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SOME RESULTS ON THE LIL IN BANACH SPACE WITH APPLICATIONS TO WEIGHTED EMPIRICAL PROCESSES

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We examine the cluster set of S_n/a_n for Banach space valued random variables, and investigate the relationship between the central limit theorem and the law of the iterated logarithm in this setting. In the case of Hilbert space valued random variables, necessary and sufficient conditions are given for the law of the iterated logarithm. Some interesting examples are also included. We then apply our results to weighted empiricals both in the supremum norm and the $L^2[0, 1]$ norm.

1. Introduction. Let $X, X_1, X_2, \cdots$ be independent identically distributed random variables and as usual let $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. We write Lx to denote the function $\max(1, \log x)$, and use L_2x to denote $L(Lx)$. Further, let $a_n = \sqrt{2nL_2n}$ for $n \geq 1$.

In case X is real valued, it is well known that the following are equivalent:

$$E(X) = 0, \qquad E(X^2) = \sigma^2 < \infty. \tag{1.1}$$

$$\mathscr{L}(S_n/\sqrt{n}) \to \mathscr{L}(Z) \tag{1.2}$$

where Z is a mean zero Gaussian random variable.

$$\limsup_n |S_n|/a_n < \infty \quad \text{with probability one.} \tag{1.3}$$

$$P\left(\omega : C\left(\left\{\frac{S_n(\omega)}{a_n}\right\}\right) = [-\sigma, \sigma]\right) = 1 \qquad \text{and} \quad \limsup_n |S_n|/a_n = \sigma \tag{1.4}$$

with probability one. In (1.4) the notation $C(\{x_n\})$ denotes all limit points of the sequence $\{x_n\}$ and is called the cluster set of $[x_n]$.

The obvious analogues of (1.1)–(1.4) are also equivalent if X takes values in a finite dimensional space. However, in case X has values in an infinite dimensional Banach space the situation is much more complicated, and though much is known it is the purpose of this paper to further the investigation of some of the various possible implications.

With the exception of the results concerning $D[0, 1]$ with the supremum norm given in Sections 5 and 6, we assume X takes values in B where B is a real separable Banach space with topological dual B^* and norm $\| \ \|$. Recall that we say X satisfies the bounded law of the iterated logarithm (LIL) if

$$P(\limsup_n \| S_n/a_n \| < \infty) = 1, \tag{1.5}$$

and that X satisfies the Central Limit Theorem (CLT) if there is a mean zero Gaussian

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random variable W with values in B such that

$$\mathscr{L}(S_n/\sqrt{n}) \to \mathscr{L}(W). \tag{1.6}$$

Finally, we say that X satisfies the compact LIL if there is a compact set D such that

$$P(\lim_n d(S_n/a_n, D) = 0) = 1 \tag{1.7}$$

and

$$P\left(C\left(\left\{\frac{S_n}{a_n}\right\}\right) = D\right) = 1. \tag{1.8}$$

Here $d(x, D) = \inf_{y\in D} \| x - y \|$ and we call D the limit set in the compact LIL. Of course, the condition (1.1) is $E(X) = 0$ and $E \| X \|^2 < \infty$, condition (1.2) says X satisfies the CLT, (1.3) is that X satisfies the bounded LIL, and (1.4) says X satisfies the compact LIL (with limit set $D = [-\sigma, \sigma]$).

To indicate, to some extent anyway, what is known regarding the implications between (1.1)–(1.4) let us recall a sampling of recent results. First of all, it is well known that the moment conditions $E(X) = 0$ and $E \| X \|^2 < \infty$ are neither necessary nor sufficient for the CLT or the bounded LIL in the infinite dimensional setting (see Kuelbs (1976a) and Pisier and Zinn (1978) for details as well as further references in this regard). However, in case $E(X) = 0$ and $E \| X \|^2 < \infty$, then X satisfying the CLT implies X also satisfies the compact LIL (Pisier, 1975), but not conversely (Kuelbs, 1976b). In addition, under the assumption that $E(X) = 0$ and $E \| X \|^2 < \infty$ Kuelbs (1977) has shown that the compact (bounded) LIL is equivalent to the sequence of probability measures $\{\mathscr{L}(S_n/a_n) : n \geq 1\}$ being uniformly tight on compact (bounded) sets of B. Finally, we mention that it is possible for X to satisfy the bounded LIL but not the compact LIL, and the first such examples were due to Pisier (1975) in the Banach space c_o. We will produce further examples in this paper even in the case of Hilbert space, and will investigate the cluster set of such examples though they fail the compact LIL.

In the broadest of terms our main results are Theorem 3.1 which studies the cluster set of $\{S_n/a_n\}$, Theorems 4.1 and 5.1 which state that, if X satisfies the CLT and $E(\| X \|^2/L_2 \| X \|^2) < \infty$, then X satisfies the compact LIL, Theorem 4.2 which provides necessary and sufficient conditions for the LIL in Hilbert space, Theorem 4.3 which provides necessary and sufficient conditions for the CLT in smooth norm spaces, and finally some interesting examples in Section 7.

More precisely, if $\{\alpha_n\}$ is any sequence of nonzero constants, then it is an easy consequence of the Hewitt-Savage zero-one law that with probability one the cluster set $C(\{S_n/\alpha_n\})$ is a nonrandom set A depending only on $\{\alpha_n\}$ and the law of X; see Lemma 1 of Kuelbs (1979) for details. As in (1.4), $C(\{x_n\})$ denotes all limit points of $\{x_n\}$, so it is a closed set, and if $\{\alpha_n\}$ is such that $\limsup_n \| S_n/\alpha_n \| = 0$, then we immediately have $A = \{0\}$. However, if $0 < \limsup_n \| S_n/\alpha_n \|$, the nature of the cluster set A is much less obvious. The situation of interest to us throughout the paper is the case of the LIL, i.e. when $\alpha_n = a_n$ for $n \geq 1$. Of course, if X is not the zero random variable, we always have $0 < \limsup_n \| S_n/a_n \|$ in this situation.

To put our results in perspective, it is useful to point out two well-known necessary conditions required of X if X satisfies the bounded LIL. We first observe that if (1.5) holds, then there exists a constant Γ such that

$$P(\| X_n/a_n \| > \Gamma \text{ i.o.}) = 0.$$

Hence by the Borel-Cantelli lemma we have $\sum_n P(\| X \| > \Gamma a_n) < \infty$, and therefore $E(\| X \|^2/L_2 \| X \|) < \infty$. Now (1.5) also implies

$$\limsup_n | f(S_n/a_n) | \leq \| f \|_{B^*} \limsup_n \| S_n/a_n \| < \infty \qquad f \in B^*, \tag{1.9}$$

and since f is linear (1.9) implies $f(X)$ satisfies the bounded LIL on the line. Hence it is also necessary that $Ef(X) = 0$ and $Ef^2(X) < \infty$ for all $f \in B^*$; see Stout (1974), page 297.

This last consequence of (1.5) we summarize by saying X is WM_0^2, i.e., X is weakly square integrable with weak mean zero if $Ef(X) = 0$ and $Ef^2(X) < \infty$ for all $f \in B^*$. If $\mu = \mathscr{L}(X)$ we also will say μ is WM_0^2 when that is more appropriate.

If $Ef(X) = 0$ for $f \in B^*$ we define the covariance function of X (or $\mu = \mathscr{L}(X)$) to be

$$T(f, g) = E(f(X)g(X)) \qquad f, g \in B^*. \tag{1.10}$$

Of course, it is immediate that the covariance function for X (or $\mu = \mathscr{L}(X)$) exists iff X (or $\mu = \mathscr{L}(X)$) is WM_0^2.

Hence we see that $E(\|X\|^2/L_2\|X\|) < \infty$ and X being WM_0^2 are minimal conditions for X to satisfy the bounded LIL, and it is under such conditions that we investigate the cluster set A. Of course, A being a cluster set implies A is always closed, and if X is WM_0^2, then there is a canonical set K, depending only on the covariance function $T(f, g)$, such that we always have

$$A \subseteq K. \tag{1.11}$$

For the definition of K, we refer the reader to Section 2 and Lemma 2.1. The proof that $A \subseteq K$ easily follows from the method used to prove (3.2) of Kuelbs (1976a). Furthermore, Lemma 2.1v implies K is compact iff the covariance function of X is weak-star sequentially continuous, so A is also compact in this case.

If X satisfies the compact LIL with limit set D, then (1.7) implies (1.5), and hence X is WM_0^2 with $E(\|X\|^2/L_2\|X\|) < \infty$. Furthermore, (1.7) also implies

$$P(\{S_n/a_n\} \text{ conditionally compact in } B) = 1 \tag{1.12}$$

and, of course, (1.11) and (1.8) together imply $D \subseteq K$. Conversely, if (1.12) holds and $A = C(\{S_n/a_n\})$, then it is easy to see that A is compact in B, and

$$P(\lim_n d(S_n/a_n, A) = 0) = 1. \tag{1.13}$$

It also is the case that (1.12) actually implies $A = K$, and this is proved in Kuelbs (1976a), Corollary 3.1, under the assumptions $EX = 0$ and $E\|X\|^2 < \infty$. However, the same method suffices if X is only WM_0^2 and (1.12) holds. Thus we see that the limit set, and hence the cluster set, in the compact LIL is always the canonical set K constructed in Lemma 2.1. It remains to study the situation when X does not satisfy the compact LIL.

The first reasonable conjecture regarding A, in the case X does not satisfy the compact LIL, is perhaps that A is the empty set. This indeed is the case provided X is a regular example of Pisier; see Theorem 1 of Kuelbs (1979). However, in the same paper there is also an example of a random variable X satisfying the bounded LIL, but not the compact LIL, and such that $A = K$ is compact and nonempty. Hence this first conjecture is immediately false.

One possible conjecture to unify our view of these matters to some degree is that if X is WM_0^2, then $A = C(\{S_n/a_n\})$ is empty or is the canonical set K. If this conjecture is true, it still remains to be decided when we have the cluster set empty and when it is K. Our Theorem 3.1 is in this direction, and gives sufficient conditions that $A = K$. In the remarks that follow the statement of Theorem 3.1 we discuss another possible conjecture.

As mentioned previously, Pisier (1975) proved that if $EX = 0$, $E\|X\|^2 < \infty$, and X satisfies the CLT, then X satisfies the compact LIL. Some recent work of Heinkel (1978a, b) shows that if $EX = 0$ and X satisfies the CLT, then one can weaken the assumption $E\|X\|^2 < \infty$ to $(E\|X\|^2L_3\|X\|/L_2\|X\|) < \infty$ and still obtain the compact LIL for X. Hence it is natural to conjecture that in the presence of the CLT, the necessary conditions $EX = 0$ and $E(\|X\|^2/L_2\|X\|) < \infty$ alone imply the compact LIL for X. This is our Theorem 4.1, and an independent proof of this is also due to Heinkel (1979a, b). The corresponding result for $D[0, 1]$ random variables is given in Theorem 5.1. In Theorem 4.2 we prove that X being WM_0^2 and $E(\|X\|^2/L_2\|X\|) < \infty$ are necessary and sufficient conditions for the bounded LIL in Hilbert space, and Corollary 4.1 deals with the compact LIL. In Theorem 4.3 we show that the necessary and sufficient conditions obtained by Pisier and Zinn

(1978), Theorem 5.1, for the CLT in $\ell_p (2 < p < \infty)$ also hold in smooth norm spaces. Our method gives a new proof of these ℓ_p results as well as extends to smooth norm spaces. Combined with Theorem 4.1 it also gives the best known sufficient conditions for X to satisfy the compact LIL in a smooth norm space.

In Section 5 we prove (Theorem 5.1) the $(D[0, 1], \|\cdot\|_\infty)$ analogue of Theorem 4.1 and in Section 6 we consider the special case of weighted empiricals. As a consequence of Theorem 5.1 we obtain from a weak convergence result of O'Reilly (1974) a theorem similar to an LIL of James (1975). We also obtain some LIL's for Cramer-von Mises statistics as a consequence of Theorem 4.2.

In Section 7 we give some examples involving Hilbert space random variables. Our first example accomplishes a number of things but perhaps its most important property is that it satisfies the bounded LIL yet it has a non-compact cluster set. This is the first such example of a non-compact cluster set under the WM_0^2 assumption, and the various properties of this example are contained in Theorem 7.1.

Theorem 7.2 exhibits a random variable which shows that the condition $\sup_{t>0} t^2 P(\|X\| > t) < \infty$ is not necessary for the LIL even in Hilbert space whereas Theorem 8 of Jain (1977) shows that this condition is always required for the CLT. This example also shows that norm conditions alone will not allow one to prove an analogue of Lemma 4.3, and that the existence of the covariance function is necessary for (4.22) to hold.

2. Some remarks on the limit set. In Kuelbs (1976a), Lemma 2.1, the limit set in the compact law of the iterated logarithm for a B-valued random variable X was carefully examined under the assumption the $E\|X\|^2 < \infty$. Subsequent investigations have shown that this assumption is too stringent and the purpose of our next lemma is to record the necessary facts in the most general case. Various portions of the lemma are not new, but are included for the sake of completeness. For example, Theorem 6 of Jain (1977) covers some of the same material, but the approach we use here proceeds in a manner which parallels Kuelbs (1976a). In fact, we show that the Bochner integral in Lemma 2.1 of Kuelbs (1976a) need only be replaced by a Pettis integral and much of the lemma holds as before. We also determine necessary and sufficient conditions for the limit set to be compact and this aspect is new. It is of importance since there are examples of non-compact limit sets, and an interesting example will be given in Section 7.

One aspect which Theorem 6 of Jain (1977) includes, and which we do not mention here, is that the expansions in (2.5) converge to x in the WM_0^2 sense, and also with probability one if the sequence $\{\alpha_k\}$ is chosen such that $\{\alpha_k(X) : k \geq 1\}$ is a martingale difference and $E\|X\| < \infty$.

Let $(M, \mathscr{F}, \lambda)$ be a σ-finite measure space and assume $Z : M \to B$ is Borel measurable with $g(Z) \in L^1(M, \mathscr{F}, \lambda)$ for all $g \in B^*$. Then it is easy to prove that there exists a $z \in B^{**}$ such that

$$z(g) = \int_M g(Z)\, d\lambda \qquad g \in B^*,$$

and we call $z \in B^{**}$ the weak integral of Z. We say Z has Pettis integral z if z actually is in B, and will write

$$z = (w) \int_M Z\, d\lambda \quad \text{or} \quad z = (P) \int_M Z\, d\lambda$$

according as Z is weakly integrable or Pettis integrable.

LEMMA 2.1. *Let μ denote a Borel probability measure on B such that μ is WM_0^2. For each $f \in B^*$ let Sf denote the weak integral of $xf(x)$ with respect to the mesure μ, i.e.*

$$Sf = (w) \int_B x f(x)\, d\mu(x) \qquad f \in B^*. \tag{2.1}$$

Then: (i) *We have*

$$\sigma(\mu) \equiv \sup_{\|f\|_{B^*}\leq 1}\left(\int_B f^2(x)\, d\mu(x)\right)^{1/2} \leq \infty. \tag{2.2}$$

(ii) *The weak integral* Sf, $f \in B^*$, *is a Pettis integral and hence* S *is a linear map from* B^* *into* B. *Furthermore,* S *is a bounded operator from* B^* *into* B.

(iii) *If* H_μ *denotes the completion of the range of* S *with respect to the norm obtained from the inner product*

$$(Sf, Sg)_\mu = \int_B f(x)g(x)\, d\mu(x), \tag{2.3}$$

then H_μ *can be viewed as a subset of* B *and for* $x \in H_\mu$

$$\| x \| \leq \sigma(\mu) \| x \|_\mu. \tag{2.4}$$

(iv) *Let* $\{f_k : k \geq 1\}$ *be a weak star dense subset of the unit ball of* B^*. *Let* $\{\alpha_k : k \geq 1\}$ *be an orthonormal sequence obtained from the sequence* $\{f_k\}$ *by the usual Gramm-Schmidt orthogonalization method with respect to the inner product given by the right side of* (2.3). *Then each* $\alpha_k \in B^*$, *and* $\{S\alpha_k : k \geq 1\}$ *is a complete orthonormal sequence in* $H_\mu \subseteq B$. *Further the linear operators*

$$\Pi_N(x) = \textstyle\sum_{k=1}^N \alpha_k(x)S\alpha_k \quad \text{and} \quad Q_N(x) = x - \Pi_N(x) \qquad N \geq 1 \tag{2.5}$$

are continuous from B *into* B *where by* $\alpha_k(x)$ *we mean the linear functional* α_k *applied to* x. Π_N *and* Q_N, *when restricted to* H_μ, *are orthogonal projections onto their ranges.*

(v) *If* K *is the unit ball of* H_μ, *then* K *is a closed symmetric convex subset of* B *and for each* $f \in B^*$

$$\sup_{x\in K} f(x) = \left(\int_B f^2(y)\, d\mu(y)\right)^{1/2}. \tag{2.6}$$

Further, K *is a compact subset of* B *iff the covariance function* $T(f, g)$ *for* μ *is weak-star sequentially continuous,* i.e., iff *for all sequences* $\{f_n\}$ *and* $\{g_n\}$ *in* B^* *such that* $f_n \to f$ *and* $g_n \to g$ *in the weak-star sense, we have* $T(f_n, g_n) \to T(f, g)$.

(vi) *If* μ *and* ν *are two measures on* B *satisfying the basic hypothesis of the lemma and having common covariance function, then* $H_\mu = H_\nu$.

Remarks. (1) If $\int_B \| x \|^2 d\mu(x) < \infty$ and $\int_B x\, d\mu(x) = 0$, then the covariance function $T(f, g)$ of μ is easily seen to be weak-star sequentially continuous. Hence K, the unit ball of H_μ, is compact in B in this case. This was previously shown in Lemma 2.1 of Kuelbs (1976a).

(2) The linear map S defined in (2.1) is equivalent to U^*U in Theorem 6 of Jain (1977).

(3) If X takes values in a Hilbert space H and we identify H and H^* as usual, then the operator S is often called the covariance operator of X. From Lemma 2.1ii it is easy to see that the operator S is bounded iff X is WM_0^2, and from the proof of Lemma 2.1v we have S compact iff K is compact.

Proof. To prove (i) we simply observe that since μ is WM_0^2 the linear map $Af = f$ takes B^* into $L^2(B, \mu)$. Now A is continuous by an application of the closed graph theorem, and hence (2.2) and (i) holds.

