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MARKOV FUNCTIONS

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A simple condition sufficient to ensure that a function of a time-homogeneous Markov process is again a time-homogeneous Markov process is proved. This result is then used to study a number of diffusions; in particular, an extension of a result of Pitman is proved, from which it is possible easily to deduce the path decompositions of Williams.

1. Introduction. The focal point of this paper is the following theorem, which implies the path decompositions of Williams (1974).

Theorem 1. *Let* $(B_t, t \geq 0)$ *be a Brownian motion on the line with drift* μ *and* $B_0 = 0$. *Let*

$$M_t = \max_{0 \leq s \leq t} B_s, \; Y_t = 2M_t - B_t.$$

Then the process $(Y_t, t \geq 0)$ *is a time homogeneous diffusion identical in law to the radial part of a three dimensional Brownian motion with drift of magnitude* $|\mu|$, *started at the origin.*

The special case of Theorem 1 with no drift was established by Pitman (1975) using random walk approximations, and this result was recently reproved by Jeulin (1979) using techniques from the theory of enlargement of filtrations. Here we give a short proof of Theorem 1 based on general criteria for when a function of a Markov process is again Markov. These criteria, which are presented in Section 2, complement those of Dynkin (1965), Kemeny and Snell (1960) and Rosenblatt (1971), none of which is applicable to the example at hand.

2. Criteria for a function of a Markov process to be Markov. Throughout this section, let $X = (X_t, t \geq 0)$ be a continuous time Markov process defined on a probability space $(\Omega, \mathscr{F}, P)$, with measurable statespace $(S, \mathscr{S})$, initial distribution λ, and transition semigroup $(P_t, t \geq 0)$ with P_0 the identity. Let $(S', \mathscr{S}')$ be a second measurable space, and let $\phi: S \to S'$ be a measurable transformation.

Consider the question of when $\phi \circ X$ is a Markov process. Dynkin (1965), Kemeny and Snell (1960), Rosenblatt (1971) give conditions for $\phi \circ X$ to be Markov, either for all initial distributions λ, or for an invariant λ. But we are particularly concerned here with the situation when there may be no invariant initial distribution, and $\phi \circ X$ may be Markov for some, but not all, initial laws λ.

We shall formulate a condition in terms of a Markov kernel Λ from S' to S, that is, a map

$$\Lambda: (y, A) \to \Lambda(y, A), y \in S', A \in \mathscr{S}$$

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such that for each $y \in S'$, $\Lambda(y, \cdot)$ is a probability on S, and for each $A \in \mathscr{S}$, $\Lambda(\cdot, A) \in b\mathscr{S}'$, the space of bounded measurable functions on S'. The kernel Λ will also be viewed as an operator taking $f \in b\mathscr{S}$ to $\Lambda f \in b\mathscr{S}'$, where $\Lambda f(y)$ is the $\Lambda(y, \cdot)$ integral of f.

The role to be played by Λ is made plain by the following simple observation. If for each $t \geq 0$, $A \in \mathscr{S}$,

$$P(X_t \in A \mid \phi \circ X_s, 0 \leq s \leq t) = \Lambda(\phi \circ X_t, A) \quad \text{a.s.,} \tag{1}$$

then $\phi \circ X$ is Markov with transition kernels Q_t defined by

$$Q_t f = \Lambda P_t(f \circ \phi), \qquad f \in b\mathscr{S}'. \tag{2}$$

In more compact notation using composition of kernels

$$Q_t = \Lambda P_t \Phi, \tag{3}$$

where Φ is the Markov kernel from S to S' which is induced by ϕ according to the formula

$$\Phi f = f \circ \phi, \qquad f \in b\mathscr{S}'. \tag{4}$$

The criterion (1) is unsatisfactory in two respects. Firstly, one has to be able to calculate the conditional distribution of X_t given the whole history of $\phi \circ X$ up to time t, and secondly, the resultant kernels Q_t may not form a semigroup because the Chapman-Kolmogorov equations need only be satisfied with exceptional null sets. These difficulties are overcome by the following theorem.

