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A NOTE ON THE SURVIVAL OF THE LONG-RANGE CONTACT PROCESS

BY MAURY BRAMSON AND LAWRENCE GRAY

University of Minnesota

The purpose of this note is to demonstrate survival for the long-range contact process. This process was introduced by Spitzer [7]; it possesses the same basic evolutionary rule as does the contact process on the integers, except that particles may appear at large distances from already extant particles instead of just as immediate neighbors. We consider here two variants of this model, and show that in both cases the system will survive if (i) it commences from a reasonably dense initial state and (ii) the birth rate for particles is moderately greater than the corresponding death rate. The methodology consists primarily of an energy argument, which provides a lower bound for the particle density of the system.

1. Introduction. The long-range contact process is a member of a class of stochastic processes known as interacting particle systems. Considerable interest in this subject has been generated over the past decade; see Liggett [6] and Griffeath [2] for references. Such systems are Markov processes on the state space $S = \{\text{all subsets of } \mathbb{Z}^d\}$. The state (or configuration) $A \in S$ is interpreted as the (typically infinite) set of *sites* occupied by particles, and the process (ξ^μ) represents the evolution in time starting from an initial distribution μ on S.

In the case of the long-range contact process, $d = 1$ and the time parameter is continuous. Change of state occurs through birth and death of particles, the dynamics of which are described in terms of *birth rates* and *death rates*. There are two types of models that we will consider. In both cases, the birth rate at an unoccupied site x depends on the state A of the system. Let $\lambda \geq 0$ be a parameter and define $\ell_x(A) = \min\{x - y : y \in A \text{ and } y \leq x\}$ and $r_x(A) = \min\{y - x : y \in A \text{ and } y > x\}$. In the *uniform birth model*, the birth rate at x is $\lambda/(\ell_x(A) + r_x(A) - 1)$. In the *centered birth model*, the birth rate at x is λ if $\ell_x(A) = r_x(A)$, $\lambda/2$ if $\ell_x(A) = r_x(A) \pm 1$, and 0 otherwise. For both models, the death rate is 1. Note that in both cases the total birth rate at sites in an empty interval between two particles is λ. The difference is that in the first model, births are distributed uniformly throughout the interval, while in the second model, they only occur halfway between occupied sites.

These models along with several others were first introduced by Spitzer [7]. Gray [1] proved the existence of such models and the fact that they are uniquely defined by specifying the birth and death rates. The reader who desires more precise formulation should consult [1].

The evolutionary rules for long-range contact processes are similar to those for nearest neighbor contact processes (see Harris [3]), in which the death rates are also 1 but births occur only at sites immediately neighboring extant particles. Another difference is that the nearest neighbor contact processes possess so-called "dual processes", whereas the long-range models do not. (See Griffeath [2] for definitions.) These differences have hampered investigation of the long-range contact processes, particularly concerning asymptotic behavior. One basic question that has not been answered up to this point is whether these long-range systems will under any circumstances *survive*. Survival means that particle

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density remains bounded above 0 for all time. Elementary reasoning shows that the process will die out for $\lambda \le 1$. Intuition says that survival must be the case for large enough λ (maybe for $\lambda > 1$) and a dense enough initial distribution, but this does not seem to be immediately translatable into a concrete argument. While there seems to be no way to apply the techniques that have been successful for nearest neighbor contact processes, it is interesting to note that consideration of the question of survival for the long-range contact process led Holley and Liggett [4] to their renewal proof asserting survival of the nearest neighbor models for $\lambda \ge 4$.

It is the purpose of this note to demonstrate survival of the long-range contact process for sufficiently large birth parameter λ and sufficiently dense translation invariant initial distribution μ. Our method is to analyze the following version of *energy*: for $\beta > 0$, let

$$\mathscr{E}^{\mu}_{\beta}(t) = \beta\text{-energy of } \xi^{\mu}_t = E[(\ell_0(\xi^{\mu}_t) + r_0(\xi^{\mu}_t))^{\beta}]$$

It is shown below (Lemma 1) that the distribution of ξ^{μ}_t is translation invariant, so that survival is implied by

$$\sup_{t\ge 0} \mathscr{E}^{\mu}_{\beta}(t) < \infty \quad \text{for some} \quad \beta > 0. \tag{1}$$

Our principle results concerning the β-energy and survival are:

