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MULTIPARAMETER GROUPS OF MEASURE-PRESERVING TRANSFORMATIONS: A SIMPLE PROOF OF WIENER'S ERGODIC THEOREM

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A simple proof of Wiener's multiparameter ergodic theorem is given in this paper. At the same time it is shown that two of the hypotheses of an existing proof can be dispensed with.

1. Introduction. Norbert Wiener's ergodic theorem (1939) has found its foremost applications in the foundations of Random Geometry. R. E. Miles (1961, 1964) and R. Cowan (1978) have made extensive use of the theorem in that field. In those papers, however, reference is made to the original source where the proof, as it often happens with pioneering work, is rather involved. The purpose of this note is twofold: in the first place we wish to offer a very simple proof of Wiener's theorem along the lines of Calderón (1968) and Fava (1973) which lends itself to fruitful generalizations; secondly, we show that two of the hypotheses used in the last paper can be dispensed with.

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. By an n-parameter group of measure-preserving transformations we mean a system of mappings $(\theta_t,\ t \in R^n)$ of X into itself having the following properties:

(i) $\theta_t(\theta_s x) = \theta_{t+s}x$; $\theta_0 x = x$ for every t and s in R^n and every x in X.

(ii) For every measurable subset E of X, $\theta_t(E)$ is measurable and its measure equals the measure of E, for any t in R^n.

(iii) For any function f measurable on X, the function $f(\theta_t x)$ is measurable on the product space $R^n \times X$, where the euclidean space R^n is endowed with Lebesgue measure. In what follows we give sufficient conditions for the almost everywhere convergence of the averages

$$A_\alpha f(x) = |U_\alpha|^{-1} \int_{U_\alpha} f(\theta_t x)\, dt \qquad \text{as } \alpha \to \infty, \tag{1}$$

where f is any function in $L^1(\mu)$, $(U_\alpha,\ \alpha > 0)$ is an increasing family of open sets in R^n containing the origin and depending on the positive real parameter α, and vertical bars denote Lebesgue measure. The most important example of such a family (U_α) is obtained by taking U_α to be the ball of radius α with center at the origin. More precisely, we prove that the averages (1) converge almost everywhere to a finite limit $f^*(x)$ provided that the following conditions are satisfied:

(A) The Hardy-Littlewood maximal operator associated with the family (U_α), namely

$$mg(t) = \sup_{\alpha>0} |U_\alpha|^{-1} \int_{U_\alpha} |g(t+s)|\, ds$$

is of weak type (1, 1), which means the existence of a constant C independent of g and of $\lambda > 0$, such that

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$$|\{t \in R^n : mg(t) > \lambda\}| \le (C/\lambda) \cdot \|g\|_{L^1(R^n)}.$$

(B) For each t in R^n

$$\lim_{\alpha\to\infty} (|(t + U_\alpha)\Delta\, U_\alpha| / |U_\alpha|) = 0,$$

where Δ denotes the symmetric difference.

The fact that (A) is satisfied when U_α is the ball of radius α, is the content of the now classical Hardy-Littlewood maximal theorem and its proof can be found, for example, in the textbook by Wheeden and Zygmund (1977). For more general conditions which guarantee the validity of (A) see Rivière (1971).

For a general ergodic theorem concerning discrete groups of measure-preserving transformations, we refer to Nguyen and Zessin (1979) and to Nguyen (1979).

2. Statements and proofs. We start with the following theorem.

THEOREM 1. (Maximal ergodic theorem). *Let $(U_\alpha, \alpha > 0)$ be an increasing family of open sets in R^n containing the origin, depending on the positive real parameter α and subject to Condition* (A). *If we denote by M the maximal ergodic operator*

$$Mf(x) = \sup_{\alpha>0} |U_\alpha|^{-1} \int_{U_\alpha} |f(\theta_t x)|\, dt$$

then there is a positive number C such that, for each positive λ,

$$\mu(\{x : Mf(x) > \lambda\}) \le (C/\lambda) \cdot \|f\|_{L^1(\mu)}.$$

PROOF. For any function $g(t)$ integrable over R^n, and for each positive integer k, we write