THEOREM 2. *Suppose there is a Markov kernel Λ from S' to S such that*

(5)
(a) $\Lambda\Phi = I$, *the identity kernel on S',*
(b) *for each $t \geq 0$ the Markov kernel $Q_t = \Lambda P_t \Phi$ from S' to S' satisfies the identity $\Lambda P_t = Q_t \Lambda$.*

Let X be Markov with semigroup (P_t) and initial distribution $\lambda = \Lambda(y, \cdot)$, where $y \in S'$. Then (1) holds, and $\phi \circ X$ is Markov with starting state y and transition semigroup (Q_t).

PROOF. Condition (b) implies that the family (Q_t) of kernels is a semigroup. Condition (a) implies that for $f \in b\mathscr{S}'$, $g \in b\mathscr{S}$,

$$\Lambda(\Phi f)g = f\Lambda g, \tag{6}$$

and together with (b) this implies

$$\Lambda P_t(\Phi f)g = Q_t f \Lambda g, \qquad t \geq 0. \tag{7}$$

Note that the concatenations of kernels and functions now appearing should be read from right to left, unless indicated by brackets. For example, the left side of (6) is the function in $b\mathscr{S}'$ obtained by multiplying g by Φf, and then operating with Λ. An obvious induction using (7) now implies that for $t_1, \cdots, t_n \geq 0$, $f_1, \cdots, f_n \in b\mathscr{S}'$, and $g \in b\mathscr{S}$,

$$\Lambda P_{t_1}(\Phi f_1) P_{t_2}(\Phi f_2) \cdots P_{t_n}(\Phi f_n) g = Q_{t_1} f_1 Q_{t_2} f_2 \cdots Q_{t_n} f_n \Lambda g,$$

and the conclusions of the theorem are evident.

REMARKS.

(i) Apart from forcing ϕ to be onto, condition (a) is a natural regularity condition for Λ acting as a conditional distribution for X_t given $\phi \circ X_t$. Assuming that singleton sets $\{y\}$ are in $\mathscr{S}'$, (a) simply says that

$$\Lambda(y, \phi^{-1}\{y\}) = 1, \qquad y \in S'.$$

(ii) As is plain from the identity (7), when (a) holds, condition (b) is equivalent to the following condition:

$$P(X_t \in A \mid \phi \circ X_t) = \Lambda(\phi \circ X_t, A) \qquad \text{a.s.}$$

whenever X_0 has distribution $\lambda = \Lambda(y, \cdot)$ for some $y \in S'$, $A \in \mathscr{S}$, $t \geq 0$.

(iii) If instead of (5b) one has simply

$$P_t\Phi = \Phi Q_t, \qquad t \geq 0 \tag{8}$$

for some kernel Q_t on S', that is to say, if the $P_t(x, \cdot)$ distribution of ϕ depends only on the value of $\phi(x)$, then $\phi \circ X$ is Markov with semigroup (Q_t) for all initial distributions λ. This is the well known criterion of Dynkin (1965).

(iv) Both Dynkin's criterion and the criterion of Theorem 2 have obvious extensions to inhomogeneous Markov processes, by the trivial device of considering the space-time chain.

(v) When Dynkin's criterion applies, the process $\phi \circ X$ is Markov with respect to the larger σ-fields generated by the underlying process X, meaning that at each instant of time, the past of X and the future of $\phi \circ X$ are conditionally independent given the present of $\phi \circ X$. This does not usually happen when Theorem 2 applies; however, $\phi \circ X$ is Markov with respect to the σ-fields of X when time is run backwards, for (1) implies that the future of X and the past of $\phi \circ X$ are conditionally independent given the present of $\phi \circ X$. Kelly (1981) identifies the condition of Theorem 2 as the time reversal of Dynkin's condition in the case of inhomogeneous discrete time Markov chains, and develops these ideas.

(vi) The difference between Dynkin's criterion and that of Theorem 2 is well illustrated by the following example from renewal theory, which was pointed out to us by Martin Jacobsen.

Let $L_1, L_2, \cdots$ be independent identically distributed nonnegative random variables, to be thought of as lifetimes. Let $S_n = \sum_{j=1}^{n} L_j$. Define the age process A and the residual lifetime process R by

$$A_t = t - \max\{S_i;\, S_i \leq t\}, \qquad t \geq 0;$$

$$R_t = \min\{S_i;\, S_i > t\} - t, \qquad t \geq 0.$$

Consider also the bivariate process $(A, R) = ((A_t, R_t), t \geq 0)$. It is well known that each of the three processes A, R, and (A, R), is Markov with stationary transition probabilities. Viewing A and R as functions of (A, R), R is Markov by Dynkin's criterion, while A is Markov by the criterion of Theorem 2, with $\Lambda(a, \cdot)$ the conditional distribution of $(a, L_1 - a)$ given $(L_1 > a)$.