THEOREM 1. *Let* (ξ^{μ}_t) *denote the long-range contact process with uniform births and translation invariant initial distribution* μ. *Assume that for some* $\beta > 0, \mathscr{E}^{\mu}_{\beta}(0) < \infty$, *and that the birth parameter* λ *is greater than* $\lambda_0 = 4 \log 2$. *Then* (1) *holds, and hence* (ξ^{μ}_t) *survives.*

THEOREM 1′. *Let* (ξ^{μ}_t) *denote the long-range contact process with centered births and translation invariant initial distribution* μ. *Assume that for some* $\beta > 0, \mathscr{E}^{\mu}_{\beta}(0) < \infty$, *and that the birth parameter* λ *is greater than* $\lambda_0 = 2$. *Then* (1) *holds, and hence* (ξ^{μ}_t) *survives.*

A simple consequence of Theorem 1 and a standard monotonicity argument (based on the "basic coupling" defined in Liggett [6]) is that if $(\xi^{\mathbb{Z}}_t)$ is the uniform birth model with initial state $\mathbb{Z}$, and if $\lambda > \lambda_0$, then the distribution of $\xi^{\mathbb{Z}}_t$ converges weakly to a non-trivial distribution μ_∞ as $t \to \infty$. The measure μ_∞ is stationary for the process. This same argument does not apply to the centered birth model, which is not monotone. We also note in passing that our techniques apply equally well to analogous discrete time models to produce results similar to Theorem 1 and Theorem 1′.

2. Proofs of results. We begin this section by introducing some terminology. It will be helpful for us to visualize a state $A \in S$ as a partition of $\mathbb{Z}$ into a collection of intervals $[x_i(A), x_{i+1}(A))$, $i \in \mathbb{Z}$, which are devoid of particles except at the left boundary, i.e., $x_i(A) \in A$ but $x \notin A$ for $x_i(A) < x < x_{i+1}(A)$. In our case, these intervals will always be of finite length. We number these sites in such a way that $x_0(A) = \max\{x \le 0 : x \in A\}$. Let $L_k(A) = x_{k+1}(A) - x_k(A)$, $k \in \mathbb{Z}$. Thus $L_k(A)$ denotes the length of the interval k intervals to the right of the interval containing the point 0 (perhaps as its left boundary). By definition,

$$\mathscr{E}^{\mu}_{\beta}(t) = E[(L_0(\xi^{\mu}_t))^{\beta}].$$

In order to analyze $\mathscr{E}^{\mu}_{\beta}(t)$, we need some control on the β-moment of the lengths $L_k(\xi^{\mu}_t)$. The first step is the following lemma which involves a "bus-stop paradox" of sorts.

LEMMA 1. *Let* (ξ^{μ}_t) *be either long-range contact process with translation invariant initial distribution* μ. *Then for all* $t \ge 0$, *the distribution of* ξ^{μ}_t *is translation invariant, and for all* $\beta > 0$ *and* $n = 0, 1, 2, \cdots$,

$$\textstyle\sum_{k=0}^{n} E[(L_{-k}(\xi^{\mu}_t) + \cdots + L_{n-k}(\xi^{\mu}_t))^{\beta}] \le (n+1)^{\beta+1} E[(L_0(\xi^{\mu}_t))^{\beta}]. \tag{2}$$

(If one is concerned only with the case $\beta = 1$, one may instead prove the more transparent inequality $E[L_n(\xi^{\mu}_t)] \leqq E[(L_0(\xi^{\mu}_t)]$ for $n \in \mathbb{Z}$. In any case, the inequalities follow from translation invariance rather than any other special properties of the models.)

PROOF. Fix $t \geq 0$ and let μ_t be the distribution of ξ_t^μ. By the uniqueness theorem in Gray [1], μ_t is uniquely determined by the birth and death rates (which are translation invariant) and by μ. It follows that μ_t is also translation invariant.

