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BOCHNER'S THEOREM ON MEASURABLE LINEAR FUNCTIONALS OF A GAUSSIAN MEASURE

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Bochner's theorem formulated by Xia Dao-Xing is established for an abstract Wiener space. Let (ι, H, E) be an abstract Wiener space. Then for every continuous cylinder set measure ν on E', the image $\iota'(\nu)$ is a Radon measure on H'.

1. Introduction. Let E be a locally convex space and μ be a Radon measure on E. The measurable linear functionals on (E, μ) are the elements of $H_0(\mu)$ = the closure of E' in $L^0(E, \mu)$. The Bochner problem, formulated by Xia [6] as follows, is investigated.

Bochner Problem. For every continuous cylinder set measure ν on E', is the image $R(\nu)$ a Radon measure on $H_0(\mu)$?

Here $R : E' \to H_0(\mu)$ is the natural mapping defined by $R(x') = \langle \cdot, x' \rangle$ and the continuity of the cylinder set measure ν means the continuity of the Fourier transform $\hat{\nu}(x) = \int_{E'} \exp i\langle x, x' \rangle)\, d\nu(x')$ on E.

In this paper we shall restrict ourselves to the case μ is a centered Gaussian Radon measure and show the Bochner problem is valid in this case.

2. Bochner problem for Gaussian case. In case μ is a centered Gaussian Radon measure, the Bochner problem can be reduced to the following form (see Sato and Okazaki [5]). Let (ι, H, E) be a triple such that H is a separable Hilbert space, E is a locally convex space and $\iota : H \to E$ is a one-to-one continuous linear mapping with dense range satisfying that the image $\mu = \iota(\gamma_H)$ is a Radon measure on E, where γ_H is the canonical Gaussian cylinder set measure on H with the Fourier transform $\hat{\gamma}_H = exp(-\| h \|_H^2)$.

Bochner Problem for measurable linear functionals of a Gaussian measure. For every continuous cylinder set measure ν on E', is the image $\iota'(\nu)$ a Radon measure on H'? ($\iota' : E' \to H'$ is the transpose of ι).

The answer to this problem is "Yes" as we shall prove in the next theorem.

THEOREM. *The Bochner Problem for a Gaussian measure is affirmative.*

PROOF. Let ν be a continuous cylinder set measure on E'. By the manner similar to Maurey [3], Théorème 4, it can be assumed $\int_{E'} | \langle x, x' \rangle |^{1/2}\, d\nu(x') < +\infty$ for every $x \in E$. That is, ν corresponds to a continuous random linear functional $\phi = \phi_\nu : E \to L^{1/2}(\Omega_\phi, \mathfrak{A}_\phi, \boldsymbol{P}_\phi)$, see Dudley [1].

The mapping $\phi \circ \iota : H \to L^{1/2}(\Omega_\phi)$ is 2-summing. To see this let $\{g_\iota\}$ be identically distributed independent random variables on a probability space $(\Omega, \mathfrak{A}, \boldsymbol{P})$ with the characteristic function $\exp(-| u |^2)$. Then it holds

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$$(\textstyle\sum_{i=1}^{n} \| \phi \circ \iota(x_i) \|_{L^{1/2}}^{2})^{1/2} \leqq C \displaystyle\int_{\Omega} \| \textstyle\sum_{i=1}^{n} \phi \circ \iota(x_i) g_i(\omega) \|_{L^{1/2}} \, dP(\omega)$$

for every n and $x_1, x_2, \cdots, x_n \in H$, since $L^{1/2}(\Omega_\phi)$ is of cotype 2 (see Maurey [2], Proposition 3). By the continuity of ϕ, there exists a continuous seminorm p on E so that $\| \phi(z) \|_{L^{1/2}} \leqq p(z)$ for every $z \in E$, so it holds

$$\begin{aligned}(\textstyle\sum_{i=1}^{n} \| \phi \circ \iota(x_i) \|_{L^{1/2}}^{2})^{1/2} &\leqq C \displaystyle\int_{\Omega} p(\textstyle\sum_{i=1}^{n} \iota(x_i) g_i(\omega)) \, dP(\omega) \\ &= C \displaystyle\int_{E} p(z) \, d\iota(\Gamma_2)(z)\end{aligned}$$