(vii) With suitable topological assumptions, such as those in Lemma 1 below, it is easy to deduce a strong Markov form of the above Theorem, but this is left to the reader.

To conclude this section we record two Lemmas which can save much labour in checking Condition (b).

For the first Lemma suppose that S and S' are metric spaces equipped with their Borel σ-fields.

Say a collection of probabilities $\mathscr{L}$ on a metric space is *determining* if for two bounded continuous functions f and g, $\lambda f = \lambda g$ for all $\lambda \in \mathscr{L}$ implies $f = g$.

Call a kernel *continuous* if it maps bounded continuous functions to bounded continuous functions, and say that a semigroup (P_t) *is Feller* if each P_t is continuous.

LEMMA 1. *Suppose that* (P_t) *is Feller, and that both* ϕ *and* Λ *are continuous. Let* X *have a fixed initial distribution* λ, *and let* q_t *be the distribution of* $\phi \circ X_t$. *Suppose further that*

(i) *for each* $t \geq 0$ *a conditional distribution for* X_t *given* $\phi \circ X_t = y$ *is* $\Lambda(y, \cdot)$, $y \in S'$;

(ii) *the collection* $\{q_t;\, t \geq 0\}$ *is determining.*

Then conditions (5a) *and* (5b) *hold,* $\phi \circ X_t$ *is Markov with initial law* q_0 *and transition semigroup* $(Q_t;\, t \geq 0)$, *which is the unique Feller semigroup on* S' *such that* $q_t = q_0 Q_t$, $t \geq 0$.

PROOF. According to (i), $\lambda P_t = q_t\Lambda$, and $q_t = \lambda P_t\Phi$ by definition, so

$$\lambda P_t = \lambda P_t \Phi\Lambda, \qquad t \geq 0.$$

This implies that for each bounded continuous function f on S',

$$q_t f = \lambda P_t \Phi f = \lambda P_t \Phi\Lambda\Phi f = q_t \Lambda\Phi f.$$

The continuity assumptions, condition (ii), and the metrisability of S' imply $\Lambda\Phi = I$. Proof of the condition (5b) is by repeated application of the identity $\lambda P_t = \lambda P_t\Phi\Lambda$ and the semigroup property of (P_t) to justify the following computation; for $s, t \geq 0$,

$$\begin{aligned} q_s\Lambda P_t &= \lambda P_s \Phi\Lambda P_t \\ &= \lambda P_s P_t \\ &= \lambda P_{s+t} \\ &= \lambda P_{s+t}\Phi\Lambda \\ &= \lambda P_s P_t \Phi\Lambda \\ &= \lambda P_s \Phi\Lambda P_t\Phi\Lambda \\ &= q_s Q_t \Lambda, \quad \text{where} \quad Q_t = \Lambda P_t \Phi. \end{aligned}$$

Now integrate a bounded continuous function f on S to conclude that $\Lambda P_t f = Q_t\Lambda f$ for all such f, using (ii) and the continuity assumptions. Thus (5b) holds, and the uniqueness assertion for (Q_t) is easily checked by a similar computation.

LEMMA 2. *Let* $(P_t, t \geq 0)$, ϕ *and* Λ *be such that* (5) *holds. Let* f *be a strictly positive measurable function on* S *such that for some* α

$$e^{-\alpha t} P_t f = f, \qquad t \geq 0, \tag{9}$$

and such that $h = \Lambda f$ *is everywhere finite on* S'. *Let* (P_t^*) *be the semigroup of transition operators on* S *defined by*

$$P_t^*(x, dw) = e^{-\alpha t} f(x)^{-1} P_t(x, dw) f(w),$$

and let

$$\Lambda^*(y, dx) = h(y)^{-1}\Lambda(y, dx) f(x).$$

Then (5) *holds for* (P_t^*), ϕ *and* Λ^*, *and* $Q_t^* = \Lambda^* P_t^* \Phi$ *is related to* $Q_t = \Lambda P_t \Phi$ *by*

$$Q_t^*(y, dz) = e^{-\alpha t} h(y)^{-1} Q_t(y, dz) h(z).$$

PROOF. This is quite straightforward.