Since μ is WM_0^2 the weak integral clearly exists, and hence we need only verify that $Sf \in B$. To this end, let C_n be increasing compact sets such that $\mu(C_n) > 1 - 1/n$. Define $S_n f = \int_{C_n} xf(x)\, d\mu(x)$, $n \geq 1$. Then $S_n f$ exists as a Bochner integral and $S_n f \in B$ *for* $n \geq 1$. Further,

$$\begin{aligned}\| S_n f - Sf\|_{B^{**}} &= \sup_{\|g\|_{B^*}\le 1} | S_n f(g) - Sf(g)| \\ &= \sup_{\|g\|_{B^*}\le 1}\left| \int_{B-C_n} g(x)f(x)\, d\mu(x)\right| \\ &\le \sigma(\mu)\left(\int_{B-C_n} f^2(x)\, d\mu(x)\right)^{1/2}\end{aligned}$$

and hence $\lim_n \| S_n f - Sf\|_{B^{**}} = 0$. Thus $\{S_n f\}$ is Cauchy in B and converges to Sf in B^{**} so (ii) holds. To see S is bounded from B^* to B, simply apply the uniform boundedness principle to the sequence $\{S_n\}$.

To prove (iii) first observe that $Sf = Sg$ implies $\int_B | f(x) - g(x)|^2\, d\mu(x) = 0$ so the inner product in (2.3) is well defined. Now (ii) implies $SB^* \subseteq B$ so (iii) holds if we verify (2.4) whenever $x = Sg$ for some $g \in B^*$ as such elements are dense in H_μ in the μ-norm. Now if $x = Sg$, then

$$\begin{aligned}\| x \| &= \sup_{\| f\|_{B^*}\le 1} | f(Sg)| \\ &= \sup_{\| f\|_{B^*}\le 1}\left| \int_B f(x)g(x)\, d\mu(x)\right| \\ &\le \sigma(\mu)\left(\int_B g^2(x)\, d\mu(x)\right)^{1/2} \\ &= \sigma(\mu)\cdot \| x\|_\mu,\end{aligned}$$

and hence (iii) holds.

The proofs of (iv) and (vi) are immediate, and the reader may consult Lemma 2.1 of Kuelbs (1976a) for the necessary details. Hence it remains only to verify (v).

To prove K is closed in B and to verify (2.6) proceeds exactly as in Lemma 2.1 of Kuelbs (1976a) so only the assertions regarding compactness remain.

To do this we let $Af = f$ denote the map from B^* into $L^2(B, \mu)$ as in (i). Then we have

$$T(f, g) = \langle Af, Ag\rangle$$

where $\langle \cdot, \cdot\rangle$ denotes the inner product in $L^2(B, \mu)$, and furthermore $A^*A = S$. Hence we can assert A^* maps $L^2(B, \mu)$ into B rather than only B^{**}. Furthermore, we have T weak-star sequentially continuous iff A is weak-star sequentially continuous from B^* into $L^2(B, \mu)$. That is, if T is weak-star sequentially continuous and if $\{f_n\}$ converges weak-star to zero, then

$$\lim_n \| Af_n \|_{L^2(B, \mu)} = \lim_n \langle Af_n, Af_n\rangle = \lim_n T(f_n, f_n) = 0.$$

Hence A is weak-star sequentially continuous. Conversely, if A is weak-star sequentially continuous and $\{f_n\}$ and $\{g_n\}$ converge weak-star to f, g, respectively, then Af_n and Ag_n converge in $L^2(B, \mu)$ to Af and Ag and hence

$$\lim_n T(f_n, g_n) = \lim_n \langle Af_n, Ag_n\rangle = \langle Af, Ag\rangle = T(f, g).$$

Thus T is weak-star sequentially continuous iff A is weak-star sequentially continuous.

Therefore T weak-star sequentially continuous implies A is a compact continuous linear map from B^* into $L^2(B, \mu)$. Hence A^* is also a compact linear mapping from $L^2(B, \mu)$ into B (see Dunford and Schwartz (1964), page 485), and thus K being closed in B implies K is compact in B.

On the other hand, if K is compact, then A^* is a compact operator and hence A is a compact operator. Thus $\{f_n\}$ converging weak-star to zero implies that for every subsequence $\{f_{n_k}\}$ there exists a subsequence $[f'_{n_k}]$ and g such that

$$\lim_{k\to\infty} \| Af'_{n_k} - g\|_{L^2(B,\mu)} = 0.$$

Now $\lim_n f_n(x) = 0$ for all $x \in B$ so $g = 0$ and hence $\lim_n \| Af_n \|_{L^2(B,\mu)} = 0$. Hence A is weak-star sequentially continuous and thus T is weak-star sequentially continuous as claimed. Hence (v) holds, and the lemma is proved.

3. Some results on the cluster set. In this section we will establish some results regarding the cluster set A of $\{S_n/a_n\}$.

THEOREM 3.1. *Let X be WM_0^2 and B-valued, and assume K is the unit ball of $H_{L(X)}$. Further, assume for every $\varepsilon > 0$ there exists $\gamma(\varepsilon) > 0$ such that*

$$P(\| S_n/a_n \| < \varepsilon) > \gamma(\varepsilon) \qquad n \in J(\varepsilon) \tag{3.1}$$

where for all $\rho < 1$

$$\sum_{n \in J(\varepsilon)} \frac{1}{n(Ln)^{\rho}} = \infty. \tag{3.2}$$

If for some choice of Π_N and Q_N as given in (2.5) *we have for infinitely many N that*

$$\Pi_N(X) \quad \text{and} \quad Q_N(X) \quad \text{are independent,} \tag{3.3}$$

then

$$P\left(C\left\{\left(\frac{S_n}{a_n}\right)\right\} = K\right) = 1. \tag{3.4}$$

COROLLARY 3.1. *Let X be WM_0^2 and assume K is compact. If* (3.1) *and* (3.2) *hold with* (3.3) *replaced by the assumption that there exists infinitely many N and $k(N)$ such that*

$$\Pi_N(X) \quad \text{and} \quad Q_{N+k(N)}(X) \quad \text{are independent,} \tag{3.5}$$

then

$$P\left(C\left\{\left(\frac{S_n}{a_n}\right)\right\} = K\right) = 1.$$

COROLLARY 3.2. *Let X be WM_0^2 and assume $E \,|\, (\alpha_j(X) \,|^3 < \infty$ for $j \geq 1$ where $[\alpha_j : j \geq 1\}$ is as in* (2.5). *If* (3.1), (3.2), *and* (3.5) *hold, then* (3.4) *holds.*

REMARKS. (1) If for some choice of Π_N and Q_N we have the coordinate random variables $\{\alpha_j(x) : j \geq 1\}$ m-independent, then (3.5) holds with $k(N) = m$. That this is so follows because $f(\Pi_N X)$ and $f(Q_{N+m}X)$ are independent for every $f \in B^*$. This can easily be seen since

$$f(\Pi_N X) = \textstyle\sum_{k=1}^{N} \alpha_k(X) f(S\alpha_k)$$

and

$$f(Q_{N+m}X) \doteq \textstyle\sum_{k \geq N+m+1} \alpha_k(X) f(S\alpha_k)$$

where the symbol $\doteq$ means the series converges in mean-square.

(2) If $S_n/a_n \to 0$ in probability, then (3.1) and (3.2) hold automatically. However, the generalization obtained in (3.1) and (3.2) beyond $S_n/a_n \to 0$ in probability, is not an idle one as the example of Section 4 of Kuelbs (1979) shows. Furthermore, if $J(\varepsilon)$ is a set of integers such that

$$\liminf_n \operatorname{card}(J(\varepsilon) \cap [0, n])/n = d(\varepsilon) > 0, \tag{3.6}$$

then (3.2) easily holds. To see this, choose an integer $\beta > 1$ such that $d(\varepsilon) \cdot \beta > 4$. Then from (3.6) and for $\rho < 1$ there is a constant $C > 0$ such that

$$\begin{aligned}\sum_{n\in J(\varepsilon)} \frac{1}{n(Ln)^{\rho}} &= \sum_{k=1}^{\infty} \sum_{n=\beta^k+1}^{\beta^{k+1}} I_{J(\varepsilon)}(n)/n(Ln)^{\rho} \\ &\geq \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\beta^{k+1}} I_{J(\varepsilon)}(n) - \beta^k\right)/\beta^k(L\beta^k)^{\rho} \\ &\geq C\sum_{k=1}^{\infty} \left(\frac{d(\varepsilon)}{2}\beta^{k+1} - \beta^k\right)\Big/\beta^k(L\beta^k)^{\rho} \\ &\geq C\sum_{k=1}^{\infty} \frac{1}{k^{\rho}(L\beta)^{\rho}} = \infty\end{aligned}$$

since $\rho < 1$.

(3) It is easy to see that the example of Jain (1976) which satisfies the central limit theorem, but not the bounded LIL, is such that Theorem 3.1 readily applies to it.

(4) Recall that $E(\|X\|^2/L_2\|X\|) < \infty$ is a necessary condition for X to satisfy the bounded LIL. Furthermore, if B is a type 2 Banach space, then $E(\|X\|^2/L_2\|X\|) < \infty$ implies $S_n/a_n \to 0$ in probability; see Proposition 5.1 below. Hence this necessary condition implies (3.1) and (3.2) in the case of type 2 spaces. Of course, the assumptions (3.3) and (3.5) are not at all necessary and it is these that hopefully one can remove. A natural conjecture would be that (3.1) and (3.2) imply the conclusion without (3.3) or (3.5). Of course, in view of the central limit theorem and the orthogonality of the sequence $\{\alpha_j(X): j \geq 1\}$ we easily see that $\Pi_N S_n/\sqrt{n}$ *and* $(\Pi_{N+k} - \Pi_N)(S_n/\sqrt{n})$ are asymptotically independent, and hence (3.3) holds for the limiting Gaussian random variable provided one exists.

For the proof of Theorem 3.1 we need the following lemma.

LEMMA 3.1. *Let $Y_1, Y_2, \cdots$ be independent identically distributed mean zero R^N valued random variables with covariance function $T(f, g) = (f, g)$, $(f, g \in R^N)$. Let $|\cdot|$ denote the Euclidean norm on R^N and let $b \in R^N$ satisfy $|b| < 1$. Let ρ be such that $|b| < \rho < 1$. Then for each $\varepsilon > 0$ there exists constants C_1, $C_2 > 0$ such that for all n*

$$P\left(\left|\frac{Y_1 + \cdots + Y_n}{a_n} - b\right| \leq \varepsilon\right) \geq \frac{C_1}{(Ln)^{\rho}} - nP(|Y_1| > \sqrt{n}) - \frac{C_2}{\sqrt{n}}E|Y_1 I(|Y_1| \leq \sqrt{n})|^3. \tag{3.7}$$

PROOF. For $j = 1, \cdots, n$ we define

$$Y_{j,n} = \begin{cases} Y_j & \text{if } |Y_j| \leq \sqrt{n} \\ 0 & \text{otherwise,} \end{cases}$$

and set $m_n = E(Y_{1,n})$. Let μ_n denote the mean zero Gaussian measure on R_N with covariance function

$$T_n(f, g) = E(f(Y_{1,n} - m_n)g(Y_{1,n} - m_n))$$

for f, $g \in R^N$ and, as usual, $f(x) = (x, f)$. Then for large n we have

$$\begin{aligned}P\left(\left|\frac{Y_1 + \cdots + Y_n}{a_n} - b\right| \leq \varepsilon\right) &\geq P\left(\left|\frac{Y_{1,n} + \cdots + Y_{n,n} - nm_n}{a_n} - b\right| \leq \varepsilon - n\frac{|m_n|}{a_n}\right) - nP(|Y_1| > \sqrt{n}) \\ &\geq P\left(\left|\frac{Y_{1,n} + \cdots + Y_{n,n} - nm_n}{a_n} - b\right| \leq \frac{\varepsilon}{2}\right) - nP(|Y_1| > \sqrt{n})\end{aligned} \tag{3.8}$$

since

$$\frac{n\,|\,m_n\,|}{a_n} \le \int_{|x|>\sqrt{n}} |\,x\,|^2\,d\nu(x) \to 0 \qquad \text{as} \quad n \to \infty$$

where $\nu = \mathscr{L}(Y_1)$.

Since $|\,m_n\,|^3 \le E\,|\,Y_{1,n}\,|^3$ the Berry-Esseen estimates given in Sazanov (1968) imply there is a constant C_2 such that for all n

$$\begin{aligned} &P\left(\left|\frac{Y_{1,n} + \cdots + Y_{n,n} - nm_n}{a_n} - b\right| \le \frac{\varepsilon}{2}\right) \\ (3.9)\qquad &\qquad \ge \mu_n\left(x \in R^N : |\,x - b\sqrt{2L_2 n}\,| < \left(\frac{\varepsilon}{2}\right)\sqrt{2L_2 n}\right) - C_2\frac{E\,|\,Y_{1,n}\,|^3}{n^{1/2}}. \end{aligned}$$

Hence by combining (3.8) and (3.9) we have (3.7) if we establish that there exists a constant C_0 such that

$$(3.10)\qquad \mu_n\left(x \in R^N : |\,x - b\sqrt{2L_2 n}\,| < \left(\frac{\varepsilon}{2}\right)\sqrt{2L_2 n}\right) \ge \frac{C_0}{(Ln)^{\rho}}.$$

Let $e_{1,n}, \cdots, e_{N,n}$ denote an orthonormal basis of R^N such that the coordinate functionals $x_j = (x, e_{j,n})$, $(1 \le j \le N, x \in R^N)$ are independent random variables with respect to μ_n. Let

$$E_n = \{x \in R^N : \textstyle\sum_{j=1}^N (x - b\sqrt{2L_2 n}, e_{j,n})^2 \le (\varepsilon^2/2)L_2 n\}$$

and

$$E_{j,n} = \{x_j : |\,x_j - b_j\sqrt{2L_2 n}\,| < \varepsilon\sqrt{L_2 n/2N}\}.$$

Letting

$$\sigma_{j,n}^2 = \int_{R^N} (x, e_{j,n})^2\,d\mu_n(x)$$

we then have

$$\begin{aligned} &\mu_n\left(x \in R^N : |\,x - b\sqrt{2L_2 n}\,| < \left(\frac{\varepsilon}{2}\right)\sqrt{2L_2 n}\right) \\ &\qquad = \int_{E_n} (N) \int \textstyle\prod_{j=1}^N \exp\{-x_j^2/2\sigma_{j,n}^2\}/\sqrt{2\pi\sigma_{j,n}^2}\,dx_1 \cdots dx_N \\ &\qquad \ge \textstyle\prod_{j=1}^N \displaystyle\int_{E_{j,n}} \exp\{-x_j^2/2\sigma_{j,n}^2\}/\sqrt{2\pi\sigma_{j,n}^2}\,dx_j \\ (3.11)\qquad &\qquad \ge \textstyle\prod_{j=1}^N \displaystyle\int_{|b_j|\sqrt{2L_2 n}}^{|b_j|\sqrt{2L_2 n} + \varepsilon\sqrt{L_2 n/2N}} \frac{e^{-x_j^2/2\sigma_{j,n}^2}}{\sqrt{2\pi\sigma_{j,n}^2}}\,dx_j \\ &\qquad \ge C\textstyle\prod_{j=1}^N \{\exp\{-b_j^2 L_2 n/\sigma_{j,n}^2\}\sigma_{j,n}/\sqrt{L_2 n}\} \end{aligned}$$

for some constant C since

$$\int_a^b e^{-s^2/2}\,ds \ge \frac{1}{b}\,e^{-a^2/2}\,[1 - e^{-(b^2-a^2)/2}], \quad 0 \le a < b.$$

$$= C\textstyle\prod_{j=1}^N \sigma_{j,n}/(Ln)^{\rho_n}(L_2 n)^{-N/2} \qquad \text{where } \rho_n \sum_{j=1}^N b_j^2/\sigma_{j,n}^2\,.$$

Now

$$\begin{aligned}\sigma_{j,n}^2 &= \int_{R^N} (x, e_{j,n})^2 \, d\mu_n(x) \\ &= T_n(e_{j,n}, e_{j,n}) \\ &= E(e_{j,n}^2(Y_{1,n} - m_n)) \\ &= E(e_{j,n}^2(Y_{1,n})) - e_{j,n}^2(m_n) \\ &= E(e_{j,n}^2(Y_1)) - E(e_{j,n}^2(Y_1)I(|Y_1| > \sqrt{n})) - e_{j,n}^2(m_n) \\ &\geq 1 - E(|Y_1|^2 I(|Y_1| > \sqrt{n})) - |m_n|^2\end{aligned} \tag{3.12}$$

since $E(e_{j,n}^2(Y_1)) = (e_{j,n}, e_{j,n}) = 1$ and $|e_{j,n}| = 1$. Hence $\lim_n \sigma_{j,n}^2 = 1$ uniformly in $\{e_{j,n} : 1 \leq j \leq N\}$, so there exists C_0 such that for all n

$$\mu_n\left(x \in R^N : |x - b\sqrt{2L_2 n}| \geq \left(\frac{\varepsilon}{2}\right)\sqrt{2L_2 n}\right) \geq \frac{C_0}{(Ln)^\rho}. \tag{3.13}$$

Combining (3.8), (3.9), and (3.13) there exist constants C_1 and C_2 such that (3.7) holds and the lemma is proved.

PROOF OF THEOREM 3.1. For each $\varepsilon > 0$ and $b \in K$ with $\| b \|_\mu < 1$ we first show that

$$\textstyle\sum_n P\left(\left\| \frac{S_n}{a_n} - b \right\| < \varepsilon\right) \Big/ n = \infty. \tag{3.14}$$

In view of Lemma 5 of Kuelbs (1979), (3.14) implies $b \in A$ and since A is always closed we thus have $K \subseteq A$. On the other hand, since X is WM_0^2 we know that we always have $A \subseteq K$, so it suffices to establish (3.14).

