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THE CENTRAL LIMIT THEOREM FOR MARKOV CHAINS

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A form of the central limit theorem for vector valued Markov chains is given, which is applicable to models arising in polymer chemistry.

In this note we outline the central limit theorem for Markov chains of the type considered by Freed (1981). In particular we show how to obtain this result from a functional central limit theorem for martingales.

Let $P(x, \Gamma)$ be a transition function on the surface of $S_r(0)$, the sphere in $\mathbb{R}^3$ of radius r centered at the origin. We define

$$Pf(x) = \int_{S_r(0)} f(y)P(x, dy), \tag{1}$$

and assume Pf is continuous if f is. The convergence in distribution of processes X_n to X in the Skorohod topology on $D[0, \infty)$ will be denoted $X_n \Rightarrow X$.

THEOREM. *Suppose there exists a ρ, $0 < |\rho| < 1$ such that*

$$\int yP(x, dy) = \rho x, \tag{2}$$

and a probability measure ν such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k f(x) = \int_{S_r(0)} f\, d\nu,\ x \in S_r(0), f \in C(S_r(0)). \tag{3}$$

Let $Y_0, Y_1 \cdots$ be a Markov chain with transition function $P(x, \Gamma)$, and set

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} Y_k. \tag{4}$$

Then $X_n \Rightarrow W_A$, where W_A is a Brownian motion with mean zero and covariance matrix A given by

$$A_{ij} = \frac{1+\rho}{1-\rho} \int y_i y_j \nu(dy). \tag{5}$$

PROOF. We leave verification of the following facts to the reader:

A. If $P(x, \Gamma)$ is a transition function on a compact metric space E, $P: C(E) \to C(E)$ and there exists a probability measure ν such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k f(x) = \int_E f\, d\nu,\ x \in E, f \in C(E), \tag{6}$$

then the convergence in (6) is uniform in x and if $Y_0, Y_1 \cdots$ is a Markov chain with transition function $P(x, \Gamma)$ then $n^{-1} \sum_{k=1}^{n} f(Y_k)$ converges almost surely to $\int_E f\, d\nu$.

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Note. The convergence in (6) is weak convergence in $C(E)$, so the mean ergodic theorem [e.g., Yosida (1968) page 213] implies the sequence converges in norm, that is uniformly in x. To obtain the almost sure convergence, note that

$$M_n = \sum_{k=1}^n (f(Y_k) - Pf(Y_{k-1}))$$

is a martingale and $n^{-1}M_n \to 0$ [Stout (1974), page 154] and hence

$$n^{-1}\sum_{k=1}^n f(Y_k) - n^{-1}\sum_{k=1}^n Pf(Y_k) \to 0.$$

The last limit holds with f replaced by $P^m f$ and hence

$$n^{-1}\sum_{k=1}^n f(Y_k) - n^{-1}\sum_{k=1}^n P^m f(Y_k) \to 0.$$

Averaging over m and using the uniformity in (6) gives the desired result.

B. If W is a $\mathbb{R}^d$-valued martingale and $\theta\cdot W$ is a $\mathbb{R}$-valued Brownian motion for all $\theta \in \mathbb{R}^d$, then W is an $\mathbb{R}^d$-valued Brownian motion.

Set $Z_0 = 0$ and

$$Z_m = \sum_{k=1}^m Y_k + \frac{\rho}{1-\rho} Y_m. \tag{7}$$

Then

$$\begin{aligned} E[Z_{m+1} \mid Z_m, Z_{m-1} \cdots Z_0] &= \sum_{k=1}^m Y_k + E\left[\left(1 + \frac{\rho}{1-\rho}\right) Y_{m+1} \mid Z_m, Z_{m-1} \cdots Z_0\right] \\ &= \sum_{k=1}^m Y_k + \frac{\rho}{1-\rho} Y_m \end{aligned} \tag{8}$$

by (2), and hence Z_m is a martingale. Set

$$W_n(t) = \frac{1}{\sqrt{n}} Z_{[nt]} \tag{9}$$

and note that $X_n \Rightarrow W_A$ if and only if $W_n \Rightarrow W_A$.