Now fix $\beta > 0$ and $n \in \mathbb{Z}^+$. Let

$$A_n(x; a_0, a_1, \ldots, a_n) = \{\omega: \xi_t^\mu \cap [x, x + a_0 + \ldots + a_n] = \{x, x + a_0, \ldots, x + a_0 + \ldots + a_n\}\},$$

where $x \in \mathbb{Z}$ and $a_j \in \mathbb{Z}^+$, $0 \leq j \leq n$. $A_n(x; a_0, a_1, \ldots, a_n)$ is the set of configurations with particles at x, $x + a_0, \ldots, x + a_0 + \ldots + a_n$, and vacancies in between these sites. Let

$$\mathscr{A}_{n,k} = \{A_n(x; a_0, \ldots, a_n): x + \textstyle\sum_{j=0}^{k-1} a_j \leq 0 < x + \sum_{j=0}^{k} a_j\}$$

for $k = 0, 1, \ldots, n$. For each choice of n and k, $\mathscr{A}_{n,k}$ forms a partition on the probability space of ξ_t^μ, with a_j being identified with $L_{j-k}(\xi_t^\mu)$, $0 \leq j \leq n$. Let

$$p_n(x; a_0, a_1, \ldots, a_n) = P(A_n(x; a_0, a_1, \ldots, a_n)).$$

For each $k = 0, 1, \ldots, n$, let $\bar{a}_k = \sum_{j=0}^{k} a_j$. Then

$$\begin{aligned} &E[(L_{-k}(\xi_t^\mu) + \ldots + L_{n-k}(\xi_t^\mu))^\beta] \\ &\quad = \textstyle\sum_{j=0}^{n} \sum_{a_j \in Z^+} \sum_{-\bar{a}_k \leq x < -\bar{a}_{k-1}} (a_0 + \ldots + a_n)^\beta p_n(x; a_0, \ldots, a_n), \end{aligned} \tag{3}$$

which, by the translation invariance of μ_t, equals

$$\textstyle\sum_{j=0}^{n} \sum_{a_j \in Z^+} a_k (a_0 + \ldots + a_n)^\beta p_n(0; a_0, \ldots, a_n). \tag{4}$$

Thus, the left side of (2) is equal to

$$\textstyle\sum_{j=0}^{n} \sum_{a_j \in Z^+} (a_0 + \ldots + a_n)^{\beta+1} p_n(0; a_0, \ldots, a_n). \tag{5}$$

For $\beta, a_0, \ldots, a_n \geqq 0$, it is elementary to show that

$$(a_0 + \ldots + a_n)^{\beta+1} \leqq (n+1)^\beta (a_0^{\beta+1} + \ldots + a_n^{\beta+1}).$$

Therefore, (5) is at most

$$\begin{aligned} &(n+1)^\beta \textstyle\sum_{k=0}^{n} \sum_{a_k \in Z^+} a_k^{\beta+1} \sum_{\substack{j=0 \\ j \neq k}}^{n} \sum_{a_j \in Z^+} p_n(0; a_0, \ldots, a_n) \\ &= (n+1)^\beta \textstyle\sum_{k=0}^{n} \sum_{a_k \in Z^+} a_k^{\beta+1} \sum_{\substack{j=0 \\ j \neq k}}^{n} \sum_{a_j \in Z^+} p_n(-(a_0 + \ldots + a_{k-1}); a_0, \ldots, a_n) \\ &= (n+1)^\beta \textstyle\sum_{k=0}^{n} \sum_{a_k \in Z^+} a_k^{\beta+1} p_0(0; a_k) \\ &= (n+1)^{\beta+1} \textstyle\sum_{a_0 \in Z^+} a_0^{\beta+1} p_0(0; a_0) \\ &= (n+1)^{\beta+1} E[(L_0(\xi_t^\mu))^\beta]. \qquad \square \end{aligned}$$

We will now use Lemma 1 to prove

LEMMA 2. *If μ is translation invariant and $\mathscr{E}_\beta^\mu(0) < \infty$, then for all $t > 0$, $\mathscr{E}_t^\mu(t) < \infty$.*

PROOF. Fix $t > 0$. Let

$$\begin{aligned} N_r(t) = 0 \vee \max\{k > 0&: \text{each of the particles at } x_1(\xi_0^\mu), \ldots, \\ &x_k(\xi_0^\mu) \text{ has died at some time in } [0, t]\} \\ N_\ell(t) = 0 \wedge \min\{k < 0&: \text{each of the particles at } x_{k+1}(\xi_0^\mu), \ldots, \\ &x_0(\xi_0^\mu) \text{ has died at some time in } [0, t]\}. \end{aligned}$$