Since $b \in K \subseteq H$ and (2.4) holds, there exists an N such that $\| Q_N b \| < \varepsilon/3$ and (3.3) holds. Hence

$$\begin{aligned}P\left(\left\| \frac{S_n}{a_n} - b \right\| < \varepsilon\right) &\geq P\left(\left\| \Pi_N\left(\frac{S_n}{a_n} - b\right) \right\| < \frac{\varepsilon}{3}, \left\| Q_N \frac{S_n}{a_n} \right\| < \frac{\varepsilon}{3}\right) \\ &= P\left(\left\| \Pi_N\left(\frac{S_n}{a_n} - b\right) \right\| < \frac{\varepsilon}{3}\right) P\left(\left\| Q_N \frac{S_n}{a_n} \right\| < \frac{\varepsilon}{3}\right),\end{aligned} \tag{3.15}$$

and since $\Pi_N(S_n/a_n)$ converges in probability to zero as $n \to \infty$ we have from (3.1) that for all sufficiently large n in $J(\varepsilon/6)$ that

$$P\left(\left\| Q_N\left(\frac{S_n}{a_n}\right) \right\| < \frac{\varepsilon}{3}\right) > \gamma\left(\frac{\varepsilon}{6}\right). \tag{3.16}$$

Combining (3.15) and (3.16) we thus have

$$P\left(\left\| \frac{S_n}{a_n} - b \right\| < \varepsilon\right) \geq \gamma\left(\frac{\varepsilon}{6}\right) P\left(\left\| \Pi_N\left(\frac{S_n}{a_n} - b\right) \right\| < \frac{\varepsilon}{3}\right) \tag{3.17}$$

for all $n \in J(\varepsilon/6)$ sufficiently large.

To verify (3.14) we now turn to the estimation of $P(\| \Pi_N(S_n/a_n - b) \| < \varepsilon/3)$. Now N is fixed, and since all norms on finite dimensional spaces are equivalent there exists a $\delta > 0$ such that $\{x : \| \Pi_N(x - b) \| < \varepsilon/3\} \supseteq \{x : \| \Pi_N(x - b) \|_\mu < \delta\}$. Hence for $n \geq 1$ we have

$$P\left(\left\| \Pi_N\left(\frac{S_n}{a_n} - b\right) \right\| < \frac{\varepsilon}{3}\right) \geq P\left(\left\| \Pi_N\left(\frac{S_n}{a_n} - b\right) \right\|_\mu < \delta\right) \tag{3.18}$$

and we will estimate the latter.

Now $\Pi_N B$ with norm $\|\cdot\|_\mu$ is isometric to R^N with the usual Euclidean norm $|\cdot|$ on R^N, i.e. take the map

$$\psi(\Pi_N x) = (\alpha_1(x), \cdots, \alpha_N(x)) \qquad x \in B$$

to yield the isometry. Further, we have $\|\Pi_N b\|_\mu \le \|b\|_\mu < 1$, so applying Lemma 3.1 with $\|b\|_\mu < \rho < 1$ and

$$Y_j = \psi(\Pi_N X_j) \qquad j \ge 1$$

we obtain constants $C_1, C_2 > 0$ such that

$$P\left(\left\|\Pi_N\left(\frac{S_n}{a_n} - b\right)\right\|_\mu < \delta\right) \ge C_1/(Ln)^\rho - nP(\|\Pi_N X_1\|_\mu > \sqrt{n}) - \frac{C_2}{n^{1/2}} E\|\Pi_N X_{1,n}\|_\mu^3 \tag{3.19}$$

where

$$\Pi_N X_{1,n} = \begin{cases} \Pi_N X_1 & \text{if } \|\Pi_N X_1\|_\mu \le \sqrt{n} \\ 0 & \text{otherwise.} \end{cases}$$

Combining (3.17), (3.18), and (3.19) we easily have (3.14) since (3.2) holds for $J(\varepsilon/6)$ and

$$\text{(3.20)} \qquad \begin{aligned} &\text{(i)} \ \textstyle\sum_n P(\|\Pi_N X_1\|_\mu > \sqrt{n}) < \infty \\ &\text{(ii)} \ \textstyle\sum_n E\|\Pi_N X_{1,n}\|_\mu^3 / n^{3/2} < \infty. \end{aligned}$$

That is, the convergence of (3.20i) follows since $E\|\Pi_N X_1\|_\mu^2 < \infty$, and to obtain (3.20ii) note that

$$\begin{aligned} \textstyle\sum_n E\|\Pi_N X_{1,n}\|_\mu^3/n^{3/2} &\le \textstyle\sum_{n=1}^\infty n^{-3/2} \sum_{k=1}^n k^{3/2} P(k-1 < \|\Pi_N X_1\|_\mu^2 \le k) \\ &\le \textstyle\sum_{k=1}^\infty k^{3/2} P(k-1 < \|\Pi_N X_1\|_\mu^2 \le k) \sum_{n\ge k} n^{-3/2} \\ &= O(\textstyle\sum_{k=1}^\infty kP(k-1 < \|\Pi_N X_1\|_\mu^2 \le k)) \\ &= O(E\|\Pi_N X_1\|_\mu^2) \\ &< \infty. \end{aligned}$$

Hence (3.20) holds and Theorem 3.1 is proved.

PROOF OF COROLLARY 3.1. The proof starts as in the proof of Theorem 3.1 except that since K is compact N is chosen so that $Q_N K \subseteq \{x : \|x\| < \varepsilon/8\}$, and (3.1) and (3.2) hold with ε replaced by $\varepsilon/4$ throughout. Then we have

$$\text{(3.21)} \qquad \begin{aligned} P\left(\left\|\frac{S_n}{a_n} - b\right\| < \varepsilon\right) &\ge P\left(\left\|\Pi_N\left(\frac{S_n}{a_n} - b\right)\right\| < \frac{\varepsilon}{4}, \ \left\|(\Pi_{N+k(N)} - \Pi_N)\frac{S_n}{a_n}\right\| < \frac{\varepsilon}{4}, \right. \\ &\qquad \left. \left\|Q_{N+k(N)}\frac{S_n}{a_n}\right\| < \frac{\varepsilon}{4}\right) \\ &\ge P\left(\left\|\Pi_N\left(\frac{S_n}{a_n} - b\right)\right\| < \frac{\varepsilon}{4}, \ \left\|Q_{N+k(N)}\frac{S_n}{a_n}\right\| < \frac{\varepsilon}{4}\right) \\ &\quad - P\left(\left\|(\Pi_{N+k(N)} - \Pi_N)\frac{S_n}{a_n}\right\| \ge \frac{\varepsilon}{4}\right). \end{aligned}$$

Arguing as in Theorem 3.1 we thus have a complete proof if

$$\sum_{n=1}^{\infty} P\left(\left\| (\Pi_{N+k(N)} - \Pi_N)\left(\frac{S_n}{a_n}\right)\right\| \geq \frac{\varepsilon}{4}\right) \Big/ n < \infty. \tag{3.22}$$

To prove (3.22) we first observe that since $Q_N K \subseteq \{x : \| x \| < \varepsilon/8\}$ we have

$$P\left(\left\| (\Pi_{N+k(N)} - \Pi_N) \frac{S_n}{a_n}\right\| > \frac{\varepsilon}{4}\right) \leq P\left(\left\| (\Pi_{N+k(N)} - \Pi_N) \frac{S_n}{a_n}\right\|_\mu \geq 2\right). \tag{3.23}$$

For economy of notation let $W = (\Pi_{N+k(N)} - \Pi_N)X_1$. Then to estimate the right hand probability in (3.23) we proceed as in the proof of Lemma 3.1 obtaining a constant C such that for all $n \geq 1$,

$$\begin{aligned}
P&\left(\left\| (\Pi_{N+k(N)} - \Pi_N) \frac{S_n}{a_n}\right\|_\mu > 2\right) \\
&\leq \mu_n(x : \| (\Pi_{n+k(N)} - \Pi_N)(x) \|_\mu > 2\sqrt{2L_2 n}) \\
&\quad + nP(\| W\|_\mu \geq \sqrt{n}) + \frac{C}{n^{1/2}} E \| WI(\| W\|_\mu > \sqrt{n}) \|_\mu^3 \\
&= \int k(N) \int_{\{u:|u|>2\sqrt{2LLN}\}} \frac{\exp\{-\sum_{i=1}^{k(N)} u_i^2/2\sigma_{i,n}^2\}\, du_1 \cdots du_{k(N)}}{\sqrt{(2\pi)^{k(N)}}\sigma_{1,n}^2 \cdots \sigma_{k(N),n}^2} \\
&\quad + nP(\| W\|_\mu > \sqrt{n}) + \frac{C}{n^{1/2}} E \| WI(\| W\|_\mu > \sqrt{n}) \|_\mu^3.
\end{aligned} \tag{3.24}$$

Since $\sigma_{i,n}^2 \leq 1$, and $\sigma_{i,n}^2$ converges uniformly to 1 as n tends to ∞ as in (3.12), we have a constant C' such that the integral term in (3.23) is easily dominated by

$$\begin{aligned}
C' &\int_{\{u:|u|>2\sqrt{2L_2 n}\}} k(N) \int \exp\{-\textstyle\sum_{i=1}^{k(N)} u_i^2/2\}/\sqrt{(2\pi)^{k(N)}}\, du_1 \cdots du_{k(N)} \\
&\leq C' \exp\{-2L_2 n\} \int_{-\infty}^{\infty} k(N) \int_{-\infty}^{\infty} \frac{\exp\{-\sum_{i=1}^{k(N)} u_i^2/4\}}{\sqrt{(2\pi)^{k(N)}}} du_1 \cdots du_{k(N)} \\
&= \frac{C' 2^{k(N)}}{(Ln)^2}.
\end{aligned}$$

Since $\sum_n 1/n(Ln)^2 < \infty$ arguing as in (3.20i) and (3.20ii) we thus have (3.22) holding and the corollary is proved.

PROOF OF COROLLARY 3.2. The proof starts as in the proof of Theorem 3.1 and Corollary 3.1. That is, we choose N, $k(N)$ such that $\| Q_N b \| < \varepsilon/4$ and (3.5) holds. Then as in (3.21) we have

$$\begin{aligned}
P\left(\left\|\frac{S_n}{a_n} - b\right\| < \varepsilon\right) &\geq P\left(\left\|\Pi_N\left(\frac{S_n}{a_n} - b\right)\right\| < \frac{\varepsilon}{4},\ \left\|Q_{N+k(N)} \frac{S_n}{a_n}\right\| < \frac{\varepsilon}{4}\right) \\
&\quad - P\left(\left\|\Pi_N\left(\frac{S_n}{a_n} - b\right)\right\| < \frac{\varepsilon}{4},\ \left\|(\Pi_{N+k(N)} - \Pi_N)\left(\frac{S_n}{a_n}\right)\right\| \geq \frac{\varepsilon}{4}\right).
\end{aligned} \tag{3.25}$$

Since (3.1), (3.2), and (3.5) hold we have as in Theorem 3.1 that for all $n \in J(\varepsilon/8)$ sufficiently large

$$\begin{aligned}
P\left(\left\|\frac{S_n}{a_n} - b\right\| < \varepsilon\right) &\geq \gamma(\varepsilon/8)\, P\left(\left\|\Pi_N\left(\frac{S_N}{a_n} - b\right)\right\| < \frac{\varepsilon}{4}\right) \\
&\quad - P\left(\left\|\Pi_N\left(\frac{S_n}{a_n} - b\right)\right\| < \varepsilon/4,\ \left\|\left(\Pi_{N+k(N)} - \Pi_N\right)\left(\frac{S_n}{a_n}\right)\right\| \geq \frac{\varepsilon}{4}\right).
\end{aligned} \tag{3.26}$$

Furthermore, N is fixed and $E|\alpha_j(x)|^3 < \infty$ for all j. Hence by the Berry-Esseen estimates given in Bhattacharya (1972), Theorem 4.1, if Z is a mean zero Gaussian random variable in $\Pi_{N+k(N)}B$ with identity covariance, i.e.

$$E(\alpha_i(z)\alpha_j(z)) = \begin{cases} 0 & i \neq j \\ 1 & i = j, \end{cases}$$

then there is a constant C such that

$$\text{(3.27i)} \quad P\left(\left\|\Pi_N\left(\frac{S_n}{a_n} - b\right)\right\| < \frac{\varepsilon}{4}\right) \geq P\left(\|\Pi_N(Z - b\sqrt{2L_2n})\| < \left(\frac{\varepsilon}{4}\right)\sqrt{2L_2n}\right) - \frac{C}{n^{1/2}},$$

and

$$\text{(3.27ii)} \quad P\left(\left\|\Pi_N\left(\frac{S_n}{a_n} - b\right)\right\| < \frac{\varepsilon}{4}, \ \left\|(\Pi_{N+k(N)} - \Pi_N)\left(\frac{S_n}{a_n}\right)\right\| \geq \frac{\varepsilon}{4}\right)$$

$$\leq P\left(\|\Pi_N(Z - b\sqrt{2L_2n})\| < \left(\frac{\varepsilon}{4}\right)\sqrt{2L_2n}, \ \frac{\|(\Pi_{N+k(N)} - \Pi_N)(Z)\|}{2L_2n} \geq \frac{\varepsilon}{4}\right) + \frac{C}{n^{1/2}}.$$

The application of Theorem 4.1 of Bhattacharya (1972) to (3.27i) is straightforward but for (3.27ii) one must do a little work. To be precise, in the notation of Bhattacharya (1972), let $g = I_C$ where $C = A \cap B$,

$$A = \{x : \|\Pi_N(x - b\sqrt{2L_2n})\| < (\varepsilon/4)\sqrt{2L_2n}\},$$

and

$$B = \left\{x : \left\|\frac{(\Pi_{N+k(N)} - \Pi_N)}{\sqrt{2L_2n}}(x)\right\| \geq \frac{\varepsilon}{4}\right\}.$$

Then A and B^c are convex and $\partial(A \cap B) \subseteq \partial A \cup \partial B$ where ∂E denotes the boundary of the set E. Further, by Bhattacharya (1972), page 469, we thus have

$$\Phi((\partial C)^{\varepsilon}) \leq d(N + k(N))\varepsilon$$

where Φ is the law of Z, $d(n + k(N))$ is a constant, and E^{ε} is the ε-neighborhood of E.

Using (3.27) and that $\Pi_N Z$ and $(\Pi_{N+k(N)} - \Pi_N)(Z)$ are independent we have from (3.26) that for all $n \in J(\varepsilon/8)$ sufficiently large

$$\text{(3.28)} \quad \begin{aligned} P\left(\left\|\frac{S_n}{a_n} - b\right\| < \varepsilon\right) &\geq P\left(\|\Pi_N(Z - b\sqrt{2L_2n})\| \leq \left(\frac{\varepsilon}{4}\right)\sqrt{2L_2n}\right). \\ &\left[\gamma\left(\frac{\varepsilon}{8}\right) - P\left(\|(\Pi_{N+k(N)} - \Pi_N)Z\| > \left(\frac{\varepsilon}{4}\right)\sqrt{2L_2n}\right)\right] - \frac{C_3}{\sqrt{n}} \end{aligned}$$

where C_3 is some positive constant. Since

$$P(\|(\Pi_{N+k(N)} - \Pi_N)Z\| > (\varepsilon/4)\sqrt{2L_2n})$$

tends to zero as $n \to \infty$ we thus have (3.14) holding for all $b \in K$ with $\|b\|_\mu < 1$ by proceeding as in Lemma 3.1 except we do not need the truncations. Thus the corollary is proved.

4. **The LIL and the CLT.** In this section we first will prove several results regarding the LIL. Since the proofs follow a general pattern we first will state our theorems and the proofs will then follow. Necessary and sufficient conditions for the CLT in smooth normed spaces appear in Theorem 4.3 at the end of the section.

THEOREM 4.1. *Let X be a mean zero random variable such that*

(4.1) *X satisfies the* CLT,

(4.2) *and*
$$E\frac{\|X\|^2}{L_2\|X\|^2}<\infty.$$

Then X satisfies the compact LIL.

REMARK. Since the condition (4.2) is a necessary condition for the LIL, we then have a necessary and sufficient condition for the LIL in the presence of the CLT.

In the case of Hilbert space valued random variables we can prove the following result.

THEOREM 4.2. *Let X be an H-valued random variable. Then X satisfies the bounded* LIL *iff*

(4.3)
$$X \text{ is } WM_0^2,$$

and (4.2) *holds. In fact, if S is the covariance operator for X* (*see Remark* 3 *following Lemma* 2.1) *and* $X_1, X_2, \cdots$ *are independent copies of X, then* (4.2) *and* (4.3) *imply that with probability one*

(4.4)
$$\limsup_n \|S_n/a_n\| \le \sqrt{8\|S\|}.$$

REMARK. Since X is WM_0^2 we have by Lemma 2.1*ii* that the covariance operator S is bounded and hence the operator norm of S, written $\|S\|$, is finite.

COROLLARY 4.1. *Let X be H-valued. Then X satisfies the compact* LIL *iff* (4.2) *and* (4.3) *hold and the covariance operator of X is compact.*

Now we describe the notation we use to prove Theorems 4.1 and 4.2. In order to allow the reader to compare our proof of Theorem 4.1 with that of Heinkel (1978–79), we shall use a slightly modified version of the notation of Heinkel and Kuelbs and Zinn (1979).

First, however, we note that in the proof of each of these theorems one can assume that X is symmetric. This follows from Pisier (1975) or the methods of Crawford (1976). Hence X is assumed to be symmetric throughout the proof.

Let $\{X_j: j \ge 1\}$ be independent copies of X. Let $I(n) = \{2^n + 1, \cdots, 2^{n+1}\}$. Also, let $\alpha(t) = t/L_2t$, $\beta(t) = tL_2t$, and set $\alpha_n = \beta^{-1}(2^n)$, $\beta_n = \alpha^{-1}(2^n)$. Then for $j \in I(n)$ let

(4.5)
$$\begin{aligned} u_j &= X_jI(\|X_j\|^2 < \alpha_n)\\ v_j &= X_jI(\alpha_n \le \|X_j\|^2 \le \beta_n)\\ w_j &= X_jI(\beta_n < \|X_j\|^2). \end{aligned}$$

The proofs of our theorems use the following lemmas.

LEMMA 4.1. *If* (4.2) *holds, then with probability one*

(4.6)
$$\lim_k \|\textstyle\sum_{j=1}^k w_j/a_k\| = 0.$$

PROOF. Since (4.2) holds we have $\|X_k\|/a_k \to 0$ a.s. and hence (4.6) holds.