3. The Bessel process of drifting Brownian motion. For a Brownian motion X in $\mathbb{R}^k$ with no drift and arbitrary initial distribution, it is well known that the radial part $|X|$ is a diffusion on $[0, \infty)$, the *k-dimensional Bessel process*, BES(k), with transition density $q_k(t, x, y)$ which can be expressed in terms of the modified Bessel function of the first kind I_ν, where $\nu = \frac{1}{2}k - 1$ (see Itô and McKean (1965), Section 2.7). Taking $\phi(x) = |x|$, $x \in \mathbb{R}^k$, this is a typical example where the criterion (8) applies. The criterion (5) is well illustrated by the following theorem, where it is essential to start at the origin if $\mu > 0$.

THEOREM 3. *Let* X *be a Brownian motion in* $\mathbb{R}^k$ *started at the origin, with a drift of magnitude* $\mu \geq 0$. *Then the radial part* $|X|$ *is a time homogeneous diffusion process on* $[0, \infty)$ *with transition density*

$$q_{k,\mu}(t, x, y) = e^{-\mu^2 t/2} h_k(\mu x)^{-1} q_k(t, x, y) h_k(\mu y), \tag{10}$$

where

$$h_k(y) = (y/2)^{-\nu}\Gamma(\nu + 1)I_\nu(y), \qquad \nu = (k/2) - 1.$$

PROOF. Fix k, and for $\mu \geq 0$, let (P_t^μ) be the semigroup of Brownian motion in $\mathbb{R}^k$ with a drift of magnitude μ in the direction of the first coordinate axis, write P_t instead of P_t^0, and define $f: \mathbb{R}^k \to (0, \infty)$ by

$$f(x_1, x_2, \cdots, x_k) = e^{\mu x_1}.$$

Then

$$e^{-\mu^2 t/2} P_t f = f, \qquad t \geq 0,$$

and

$$P_t^\mu(x, dw) = e^{-\mu^2 t/2} f(x)^{-1} P_t(x, dw) f(w).$$

Let $(Q_t, t \geq 0)$ be the BES(k) semigroup on $[0, \infty)$, and for $y \geq 0$ let $\Lambda(y, \cdot)$ be the uniform probability distribution on the sphere of radius y in $\mathbb{R}^k$. As is obvious by symmetry,

$$\Lambda P_t = Q_t \Lambda, \qquad t \geq 0.$$

Lemma 2 implies

$$\Lambda^\mu P_t^\mu = Q_t^\mu \Lambda^\mu, \qquad t \geq 0,$$

where $\Lambda^\mu(y, \cdot)$ is the (von Mises) distribution on the sphere of radius y with density proportional to $e^{\mu x_1}$, $Q_t^\mu(x, dy) = q_{k,\mu}(t, x, y)\, dy$, and we have used the fact that the integral of $e^{\mu x_1}$ with respect to the uniform probability distribution on the surface of the sphere of radius y in k dimensions is $h_k(\mu y)$. Theorem 3 now follows from Theorem 2.

Call a diffusion process on $[0, \infty)$ with transition density $q_{k,\mu}$ and starting state r a BES$^r(k, \mu)$. The above argument goes on to show that the von Mises distribution $\Lambda^\mu(r, \cdot)$ on the sphere of radius r is the hitting distribution of that sphere for the drifting Brownian motion X started at the origin, and that if X is given initial distribution $\Lambda^\mu(r, \cdot)$ then $|X|$ is a BES$^r(k, \mu)$. See Pitman and Yor (1978) for further development including a Brownian motion proof of the result of Hartman and Watson (1974) that the von Mises distribution on the circle is a mixture of wrapped normal distributions.

