infty |d_k|^2) \ge E(\sum_{k=1}^\infty |e_k|^2) = \|g\|_2^2$, which implies (5.1).

(ii) There are other kinds of martingale transforms that may be studied for their intrinsic interest and their role in applications. For example, consider f and g, with $f_n = \sum_{k=1}^n d_k$ and $g_n = \sum_{k=1}^n v_k d_k$, where v is a B-valued predictable sequence and d is a real-valued martingale difference sequence. Here, also, the analogue of Theorem 1.1 holds and has the same proof. It is not hard to show that l^r is well-behaved for transforms of this type (for example, the analogue of (1.3) holds for l^r) if and only if $2 \le r < \infty$. (To prove the "if" part, one may assume that $p = r$ and the proof of $\|g\|_r \le c_r \|f\|_r$ reduces to an application of the square function inequality to real-valued martingales.) In fact, Pisier has shown (personal communication) that a Banach space B is well-behaved for transforms of this type if and only if B is isomorphic to a 2-smooth space. The "only if" part is an easy consequence of Corollary 3.1 of [18]. In the other direction, Proposition 2.4 of [18] gives $\|g\|_2^2 \le cE \sum_{k=1}^{\infty} |v_k d_k|^2$ and, since $|v_k(\omega)| \le 1$ and f is a real-valued martingale, the inequality $\|g\|_2 \le c\|f\|_2$ follows.

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