LEMMA 4.2. *Let X be a symmetric B-valued random variable such that* (4.1) *and* (4.2) *hold, and assume* $\{v_j: j \ge 1\}$ *is as in* (4.5). *Then with probability one*

(4.7)
$$\lim_k \textstyle\sum_{j=1}^k v_j/a_k = 0$$

PROOF. Let $T_n = \sum_{j\in I(n)} v_j$ for $n \ge 1$. Then as in Lemma 3.3 of Heinkel (1978b), or the method employed in Stout (1974), page 159, we have (4.7) if and only if with probability one

(4.8)
$$\lim_n \frac{T_n}{a_{2^n}} = 0.$$

To prove (4.8), we set $Z_j = 2^n v_j/a_{2^n}$ for $j \in I(n)$ and all $n \geq 1$. Then $\{Z_j: j \geq 1\}$ is a sequence of independent symmetric random variables, and we have

$$\frac{T_n}{a_{2^n}} = \sum_{j\in I(n)} Z_j/2^n. \tag{4.9}$$

Hence to prove (4.8) it suffices to prove

$$\lim_n \sum_{j\in I(n)} Z_j/2^n = 0 \tag{4.10}$$

with probability one. Now to accomplish this we prove $\{Z_j: j \geq 1\}$ satisfies the conditions of Theorem 1 of Kuelbs and Zinn (1979), and hence by applying the proof of Theorem 1 of Kuelbs and Zinn (1979) we have (4.10) and the lemma will be proved.

To apply the proof of Theorem 1 of Kuelbs and Zinn (1979) it suffices to show

$$\frac{Z_j}{j} \to 0 \text{ a.s.}, \tag{4.11}$$

$$\sum_{n\geq 1} [\Lambda(n)]^2 < \infty \tag{4.12}$$

where

$$\Lambda(n) = \sum_{j\in I(n)} E\|Z_j\|^2/4^n, \tag{4.13}$$

and that

$$\sum_{j=1}^k Z_j/k \to 0 \text{ in probability.} \tag{4.14}$$

Now (4.11) holds since for $j \in I(n)$

$$\frac{\|Z_j\|}{j} \leq \frac{\|v_j\|}{a_{2^n}} \leq \frac{\|X_j\|}{a_{2^n}},$$

and $\dfrac{X_n}{a_n} \to 0$ a.s. since this is equivalent to $\sum_{n\geq 1} P(\|X_n\| > \varepsilon\sqrt{n\, L_2 n}\,) < \infty$ for all $\varepsilon > 0$ and the latter is equivalent to (4.2).

To prove (4.12) note that

$$\Lambda(n) = E\left\{\frac{\|X\|^2 I(\alpha_n \leq \|X\|^2 \leq \beta_n)}{2L_2 2^n}\right\},$$

and hence if Y is an independent copy of X we have

$$\begin{aligned}
\sum_n [\Lambda(n)]^2 &= \sum_n E\left\{\frac{\|X\|^2 I(\alpha_n \leq \|X\|^2 \leq \beta_n)}{2L_2 2^n}\right\} E\left\{\frac{\|Y\|^2 I(\alpha_n \leq \|Y\|^2 \leq \beta_n)}{2L_2 2^n}\right\} \\
&\leq \sum_n E\left\{\frac{\|X\|^2 I(\alpha_n \leq \|X\|^2 \leq \beta_n)}{L_2\|X\|^2} \frac{\|Y\|^2 I(\alpha_n \leq \|Y\|^2 \leq \beta_n)}{L_2\|Y\|^2}\right\} \\
&= \sum_n E\left\{\frac{\|X\|^2\|Y\|^2}{L_2\|X\|^2 L_2\|Y\|^2}\right. \\
&\qquad \left.\cdot\, I\left(\frac{\|X\|^2}{L_2\|X\|^2} \leq 2^n \leq \|X\|^2 L_2\|X\|^2; \frac{\|Y\|^2}{L_2\|Y\|^2} \leq 2^n \leq \|Y\|^2 L_2\|Y\|^2\right)\right\} \\
&= E\left\{\frac{\|X\|^2}{L_2\|X\|^2}\frac{\|Y\|^2}{L_2\|Y\|^2} \sum_n I(M \leq 2^n \leq m)\right\}
\end{aligned} \tag{4.15}$$

where

$$M = \max\left\{\frac{\|X\|^2}{L_2\|X\|^2}, \frac{\|Y\|^2}{L_2\|Y\|^2}\right\}$$

and

$$m = \min\{\|X\|^2 L_2\|X\|^2, \|Y\|^2 L_2\|Y\|^2\}.$$

Hence

$$\tag{4.16} \begin{aligned} \textstyle\sum_n[\Lambda(n)]^2 &\le E\left\{\frac{\|X\|^2}{L_2\|X\|^2}\frac{\|Y\|^2}{L_2\|Y\|^2}(Lm - LM)^+/\log 2\right\} \\ &\le \frac{2}{\log 2}E\left\{\frac{\|X\|^2}{L_2\|X\|^2}\frac{\|Y\|^2}{L_2\|Y\|^2}(Lm - LM)^+ I(\|X\| \le \|Y\|)\right\}. \end{aligned}$$

We now estimate the quantity

$$\tag{4.17} \begin{aligned} D &= (Lm - LM)^+ I(\|X\| \le \|Y\|) \\ &= [L(\|X\|^2 L_2\|X\|^2) - L(\|Y\|^2/L_2\|Y\|^2)]^+ \\ &\quad \cdot I\left(\|X\| \le \|Y\|;\ \|X\|^2 L_2\|X\|^2 \ge \frac{\|Y\|^2}{L_2\|Y\|^2}\right). \end{aligned}$$

Since there is a constant $c < \infty$ such that $L_2\|Y\|^2 < c\, L_2\|X\|$ under the above restrictions we have

$$\tag{4.18} D \le 2L_3\|Y\|^2 I(\|X\| \le \|Y\| \le c\,\|X\|\, L_2\,\|X\|),$$

and hence by combining (4.16) and (4.18)

$$\tag{4.19} \begin{aligned} \textstyle\sum_n[\Lambda(n)]^2 &\le \frac{4}{\log 2}E_X\left[\frac{\|X\|^2}{L_2\|X\|^2}E_Y\left[\frac{\|Y\|^2 L_3\|Y\|^2}{L_2\|Y\|^2}I(\|X\| \le \|Y\| \le c\|X\| L_2\|X\|)\right]\right] \\ &\le \frac{4}{\log 2}E_X\left\{\frac{\|X\|^2}{L_2\|X\|^2}\right\}\sup_\lambda E_Y\left[\frac{\|Y\|^2 L_3\|Y\|^2}{L_2\|Y\|^2}I(\lambda \le \|Y\| \le c\lambda\, L_2\lambda)\right]. \end{aligned}$$

Let $\gamma(t) = t\, L_3t/L_2t$. Then $\gamma'(t) \le L_3t/L_2t$ for $L_3t > 1$ and hence

$$\tag{4.20} \begin{aligned} &\sup_\lambda E\left[\frac{\|X\|^2 L_3\|X\|^2}{L_2\|X\|^2}I(\lambda \le \|X\| \le c\lambda L_2\lambda)\right] \\ &\quad = \sup_\lambda \int_\lambda^{c\lambda L_2\lambda} \gamma(t^2)P_{\|X\|}(dt) \\ &\quad = \sup_\lambda\left[\gamma(\lambda^2)P(\|X\| > \lambda) + \int_\lambda^{c\lambda L_2\lambda} 2t\gamma'(t^2)P(\|X\| > t)\, dt\right]. \end{aligned}$$

Since X satisfies the CLT by (4.1), it follows from Theorem 8 of Jain (1977) that

$$\tag{4.21} \sup_{t>0} t^2 P(\|X\| > t) < \infty.$$

Now using (4.21) in (4.20) we see that there is a finite constant c_1 such that

$$\textstyle\sum_n[\lambda(n)]^2 \le c_1 \sup_\lambda\left[\frac{L_3\lambda^2}{L_2\lambda^2} + \int_\lambda^{c\lambda L_2\lambda} t\frac{L_3t^2}{L_2t^2}\cdot\frac{1}{t^2}\, dt\right] < \infty.$$

Hence (4.12) holds and it remains only to verify (4.14). Now for $\varepsilon > 0$,

$$P(\|\textstyle\sum_{j=1}^k Z_j\|/k > \varepsilon) \le E\|\sum_{j=1}^k Z_j/k\|/\varepsilon,$$

and by the comparison principle in Lemma 4.1 of Hoffman-Jorgensen (1974) we have for $r = \max\{j : 2^j \le k\}$ that

$$E\|\sum_{j=1}^{k} Z_j\| = E\|\sum_{n=1}^{r-1} \sum_{j\in I(n)} \frac{2^n v_j}{a_{2^n}} + \sum_{j=2^r+1}^{k} \frac{2^r}{a_{2^r}} v_j\|$$
$$\le \sup_{1\le n\le r} \frac{2^n}{a_{2^n}} E\|\sum_{j=1}^{k} v_j\|$$
$$= \frac{2^r}{a_{2^r}} E\|\sum_{j=1}^{k} v_j\|.$$

Hence there is a finite constant c such that

$$P(\|\sum_{j=1}^{k} Z_j\|/k > \varepsilon) \le \frac{2^r}{k\varepsilon a_{2^r}} E\|\sum_{j=1}^{k} v_j\|$$
$$\le \frac{c}{\varepsilon} E\|\sum_{j=1}^{k} v_j\|/a_k.$$

Now since we have (4.1) the argument used in Lemma 2.3 of Kuelbs and Zinn (1979) is easily modified to prove that

$$\lim_k E\|\sum_{j=1}^{k} v_j\|/a_k = 0,$$

so (4.14) holds as claimed and our proof is complete.

In the case of Hilbert space valued random variables we have the following which improves Lemma 4.2.

LEMMA 4.3. *Let X be a symmetric H-valued random variable such that* (4.2) *and* (4.3) *hold. Further, assume $\{v_j : j \ge 1\}$ is as in* (4.5). *Then with probability one*

$$\lim_k \sum_{j=1}^{k} v_j/a_k = 0. \tag{4.22}$$

PROOF. As in Lemma 4.2 it suffices to prove that $\lim_n T_n/a_{2^n} = 0$ where $T_n = \sum_{j\in I(n)} v_j$ *for* $n \ge 1$.

Let $D_k = \sum_{j=2^n+1}^{k} v_j$ for $k \in I(n)$. Then

$$\begin{aligned} \|T_n\|^2 &= \sum_{i,j\in I(n)} \langle v_i, v_j\rangle \\ &= \sum_{i\in I(n)} \|v_i\|^2 + 2\sum_{i\in I(n)} \langle v_i, D_{i-1}\rangle, \end{aligned} \tag{4.23}$$

and since $\sum_{i\in I(n)} \|v_i\|^2 \le \sum_{j=1}^{2^{n+1}} \|X_j\|^2$ we have by the argument in Pisier and Zinn (1978, page 294) that

$$\lim_n \sum_{i\in I(n)} \|v_i\|^2/a_{2^n}^2 = 0. \tag{4.24}$$

Thus

$$\limsup_n \frac{\|T_n\|^2}{a_{2^n}^2} = 2\limsup_n \sum_{i\in I(n)} \langle v_i, D_{i-1}\rangle/a_{2^n}^2, \tag{4.25}$$

and since the random variables $w_j = \langle v_j, D_{j-1}\rangle$ are square integrable and orthogonal, by Chebyshev's inequality we have for each $\varepsilon > 0$ that

$$P(|\sum_{i\in I(n)} \langle v_i, D_{i-1}\rangle| > \varepsilon\, a_{2^n}^2) \le \frac{1}{\varepsilon^2 a_{2^n}^4} \sum_{i\in I(n)} E\langle v_i, D_{i-1}\rangle^2$$
$$= \frac{1}{\varepsilon^2 a_{2^n}^4} \sum_{i,j\in I(n), i<j} E\langle v_i, v_j\rangle^2$$
$$\le \frac{4^n}{\varepsilon^2 a_{2^n}^4} E\{\langle X, Y\rangle^2 I(\alpha_n \le \|X\|^2, \|Y\|^2 \le \beta_n)\}$$

where Y is an independent copy of X. Hence there is a finite constant c such that

$$J \equiv \sum_n P(|\sum_{i\in I(n)} \langle v_i, D_{i-1}\rangle| > \varepsilon\, a_{2^n}^2)$$
$$\le (1/\varepsilon^2)\sum_n E\{\langle X, Y\rangle^2 I(\alpha_n \le \|X\|^2, \|Y\|^2 \le \beta_n)/(L_2 2^n)^2\}$$
$$\le \frac{c}{\varepsilon^2}\sum_n E\left\{\frac{\langle X, Y\rangle^2}{L_2\|X\|^2 L_2\|Y\|^2} I(\alpha_n \le \|X\|^2, \|Y\|^2 \le \beta_n)\right\}$$
$$= \frac{c}{\varepsilon^2} E\left\{\frac{\langle X, Y\rangle^2}{L_2\|X\|^2 L_2\|Y\|^2}\sum_n I(\alpha_n \le \|X\|^2, \|Y\|^2 \le \beta_n)\right\}.$$

Thus arguing as in (4.15), (4.16), (4.17), and (4.18) we have another constant $d < \infty$ such that

$$J \le \frac{d}{\varepsilon^2} E\left\{\frac{\langle X, Y\rangle^2}{L_2\|X\|^2 L_2\|Y\|^2} L_3\|Y\|^2\right\}$$
$$\le \frac{d}{\varepsilon^2} E_X\left\{E_Y\left[\frac{\langle X, Y\rangle^2}{L_2\|X\|^2}\right]\right\}$$
$$= \frac{d}{\varepsilon^2} E_X\left\{\frac{\langle SX, X\rangle}{L_2\|X\|^2}\right\}$$
$$\le \frac{d\|S\|}{\varepsilon^2} E\left\{\frac{\|X\|^2}{L_2\|X\|^2}\right\} < \infty$$

where S is the covariance operator of X and the norm of S, $\|S\|$, is finite by Lemma 2.1 since X satisfies(4.3). Therefore $J < \infty$ and by the Borel-Cantelli lemma we have the right hand side of(4.25) less than ε. Thus $\lim_n T_n/a_{2^n} = 0$ a.s. and the lemma is proved.

PROOF OF THEOREM 4.1. As in Heinkel (1978, 1979) and Pisier (1975) it suffices to show that

$$\sup_n E\|S_n/\sqrt{n}\| < \infty$$

and

$$E\left\{\frac{\|X\|^2}{L_2\|X\|^2}\right\} < \infty$$

together imply that $\sup_n \|S_n/a_n\| < \infty$ with probability one.

In Proposition 4.3 of Pisier (1975) it is shown that the hypotheses of the theorem imply

$$\sup_n \|\textstyle\sum_{j=1}^n u_j\|/a_n < \infty \tag{4.26}$$

with probability one, and hence in view of (4.7) and Lemmas 4.1 and 4.2 the proof is complete.

REMARK. For another approach to the proof of Theorem 4.1 see the proof of Theorem 5.1. Note also that since $C[0, 1]$ is a closed subspace of $D[0, 1]$ and any separable Banach space can be viewed as a closed subspace of $C[0, 1]$, our Theorem 5.1 actually implies Theorem 4.1.

PROOF OF THEOREM 4.2. By considering $\{X_j/\|S\|^{1/2}: j \ge 1\}$ it suffices to handle the situation when $\|S\| = 1$. Hence in view of Lemmas 4.1 and 4.3, to prove that X satisfies the bounded LIL it suffices to prove (4.26) holds. To verify (4.26) and obtain the bound claimed in (4.4), we prove the following proposition completing the proof.

PROPOSITION 4.1. *Let X be a symmetric H-valued random variable such that* (4.2) *and* (4.3) *hold and $\|S\| = 1$ where $\|S\|$ is the norm of the covariance operator S for X. Further, assume $\{u_j: j \ge 1\}$ is as in* (4.5). *Then*

$$\limsup_n \|\textstyle\sum_{j=1}^n u_j/a_n\| \le \sqrt{8}. \tag{4.27}$$

PROOF OF PROPOSITION 4.1. To prove (4.27) we proceed as in Lemma 4.3. That is, here we set $T_n = \sum_{j=1}^{n} u_j$ for $n \geq 1$ and note that

$$\| T_n \|^2 = \sum_{j=1}^{n} \| u_j \|^2 + 2 \sum_{j=1}^{n-1} \langle u_j, T_{j-1} \rangle. \tag{4.28}$$

Hence as in Lemma 4.3 we have

$$\lim \sum_{j=1}^{n} \| u_j \|^2 / a_n^2 = 0, \tag{4.29}$$

and to establish (4.27) we define

$$M_n = \sum_{j=1}^{n} \langle u_j, T_{j-1} \rangle I(| \langle u_j, T_{j-1} \rangle | \leq j) \qquad n \geq 1, \tag{4.30}$$

and set

$$s_n^2 = \sum_{j=1}^{n-1} (\| S^{1/2} T_j \|^2 \vee a_j^2) \qquad n \geq 2. \tag{4.31}$$

Then (4.27) holds if we prove that with probability one

$$\limsup_n \frac{M_n}{s_n \sqrt{2 L_2 s_n}} \leq \sqrt{2}, \tag{4.32}$$

and

$$P(| \langle u_j, T_{j-1} \rangle | > j \text{ i.o.}) = 0. \tag{4.33}$$

To see that (4.32) and (4.33) actually imply (4.27) first observe that (4.28), (4.29), (4.32), and (4.33) imply

$$\limsup_n \frac{\| T_n \|^2}{s_n \sqrt{2 L_2 s_n}} \leq 2\sqrt{2} \qquad \text{a.s.} \tag{4.34}$$

Further, we have $s_n^2 \geq \frac{1}{\theta} n^2 L_2 n$ for n large provided $\theta > 1$ is arbitrary, and hence

$$s_n \sqrt{2 L_2 s_n} \geq \frac{1}{\sqrt{\theta}} n \sqrt{L_2 n} \sqrt{2 L_2 (1/\theta \, n^2 L_2 n)} \geq \frac{1}{\theta} a_n^2. \tag{4.35}$$

Thus $\theta > 1$ arbitrary and (4.34) and (4.35) combine to imply

$$\limsup_n \frac{\| T_n \|^2 \vee a_n^2}{s_n \sqrt{2 L_2 s_n}} \leq 2\sqrt{2} \qquad \text{a.s.} \tag{4.36}$$

Now recall $\| S \| = 1$. Thus a.s. for n sufficiently large we have

$$\begin{aligned} (\| T_n \|^2 \vee a_n^2)^2 &\leq 16 \theta s_n^2 L_2 s_n \\ &= 16 \theta \sum_{j=1}^{n-1} (\| S^{1/2} T_j \|^2 \vee a_j^2) L_2 (\textstyle\sum_{j=1}^{n-1} (\| S^{1/2} T_j \|^2 \vee a_j^2)) \\ &\leq 16 \theta^2 \sum_{j=1}^{n-1} (\| T_j \|^2 \vee a_j^2) L_2 (\textstyle\sum_{j=1}^{n-1} (\| T_j \|^2 \vee a_j^2)) . \end{aligned} \tag{4.37}$$

The equation (4.27) is now an immediate consequence of (4.37) and the following lemma since $\theta > 1$ is arbitrary.