$$m_k g(t) = \sup_{\delta(U_\alpha)<k} |U_\alpha|^{-1} \int_{U_\alpha} |g(t+s)|\, ds$$

if $|t| \le k$, $m_k g(t) = 0$ otherwise, where $\delta(U_\alpha)$ denotes the diameter of U_α, while as before

$$mg(t) = \sup_{\alpha>0} |U_\alpha|^{-1} \int_{U_\alpha} |g(t+s)|\, ds,$$

so that $m_k g(t) \le mg(t)$ and $\lim_{k\to\infty} m_k g(t) = mg(t)$.

From (A) we derive the inequalities

$$|\{t : m_k g(t) > \lambda\}| \le |\{t : mg(t) > \lambda\}| \le (C/\lambda) \int_{R^n} |g(t)|\, dt.$$

Let us define the function $F(t, x) = f(\theta_t x)$ if $|t| \le 2k$, $F(t, x) = 0$ otherwise. It follows from Fubini's theorem that $F(t, x)$ is an integrable function of t for almost all x. For a given $\lambda > 0$ consider the set E of all pairs (t, x) such that $m_k F(t; x) > \lambda$ and its sections $E_t = \{x : (t, x) \in E\}$; $E^x = \{t : (t, x) \in E\}$. We observe that for $|t| \le k$, $m_k F(t, x) = m_k F(0, \theta_t x)$, and therefore $E_t = \theta_t^{-1}(E_0)$ for $|t| \le k$, while $E_t = \varnothing$ if $|t| > k$. If we denote by ρ the product of Lebesgue measure with the measure μ, then

$$\rho(E) = \int_{R^n} \mu(E_t)\, dt = \int_{|t|\le k} \mu(E_t)\, dt = \omega_n k^n \mu(E_0),$$

where ω_n is the measure of the unit ball in R^n.

On the other hand

$$\rho(E) = \int_X |E^x| \, d\mu \le \int_X d\mu (C/\lambda) \int_{|t|\le 2k} |f(\theta_t x)| \, dt$$

$$= (C/\lambda) \int_{|t|\le 2k} dt \int_X |f(\theta_t x)| \, d\mu = (C/\lambda)\omega_n (2k)^n \|f\|_{L^1(\mu)}.$$

Therefore

$$\mu(E_0) \le (C/\lambda) 2^n \|f\|_{L^1(\mu)}$$

and Theorem 1 follows from the last inequality by letting $k \to \infty$. The operator M also satisfies an inequality, whose verification is immediate:

$$\|Mf\|_\infty \le \|f\|_\infty,$$

which allows us to prove the following result.

COROLLARY 1. (Wiener's inequality). *There exists a positive constant C, such that for each $\lambda > 0$ and f in $L^p(\mu)$, $1 \le p < \infty$;*

$$\mu(\{Mf > \lambda\}) \le (C/\lambda) \int_{|f|>\lambda/2} |f| \, d\mu,$$

with C independent of f and λ.

PROOF. Let f be an element of $L^p(\mu)$. For each $\lambda > 0$ we write

$$f^\lambda = f \cdot \chi_{\{|f|>\lambda\}}; \; f_\lambda = f \cdot \chi_{\{|f|\le\lambda\}},$$

where χ_E stands for the characteristic function of the set E. Then

$$Mf \le Mf^\lambda + Mf_\lambda \le Mf^\lambda + \lambda;$$

hence,

$$\{Mf > 2\,\lambda\} \subset \{Mf^\lambda > \lambda\}.$$

Noting that $f^\lambda \in L^1(\mu)$, from Theorem 1 we conclude

$$\mu(\{Mf > 2\lambda\}) \le \mu(\{Mf^\lambda > \lambda\}) \le (C/\lambda) \int_X |f^\lambda| \, d\mu = (C/\lambda) \int_{|f|>\lambda} |f| \, d\mu,$$

and Corollary 1 follows by replacing λ by $\lambda/2$ in the last inequalities.