The generator G_k of BES(k) with no drift is the radial part of ½Δ,

$$G_k = \frac{1}{2}\frac{d^2}{dr^2} + \frac{k-1}{2r}\frac{d}{dr},$$

and the generator $G_{k,\mu}$ of the BES(k, μ) can be calculated from (10) as

$$G_{k,\mu} = G_k + h_k(\mu r)^{-1}\left[\frac{d}{dr} h_k(\mu r)\right]\frac{d}{dr}. \tag{11}$$

For odd dimensions k the Bessel function I_ν can be expressed in terms of hyperbolic functions, and in particular for $k = 3$ one finds that

$$\begin{aligned} h_3(y) &= y^{-1}\sinh y \qquad \text{if } y \neq 0 \\ &= 1 \qquad \text{if } y = 0, \end{aligned} \tag{12}$$

whence for $\mu > 0$

$$G_{3,\mu} = \frac{1}{2}\frac{d^2}{dr^2} + \mu \coth \mu r \frac{d}{dr}. \tag{13}$$

The transition density $q_{3,\mu}$ of BES($3, \mu$) can be explicitly obtained from (10) and (12) together with the following well known formulae for q_3:

$$q_3(t, x, y) = \begin{cases} 2yg(y, t) & \text{if} \quad x = 0 \\ x^{-1}p^0(t, x, y)y, & \quad x > 0 \end{cases} \tag{14}$$

where

$$g(y, t) = (2\pi t^3)^{-1/2} y \exp(-y^2/2t), \tag{15}$$

$$p(t, x, y) = (2\pi t)^{-1/2} \exp(-(x-y)^2/2t)$$

is the Brownian transition density, and

$$p^0(t, x, y) = p(t, x, y) - p(t, x, -y)$$

is the transition density for Brownian motion killed at zero. As observed by McKean (1963), Knight (1969), and Williams (1974), (14) can be interpreted by saying that for $\mu = 0$, BES(3, μ) is Brownian motion with drift μ on $[0, \infty)$ conditioned to hit ∞ before 0, where the conditioning is to be understood in the sense of Doob (1957). It is easy to show, using the above formulae, that the same is true also for $\mu > 0$, where the conditioning can now be interpreted in an elementary manner provided that the motion starts at a level $r > 0$.

4. Proof of Theorem 1. Let $B = (B_t, t \geq 0)$ be defined on a space $(\Omega, \mathscr{F})$ equipped with probabilities $\mathbb{P}_\mu$, $\mu \in \mathbb{R}$, such that under $\mathbb{P}_\mu$ the process B is a Brownian motion with drift μ started at zero. Let

$$M_t = \max_{0 \leq s \leq t} B_s, \quad Y_t = 2M_t - B_t,$$

$$X_t = (M_t - B_t, M_t), \quad \text{so } Y_t = \phi \circ X_t, \quad \text{where } \phi(u, v) = u + v.$$

As is easily verified, the process $(X_t, t \geq 0)$ with statespace $[0, \infty)^2$ is Markov under $\mathbb{P}_\mu$ with Feller semigroup $(P_t^\mu, t \geq 0)$ which may be described thus: for $u, v \geq 0$, $P_t^\mu((u, v), \cdot)$ is the $\mathbb{P}_\mu$ distribution of

$$((M_t \vee u) - B_t, v + (M_t - u)^+).$$

For $\mu = 0$ the joint distribution of M_t and B_t may be obtained using the reflection principle (see for example Freedman (1971)), and one finds that for $u, v \geq 0$, and $g(y, t)$ as in (15),

$$\mathbb{P}_0(M_t - B_t \in du, M_t \in dv) = 2g(u + v, t)\,du\,dv. \tag{16}$$

But, according to the Cameron-Martin formula,

$$\frac{d\mathbb{P}_\mu}{d\mathbb{P}_0} = \exp(\mu B_t - \mu^2 t/2) \quad \text{on } \mathscr{F}_t,$$

where $\mathscr{F}_t$ is the σ-field generated by B_s, $0 \leq s \leq t$ (see McKean (1969), page 97, or Freedman (1971) Section 1.11). Thus (16) implies

$$\mathbb{P}_\mu(M_t - B_t \in du, M_t \in dv) = 2g(u + v, t)e^{\mu(v-u) - \mu^2 t/2}\,du\,dv. \tag{17}$$