LEMMA 4.4. *If $\{c_n\}$ is a sequence of positive numbers such that for n sufficiently large*

$$c_n^2 \leq \rho \sum_{k=1}^{n-1} c_k L_2 (\textstyle\sum_{k=1}^{n-1} c_k) \qquad p > 0, \tag{4.38}$$

then

$$\limsup_n \frac{c_n}{a_n^2} \leq \rho / 2. \tag{4.39}$$

REMARK. To finish the proof of Proposition 4.1 it remains to prove that (4.32), (4.33) and Lemma 4.4 hold.

PROOF OF (4.32). Set $\mathscr{F}_n = \sigma(u_1, \ldots, u_n)$. Since

$$M_n - M_{n-1} = \langle u_n, T_{n-1}\rangle > I(|\langle u_n, T_{n-1}\rangle| \le n)$$

we have

$$E(M_n - M_{n-1} | \mathscr{F}_{n-1}) = 0$$

and

$$E((M_n - M_{n-1})^2 | \mathscr{F}_{n-1}) \le \langle ST_{n-1}, T_{n-1}\rangle = \|S^{1/2}T_{n-1}\|^2$$

with probability one.

Hence $\{M_n, \mathscr{F}_n, n \ge 1\}$ is a martingale and by the proof of Lemma 5.4.1, page 299 of Stout (1974), we have that the process

$$V_n = e^{\lambda M_n - \lambda^2(1+c\lambda)s_n^2/2}$$

is a supermartingale for $n \le c$ provided $\lambda c \le 1$. Hence Corollary 5.4.1 of Stout (1974) page 299, applied to the supermartingale $V_{n\wedge\tau}$, where τ is a stopping rule, yields the inequality

$$P(\max_{n\le c} V_{n\wedge\tau} \ge \alpha) \le 1/\alpha \tag{4.40}$$

for λ such that $\lambda c \le 1$.

To prove (4.32) it suffices to show that for $\theta > 1$

$$P(M_n \ge \theta\sqrt{2}s_n\sqrt{2L_2s_n} \text{ i.o.}) = 0. \tag{4.41}$$

For each positive integer k set

$$\tau_k(\omega) = \inf\{t: s_{t+1}^{(\omega)} \ge \theta^{2k}\}.$$

Then each τ_k is a stopping rule with respect to $\{\mathscr{F}_n\}$ and $\tau_k \to +\infty$ a.s. as $k \to \infty$. Further, the probability in (4.41) can be written as (and dominated by)

$$\begin{aligned} &P(\max_{\tau_{k-1}<n\le\tau_k}[M_n - \theta\sqrt{2}\, s_n\sqrt{2L_2s_n}] \ge 0 \text{ i.o. in } k) \\ &\qquad \le P(\max_{\tau_{k-1}<n\le\tau_k} M_n \ge \theta\sqrt{2}\, s_{\tau_{k-1}+1}\sqrt{2L_2s_{\tau_{k-1}+1}} \text{ i.o. in } k). \end{aligned} \tag{4.42}$$

Hence it suffices to show that

$$\textstyle\sum_k P(\max_{n\le\tau_k} M_n \ge \theta\sqrt{2}\, s_{\tau_{k-1}+1}\sqrt{2L_2s_{\tau_{k-1}+1}}) < \infty. \tag{4.43}$$

By construction $s_{n+1}^2 \ge \sum_{k=1}^n 2kL_2k$ so for n large $s_n^2 \ge \frac{1}{\theta}\, n^2L_2n$. On the other hand, $s_{\tau_k}^2 \le \theta^{2k}$ so that

$$\tau_k^2 L_2\tau_k \le \theta^{2k+1},$$

and hence for k sufficiently large

$$\tau_k \le \frac{\theta^{k+1}}{\sqrt{L_2\theta^{2k}}}.$$

We take c to be the greatest integer in $\theta^{k+1}/\sqrt{L_2\theta^{2k}}$ for (4.40) and set $\lambda = \sqrt{L_2\theta^{2k}}\,/\theta^{k+1}$. Hence (4.40) implies

$$P(\max_{n\le\tau_k} e^{\lambda M_n - \lambda^2 s_n^2} \ge e^x) \le e^{-x}, \tag{4.44}$$

which implies

$$P(\max_{n\le\tau_k} M_n \le \frac{x}{\lambda} + \lambda s_{\tau_k}^2) \le e^{-x}. \tag{4.45}$$

Setting $x = \theta L_2\theta^{2k}$ and recalling our choice of λ we then have

$$\textstyle\sum_k P(\max_{n\le\tau_k} M_n \ge \theta^{k+2}\sqrt{L_2\theta^{2k}} + \theta^{k-1}\sqrt{L_2\theta^{2k}}) < \infty. \tag{4.46}$$

Now

$$P(\max_{n\le\tau_k} M_n \ge \theta^{k+2}\sqrt{L_2\theta^{2k}} + \theta^{k-1}\sqrt{L_2\theta^{2k}}) \ge P(\max_{n\le\tau_k} M_n \ge 2\theta^{k+2}\sqrt{L_2\theta^{2k}}),$$

and by construction $s_{\tau_{k-1}+1} \ge \theta^{k-1}$, so the above probability dominates

$$P(\max_{n\le\tau_k} M_n \ge 2\theta^3 s_{\tau_{k-1}+1}\theta\sqrt{L_2 s^2_{\tau_{k-1}+1}}) \tag{4.47}$$

for k sufficiently large. Since $\theta > 1$ is arbitrary (4.46) thus implies (4.43), and hence the proof of (4.32) is complete.

PROOF OF (4.33). By Markov's inequality

$$\begin{aligned} P(|<u_j, T_{j-1}>|>j) &\le E[<u_j, T_{j-1}>^4]/j^4 \\ &= \frac{1}{j^4} E[E((\textstyle\sum_{k=1}^{j-1} <u_k, u_j>)^4 \mid u_j)]. \end{aligned} \tag{4.48}$$

Since the random variables $\{<u_k,u_j>: 1 \le k \le j-1\}$ are conditionally independent, symmetric and identically distributed given u_j we have

$$\begin{aligned} &E(<u_j, u_k> \mid u_j) = 0 \\ &E(<u_j, u_k>^2 \mid u_j) \le E(<X_k, u_j>^2 \mid u_j) \\ &\qquad = <Su_j, u_j> = \|S^{1/2}u_j\|^2 \end{aligned} \tag{4.49}$$

for $1 \le k \le j-1$. In addition, since

$$(\textstyle\sum_{k=1}^{j-1} <u_j, u_k>)^4 = \sum_{k=1}^{j-1} <u_j, u_k>^4 + \sum_{1\le k<\ell\le j-1} <u_j, u_k>^2<u_j, u_\ell>^2$$
$$+ \text{ expressions involving cubes or linear terms,}$$

the conditional independence and (4.49) imply that

$$\begin{aligned} E(E((\textstyle\sum_{k=1}^{j-1} <u_j, u_k>)^4 \mid u_j) &= E(E[\textstyle\sum_{k=1}^{j-1} <u_j, u_k>^4 \\ &\quad + \textstyle\sum_{1\le k<\ell\le j-1} <u_j, u_k>^2<u_j, u_\ell>^2 \mid u_j]) \\ &\le \textstyle\sum_{k=1}^{j-1} E(E[<X_j, X_k>^4 I[|<X_j, X_K>| \\ &\quad \le j/L_2 j] \mid u_j]) + j(j-1)E\|S^{1/2}u_j\|^4 \\ &\le (j-1)E[<X_1, X_2>^4 I[|<X_1, X_2>| \le j/L_2 j]] \\ &\quad + j^2\|S\|^2 E[\|X_1\|^4 I[\|X_1\| \le \sqrt{j/L_2 j}]]. \end{aligned} \tag{4.50}$$

Combining (4.48) and (4.50) we have (4.33) provided the two series

$$I = \textstyle\sum_{j\ge1} \frac{1}{j^3} E\{<X_1, X_2>^4 I[|<X_1, X_2>| \le j/L_2 j]\} \tag{4.51}$$

and

$$II = \textstyle\sum_j \frac{1}{j^2} E[\|X_1\|^4 I[\|X_1\| \le \sqrt{j/L_2 j}]] \tag{4.52}$$

both converge.

Now

$$\begin{aligned} I &\le E[<X_1, X_2>^4 \textstyle\sum_{j\ge1} \frac{1}{j^3} I[j \ge |<X_1, X_2>|L_2|<X_1, X_2>|]] \\ &\le E[<X_1, X_2>^4/<X_1, X_2>^2(L_2|<X_1, X_2>|)^2] \\ &= E\left\{\frac{<X_1, X_2>^2}{(L_2|<X_1, X_2>|)^2}\right\}, \end{aligned}$$

and since the function $t \to t^2/(L_2 t)^2$ is monotone increasing for $t \geq 0$ we have

$$\begin{aligned} I \leq E\left\{\frac{\|X_1\|^2\|X_2\|^2}{(L_2\|X_1\|\,\|X_2\|)^2}\right\} &\leq E\left\{\frac{\|X_1\|^2\|X_2\|^2}{(L_2\|X_1\|\,\|X_2\|)^2} I(\|X_1\| \wedge \|X_2\| \leq 1)\right\} \\ &\quad + E\left\{\frac{\|X_1\|^2\|X_2\|^2}{(L_2\|X_1\|\,\|X_2\|)^2} I(\|X_1\| \wedge \|X_2\| \geq 1)\right\} \\ &\leq 2E\left\{\frac{\|X_1\|^2\|X_2\|^2}{(L_2\|X_1\|\,\|X_2\|)^2} I(\|X_1\| \leq 1)\right\} \\ &\quad + E\left\{\frac{\|X_1\|^2}{L_2\|X_1\|}\right\}\cdot E\left\{\frac{\|X_2\|^2}{L_2\|X_2\|}\right\} \\ &\leq 2E\left\{\frac{\|X_2\|^2}{(L_2\|X_2\|)^2}\right\} + \left\{E\left(\frac{\|X_1\|^2}{L_2\|X_1\|}\right)\right\}^2 < \infty. \end{aligned}$$

In addition

$$\begin{aligned} II &\leq E\left(\|X_1\|^4 \textstyle\sum_{j\geq 1} \frac{1}{j^2} I(j \geq \|X_1\|^2 L_2\|X_1\|^2)\right) \\ &\leq 2E\left\{\frac{\|X_1\|^2}{L_2\|X_1\|^2}\right\} < \infty. \end{aligned}$$

Hence (4.33) holds so it suffices to prove Lemma 4.4.

PROOF OF LEMMA 4.4. Suppose that the inequality (4.38) holds for $n \geq N$. Set $M = \max_{k<N} c_k^2$. Then for all positive integers n,

$$c_n^2 \leq \rho \textstyle\sum_{k=1}^{n-1} c_k L_2(\sum_{k=1}^{n-1} c_k) + M. \tag{4.53}$$

Without loss of generality we may assume $\{c_n\}$ is monotone increasing since $\{\max_{k\leq n} c_k : n \geq 1\}$ also satisfies (4.53). From (4.53) we therefore obtain

$$c_n \leq \rho \textstyle\sum_{k=1}^{n-1} \frac{c_k}{c_n} L_2(\sum_{k=1}^{n-1} c_k) + M/c_n \leq \rho(n-1)L_2(\sum_{k=1}^{n-1} c_k) + M/c_n,$$

and hence

$$\textstyle\sum_{n=1}^{m} c_n \leq \frac{\rho}{2} m^2 L_2(\sum_{k=1}^{m-1} c_k) + mM/c_1.$$

Thus if $\theta > 1$ is arbitrary then for m sufficiently large

$$\textstyle\sum_{n=1}^{m} c_n \leq \frac{\rho\theta}{2} m^2 L_2(\sum_{k=1}^{m} c_k),$$

and for $x = \sum_{n=1}^{m} c_n$ the above implies

$$\frac{x}{L_2 x} \leq \frac{\rho\theta}{2} m^2.$$

For m large this implies $x \leq (\rho/2)\theta^2 m^2 L_2 m^2$ and hence for m large

$$\textstyle\sum_{k=1}^{m} c_k \leq \frac{\rho\theta^2}{2} m^2 L_2 m^2.$$

Recall that $c_n \leq \rho(n-1)L_2(\sum_{k=1}^{n-1} c_k) + M/c_n$ for all n, so for n sufficiently large

$$c_n \leq \rho(n-1)L_2(\rho(\theta^2/2)n^2 L_2 n^2) + M/c_n \leq \theta\rho n L_2 n \leq \theta \frac{\rho}{2} a_n^2.$$

Hence the lemma is proved and Proposition 4.1 is verified.

PROOF OF COROLLARY 4.1. If X satisfies the compact LIL then (1.7) holds, and as described in Section 1 the canonical set K must be compact. Hence from the remarks following Lemma 2.1 we have the covariance operator S compact.

Now S is always symmetric and non-negative, and if S is compact, then there is an orthonormal basis $\{e_k\}$ and constants $\{\lambda_k\}$ tending to zero such that

$$Sx = \sum_k \lambda_k (x, e_k) e_k \qquad x \in H.$$

Fixing $\varepsilon > 0$ and choosing N so that $\sup_{k \geq N} \lambda_k < \varepsilon$ we define the projection $\tau(x) = \sum_{k \leq n}^{\text{?}} (x, e_k) e_k$. Then the random variable $X - \tau(X)$ satisfies the conditions of Theorem 4.2 and, in fact,

$$\lim\sup \| \sum_{j=1}^n (X_j - \tau(X_j))/a_n \| \leq 8\varepsilon.$$

Since $\tau(X)$ is finite dimensional with mean zero and finite second moment, and $\varepsilon > 0$ was arbitrary, we easily see that (1.7) holds. Hence X satisfies the compact LIL and the corollary is proved.

For the remainder of the section we shall assume (A): B is a real separable Banach space, whose norm is twice directionally differentiable and such that the second directional derivative, D_x^2, is Lip(α) away from zero for some $\alpha > 0$ and such that

$$\sup_{\|x\|=1} \| D_x^2 \| < \infty.$$

(We refer the reader to Kuelbs (1974) for the definitions of the terms used in (A).)

We will now use the results in Kuelbs (1974) to obtain necessary and sufficient conditions for the CLT for Banach spaces satisfying (A).

THEOREM 4.3. *Let B satisfy* (A) *and let X be a mean zero B-valued* rv. *Then X satisfies the* CLT *iff the following two conditions hold.*

(4.54) $$t^2 P(\|X\| \geq t) \to 0 \quad \text{as } t \to \infty,$$

(4.55) $$X \text{ is pre-Gaussian.}$$

PROOF. By Pisier (1975) we may assume X is symmetric. Then by Theorem 2.1 Kuelbs (1974), for any $\beta > 0$

$$P(\|S_n\| \geq \sqrt{n}\, t) \leq nP(\|X\| \geq \sqrt{n}) + \mu_n(\|x\| \geq t - \beta) + Cn^{-\alpha/2} E(\|X\|^{2+\alpha} I(\|X\| \leq \sqrt{n}))$$

for some $C = C(\beta) < \infty$ and all $t \geq 2\beta$, where μ_n is the Gaussian measure with the same covariance as $XI(\|X\| \leq \sqrt{n})$. But, if $\sup_{t \leq A} t^2 P(\|X\| \geq t) \leq \delta$,

$$\begin{aligned} E\|X\|^{2+\alpha} I(\|X\| \leq \sqrt{n}) &= \int_0^\infty P(\|X\|^{2+\alpha} \geq t, \|X\| \leq \sqrt{n})\, dt \\ &\leq A^{2+\alpha} + \int_A^{\sqrt{n}} (2+\alpha) t^{1+\alpha} P(\|X\| \geq t)\, dt \\ &\leq A^{2+\alpha} + (2+\alpha)\delta \int_0^{\sqrt{n}} t^{\alpha-1}\, dt \leq A^{2+\alpha} + \frac{\delta(2+\alpha)}{\alpha} n^{\alpha/2}. \end{aligned}$$

Also, by T. W. Anderson's inequality (1955) for every convex symmetric set $C \subseteq B$, we have $\mu(C) \leq \mu_n(C)$, when μ is the Gaussian measure with the same covariance as X.

Hence, since (4.54) implies $nP(\|X\| \geq \sqrt{n})$ is small for large n, we have

$$P(\|S_n\| \geq t\sqrt{n}) \leq \varepsilon \qquad \text{for } n \text{ and } t \text{ sufficiently large.}$$

By choosing a simple function τ such that

$$E\tau(X) = 0,$$

$$\sup_{t>0} t^2 P(\|X - \tau(X)\| \geq t) \leq \varepsilon/[C(\varepsilon) + 1]$$

and

$$\mu\left(\|x - \tau(x)\| \geq \frac{\varepsilon}{2}\right) \leq \varepsilon,$$

and then applying the above to $X - \tau(X)$, we see that $\{S_n/\sqrt{n}\}$ is tight, which suffices.

Combining Theorems 4.1 and 4.3, we have

COROLLARY 4.2. *Let B satisfy* (A) *and let X be a mean zero B-valued* rv. *Then if X satisfies* (4.2), (4.54) *and* (4.55), *then X satisfies the compact* LIL.

REMARKS. (i) Corollary 4.2 applies to L^p, $2 \leq p < \infty$, by Kuelbs (1974). (ii) Pisier has shown (personal communication) that if a mean zero rv in ℓ^p, $2 < p < \infty$ satisfies (4.2) and for $X = (X^k)$, $Y = (Y^k)$ i.i.d.

$$E\{(\textstyle\sum_k |X^k Y^k|^{p/2})^{4/p}\} < \infty \tag{4.56}$$

then $X \in$ LIL.

5. The LIL in D[0, 1]. In this section we extend Theorem 4.1 to $D[0, 1]$ valued random variables. Since $D[0, 1]$ is non-separable in the supremum norm, $\|\cdot\|$, we first review convergence of laws and the LIL in such a space. The following proposition is very useful.