COROLLARY 2. *For each $p > 1$, there exists a finite constant C_p, such that*

$$\|Mf\|_{L^p(\mu)} \le C_p \|f\|_{L^p(\mu)} \qquad 1 < p < \infty$$

with C_p depending only on p and on the dimension n.

PROOF. Assuming that f is in $L^p(\mu)$, by virtue of Corollary 1 and Fubini's theorem

$$\int_X |Mf|^p \, d\mu = p \int_0^\infty \mu(\{Mf > \lambda\}) \lambda^{p-1} \, d\lambda$$

$$\le p \int_0^\infty d\lambda \lambda^{p-1} (C/\lambda) \int_{|f|>\lambda/2} |f| \, d\mu = p \cdot C \int_0^\infty d\lambda \lambda^{p-2} \int_X |f| \, \chi_{\{|f|>\lambda/2\}} \, d\mu$$

$$= p\cdot C\int_0^\infty d\mu\, |f| \int_0^{2|f|} \lambda^{p-2}\, d\lambda = (C\cdot p\cdot 2^{p-1}/p-1)\int_X |f|^p\, d\mu. \qquad \square$$

The following theorem represents a generalization of Wiener's multiparameter ergodic theorem.

THEOREM 2. *If the family of regions U_α satisfies conditions* (A) *and* (B), *then for any f in $L^1(\mu)$ the averages*

$$A_\alpha f(x) = |U_\alpha|^{-1}\int_{U_\alpha} f(\theta_t x)\, dt$$

converge almost everywhere in X as $\alpha \to \infty$.

PROOF. Let us consider the set of all functions h which can be represented in the form

$$h(x) = f(x) - f(\theta_s x),$$

where f is a bounded function having support of finite measure and s is any point in R^n. For any function h of this form, we have

$$A_\alpha h(x) = |U_\alpha|^{-1}\int_{U_\alpha} h(\theta_t x)\, dt = |U_\alpha|^{-1}\int_{U_\alpha} (f(\theta_t x) - f(\theta_{t-s}x))\, dt$$

$$= |U_\alpha|^{-1}\left(\int_{U_\alpha} f(\theta_t x)\, dt - \int_{s+U_\alpha} f(\theta_t x)\, dt\right).$$

Therefore

$$|A_\alpha h(x)| \le |U_\alpha|^{-1}\int_{U_\alpha \Delta (s+U_\alpha)} |f(\theta_t x)|\, dt.$$

Since $f(\theta_t x)$ is a bounded function of t for almost all x, and the family (U_α) satisfies (B), we see at once that $A_\alpha h(x)$ tends to zero for almost all x as $\alpha \to \infty$.

We will say that a function $1(x)$ in $L^2(\mu)$ is invariant if for every t, $1(\theta_t x) = 1(x)$ for almost all x. If 1 is an invariant function, for almost all x, we have $1(\theta_t x) = 1(x)$ for almost all t. Therefore

$$|U_\alpha|^{-1}\int_{U_\alpha} 1(\theta_t x)\, dt = 1(x)$$

for all α, almost everywhere in X.

We conclude that the averages $A_\alpha f(x)$ converge almost everywhere if f is in the linear span S of the functions h and 1. Our second step in the proof is to show that S is dense in $L^2(\mu)$. For this purpose, let us assume that a certain function $\bar{g}$ in $L^2(\mu)$ is orthogonal to all functions of S. Therefore

$$0 = \int_X h(x)\bar{g}(x)\, d\mu = \int_X (f(x) - f(\theta_s x))\bar{g}(x)\, d\mu$$

$$= \int_X f(x)(\bar{g}(x) - \bar{g}(\theta_{-s}x))\, d\mu,$$

for any bounded function f with support of finite measure and for any s in R^n. This implies that the function $\bar{g}$ is invariant. Since $\bar{g}$ is orthogonal to all invariant functions, we deduce that $\bar{g} = 0$ a.e., which proves that the linear span S is dense in $L^2(\mu)$.