For $y > 0$ let $\Lambda_\mu(y, \cdot)$ be the probability on $[0, \infty)^2$ which concentrates on the line segment

$$\{(u, v): u + v = y,\ u, v \geq 0\}$$

with density at (u, v) proportional to $e^{\mu(v-u)}$, and let $\Lambda_\mu(0, \cdot)$ be a unit mass at (0, 0). As is obvious from (17), $\Lambda_\mu(y, \cdot)$ serves as a $\mathbb{P}_\mu$ conditional distribution for $X_t = (M_t - B_t, M_t)$ given $Y_t = y$, and an easy integration using (17) shows that the $\mathbb{P}_\mu$ distribution of $Y_t = (M_t - B_t) + M_t$ is

$$q_t^\mu(dy) = 2yh_3(\mu y)g(y, t)e^{-\mu^2 t/2}dy, \tag{18}$$

where $h_3(y)$ was defined in (12). But the uniqueness of Laplace transforms implies that for each μ the laws $\{q_t^\mu, t \geq 0\}$ on $[0, \infty)$ are determining, so Lemma 1 and Theorem 2 of the previous section imply that under $\mathbb{P}_\mu$ the process $(Y_t, t \geq 0)$ is Markov with the unique Feller semigroup $(Q_t^\mu, t \geq 0)$ on $[0, \infty)$ such that

$$Q_t^\mu(0, dy) = q_t^\mu(dy), \qquad t \geq 0.$$

The proof is now completed by an inspection of (10), (14) and (18).

5. Consequences of Theorem 1. Call a one dimensional Brownian motion with drift μ a BM(μ), and call the radial part of a three dimensional Brownian motion with drift $\mu \geq 0$ a BES(3, μ), where the starting position may be indicated by a superscript. Theorem 1 implies that corresponding to every relation between BM(μ) and BES(3, μ) described in Williams (1974) and Pitman (1975) for $\mu = 0$, there is an analogous relation for every $\mu \geq 0$.

To start off, let B be an arbitrary process with continuous paths, and let $Y = 2M - B$ be defined as in Theorem 1. The transformation from B to Y can be interpreted geometrically as reflection in the level of the past maximum. Thus if the value of $M_\infty = \sup_t M_t$ is known, the transformation can be inverted according to the formulae

(19) $$M_t = \inf\{M_\infty, Y_s, s \geq t\}$$

(20) $$B_t = 2M_t - Y_t.$$

COROLLARY 1. *Fix* $\mu \geq 0$, *suppose that* Y *is a* BES^0(3, μ), and define

$$M_t^+ = \inf\{Y_s, s \geq t\}, \qquad B_t^+ = 2M_t^+ - Y_t,$$
$$\sigma_b = \sup\{s : Y_s \leq b\}, \qquad \tau_b = \inf\{s : B_s^+ > b\}.$$

(i) B^+ *is a* BM^0(μ);
(ii) $M_t^+ = \sup\{B_s, 0 \leq s \leq t\}$;
(iii) $\sigma_b = \tau_b$ *for all* $b \geq 0$, a.s.;
(iv) *For each* $b > 0$ *the post-*σ_b *process* $\tilde{Y}$ *defined by*

$$\tilde{Y}(u) = Y(\sigma_b + u) - b, \qquad u \geq 0$$

is a BES^0(3, μ) *independent of the pre-*σ_b *process*

$$(Y(t), 0 \leq t \leq \sigma_b)$$

which has the same law as

$$(b - B^+(\tau_b - t), 0 \leq t \leq \tau_b).$$

PROOF. It evidently suffices to prove these assertions for any particular BES^0(3, μ) process Y. So appeal to Theorem 1 and take $Y = 2M - B$ where M is the past maximum process of a BM^0(μ) process B. Since $\mu \geq 0$, $M_\infty = \infty$ a.s., so (19) and (20) imply that the processes M^+ and B^+ defined in (i) are a.s. identical to M and B respectively. This proves (i) and (ii), and (iii) follows easily. Turning to (iv), the strong Markov property of B at time τ_b implies that the post-τ_b process. $\tilde{B}$ defined by

$$\tilde{B}(u) = B(\tau_b + u) - b, \qquad u \geq 0,$$

is a BM$^\circ$(μ) independent of the pre-τ_b process

$$(B(t), 0 \leq t \leq \tau_b).$$

Since $\tilde{Y} = 2\tilde{M} - \tilde{B}$, where $\tilde{M}$ is the past maximum process of $\tilde{B}$, Theorem 1 implies that Y is a BES^0(3, μ), independent of $(B(t), 0 \leq t \leq \tau_b)$ and hence of $(Y(t), 0 \leq t \leq \sigma_b)$. The last assertion can be deduced from Theorem 1 using a time reversal argument involving the process with stationary independent increments $(\tau_a, 0 \leq a \leq b)$ and the excursions of

B below M, but it seems to be more trouble than it is worth to try and make this argument rigorous. Rather it is simpler to mimic the proof of Theorem 3.4 in Williams (1974), which is the present result for $\mu = 0$.