where Γ_2 is the Gaussian measure on H with $\hat{\Gamma}_2 = \exp(-\sum_{i=1}^{n} | \langle x_i, h \rangle |^2)$. By Maurey [2], Théorème 1, it holds

$$(\textstyle\sum_{i=1}^{n} \| \phi \circ \iota(x_i) \|_{L^{1/2}}^{2})^{1/2} \leqq C \displaystyle\int_{E} p(z) \, d\iota(\gamma_H)(z) \cdot (\| \gamma_H \|_1^*)^{-1} \cdot \| \Gamma_2 \|_1^*,$$

where $\| \sigma \|_1^* = \sup_{\|h\| \leqq 1} \int_H | \langle x, h \rangle | \, d\sigma(x)$ for a cylinder set measure σ on H. We have $\| \Gamma_2 \|_1^* = \| \gamma_H \|_1^* \cdot \sup_{\|h\| \leqq 1} (\sum_{i=1}^{n} | \langle x_i, h \rangle |^2)^{1/2}$, so it holds

$$(\textstyle\sum_{i=1}^{n} \| \phi \circ \iota(x_i) \|_{L^{1/2}}^{2})^{1/2} \leqq C \displaystyle\int_{E} p(z) \, d\iota(\gamma_H)(z) \cdot \sup_{\|h\|_H \leqq 1} (\textstyle\sum_{i=1}^{n} | \langle x_i, h \rangle |^2)^{1/2}$$

for every $x_1, x_2, \cdots, x_n \in H$. Remark that $\int_E p(z) \, d\iota(\gamma_H)(z) < +\infty$.

By Pietsch's factorization theorem (Pietsch [4], Pietsch's theorem is valid for a p-summing operator $u : X \to Y$, where X is a Banach space and Y is a quasi-normed space such as $Y = L^{1/2}(\Omega)$) $\phi \circ \iota$ can be decomposed as

$$\begin{array}{ccccc} H & \xrightarrow{\ \iota\ } & E & \xrightarrow{\ \phi\ } & L^{1/2}(\Omega_\phi) \\ & {}_{U}\searrow & & \nearrow_{V} & \\ & & H_1 & & \end{array}$$

where H_1 is a Hilbert space, U is a Hilbert-Schmidt operator and V is continuous. Let ν_0 be the cylinder set measure on $(L^{1/2}(\Omega_\phi))^a$ (the algebraic dual) which corresponds to the identity random linear functional $id : L^{1/2}(\Omega_\phi) \to L^{1/2}(\Omega_\phi)$. The image $V'(\nu_0)$ can be regarded as a continuous cylinder set measure on H_1', hence $U'(V'(\nu_0))$ is a Radon measure on H' and we can see $U'(V'(\nu_0)) = \iota'(\nu)$.

This completes the proof.

REFERENCES

[1] Dudley, R. M. (1970). Random linear functionals. *Trans. Amer. Math. Soc.* **152** 273–298.

[2] Maurey, B. (1972a). Espace de cotype p, $0 < p \leqq 2$. *Séminaire Maurey-Schwartz* 1972–1973 No. 7. Ecole Polytechniques.

[3] Maurey, B. (1972b). Théoremes de Nikishin: Théoremes de factorisation pour les applications linéaires à valeurs dans un espace $L^0(\Omega, \mu)$. *Séminaire Maurey-Schwartz* 1972–1973 No. 10, 11 *et* 12. Ecole Polytechnique.

[4] Pietsch, A. (1967). Absolut P-summierende Abbildungen in normierten Räumen. *Studia. Math.* **28** 333–353.

[5] Sato, H. and Okazaki, Y. (1975). Separabilities of a Gaussian Radon measure. *Ann. Inst. H. Poincaré* **11** 287–298.

[6] Xia, Dao-Xing. (1972). *Measure and Integration Theory on Infinite-Dimensional Spaces.* Academic, New York.

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