PROPOSITION 5.1. (Dudley (1976), Proposition 23.6). *In $D[0, 1]$ the σ-field $\mathscr{B}_1$ generated by the balls* (*in the supremum norm*) *is the smallest σ-field for which all coordinate functions are measurable.*

Hence by a $D[0, 1]$ valued random variable Y we mean $Y(t, \omega)$ is a real random variable for each $t \in [0, 1]$, and for each ω, $Y(\cdot, \omega)$ is in $D[0, 1]$. Thus the sum of $D[0, 1]$ valued random variables is a $D[0, 1]$ valued random variable and hence both the bounded and compact LIL have immediate formulations in $(D[0, 1], \|\cdot\|)$. To identify the limit set in the compact LIL as the unit ball of some Hilbert space determined by a covariance structure can be done as in Kuelbs (1976a, pages 747–748). One can also apply the results of Section 2 provided there is a tight Borel measure μ on $(D[0, 1], \|\cdot\|)$ with the given covariance structure and this is always the situation in the cases we study.

By the law of a $D[0, 1]$ valued random variable Y, we mean the measure on $\mathscr{B}_1$ induced by Y.

DEFINITION (Dudley, 1976). A sequence of laws $\{\mu_n\}$ is said to converge to the law $\mu(\mu_n \Rightarrow \mu)$ if

$$\lim_n \int f\, d\mu_n = \int f\, d\mu$$

for all real-valued, bounded, and continuous functions f on $D[0, 1]$ which are $\mathscr{B}_1$-measurable.

REMARK. The reader should compare this definition and the results on tightness that follow with Theorem 2.3 and Remark 2.10 of Erickson and Fabian (1975).

Since in our theorem the limit measure is Gaussian, the following proposition is useful. For the proof see, for example, Jain and Kallianpur (1972).

PROPOSITION 5.2. *If a mean zero Gaussian measure* μ *on* $(D[0, 1], \mathscr{B}_1)$ *has a continuous covariance function, then* $\mu(C[0, 1]) = 1$.

As a consequence of Theorem 5.1 and Theorem 15.5 of Billingsley (1968) one can easily prove the following proposition.

PROPOSITION 5.3. *Let* μ_n, μ *be probability measures on* $(D[0, 1], \mathscr{B}_1)$ *with* $\mu(C[0, 1]) = 1$. *Then* $\mu_n \Rightarrow \mu$ *iff*

(i) $\lim_{a\to\infty} \sup_n \mu_n(x:|x(0)| > a) = 0.$

(5.1) (ii) *For each* $\varepsilon > 0$, $\lim_{\delta\downarrow 0} \limsup_n \mu_n(x:\sup_{|s-t|<\delta} |x(s) - x(t)| > \varepsilon) = 0$

(iii) $\mu_n \circ \pi_T^{-1} \Rightarrow \mu \circ \pi_T^{-1}$ *for every finite set* $T \subseteq [0, 1]$.

We can now state the main result of this section.

THEOREM 5.1. *Let* Y *be a* $D[0, 1]$ *valued random variable. Assume that* $EY(t) = 0$, $EY^2(t) < \infty$ *for all* $t \in [0, 1]$ *and that* $R(s, t) = EY(s)Y(t)$ *is continuous. Then, if* Y *satisfies the* CLT *with limiting Gaussian measure* μ *and* $E\|Y|^2/L_2\|Y\|^2 < \infty$, *then* Y *satisfies the compact* LIL *in* $(D[0, 1], \|\cdot\|)$.

REMARK. There are a substantial number of results regarding the CLT in $D[0, 1]$ and $C[0, 1]$ (see e.g., Strassen, Dudley (1969), and Giné (1974), Dudley (1974), Jain, Marcus (1975), Hahn (1976, 1978)). Hence our Theorems 4.1 and 5.1 can be applied (provided the covariance is continuous.)

PROOF. Using the argument in Lemma 3.16 of Dudley and Kuelbs (1978) or Corollary 7.2 of Crawford (1976) it suffices to assume Y is symmetric.

Now let $Y_1, Y_2, \cdots$ be independent copies of Y. Then by Theorem 4.2 of Kuelbs (1976a) it suffices to prove that

$$P(\lim_{\delta\downarrow 0} \limsup_n \sup_{|s-t|<\delta} |\textstyle\sum_{j=1}^n (Y_j(s) - Y_j(t))/a_n| = 0) = 1. \tag{5.2}$$

To establish (5.2) fix $\varepsilon > 0$, $0 < \varepsilon < 1$ and define

$$u_j = Y_j I(\|Y_j\|^2 < \varepsilon\alpha_n)$$
$$v_j = Y_j I(\varepsilon\alpha_n \le \|Y_j\|^2 \le \beta_n)$$
$$w_j = Y_j I(\beta_n < \|Y_j\|^2)$$

for $j \in I(n)$ where $I(n)$, α_n, and β_n are as in (4.5). Then by the proofs of Lemma 4.1 and Lemma 4.2 we have with probability one that

$$\limsup_n \|\textstyle\sum_{j=1}^n v_j/a_n\| = 0$$

and

$$\limsup_n \|\textstyle\sum_{j=1}^n w_j/a_n\| = 0.$$

Since $\varepsilon > 0$ was arbitrary and $\sup_{|s-t|<\delta} |f(s) - f(t)| \le 2\|f\|$ we will have (5.2) established if we show there is a $\delta > 0$ sufficiently small such that with probability one

$$\limsup_n \sup_{|s-t|<\delta} |\textstyle\sum_{j=1}^n (u_j(s) - u_j(t)/a_n| < \varepsilon. \tag{5.3}$$

If $S_n = \sum_{j=1}^n Y_j$, then Propositions 5.2 and 5.3 and Y satisfying the CLT with continuous covariance implies that for every $\varepsilon > 0$, $0 < \varepsilon < 1$, and $a > 0$ there exists $r = r(\varepsilon)$ such that

(5.4) and

(i) $\sup_n P(|S_n(0)| > n^{1/2}a) < \varepsilon$

(ii) $\sup_n P(\|S_n\|_\ell > n^{1/2}\varepsilon/2) < \varepsilon/2 \qquad 1 \le \ell \le r$

where $\|f\|_\ell = \sup_{(\ell-1)/r<t\le \ell/r} |f(t) - f(\ell/r|$.

Using (5.4) we now follow Jain (1977) and Pisier (1975) to obtain a bound on $E\|S_m\|_\ell$.

By using the inequalities $1 - t \leq e^{-t}$ and $1 - e^{-t} \geq t/(1+t)$ and Levy's inequality we have that

$$\frac{\sum_{k=1}^{n} P(\| S_{km} - S_{(k-1)m} \|_{\ell} > m^{1/2}n^{1/2}\varepsilon)}{1 + \sum_{k=1}^{n} P(\| S_{km} - S_{(k-1)m} \|_{\ell} > m^{1/2}n^{1/2}\varepsilon)} \leq P(\max_{k \leq n} \| S_{km} - S_{(k-1)m} \|_{\ell} > m^{1/2}n^{1/2}\varepsilon)$$

$$\leq P(\max_{k \leq n} \| S_{km} \|_{\ell} > m^{1/2}n^{1/2}\varepsilon/2)$$

$$\leq 2\, P(\| S_{nm} \|_{\ell} > m^{1/2}n^{1/2}\varepsilon/2) < \varepsilon/2.$$

Hence we have for $1 \leq \ell \leq r$ that

$$nP(\| S_m \|_{\ell} > m^{1/2}n^{1/2}\varepsilon) \leq \frac{\varepsilon/2}{1 - \varepsilon/2} < \varepsilon \tag{5.5}$$

since $0 < \varepsilon < 1$. Thus for $n \geq 1$ and any t such that

$$n^{1/2}\, \varepsilon \leq t \leq (n+1)^{1/2}\varepsilon$$

we have for $1 \leq \ell \leq r$ that

$$P(\| S_m \|_{\ell} > m^{1/2}t) \leq 2\varepsilon^3/t^2 \qquad t \geq \varepsilon. \tag{5.6}$$

But then for any integer α and $\eta_j = Y_j - u_j$ we have by symmetry that for $1 \leq \ell \leq r$

$$\begin{aligned} P(\| \textstyle\sum_{\alpha < j \leq \alpha+m} u_j \|_{\ell} > m^{1/2}t) &= P(\| \textstyle\sum_{\alpha < j \leq \alpha+m} (u_j - \eta_j + \eta_j + u_j) \|_{\ell} > 2m^{1/2}t) \\ &\leq 2\, P(\| \textstyle\sum_{\alpha < j \leq \alpha+m} Y_j \|_{\ell} > m^{1/2}t) \\ &\leq 4(\varepsilon^3/t^2) \qquad t \geq \varepsilon. \end{aligned} \tag{5.7}$$

Hence for $1 \leq \ell \leq r$

$$E\| m^{-1/2} \textstyle\sum_{\alpha < j \leq \alpha+m} u_j \|_{\ell} \leq \varepsilon + \displaystyle\int_{\varepsilon}^{\infty} P(\| \textstyle\sum_{\alpha < j \leq \alpha+m} u_j \|_{\ell} > m^{1/2}t)\, dt \leq 5\varepsilon. \tag{5.8}$$

From (5.8) and the proof of Proposition 4.3 of Pisier (1975) there exists a fixed constant A such that for β_k as prior to (4.5) we have

$$P(\| \textstyle\sum_{j=1}^{2^k} u_j \|_{\ell} > A\beta_k^{1/2}\, \varepsilon) \leq \dfrac{1}{(k \ell n\, 2)^2} \qquad 1 \leq \ell \leq r. \tag{5.9}$$

Finally by Levy's inequality (5.9) implies

$$P(\max_{1 \leq m \leq 2^k} \| \textstyle\sum_{j=1}^{m} u_j \|_{\ell} > A\, \beta_k^{1/2}\, \varepsilon) < 2/(k \ell n 2)^2$$

and hence by standard arguments we have a fixed constant L such that

$$\limsup_m \| \textstyle\sum_{j=1}^{m} u_j \|_{\ell/a_m} \leq L\, \varepsilon \qquad 1 \leq \ell \leq r. \tag{5.10}$$

Since $\varepsilon > 0$, $0 < \varepsilon < 1$, was arbitrary and L is fixed (5.10) easily gives (5.3) so the theorem is proved.

6. Applications to Weighted Empiricals. In this section we apply Theorem 5.1, Theorem 4.2 and Corollary 4.1 to processes of the form

$$Y(t) = \begin{cases} \omega(t)[I(U \leq t) - t] & \text{if } t \in (0, 1) \\ 0 & \text{if } t = 0 \text{ or } 1 \end{cases} \tag{6.1}$$

where U is uniformly distributed on [0, 1]. Since we will compare our results with those of James (1975), we need to discuss the relationships between the various conditions on the weight function ω given in (6.1). This will be done in a sequence of lemmas which will follow the statements of a theorem of O'Reilly (1974) on the CLT for Y (see also Shorack (1979)), a theorem of James (1975) on the compact LIL for Y, and our theorem concerning the compact LIL for Y. These results are all taken with respect to the supremum norm. After this we switch our attention to $L^2[0, 1]$. A direct application of Theorem 4.2 and Corollary 4.1 yields some results on the Cramer-von Mises statistics. However there is no known condition on a covariance function which is necessary and sufficient for the associated covariance operator from $L^2[0, 1]$ to $L^2[0, 1]$ to be bounded. Hence for ease of applications we give some conditions on ω which are sufficient to ensure the existence of a bounded covariance operator and which are close to being necessary.

Now for the sup-norm results.

THEOREM A. (O'Reilly (1974)). *Let ω satisfy the following conditions*:

(6.2)
(i) *ω is a continuous strictly positive function on* $(0, 1)$;
(ii) *for some $\gamma > 0$, we have ω is nonincreasing (nondecreasing) on* $(0, \gamma]([1-\gamma, 1))$.

Then a necessary and sufficient condition for Y to satisfy the CLT *in* $(D[0, 1], \|\cdot\|)$ *is*

$$\int_0^1 t^{-1}\exp\left(-\frac{\varepsilon}{k_i(t)}\right) dt < \infty, \qquad \text{for all} \quad \varepsilon > 0, i = 1, 2, \tag{6.3}$$

where $k_1(t) = t\omega^2(t)$ *and* $k_2(t) = (1-t)\omega^2(t)$.

THEOREM B. (James (1975)). *Let ω satisfy the following conditions*

(6.4)
(i) *ω is a non-negative function on* $(0, 1)$,
(ii) *for some $\gamma > 0$ we have ω is bounded on* $[\gamma, 1-\gamma]$,
(iii) *$t\omega^2(t)((1-t)\omega^2(t))$ is monotone increasing (decreasing) on* $(0, \gamma]([1-\gamma, 1))$.

Let $K = \{f: f(t) = \int_0^t g(s)\, ds, \int_0^1 g^2(s)\, ds \le 1$, and $\int_0^1 g(s)\, ds = 0\}$. Then a necessary and sufficient condition for Y to satisfy the compact LIL *in the sup-norm with limit set ωK is*

$$\int_0^1 \frac{\omega^2(t)}{L_2\left(\frac{1}{t(1-t)}\right)} dt < \infty. \tag{6.5}$$

REMARK. If ω is not in $D[0, 1]$ then the compact LIL is in the Banach space of bounded functions on $[0, 1]$ with sup-norm.

THEOREM 6.1. *Let ω satisfy* (6.2) *and* (6.3) *(the conditions of O'Reilly's Theorem). Then Y satisfies the compact* LIL *in* $(D[0, 1], \|\cdot\|)$ *if and only if* (6.5) *(James' integrability condition) holds.*

We now give the promised sequence of lemmas.

LEMMA 6.1. *If ω satisfies* (6.2) *and* (6.3) *(the conditions of O'Reilly's Theorem), then*

$$\text{(i)}\ \lim_{t\downarrow 0} t\omega^2(t) = 0$$

(6.6) *and*

$$\text{(ii)}\ \lim_{t\uparrow 1} (1-t)\omega^2(t) = 0.$$

PROOF. We only prove (6.6i), the proof of (6.6ii) being similar. By (6.3), as $u \to 0$

$$0 \leftarrow \int_0^u \frac{1}{t}\exp\left(-\frac{1}{t\omega^2(t)}\right) dt \ge \int_0^u \frac{1}{t}\exp\left(-\frac{1}{t\omega^2(u)}\right) dt \qquad \text{(by (6.2))}$$

$$= \int_0^{u\omega^2(u)} \frac{1}{s} \exp\left(-\frac{1}{s}\right) ds.$$

Hence (6.6i) holds.

LEMMA 6.2. *Assume that* (*the conditions of O'Reilly's Theorem*) (6.2) *and* (6.3) *hold. Then* $E\|Y\|^2/L_2\|Y\|^2 < \infty$ iff (*the integrability condition of James*) (6.5) *holds.*

PROOF. First we note that a.s.

$$\|Y\| = \max\{\sup_{0<t<U} t\omega(t), \sup_{U\le t<1} (1-t)\omega(t)\}.$$

Now the function $\alpha(t) = t/L_2 t$ is both increasing and subadditive. Hence, $E\|Y\|^2/L_2\|Y\|^2 < \infty$ iff

$$\int_0^1 \frac{\ell_i(u)}{L_2(\ell_i(u))} du < \infty, \qquad i = 1, 2 \tag{6.7}$$

where

$$\ell_1(u) = \sup_{0<t<u} t^2\omega^2(t)$$

and

$$\ell_2(u) = \sup_{u\le t<1} (1-t)^2\omega^2(t).$$

We now use (6.2) and (6.3) to simplify (6.7). If $u \ge 1-\gamma$, then

$$\begin{aligned}\ell_1(u) &\le \sup_{0<t<1-\gamma} t^2\omega^2(t) + \sup_{1-\gamma\le t<u} t^2\omega^2(t)\\ &\le \ell_1(1-\gamma) + \omega^2(u) \quad \text{by (6.2ii)}.\end{aligned}$$

Also we clearly have,

$$\ell_1(u) \ge \sup_{1-\gamma\le t<u} t^2\omega^2(t) \ge (1-\gamma)^2\omega^2(u).$$

Hence by the properties of α,

$$\int_{1-\gamma}^1 \frac{\ell_1(u)}{L_2\ell_1(u)} du < \infty \quad \text{iff} \quad \int_{1-\gamma}^1 \frac{\omega^2(u)}{L_2\omega^2(u)} du < \infty.$$

On the other hand, by Lemma 6.1, ℓ_1 is bounded on $(0, 1-\gamma]$. Hence

$$\int_0^1 \frac{\ell_1(u)}{L_2\ell_1(u)} du < \infty \quad \text{iff} \quad \int_{1-\gamma}^1 \frac{\omega^2(u)}{L_2\omega^2(u)} du < \infty.$$

Similarly,

$$\int_0^1 \frac{\ell_2(u)}{L_2\ell_2(u)} du < \infty \quad \text{iff} \quad \int_0^{\gamma} \frac{\omega^2(u)}{L_2\omega^2(u)} du < \infty.$$

Since ω is bounded on $[\gamma, 1-\gamma]$, $E\dfrac{\|Y\|^2}{L_2\|Y\|^2} < \infty$ iff

$$\int_0^1 \frac{\omega^2(u)}{L_2\omega^2(u)} du < \infty. \tag{6.8}$$

Now, if $\tau \le \gamma$ is such that for $t \le \tau$ and $t \ge 1-\tau$, $\omega^2(t) \ge e^{e^2}$:

$$\int_0^{\tau} \frac{\omega^2(t)}{L_2\omega^2(t)} dt = \int_0^{\tau} \frac{\omega^2(t)}{L\left(\log(t^2\omega^2(t)) + \log\frac{1}{t^2}\right)} dt \ge \int_0^{\tau} \frac{\omega^2(t)}{L\left(\log(\ell_1(\tau)) + \log\left(\frac{1}{t^2}\right)\right)} dt.$$

Hence $\int_0^\tau \frac{\omega^2(t)}{L_2\omega^2(t)}\,dt < \infty$ implies $\int_0^\tau \frac{\omega^2(t)}{L_2\left(\frac{1}{t(1-t)}\right)}\,dt < \infty$. Similarly,

$$\int_{1-\tau}^1 \frac{\omega^2(t)}{L_2\omega^2(t)}\,dt < \infty \quad \text{implies} \quad \int_{1-\tau}^1 \frac{\omega^2(t)}{L_2\left(\frac{1}{t(1-t)}\right)}\,dt < \infty.$$

Hence by the boundedness of ω on $[\gamma, 1-\gamma]$,

$$\int_0^1 \frac{\omega^2(t)}{L_2\omega^2(t)}\,dt < \infty \quad \text{implies} \quad \int_0^1 \frac{\omega^2(t)}{L_2\left(\frac{1}{t(1-t)}\right)}\,dt < \infty.$$

For the other direction we let

$$A = \left\{t \in [0,\tau]\colon \exp\left\{\left(L\frac{1}{t}\right)^{1/2}\right\} \le \omega^2(t)\right\}.$$

Then,

$$\int_A \frac{\omega^2(t)}{L_2\omega^2(t)}\,dt \le \int_A \frac{2\omega^2(t)}{L_2\frac{1}{t}}\,dt < \infty.$$

Also by the properties of α,

$$\int_{[0,\tau]\setminus A} \frac{\omega^2(t)}{L_2\omega^2(t)}\,dt \le \int_{[0,\tau]\setminus A} \frac{2\exp\left\{\left(L\frac{1}{t}\right)^{1/2}\right\}}{L_2\frac{1}{t}}\,dt$$

$$\le 2\int_0^\tau \frac{\exp\left\{\left(L\frac{1}{t}\right)^{1/2}\right\}}{L_2\frac{1}{t}}\,dt = 2\int_{\sqrt{L\frac{1}{\tau}}}^\infty 2u\frac{\exp(u)}{2Lu}\exp(-u^2)\,du < \infty.$$

Hence

$$\int_0^\tau \frac{\omega^2(t)}{L_2\left(\frac{1}{t(1-t)}\right)}\,dt < \infty \quad \text{implies} \quad \int_0^\tau \frac{\omega^2(t)}{L_2\omega^2(t)}\,dt < \infty.$$

The remainder of the proof follows similarly.