REMARKS.

(a) The time reversal of excursions linking Williams' Theorem 3.4 and Theorem 1 of this paper was what originally suggested the results of Pitman (1975).

(b) Part (iv) shows that the last exit process $(\sigma_b,\ b \geq 0)$ associated with $\mathrm{BES}^0(3, \mu)$ process Y has stationary independent increments, and one can easily obtain formulae for the distribution of σ_b from known results for τ_b. For example, the expectation of σ_b is b/μ.

(c) To obtain a more general formulation of the last exit decomposition (iv) for a transient diffusion Y see Theorems 2.4 and 2.5 of Williams (1974). Assuming that Y starts at zero and escapes to $+\infty$, the last exit process $(\sigma_b,\ b \geq 0)$ still has independent increments. Moreover, the last exit process has stationary increments only if Y is a $\mathrm{BES}^0(3, \mu)$ after a rescaling of time. To see this, consider the reversal of Y from σ_b, defined by

$$\bar{Y}_t = b - Y_{(\sigma_b - t) \vee 0} \qquad t \geq 0,$$

for some fixed $b > 0$. Then the last exit process of Y becomes the first hitting process of $\bar{Y}$ and the fact that $\bar{Y}$ is a time-homogeneous strong Markov process, coupled with the fact that the first hitting process of $\bar{Y}$ is a Lévy process, implies that $\bar{Y}$ is a $\mathrm{BM}(\mu)$ stopped on first hitting b, for some $\mu \geq 0$. Since b was arbitrary, this determines Y, and the fact that a $\mathrm{BES}^0(3, \mu)$ reversed from σ_b is a stopped $\mathrm{BM}(\mu)$ (Corollary 1 (iv)) proves the assertion.

The next consequence of Theorem 1 is a path decomposition of downward drifting Brownian motion at its maximum, which is a slight refinement of Theorem 2.1 of Williams (1974). For more general but less explicit decomposition of Markov processes in a similar vein see Millar (1977) and (1978).

COROLLARY 2. *Let B be a* $\mathrm{BM}^0(-\mu)$, *where* $\mu > 0$. *For* $0 \leq t \leq \infty$ *let* $M_t = \sup_{0 \leq s < t} B_s$, *and put* $\rho = \sup\{t: B_t = M_\infty\}$. *Then*

(i) M_∞ *is exponentially distributed with rate* 2μ, *independently of the* $\mathrm{BES}^0(3, \mu)$ *process* $Y = 2M - B$.

(ii) *Let* $M_t^+ = \inf\{Y_s,\ s \geq t\}$, $B_t^+ = 2M_t^+ - Y_t$. *Then* B^+ *is a* $\mathrm{BM}^0(\mu)$ *independent of* M_∞, $\rho = \inf\{t: B_t^+ = M_\infty\}$ a.s., *and the processes* $(B_t,\ 0 \leq t \leq \rho)$ *and* $(B_t^+,\ 0 \leq t \leq \rho)$ *are* a.s. *identical.*

(iii) *The process* $(M_\infty - B_{\rho+u},\ u \geq 0)$ *is a* $\mathrm{BES}^0(3, \mu)$ *independent of* $(B_t,\ 0 \leq t \leq \rho)$.

PROOF. It was shown in the last section that for B, M and Y as above, the conditional distribution of M_t given $(Y_s,\ 0 \leq s \leq t)$ has density proportional to $e^{-2\mu x}$ for x in the interval $[0, Y_t]$. Since $Y_t \to \infty$ a.s., (i) results by martingale convergence, and (ii) and (iii) follow quickly using Corollary 1.