Fix $\theta \in \mathbb{R}^3$ and consider $\theta\cdot W_n$. We apply Theorem 1 of Gänssler and Häusler (1979) which is a refinement of a result of McLeish (1974). In our setting

$$\tau_n(t) = [nt] \quad \text{and} \quad X_{ni} = \frac{1}{\sqrt{n}}\, \theta\cdot\left(Y_i + \frac{\rho}{1-\rho}(Y_i - Y_{i-1})\right).$$

Clearly $\{\max_i |X_{ni}|\}$ is uniformly integrable and (2) and (3) imply

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{[nt]} X_{ni}^2 &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{[nt]} \left(\theta\cdot\left(Y_i + \frac{\rho}{1-\rho}(Y_i - Y_{i-1})\right)\right)^2 \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{[nt]} \left[\frac{1}{(1-\rho)^2}(\theta\cdot Y_i)^2 \right. \\ &\qquad \left. - \frac{2\rho}{(1-\rho)^2}(\theta\cdot Y_i)(\theta\cdot Y_{i-1}) + \frac{\rho^2}{(1-\rho)^2}(\theta\cdot Y_{i-1})^2\right] \\ &= t\left[\frac{1+\rho^2}{(1-\rho)^2}\int (\theta\cdot y)^2 \nu(dy) \right. \\ &\qquad \left. - \frac{2\rho}{(1-\rho)^2}\int\int (\theta\cdot x)(\theta\cdot y) P(y, dx)\nu(dy)\right] \\ &= t\frac{1+\rho}{1-\rho}\int (\theta\cdot y)^2 \nu(dy). \end{aligned} \tag{10}$$

Comparing (10) to (5), Theorem 1 of Gänssler and Häusler implies $\theta \cdot W_n \Rightarrow \theta \cdot W_A$.

Tightness for $\{W_n\}$ is easy to verify, and the uniform integrability of $\{W_n(t)\}$ for each t ($\sup_n E[\,|\, W_n(t) \,|\,^2] < \infty$) and the fact that W_n is a martingale imply any limit point of $\{W_n\}$ must be a martingale. If W_0 is a limit point of $\{W_n\}$, then W_0 is a martingale with $\theta \cdot W_0$ having the same distribution as $\theta \cdot W_A$ for all θ, that is $\theta \cdot W_0$ is a Brownian motion with variance given by (10). This in turn implies W_0 is a Brownian motion with covariance given by (5), that is W_0 has the same distribution as W_A and hence $W_n \Rightarrow W_A$. □

REMARKS. (a) If ν is normalized surface measure, then $A = \sigma^2 I$ with $\sigma^2 = \dfrac{r^2(1+\rho)}{3(1-\rho)}$. Note that in Freed $\rho = -\cos\theta$ where θ is the bond angle.

(b) The proof given above is close to that of Maigret (1978) and to Heyde (1974). For other recent approaches to the central limit theorem for Markov chains see Chung (1967, page 99), Rosenblatt (1971, page 217) and Lifshits (1978). Other versions of the central limit theorem for martingales include Rebolledo (1980), Rootzen (1977, 1980) and Helland (1980).

(c) The proof used above extends easily to other situations. For example let $Y_0, Y_1, \cdots$ be any sequence of random variables. If the limit

$$\Gamma_m = \lim_{N\to\infty} E[\textstyle\sum_{k=m+1}^{N} Y_k \mid Y_m, Y_{m-1} \cdots Y_0] \tag{11}$$

exists in L_1, then

$$Z_m = \textstyle\sum_{k=1}^{m} Y_k + \Gamma_m \tag{12}$$

is a martingale. If $Y_0 Y_1 \cdots$ is a positive recurrent Markov chain with discrete state space E and stationary distribution ν, $\int g\, d\nu = 0$, and for some fixed $z \in E$, $\tau_z = \min\{k > 0 : Y_k = z\}$ satisfies

$$E[\textstyle\sum_{k=1}^{\tau_z} |g(Y_k)| \mid Y_0 = y] < \infty \qquad \text{all} \quad y \in E, \tag{13}$$

then setting

$$f(y) = E[\textstyle\sum_{k=1}^{\tau_z} g(Y_k) \mid Y_0 = y] \tag{14}$$

we have that

$$Z_m = \textstyle\sum_{k=1}^{m} g(Y_k) + f(Y_m) \tag{15}$$

is a martingale.

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