Note that

$$\mathscr{E}^{\mu}_{\beta}(t) \le E\left[\left(\sum_{k=N_{\ell}(t)}^{N_r(t)} L_k(\xi^{\mu}_0)\right)^{\beta}\right]$$

$$\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1-e^{-t})^{i+j}E\left[\left(\sum_{k=-i}^{j} L_k(\xi^{\mu}_0)\right)^{\beta}\right]$$

since the deaths of the particles at the sites $x_k(\xi^{\mu}_0)$ occur at exponential times with mean 1 which are independent of each other and of ξ^{μ}_0. With the change of variable $n = i + j$, this inequality may be rewritten as

$$\mathscr{E}^{\mu}_{\beta}(t) \le \sum_{n=0}^{\infty}(1-e^{-t})^{n}\sum_{k=0}^{n} E[(L_{-k}(\xi^{\mu}_t) + \cdots + L_{n-k}(\xi^{\mu}_t))^{\beta}].$$

By Lemma 1, this is at most

$$\sum_{n=0}^{\infty}(n+1)^{\beta+1}(1-e^{-t})^{n}\mathscr{E}^{\mu}_{\beta}(0),$$

which is finite. □

We now prove the theorems. The technique is quite simple. We use the martingale approach of Holley and Stroock [5] to get a differential inequality involving $\mathscr{E}^{\mu}_{\beta}(t)$.

Proof of Theorem 1. (Uniform birth model). When the existence of (ξ^{μ}_t) is proved in Gray [1] it is shown that if $f: S \to \mathscr{R}$ is any function in the class $\mathscr{F} = \{f: \exists \text{ finite } A_0 \in S \text{ with } f(A) = f(A \cap A_0) \text{ for all } A \in S\}$, then for all $0 \le s < t$,

$$Ef(\xi^{\mu}_t) - Ef(\xi^{\mu}_s) = \int_s^t E[Gf(\xi^{\mu}_u)]\,du, \tag{6}$$

where G is the *pregenerator* of (ξ^{μ}_t) defined by

$$Gf(A) = \sum_{x\in A}(f(A\backslash\{x\}) - f(A)) + \sum_{x\notin A}\frac{\lambda(f(A\cup\{x\}) - f(A))}{\ell_x(A) + r_x(A) - 1}. \tag{7}$$

(In fact, the infinitesimal generator of (ξ^{μ}_t) is an extension of G.) Since $\mathscr{E}^{\mu}_{\beta}(t) = E[(L_0(A))^{\beta}]$, it would be nice to apply (6), with $f(A) = (L_0(A))^{\beta}$, to obtain estimates for the rate of growth of $\mathscr{E}^{\mu}_{\beta}(t)$. Unfortunately, $(L_0(\cdot))^{\beta} \notin \mathscr{F}$. This difficulty is overcome by approximating with functions in $\mathscr{F}$. Define $f_n(A) = (L_0(A) \wedge n)^{\beta}$ for $n = 1, 2, 3, \cdots$. We will use (7) to obtain bounds on Gf_n. Applying these bounds in (6) and letting $n \to \infty$ will enable us to estimate the growth of $\mathscr{E}^{\mu}_{\beta}(t)$.

Therefore, consider (7) with ξ^{ξ}_u and f_n in place of A and f. Since $0 \le f_n(\xi^{\mu}_u) \le (L_0(\xi^{\mu}_u))^{\beta}$, it is easy to see that

$$-\lambda(L_0(\xi^{\mu}_u))^{\beta} \le Gf_n(\xi^{\mu}_u).$$

Equation (6) implies that

$$-\lambda\int_s^t \mathscr{E}^{\mu}_{\beta}(u)\,du \le Ef_n(\xi^{\mu}_t) - Ef_n(\xi^{\mu}_s). \tag{8}$$

Since $\mathscr{E}^{\mu}_{\beta}(0) < \infty$, Lemma 2 allows us to apply monotone convergence to the right side of (8) to obtain

$$-\lambda\int_s^t \mathscr{E}^{\mu}_{\beta}(u)\,du \le \mathscr{E}^{\mu}_{\beta}(t) - \mathscr{E}^{\mu}_{\beta}(s). \tag{9}$$

We also need an upper bound on $\mathscr{E}^{\mu}_{\beta}(t) - \mathscr{E}^{\mu}_{\beta}(s)$. We start again with Gf_n and note from (7) that