If for any g in $L^p(\mu)$, $1 \leq p < \infty$, we write

$$\bar{M}g(x) = \limsup_{\alpha\to\infty} A_\alpha g(x) - \liminf_{\alpha\to\infty} A_\alpha g(x),$$

then it easily follows that

(i) $\bar{M}(g_1 + g_2) \leq \bar{M}g_1 + \bar{M}g_2$,

(ii) $\bar{M}g \leq 2Mg$,

(iii) $\bar{M}g = 0$ a.e. for g in S.

Let us now choose any g in $L^2(\mu)$. By virtue of the preceding, there exists a sequence (g_n) of functions in S, such that $\| g - g_n \|_{L^2(\mu)} \to 0$ as $n \to \infty$. Since

$$\bar{M}g \leq \bar{M}(g - g_n) + \bar{M}g_n = \bar{M}(g - g_n) \qquad \text{a.e.},$$

it follows that

$$\| \bar{M}g \|_{L^2(\mu)} \leq \| \bar{M}(g - g_n) \|_{L^2(\mu)} \leq 2 \| M(g - g_n) \|_{L^2(\mu)} \leq 2C_2 \| g - g_n \|_{L^2(\mu)},$$

so that letting $n \to \infty$, we get $\bar{M}g = 0$ a.e., for any function g in $L^2(\mu)$.

To end the proof of Theorem 2, given f in $L^1(\mu)$, we select a sequence (f_n) in $L^2(\mu) \cap L^1(\mu)$, such that $\| f_n - f \|_{L^1(\mu)} \to 0$ as $n \to \infty$. Since

$$\bar{M}f \leq \bar{M}(f - f_n) \leq 2M(f - f_n),$$

for every $\lambda > 0$, we have

$$\mu(\{\bar{M}f > \lambda\}) \leq \mu(\{M(f - f_n) > \lambda/2\}) \leq (2C/\lambda) \| f - f_n \|_{L^1(\mu)}$$

and Theorem 2 follows from the last inequality by letting $n \to \infty$.

If in the last theorem we take for U_α the ball of radius α with center at the origin, we obtain Wiener's theorem as a particular case.

3. Invariance properties of the limit function f*. A measurable subset E of X will be called invariant if its indicator function is invariant. The invariant subsets of X form a σ-field that we shall denote by $\mathscr{I}$. In addition to the preceding, we have the following useful

THEOREM 3. *The limit function f^* satisfies the following properties*:

(i) $\| f^* \|_{L^1(\mu)} \leq \| f \|_{L^1(\mu)}$.

(ii) *f^* is invariant.*

(iii) *If $\mu(X) < \infty$, then $A_\alpha f$ converge to f^* in $L^1(\mu)$ and for every $E \in \mathscr{I}$ we have*

$$\int_E f^* \, d\mu = \int_E f \, d\mu.$$

PROOF. We only prove (ii) since the proof of the other assertions is standard in ergodic theory. Let us show in the first place that f^* is invariant whenever f is an element of $L^2(\mu)$. For this purpose, we select a sequence (f_n) of elements of S, converging to f in the norm of $L^2(\mu)$. Since each f_n is invariant, for every t in R^n, we have

$$\begin{aligned} \| f^*(x) - f^*(\theta_t x) \|_{L^2(\mu)} &\leq \| f^*(x) - f_n^*(x) \|_{L^2(\mu)} \\ &\quad + \| f_n^*(\theta_t x) - f^*(\theta_t x) \|_{L^2(\mu)} \leq 2 \| M(f - f_n) \|_{L^2(\mu)} \\ &\leq 2C_2 \| f - f_n \|_{L^2(\mu)}, \end{aligned}$$

and the invariance of f^* becomes evident by letting $n \to \infty$. Now, (ii) follows from (i) and the fact that $L^2(\mu) \cap L^1(\mu)$ is dense in $L^1(\mu)$.

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