A further corollary of Theorem 1 is the extension of Theorem 4.1 of Pitman (1975) to $\mathrm{BES}(3, \mu)$ processes for all $\mu \geq 0$. This implies a path decomposition at the minimum of a $\mathrm{BES}^b(3, \mu)$ process Y which extends Theorem 3.1 of Williams (1974). For $\mu > 0$ this decomposition can also be derived quickly from Corollary 2, since if B is a $\mathrm{BM}^0(-\mu)$ with maximum M_∞ as in that corollary, the remarks at the end of Section 3 show that $(b - B_t,\ t \geq 0)$ conditional on $(M_\infty \leq b)$ is a $\mathrm{BES}^b(3, \mu)$. The details are left to the reader.

6. When is 2M − B a diffusion? For which regular diffusion processes B is $2M - B$ again a time homogeneous strong Markov process? We now mention two families of examples in addition to Brownian motion with drift. Further examples can be obtained by obvious rescalings, and there is the trivial example of a diffusion B which never rises above its initial value, but recent work of Rogers (1981) proves that there are essentially no more.

EXAMPLE 1. Let B be a $\mathrm{BM}^0(-\mu)$ conditioned never to hit b, where $b > 0$, $\mu \geq 0$, and for $\mu = 0$ the conditioning should be interpreted in the sense of Doob (1957). Put another way, $B_t = b - R_t$ where R is a $\mathrm{BES}^b(3, \mu)$. Then $2M - B$ is a $\mathrm{BES}^0(3, \mu)$. For $\mu > 0$ this follows from Corollary 2(i), by conditioning $Y = 2M - B$ on $M_\infty < b$ and using independence of M_∞ and Y, and the result extends easily to $\mu = 0$ by weak convergence. Alternatively, the conclusion can be obtained for any $\mu \geq 0$ from the path decomposition of $\mathrm{BES}^b(3, \mu)$ as its minimum which was mentioned at the end of the last section.

EXAMPLE 2. Let $\mu \geq 0$, $0 \leq p \leq 1$, $q = 1 - p$. Let B be a p and q mixture of $\mathrm{BM}^0(\mu)$ and $\mathrm{BM}^0(-\mu)$. That is,

$$B_t = \Sigma B_t^+, \qquad t \geq 0,$$

where Σ is a random sign, equal to $+1$ with probability p and -1 with probability $q = 1 - p$, and B^+ is a $\mathrm{BM}^0(\mu)$ independent of Σ. It is immediate from Theorem 1 that $2M - B$ is then a $\mathrm{BES}(3, \mu)$ though it is not so obvious that B is itself a time homogeneous diffusion. To see this, consider the process X defined by $X_t = (\Sigma, B_t^+)$, $t \geq 0$, which is Markov with statespace $S = \{-1, 1\} \times \mathbb{R}$, and transition semigroup $(P_t^\mu, t \geq 0)$ derived in an obvious way from the $\mathrm{BM}(\mu)$ semigroup. Define $\phi: S \to \mathbb{R}$ by $\phi(s, b) = sb$, so $\phi(X_t) = B_t$. For $\mu = 0$ it is plain that $B = \phi(X)$ is a $\mathrm{BM}^0(0)$ independent of Σ, and therefore the condition of Theorem 2 is satisfied by $(P_t^0, t \geq 0)$ with the kernel Λ defined by

$$\Lambda(y, \cdot) = p\delta(1, y) + q\delta(-1, y),$$

where $\delta(s, b)$ is a unit mass at (s, b). For $\mu > 0$ define $f_\mu: S \to \mathbb{R}$ by $f_\mu(s, b) = e^{\mu s b}$. Since P_t^μ is derived from P_t through f_μ as in Lemma 2, that Lemma implies that when B^+ has drift $\mu > 0$, $B = \phi(X)$ is Markov with transition semigroup

$$Q^\mu(y, dz) = e^{-\mu^2 t/2} h_\mu(y)^{-1} p(t, y, z) h_\mu(z)\, dz,$$

where $p(\cdot, \cdot, \cdot)$ is the transition density of $\mathrm{BM}(0)$ and

$$h_\mu(y) = \Lambda f_\mu(y) = pe^{\mu y} + qe^{-\mu y}.$$

It follows easily that B is a regular diffusion.

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