(10) $$Gf_n(\xi_u^\mu) = f_n(\xi_u^\mu\backslash\{x_0(\xi_u^\mu)\}) + f_n(\xi_u^\mu\backslash\{x_1(\xi_u^\mu)\}) - 2f_n(\xi_u^\mu) + \varphi_n(\xi_u^\mu),$$

where

$$\varphi_n(A) = \sum_{x=x_0(A)+1}^{x_1(A)-1} \frac{\lambda(f_n(A\cup\{x\}) - f_n(A))}{x_1(A) - x_0(A) - 1}$$

for $A \in S$. Clearly, $|\varphi_n| \le \lambda f_n$. Let $\varphi = \lim_n \varphi_n$. Then if we substitute (10) into (6) and apply dominated and monotone convergence, we obtain

(11) $$\begin{aligned}\mathscr{E}_\beta^\mu(t) - \mathscr{E}_\beta^\mu(s) &\le \int_s^t E[(L_0(\xi_u^\mu\backslash\{x_0(\xi_u^\mu)\}))^\beta + (L_0(\xi_u^\mu\backslash\{x_1(\xi_u^\mu)\}))^\beta \\ &\qquad - 2(L_0(\xi_u^\mu))^\beta + \varphi(\xi_u^\mu)]\,du \\ &= \int_s^t (E[(L_{-1}(\xi_u^\mu) + L_0(\xi_u^\mu))^\beta + (L_0(\xi_u^\mu) + L_1(\xi_u^\mu))^\beta] \\ &\qquad - 2\epsilon_\beta^\mu(u) + E\varphi(\xi_u^\mu))\,du.\end{aligned}$$

By Lemma 1,

$$E[(L_{-1}(\xi_u^\mu) + L_0(\xi_u^\mu))^\beta + (L_0(\xi_u^\mu) + L_1(\xi_u^\mu))^\beta] \le 2^{\beta+1}\mathscr{E}_\beta^\mu(u),$$

so (11) becomes

(12) $$\mathscr{E}_\beta^\mu(t) - \mathscr{E}_\beta^\mu(s) \le \int_s^t ((2^{\beta+1} - 2)\mathscr{E}_\beta^\mu(u) + E\varphi(\xi_u^\mu))\,du.$$

The next step is to estimate $E\varphi(\xi_u^\mu)$. Note that on the set

$$B_{k,\ell} = \{L_0(\xi_u^\mu) = k,\ x_0(\xi_u^\mu) = -\ell\}, \qquad 0 \le \ell < k,$$

it is the case that

$$\begin{aligned}\varphi(\xi_u^\mu) &= \frac{\lambda}{k-1}\Big[\sum_{j=1}^{\ell} ((k-j)^\beta - k^\beta) + \sum_{j=\ell+1}^{k-1} (j^\beta - k^\beta)\Big] &&\text{if } k > 1 \\ &= 0 &&\text{if } k = 1.\end{aligned}$$

By Lemma 1, the distribution of ξ_u^μ is translation invariant, so $P(B_{k,\ell}) = (1/k)\,P(B_k)$, where $B_k = \{L_0(\xi_u^\mu) = k\}$. It follows that

$$\begin{aligned}E\varphi(\xi_u^\mu) &= \sum_{k=2}^\infty \frac{\lambda P(B_k)}{(k-1)k} \sum_{\ell=0}^{k-1} \Big[\sum_{j=1}^{\ell} ((k-j)^\beta - k^\beta) + \sum_{j=\ell+1}^{k-1} (j^\beta - k^\beta)\Big] \\ &= \sum_{k=2}^\infty \frac{\lambda P(B_k)}{(k-1)k} \sum_{j=1}^{k-1} \Big[\sum_{\ell=j}^{k-1} ((k-j)^\beta - k^\beta) + \sum_{\ell=0}^{j-1} (j^\beta - k^\beta)\Big] \\ &= \sum_{k=2}^\infty \frac{\lambda P(B_k)}{(k-1)k} \sum_{j=1}^{k-1} ((k-j)^{\beta+1} + j^{\beta+1} - k^{\beta+1}) \\ &\le \sum_{k=2}^\infty \frac{\lambda P(B_k)}{(k-1)k} \left(2\int_0^k x^{\beta+1}\,dx - (k-1)k^{\beta+1}\right).\end{aligned}$$

Assuming without loss of generality that $\beta \le 1$, we obtain

$$E\varphi(\xi_u^\mu) \le \frac{-\lambda\beta}{2+\beta}\,\mathscr{E}_\beta^\mu(u) + 2\lambda.$$