LEMMA 6.3. *If ω satisfies* (6.4) (*James' regularity condition*) *and* (6.5) (*James' integrability condition*), *then ω satisfies* (6.3) (*the integrability condition of O'Reilly*).

PROOF. First we note that

$$c =: \int_0^\gamma \frac{\omega^2(t)}{L_2\left(\frac{1}{t(1-t)}\right)}\,dt \ge \int_u^\gamma \frac{\omega^2(t)}{L_2\left(\frac{1}{t(1-\gamma)}\right)}\,dt$$

$$= \int_u^\gamma \left[\frac{t\omega^2(t)}{L_2\left(\frac{1}{(1-\gamma)}\frac{1}{t}\right)}\right]\frac{1}{t}\,dt \ge \frac{u\omega^2(u)}{L_2\left(\frac{1}{(1-\gamma)}\frac{1}{u}\right)} L\left(\frac{\gamma}{u}\right) \quad \text{by (6.4).}$$

Now for $t \geq 0$, $t^2/2 \leq e^t$ which implies $e^{-1/t} \leq 2t^2$. Hence

$$\int_0^\gamma \frac{1}{t} \exp\left(-\frac{\varepsilon}{t\omega^2(t)}\right) dt \leq \int_0^\gamma \frac{2}{t}\left[\frac{t\omega^2(t)}{\varepsilon}\right]^2 dt \leq \frac{2}{\varepsilon^2}\int_0^\gamma \frac{1}{t}\left[\frac{cL_2\left(\frac{1}{1-\gamma}\frac{1}{t}\right)}{L\left(\frac{\gamma}{t}\right)}\right]^2 dt$$

which is finite. The finiteness of the other integral follows similarly.

PROOF OF THEOREM 6.1. By Lemma 6.1 the covariance function of Y is continuous; hence by Theorem 5.1 and Theorem A we need see that $E\|Y\|^2/L_2\|Y\|^2 < \infty$ iff (6.5). But this is the statement of Lemma 6.2.

REMARK. Since $E\|Y\|^2/L_2\|Y\|^2 < \infty$ is a necessary condition for the bounded LIL it follows by Lemma 6.3 that with the additional hypothesis (6.2) we can reobtain James' result. On the other hand, Theorem 6.1 is applicable when the hypotheses of Theorem B are not satisfied.

We now turn to the application of Theorem 4.2 and Corollary 4.1 to Cramer-von Mises statistics. To do this we need only check the conditions of these theorems in the case the process has the form (6.1). Of course we assume ω is Lebesgue measurable on $[0, 1]$.

We first give necessary and sufficient conditions for Y to be an $L^2[0, 1]$ rv

LEMMA 6.4. *Y is an $L^2[0, 1]$ rv iff*

$$\int_0^1 \lambda^2(t)\, dt < \infty \qquad \text{where} \quad \lambda(t) = t(1-t)\omega(t). \tag{6.9}$$

PROOF. First $Y \in L^2[0, 1]$ a.s. iff $\|Y\|_2^2 = \int_0^U t^2\omega^2(t)\, dt + \int_U^1 (1-t)^2\omega^2(t)\, dt < \infty$ a.s. which happens iff (6.9) holds.

We now show that under (6.9) Y is a rv in $L^2[0, 1]$. Since $L^2[0, 1]$ is separable, its Borel field is generated by the cylinders. But, if $f \in L^2[0, 1]$,

$$\langle f, Y\rangle = \int_0^U (-t)\omega(t)f(t)\, dt + \int_U^1 (1-t)\omega(t)f(t)\, dt.$$

Now each term being the integral of an element in $L^1[0, 1]$, is a continuous function of U $(0 < U < 1)$, which is of course measurable.

THEOREM 6.2. *Assume $\lambda \in L^2[0, 1]$. Then Y satisfies the bounded* LIL *in $L^2[0, 1]$) iff the following conditions hold*:

$$\text{(i)} \quad \int_0^{1/2} tf(t)\omega(t)\left[\frac{1}{t}\int_0^t f(s)s\omega(s)\, ds\right] dt < \infty, \qquad \text{for all} \quad f \in L^2[0, 1/2];$$

$$\text{(ii)} \quad \int_0^{1/2} t\omega(1-t)f(t)\left[\frac{1}{t}\int_0^t f(s)s\omega(1-s)\, ds\right] dt < \infty, \qquad \text{for all} \quad f \in L^2[0, 1/2]; \tag{6.10}$$

and

$$\int_0^1 \frac{r_i(u)}{L_2(r_i(u))}\, du < \infty, \qquad i = 1, 2, \tag{6.11}$$

where $\quad r_1(u) = \int_0^u t^2\omega^2(t)\, dt \quad$ *and* $\quad r_2(u) = \int_u^1 (1-t)^2\omega^2(t)\, dt.$

PROOF. First we show that $Ef^2(Y) < \infty$ for all $f \in L^2[0, 1]$ iff (6.10) holds.

$$Ef^2(Y) = E\left|\int_0^1 f(t)\omega(t)[I(U \le t) - t]\,dt\right|^2 = \int_0^1\int_0^1 f(t)f(s)\omega(t)\omega(s)[t \wedge s - ts]\,dt\,ds.$$

Now the double integrals over each of the sets $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ and $[\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ are finite since $t\omega(t) \in L^2[0, \frac{1}{2}]$ and $(1 - t)\omega(t) \in L^2[\frac{1}{2}, 1]$. We also note that for the same reason

$$\int_0^{1/2}\int_0^{1/2} f(t)f(s)\omega(t)\omega(s)ts\,dt\,ds < \infty$$

$$\int_{1/2}^1\int_{1/2}^1 f(t)f(s)\omega(t)\omega(s)(1 - t)(1 - s)\,dt\,ds < \infty.$$

Hence, since $t \wedge s - ts = (1 - t) \wedge (1 - s) - (1 - t)(1 - s)$, $Ef^2(Y) < \infty$ iff

$$\int_0^{1/2}\int_0^{1/2} f(t)f(s)\omega(t)\omega(s)t \wedge s\,dt\,ds < \infty$$

(6.12) and

$$\int_{1/2}^1\int_{1/2}^1 f(t)f(s)\omega(t)\omega(s)(1 - t) \wedge (1 - s)\,dt\,ds < \infty.$$

But by symmetry and a change of variable (6.12) is seen to be equivalent to (6.10).

One can now use the properties of α to see that $E\|Y\|_2^2/L_2\|Y\|_2^2 < \infty$ iff (6.11) holds. Hence by Theorem 4.2 the proof is complete.

We now give a useful condition on ω which implies the bounded LIL. Of course, if $\int_0^1 \omega^2(t)\,dt < \infty$ then Y is a uniformly bounded random variable with values in $L_2[0, 1]$ and hence Y satisfies the compact LIL by Corollary 4.1, i.e., $E\|Y\|^2 \le \int_0^1 \omega^2(t)\,dt < \infty$ implies the covariance operator is compact by the remarks following Lemma 2.1.

COROLLARY 6.1. *Assume* $\lambda \in L^\infty[0, 1]$. *Then* Y *satisfies the bounded* LIL *in* $L^2[0, 1]$ iff (6.11) *holds. Further if O'Reilly's regularity condition* (6.2ii) *holds and* ω *is bounded on* $[\gamma, 1 - \gamma]$, *then* $\lambda \in L^\infty[0, 1]$ *is necessary for* (6.10), *i.e., for the covariance operator to be bounded.*

PROOF. To show $\lambda \in L^\infty[0, 1]$ implies (6.10) it clearly suffices to show $f \in L^2[0, 1]$ implies $1/t \int_0^t f(s)\,ds \in L^2[0, 1]$. But, this is a consequence of Hardy's inequality as given on page 72 of Rudin (1966).

To see the necessity we let $f_a(t) = \dfrac{I_{[0,a]}(t)}{\sqrt{a}}$ for $0 < a < \gamma$. Then $\|f_a\|_2 = 1$ and by (6.2ii)

$$Ef_a^2(Y) \ge \frac{1}{2}\frac{\omega^2(a)}{a}\int_0^a\int_0^t s\,ds\,dt = \frac{1}{12}a^2\omega^2(a). \tag{6.13}$$

Hence if the covariance operator is bounded $\sup_{0<a<\gamma} a^2\omega^2(a) < \infty$. By symmetry and boundedness of ω on $[\gamma, 1 - \gamma]$ we have $\lambda \in L^\infty[0, 1]$.

COROLLARY 6.2. *Assume* $\lambda \in L^\infty[0, 1]$ *and*

$$\lim_{t\to 0} \lambda(t) = \lim_{t\to 1} \lambda(t) = 0. \tag{6.14}$$

Then Y *satisfies the compact* LIL *in* $L^2[0, 1]$ iff (6.11) *holds. Further, if O'Reilly's* (6.2ii) *holds and* ω *is bounded on* $[\gamma, 1 - \gamma]$, *then the conditions* $\lambda \in L^\infty[0, 1]$ *and* (6.14) *are necessary for the compactness of the covariance operator.*

PROOF. We need only show the covariance function is weak-star sequentially continuous. Hence we will assume $f_n \to 0$ weak-star and we must show that $Ef_n^2(Y) \to 0$. But for any $g \in L^2[0, 1]$, $g \ge 0$,

$$Eg^2(Y) = \int_0^1 \int_0^1 g(t)g(s)\omega(t)\omega(s)[t \wedge s - ts]\, ds\, dt$$

$$\leq 2\left[\int_0^{1/2} g(s)s\omega(s)\, ds\right]\left[\int_{1/2}^1 g(t)(1-t)\omega(t)\, dt\right]$$

$$+ 2\int_0^{1/2} g(t)t\omega(t)\left[\frac{1}{t}\int_0^t g(s)s\omega(s)\, ds\right] dt$$

$$+ 2\int_{1/2}^1 g(t)(1-t)\omega(t)\left[\frac{1}{1-t}\int_t^1 g(s)(1-s)\omega(s)\, ds\right] dt.$$

For any $h \in L^2[0, 1]$ we also have

$$Eh^2(Y) \leq 8\{Eh^2_{[0,\delta]}(Y) + Eh^2_{(\delta,1/2]}(Y) + Eh^2_{(1/2,1-\delta)}(Y) + Eh^2_{[1-\delta,1]}(Y)\},$$

where

$$h_A(t) = h(t)I_A(t).$$

Hence by symmetry we need only show $f_n \to 0$ weak-star implies

$$\limsup_{\delta\to 0} \limsup_{n\to\infty} I_n = 0 \quad \text{and} \quad \lim_{n\to\infty} II_n = 0$$

where

$$I_n = \int_0^\delta f_n(t)t\omega(t)\left[\frac{1}{t}\int_0^t f_n(s)s\omega(s)\, ds\right] dt$$

and

$$II_n = \int_\delta^{1/2} f_n(t)t\omega(t)\left[\frac{1}{t}\int_\delta^t f_n(s)s\omega(s)\, ds\right] dt.$$

But, if we choose $\delta > 0$ so that $t\omega(t) \leq \varepsilon$

$$I_n \leq \varepsilon^2 \|f_n\|_2 \|T(|f_n|)\|_2,$$

where

$$(Tg)(t) = \frac{1}{t}\int_0^t g(s)\, ds.$$

Hence, by Hardy's inequality (see e.g. Rudin (1966), page 72)

$$I_n \leq 2\varepsilon^2 \sup_n \|f_n\|_2^2 =: 2K^2\varepsilon^2,$$

and K is finite, e.g., by the uniform boundedness principle.

On the other hand if $C(\delta) = \sup_{\delta\leq u\leq 1/2} \omega(u)$,

$$\left|\frac{1}{t}\int_\delta^t f_n(s)s\omega(s)\, ds\right| \leq \frac{C(\delta)}{t}\|f_n\|_2\left(\int_0^t s^2\, ds\right)^{1/2} I_{(\delta,1/2]}(t) = \frac{C(\delta)}{3^{1/2}} Kt^{1/2} I_{(\delta,1/2]}(t)$$

which is in $L^2[0, 1]$. Therefore by the Lebesgue dominated convergence theorem

$$\frac{1}{t}\int_\delta^t f_n(s)s\omega(s)\, ds \to 0 \qquad \text{in} \quad L^2[0, 1].$$

This then yields

$$II_n \leq 2\|\lambda\|_\infty \|f_n\|_2 \|T(f_n I_{(\delta,1/2]})\|_2 \to 0$$

and the proof of sufficiency is complete.

For the necessity we note that the functions $\{f_a\}$ used in the proof of Corollary 6.1, converge to zero weakly as $a \to 0$. Hence, if the covariance operator is compact $Ef_a^2(Y) \to 0$. (6.13) and its counterpart near 1 then imply (6.14).

REMARK. One can now show by direct calculation that for

$$\omega(t) = \frac{1}{t(1-t)\left[L\left(\frac{1}{t(1-t)}\right)L_2\left(\frac{1}{t(1-t)}\right)\right]^{1/2}}$$

Y satisfies the compact LIL.

7. Some examples in Hilbert space. Let ℓ_2 denote the Hilbert space of all real sequences with canonical basis $\{e_k\}$, i.e. $e_k = (0, \ldots, 0, 1, 0, \ldots)$ where the number one appears in the k^{th} position. Throughout Theorem 7.1

$$X = \textstyle\sum_{k \geq 1} \eta_k e_k \tag{7.1}$$

where $\{\eta_k\}$ is a sequence of independent random variables such that for $k \geq 1$

$$P(\eta_k = \pm\sqrt{c_k}) = 1/2c_k \tag{7.2}$$

$$P(\eta_k = 0) = 1 - 1/c_k$$

and $c_k = e^{e^{k^2}}$. We will prove the following result.

THEOREM 7.1. *Let X be defined as indicated in* (7.1) *and* (7.2), *and let* $X_1, X_2, \ldots$ *be independent copies of X defined on* $(\Omega, \mathscr{F}, P)$. *Then*

(i) *X is* WM_0^2.

(ii) *The covariance function of X is*

$$T(f, g) = (f, g) \qquad f, g \in \ell_2$$

and thus the Hilbert space $H_{\mathscr{L}(X)}$ *determined as in Lemma* 2.1 *is actually* ℓ_2 *with the* ℓ_2 *norm. Hence in this case K is the unit ball of* ℓ_2.

(iii) $E\left(\frac{\|X\|^2}{L_2\|X\|}\right) < \infty$.

(iv) $\frac{S_n}{a_n} \to 0$ *in probability.*

(v) *With probability one,* $C\left(\left\{\frac{S_n}{a_n}\right\}\right) = K$ *and hence is noncompact.*

(vi) *X satisfies the bounded* LIL *in* ℓ_2.

(vii) *If* $\{\lambda_k\}$ *is a sequence of positive numbers converging to zero and* Λ *denotes the linear operator*

$$\Lambda(x) = \textstyle\sum_{k=1}^{\infty} \lambda_k(x, e_k)e_k \qquad x \in \ell_2$$

then the random variable ΛX *satisfies the compact* LIL *in* ℓ_2 *with limit set* $\Lambda K = \{\{y_i\} \in \ell_2: \sum_{i=1}^{\infty} (y_i/\lambda_i)^2 \leq 1\}$.

REMARK. If, in addition to $\lambda_k \to 0$, $\{\lambda_k\}$ is such that $\sum_{k=1}^{\infty} \lambda_k^4 = \infty$, then ΛX is an example which shows the condition (3.3) of Theorem 3.1 of Pisier and Zinn (1978) is not necessary.

The following clarifies the condition (3.3) of Pisier and Zinn (1978).

PROPOSITION 7.1. *A Hilbert space valued* rv *X with* $E(X) = 0$ *satisfies* (3.3) *iff* $E[(y, X)^2] < \infty$ *for each element y of the Hilbert space and the covariance is given by an operator of Hilbert Schmidt type.*

LEMMA 7.1. *If X has a continuous covariance operator, C, and $E[(X, Y)^2] < \infty$ where X, Y are* i.i.d. *then C is Hilbert Schmidt and*

$$\| C \|_2^2 = E[(CX, X)] = E[(X, Y)^2].$$

PROOF. Let dP denote the distribution of X on H. Then (3.3) of Pisier and Zinn is equivalent to

$$\gamma \equiv \int_H \int_H (x, y)^2 \, dP(x) \, dP(y) < \infty.$$

But for each fixed y,

$$\int_H (x, y)^2 \, dP(x) = (Cy, y).$$

Hence

$$\gamma = \int_H (Cy, y) \, dP(dy) = E[(CX, X)] = E[\| C^{1/2} X \|^2].$$

But since $\| C^{1/2} X \|$ is square integrable, the covariance of the rv $C^{1/2}X$ determines a trace class operator. Now

$$\begin{aligned} E[(y, C^{1/2}X)(z, C^{1/2}X)] &= E[(C^{1/2}y, X)(C^{1/2}z, X)] \\ &= (CC^{1/2}y, C^{1/2}z) \\ &= (C^2 y, z). \end{aligned}$$

Hence, C^2 is trace class. Moreover, trace $C^2 = E[\| C^{1/2} X \|^2]$ so

$$\| C \|_2^2 = E[(CX, X)].$$

PROOF OF PROPOSITION 7.1. For $n > 0$ let X^n denote the truncation

$$X^n = \begin{cases} X & \| X \| \le n \\ 0 & \| X \| > n. \end{cases}$$

Then X^n has a continuous covariance operator C_n and the lemma implies that

$$\| C_n \|_2^2 = E[(X^n, Y^n)^2] = \int_{\|x\| \le n} \int_{\|x\| \le n} (x, y)^2 \, dP(x) \, dP(y) \le E[(X, Y)^2] = \gamma.$$

Hence, $\| C_n \|_2^2 < \gamma$ for all n. For fixed y in H, $(y, X^n)^2$ is monotone increasing in n so that

$$\lim_{n\to\infty} E[(y, X^n)^2] = E[(y, X)^2].$$

But, $E[(y, X^n)^2] = (C_n y, y) \le \| C_n \|_2 \| y \|^2 < \sqrt{\gamma} \| y \|^2$ so X determines a bounded covariance operator. The lemma then implies that the operator is of Hilbert Schmidt type.

The converse assertion follows easily. That is, if X has Hilbert Schmidt covariance C, then

$$Cx = \textstyle\sum_k \lambda_k (x, e_k) e_k$$

where $\{e_k\}$ are orthonormal and $\| C \|_2^2 = \sum_k \lambda_k^2 < \infty$. Hence

$$E(X, Y)^2 = \int_H \int_H (x, y)^2 \, dP(x) \, dP(y) = \int_H (Cy, y) \, dP(y) = \textstyle\sum_k \lambda_k^2 < \infty.$$

PROOF OF THEOREM 7.1 If $f = \{f_k\}$ and $g = \{g_k\}$ are elements of ℓ_2^* (which we identify with ℓ_2 throughout) then $\sum_{k=1}^{\infty} f_n \eta_k$ and $\sum_{k=1}^{\infty} g_k \eta_k$ both converge in $L^2(\Omega, \mathscr{F}, P)$ and it is trivial to verify that $Ef(X) = 0$ and $T(f, g) = E(f(X)g(X)) = (f, g)$. Hence (i) holds and (ii) holds if we employ the construction of $H_{\mathscr{L}(X)}$ indicated in Lemma 2.1.

To prove (iii) observe that

(7.3) $$E\left(\frac{\|X\|^2}{L_2\|X\|^2}\right) \leq \sum_{k=1}^{\infty} E\left(\frac{\eta_k^2}{L_2\eta_k^2}\right) = \sum_{k=1}^{\infty} k^{-2} < \infty.$$

Now $E\left(\frac{\|X\|^2}{L_2\|X\|}\right) < \infty$ iff $E\left(\frac{\|X\|^2}{L_2\|X\|^2}\right) < \infty$ so (iii) holds.

To verify (iv) we prove the following proposition.

PROPOSITION 7.2. *Let X be a B-valued random variable such that $EX = 0$ and $E\left(\frac{\|X\|^2}{L_2\|X\|}\right) < \infty$. If B is a type 2 space and $X_1, X_2, \ldots$ are independent copies of X, then*

(7.4) $$S_n/a_n \to 0 \textit{ in probability.}$$

PROOF. Let

$$X_{j,n} = \begin{cases} X_j & \text{if } \|X_j\| \leq a_n \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq j \leq n$. Let $m_n = E(XI(\|X\| \leq a_n))$ and fix $\varepsilon > 0$. Then for large n we have

(7.5) $$P\left(\left\|\frac{S_n}{a_n}\right\| > \varepsilon\right) \leq P\left(\left\|\sum_{j=1}^{n} \frac{(X_{j,n} - m_n)}{a_n}\right\| > \varepsilon/2\right) + nP(\|X\| > a_n)$$

since $E\left(\frac{\|X\|^2}{L_2\|X\|}\right) < \infty$ easily implies $\lim_n \frac{n}{a_n} \|m_n\| = 0$.

Now $E\left(\frac{\|X\|^2}{L_2\|X\|}\right) < \infty$ also implies $\sum_n P(\|X\| > a_n) < \infty$, and since the terms $P(\|X\| > a_n)$ decrease in n this implies $P(\|X\| > a_n) = o\left(\frac{1}{n}\right)$. Hence $\lim_n nP(\|X\| > a_n) = 0$, so (7.5) implies (7.4) if

(7.6) $$\sum_{j=1}^{n} \frac{(X_{j,n} - m_n)}{a_n} \to 0 \qquad \text{in probability.}$$

Now B being a type 2 Banach space implies that there is a constant C such that

(7.7) $$E\|\sum_{j=1}^{n} (X_{j,n} - m_n)\|^2 \leq CnE\|X_{1,n} - m_n\|^2,$$

and since $\|m_n\|^2 \leq E\|X_{1,n}\|^2$ we also have $E\|X_{1,n} - m_n\|^2 \leq 4E\|X_{1,n}\|^2$. Hence by Markov's inequality we have

(7.8) $$P(\|\sum_{j=1}^{n} (X_{j,n} - m_n)\| > \varepsilon a_n) \leq \frac{2C}{\varepsilon^2 L_2 n} E\|X_{1,n}\|^2.$$

Since $\|X\| \leq a_n$ implies $1/L_2 n \leq 2/L_2\|X\|$ for n sufficiently large, we have for such n that

(7.9) $$\frac{E\|X_{1,n}\|^2}{L_2 n} \leq 2\left\{\frac{E(\|X\|^2 I(\|X\| \leq \rho))}{L_2 n} + \frac{E\|X\|^2 I(\|X\| > \rho)}{L_2\|X\|}\right\}.$$

Letting n approach infinity and then ρ approach infinity we have (7.6). Hence the proof of Proposition 7.2 is complete.

To verify (v) is a simple application of Theorem 3.1. That is, since $H_{\mathscr{L}(X)} = \ell_2$ with both spaces having the same norm, it is easy to see from Lemma 2.1 that we can take $\prod_N(x) = \sum_{k=1}^{N} (x, e_k)e_k$ and $Q_N(x) = \sum_{k=N+1}^{\infty} (x, e_k)e_k$. Hence since $S_n/a_n \to 0$ in probability we have (3.1) and (3.2), and since the sequence $\{\eta_k\}$ consists of independent random variables we also have (3.3). Thus Theorem 3.1 implies (v).

Now if we are given (vi), and that Λ is a compact linear operator, we then have (vii) as an easy consequence of the method of proof in Corollary 3.1 of Kuelbs (1976a). Theorem 7.1 is proved once we establish (vi) and this follows immediately from Theorem 4.2.

The condition $\sup_{t>0} t^2P(\|X\| > t) < \infty$ is known to be necessary for X to satisfy the CLT and was thought to perhaps also be necessary for the LIL. That this is not the case follows from our next example. This example is also important in that it shows that no condition defined only in terms of the law of the norm will allow one to prove the "middle sum goes to zero" as in Lemma 4.3. That is, we will see that one must take into account the existence of the covariance function as well as the norm condition (4.2) to obtain the analogue of (4.22) holding even on the real line.

DEFINITION. Fix $r \geq 0$. Define the quantity 2_r^k by $2_0^k = k$ and $2_r^k = 2^{2_{r-1}^k}$.

Let $\{e_j\}$ denote the canonical basis in ℓ_2 as above, and assume $\{\delta_j\}$ and $\{Z_j\}$ are independent sequences of random variables such that

(7.10)
$$a)\ \delta_i\delta_j = 0 \qquad (i \neq j);$$
$$b)\ P(\delta_j = 1) = \frac{6}{\pi^2 j^2}, \qquad P(\delta_j = 0) = 1 - \frac{6}{\pi^2 j^2};$$
c) Z_j is symmetric with
$$Z_j^2 = \begin{cases} 2_r^k & \text{with probability } \dfrac{\pi^2}{6}\dfrac{j^2}{k^2 2_r^k} I(j \leq 2_{r-2}^k) \\ 0 & \text{otherwise.} \end{cases}$$

We now can prove the following result.

THEOREM 7.2. *Let $\{\delta_j\}$ and $\{Z_j\}$ be independent sequences as given in* (7.10) *and set*

$$Y = \textstyle\sum_{j=1}^{\infty} \delta_j Z_j e_j. \tag{7.11}$$

Then Y is a symmetric ℓ_2-valued random variable such that

(7.12)
$$a)\ E\left(\frac{\|Y\|^2}{L_2\|Y\|^2}\right) < \infty \qquad \text{for} \quad r \geq 2;$$
$$b)\ \limsup_{t\to\infty} t^2P(\|Y\| > t) = \infty \qquad \text{for} \quad r \geq 3;$$
c) *Y satisfies the compact* LIL *in ℓ_2 for $r \geq 2$.*

Further, if G is a symmetric real valued random variable such that $\mathscr{L}(G^2) = \mathscr{L}(\|Y\|^2)$ and $r > 4$ then

$$\textstyle\sum_{j=1}^{n} v_j/a_n \not\to 0 \qquad \text{a.s.} \tag{7.13}$$

where

$$v_j = G_jI(\alpha_n \leq |G_j|^2 \leq \beta_n) \qquad j \in I(n),\ n \geq 1 \tag{7.14}$$

provided $\{G_j\}$ is a sequence of indepedent copies of G.

PROOF. By construction, if $Y \neq 0$, then there is exactly one j such that $\delta_j = 1$ and hence $Y = Z_je_j$. Thus Y takes values in ℓ_2 and for $r \geq 2$, $k \geq 1$

$$P(\|Y\|^2 = 2_r^k) = \textstyle\sum_{j\geq 1} P(\delta_j = 1, Z_j^2 = 2_r^k) = \sum_{j=1}^{\infty} \dfrac{6}{\pi^2 j^2}\dfrac{\pi^2 j^2}{6k^2 2_r^k} I(j \leq 2_{r-2}^k) = 2_{r-2}^k/2_r^k k^2.$$

Thus for $r \geq 2$ (7.12a) holds since

$$E\left(\frac{\|Y\|^2}{L_2\|Y\|^2}\right) \leq \sum_{k=1}^{\infty} \frac{2_r^k}{2_{r-2}^k}\frac{2_{r-2}^k}{2_r^k k^2} < \infty.$$

Also, for $2_r^k \le t^2 < 2_r^{k+1}$,

$$P(\| Y\| > t) = P(\| Y\|^2 \ge 2_r^{k+1}) \ge \frac{2_{r-2}^{k+1}}{2_r^{k+1}(k+1)^2},$$

so that for $t^2 = 2_r^{k+1} - 1$,

$$t^2 P(\| Y\| > t) \ge \frac{1}{2}\frac{2_{r-2}^{k+1}}{(k+1)^2}.$$

Hence (7.12b) holds for $r \ge 3$.

In view of Corollary 4.1 to verify that Y satisfies the compact LIL it now remains only to show that Y has a compact covariance operator.

Now for $r \ge 2$,

$$E(\delta_j^2 Z_j^2) = E(\delta_j^2)E(Z_j^2) = \frac{6}{\pi^2 j^2}\sum_k 2_r^k \frac{\pi^2}{6}\frac{j^2}{k^2 2_r^k} I(j \le 2_{r-2}^k)$$

$$= \sum_k \frac{1}{k^2} I(j \le 2_{r-2}^k) \to 0 \qquad \text{as} \quad j \to \infty,$$

so for each $x \in \ell_2$ it is easy to prove that the series $\sum_j x_j \delta_j Z_j$ converges in mean square. Hence $\langle x, Y\rangle = \sum_j x_j \delta_j Z_j$ is square integrable for each x in ℓ_2, and Y has covariance operator

$$Sx = \sum_j E(\delta_j^2 Z_j^2)(x, e_j)e_j \qquad x \in \ell_2.$$

Since $E(\delta_j^2 Z_j^2) \to 0$ as $j \to \infty$ we have S compact, and (7.12c) holds.

Now take $r \ge 4$. We first note that there exists at most one k (called k_n) for which

$$\alpha_n \le 2_r^k \le \beta_n \tag{7.15}$$

where α_n and β_n are as in (4.5). Let

$$A = \{n \ge 1 : \exists k_n \quad \text{satisfying (7.15)}\},$$

$$J = \left\{ n \in A : 2_r^{k_n} > \alpha_n \left(\frac{2_{r-2}^{k_n}}{k_n^3}\right)^2 \right\},$$

and

$$J' = J^c \cap A.$$

If G is a symmetric real valued random variable with $\mathscr{L}(G^2) = \mathscr{L}(\| Y\|^2)$ and $\{v_j\}$ is as in (7.14) we thus have for $H_j = 2^n v_j / a_{2^n}$, $j \in$ I(n), n $\ge$ 1, that

$$\Lambda(n) \equiv \sum_{j \in I(n)} E(H_j^2)/4^n = E\left\{\frac{\| Y\|^2 I(\alpha_n \le \| Y\|^2 \le \beta_n)}{2L_2 2^n}\right\}$$

$$= \begin{cases} \dfrac{2_r^{k_n}}{2L_2 2^n} \cdot \dfrac{2_{r-2}^{k_n}}{2_r^{k_n} k_n^2} & \text{if} \quad n \in A \\ 0 & \text{if} \quad n \notin A. \end{cases} \tag{7.16}$$

Since $\alpha_n \sim \dfrac{2^n}{L_2 2^n}$ and $\beta_n \sim 2^n L_2 2^n$ one can show that

$$\sum_{n \in J} \Lambda(n) \le \sum_k \sum_{\{n : k_n = k\}} \frac{2_r^k \cdot 2_{r-2}^k}{2_r^k k^2 (2L_2 2^n)} = O\left(\sum_k \frac{Lk}{k^2}\right) < \infty, \tag{7.17}$$

and further that

$$
\begin{aligned}
\textstyle\sum_{n\in J'} \exp\{-M/\Lambda(n)\} &= \textstyle\sum_{n\in J'} \exp\left\{-\frac{2ML_2 2^n(k_n^2)}{2^{k_n}_{r-2}}\right\} \\
&\approx \textstyle\sum_{n\in J'} \exp\{-M'k_n^2\} \\
&= \textstyle\sum_k \sum_{\{n\in J':k_n=k\}} \exp\{-M'k^2\} \\
&\approx \textstyle\sum_k 2^k_{r-3} \exp\{-M'k^2\} \\
&= \begin{cases} <\infty & r\le 4, M>0 \\ \infty & r>4, M>0. \end{cases}
\end{aligned} \tag{7.18}
$$

In (7.18) we write $\sum_k c_k \approx \sum_k d_k$ to denote the fact that there are strictly positive constants ϕ_1 and ϕ_2 such that $\phi_1 \le c_k/d_k \le \phi_2$ $(k \ge 1)$.

Now let $T_n = \sum_{j\in I(n)} v_j$ for $n \ge 1$. Then as in Lemma 4.2 we have (7.13) iff

$$\lim_n \frac{T_n}{a_{2^n}} \ne 0. \tag{7.19}$$

To verify (7.19) notice that

$$T_n/a_{2^n} = \textstyle\sum_{j\in I(n)} H_j/2^n,$$

and hence as above $T_n/a_{2^n} \to 0$ a.s. iff $\sum_{j=1}^k H_j/k \to 0$ a.s. Now let

$$W_j = \begin{cases} H_j & j\in I(n), n\in J' \\ 0 & \text{otherwise} \end{cases}$$

$$R_j = H_j - W_j \qquad j \ge 1.$$

Then we will show

$$\textstyle\sum_{j=1}^k R_j/k \to 0 \qquad \text{a.s.,} \tag{7.20}$$

and

$$\textstyle\sum_{j=1}^k W_j/k \not\to 0 \qquad \text{a.s.} \tag{7.21}$$

Thus $\sum_{j=1}^k H_k/k \not\to 0$ a.s., so (7.19) holds as claimed and the theorem is proved once we establish (7.20) and (7.21).

To verify (7.20) we again need only prove

$$\textstyle\sum_{j\in I(n)} R_j/2^n \to 0 \text{ a.s.,}$$

and by the Borel-Cantelli lemma this follows if we show that for every $\varepsilon > 0$

$$\textstyle\sum_n P(|\sum_{j\in I(n)} R_j| > \varepsilon 2^n) < \infty. \tag{7.22}$$

To prove (7.22) note that

$$\textstyle\sum_n P(|\sum_{j\in I(n)} R_j| > \varepsilon 2^n) = \sum_{n\in J} P(|\sum_{j\in I(n)} H_j| > \varepsilon 2^n)$$

since $R_j = 0$ if $j \in I(n)$ and $n \in J^c$. Hence

$$\textstyle\sum_n P(|\sum_{j\in I(n)} R_j| > \varepsilon 2^n) \le \frac{1}{\varepsilon^2} \sum_{n\in J} \Lambda(n) < \infty$$

by (7.17), so (7.22) holds. Thus (7.20) is verified and it remains only to prove (7.21). Now (7.21) will be proved by applying the SLLN in Theorem 1 of Chung (1951).

To apply this result we first must show that

$$M_n \equiv \max_{j\in I(n)} |W_j| = o(\textstyle\sum_{j\in I(n)} E(W_j^2)/2^n). \tag{7.23}$$

Now (7.23) holds since

$$\max_{j \in I(n)} |W_j| = \begin{cases} 0 & n \notin J' \\ \dfrac{2^n}{a_{2^n}} (2_r^{k_n})^{1/2} \le \alpha_n^{1/2} \dfrac{2_{r-2}^{k_n}}{k_n^3} \cdot 2^n / a_{2^n} = \dfrac{O(2^n)}{k_n^3}, & n \in J', \end{cases}$$

and

$$\sum_{j \in I(n)} \frac{E(W_j^2)}{2^n} = \begin{cases} 2^n \Lambda(n) \approx \dfrac{2^n}{k_n^2} & \text{for } n \in J' \\ 0 & \text{otherwise.} \end{cases}$$

Hence by Theorem 1 of Chung (1951) we have (7.21) iff for some $\varepsilon > 0$

(7.24) $$\sum_{n \in J'} \exp\{-\varepsilon/\Lambda(n)\} = \infty.$$

Now (7.24) follows from (7.18) provided $r > 4$ and hence the theorem is proved.

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