We now substitute this estimate into (12) to reach the conclusion that

$$\mathscr{E}^{\mu}_{\beta}(t) - \mathscr{E}^{\mu}_{\beta}(s) \leq \delta(\beta) \int_s^t \mathscr{E}^{\mu}_{\beta}(u)\, du + 2\lambda(t-s), \tag{13}$$

where $\delta(\beta) = 2^{\beta+1} - 2 - \dfrac{\lambda\beta}{2+\beta}$. From the proof of Lemma 2, it follows that the integrands in (9) and (13) are bounded for any $t > s \geq 0$. Consequently, $\mathscr{E}^{\mu}_{\beta}(t)$ is continuous for $t \geq 0$.

Now fix $\lambda > 4 \log 2$. For such λ, $\delta(0) = 0$ and $\delta'(0) < 0$, so for β small enough, $\delta(\beta) < 0$. Then (13) implies that the function $\mathscr{E}^{\mu}_{\beta}(t)$ must decrease on every interval I such that $\mathscr{E}^{\mu}_{\beta}(u) > -2\lambda/\delta(\beta)$ for $u \in I$. It follows from the continuity of $\mathscr{E}^{\mu}_{\beta}(t)$ that

$$\sup_{t\geq 0} \mathscr{E}^{\mu}_{\beta}(t) \leq \max\{\mathscr{E}^{\mu}_{\beta}(0), -2\lambda/\delta(\beta)\}.$$

Thus, (1) holds and the proof is complete. □

PROOF OF THEOREM 1′. (Centered birth model). The proof here is essentially the same as the proof of Theorem 1, except for the estimation of Gf_n. For this model,

$$Gf(A) = \textstyle\sum_{x\in A}(f(A\backslash\{x\}) - f(A)) + \sum_{x\notin A}(f(A \cup \{x\}) - f(A))b_x(A)$$

for $f \in \mathscr{F}$, where $b_x(A)$ is the birth rate at x for the state A defined for the centered birth model in Section 1. Therefore,

$$Gf_n(\xi^{\mu}_u)$$
$$\leq (L_{-1}(\xi^{\mu}_u) + L_0(\xi^{\mu}_u))^{\beta} + (L_0(\xi^{\mu}_u) + L_1(\xi^{\mu}_u))^{\beta} - 2f_n(\xi^{\mu}_u) + \lambda\left(\left(\frac{L_0(\xi^{\mu}_u) + 1}{2}\right)^{\beta} - f_n(\xi^{\mu}_u)\right).$$

Assume without loss of generality that $\beta \leq 1$. Then $(L_0(\xi^{\mu}_u) + 1)^{\beta} \leq (L_0(\xi^{\mu}_u))^{\beta} + 1$. We use this fact and Lemma 1 to obtain

$$EGf_n(\xi^{\mu}_u) \leq \left(2^{\beta+1} + \frac{\lambda}{2^{\beta}}\right)\mathscr{E}^{\mu}_{\beta}(u) - (2+\lambda)\, Ef_n(\xi^{\mu}_u) + \lambda/2^{\beta}.$$

Substituting this into (6) and applying monotone convergence, we have the following analogue to (13):

$$\mathscr{E}^{\mu}_{\beta}(t) - \mathscr{E}^{\mu}_{\beta}(s) \leq \bar{\delta}(\beta) \int_s^t \mathscr{E}^{\mu}_{\beta}(u)\, du + \frac{\lambda}{2^{\beta}}(t-s), \tag{14}$$

where $\bar{\delta}(\beta) = 2^{\beta+1} - 2 - \lambda(1 - 1/2^{\beta})$. The rest follows as in the previous proof. The inequality (9) still holds in the present case, so $\mathscr{E}^{\mu}_{\beta}(t)$ is continuous for $t \geq 0$. If $\lambda > 2$, then $\bar{\delta}(\beta) < 0$ for small enough β, so by (14),

$$\sup_{t\geq 0} \mathscr{E}^{\mu}_{\beta}(t) \leq \max\{\mathscr{E}^{\mu}_{\beta}(0), -\lambda/(2^{\beta}\bar{\delta}(\beta))\}. \ \square$$

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SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
127 VINCENT HALL
206 CHURCH ST., S.E.
MINNEAPOLIS, MINNESOTA 55455