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ASYMPTOTIC PROPERTIES OF SEMIGROUPS OF MEASURES ON VECTOR SPACES

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Let (E, B) be a measurable vector space and q be a measurable seminorm on E. Suppose that $(\mu_t)_{t>0}$ is a q-continuous convolution semigroup of probability measures on (E, B). It is proved that there exists a right-continuous nonincreasing function θ such that

$$\lim_{t\to 0+} (1/t)\cdot\mu_t\{x: q(x) > s\} = \theta(s)$$

for every $s > 0$ at which θ is continuous. If μ_t, $t > 0$, are Gaussian, then $\theta \equiv 0$; if there exists a measurable linear functional f such that $f(\cdot)$ is not Gaussian (with respect to μ_1) and $q \geqslant |f|$ then $\theta \not\equiv 0$.

1. Introduction. The main result of this paper is contained in Section 4. We prove there that if q is a measurable seminorm on a measurable vector space (E, B) and $(\mu_t)_{t>0}$ is a q-continuous convolution semigroup (for definitions see Section 2) of probability measures on (E, B) then there exists a right-continuous nonincreasing function θ defined on $(0, \infty)$ such that

$$\lim_{t\to 0+} (1/t)\mu_t\{x: q(x) > s\} = \theta(s)$$

whenever $s > 0$ is a continuity point of θ. Moreover, if the μ_t are Gaussian then $\theta \equiv 0$; if $q \geqslant |f|$ for a measurable linear functional f such that $f(\cdot)$ is not Gaussian (with respect to μ_1) then $\theta \not\equiv 0$. From this result we obtain the following theorem for continuous semigroups of probability measures on separable normed spaces: if $(\mu_t)_{t>0}$ is such a semigroup then

$$\lim_{t\to 0+} (1/t)\mu_t\{x: \|x\| > s\} = \theta(s)$$

for every s at which θ is continuous; μ_1 is Gaussian if and only if $\theta \equiv 0$. Even if E is finite-dimensional the existence of this limit seems to have been unknown for non-Euclidean norms.

Results of this type were established in de Acosta [1] for stable measures and homogeneous seminorms. However, his methods are based on rather elementary inequalities (stated in our paper as Lemma 2.3) and cannot be adapted to more general situations.

Our approach, although in the spirit of de Acosta, is different and is based on two powerful tools: Lévy's and Hoffmann-Jørgensen's inequalities. It is also applicable to situations when q is not homogeneous as well as to more general semigroups of probability measures. Thus, results of this type can also be applied to more general vector spaces, e.g., to the space L_0 of all measurable functions on $[0, 1]$, with convergence in measure. This aspect also seems to be of interest because L_0 is a natural space of sample paths for measurable stochastic processes.

2. Preliminaries. In this section we introduce some terminology and notations and state some inequalities which are basic for the rest of the paper.

Throughout the paper, unless stated otherwise, we will deal with a measurable vector

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space (E, B); that is, E a real vector space and B a σ-field of subsets of E such that:

(i) The mapping $(x, y) \to x + y$ from $(E \times E, B \otimes B)$ into (E, B) is measurable.

(ii) The mapping $(\lambda, x) \to \lambda x$ from $(R \times E, \mathscr{R} \otimes B)$ into (E, B) is measurable, where $(R, \mathscr{R})$ is the real line with the Borel σ-field.

A function $q: E \to R^+$, $q(0) = 0$, will be called a seminorm if it is subadditive, that is $q(x + y) \leqslant q(x) + q(y)$ for every $x, y \in E$, and nondecreasing, that is $q(\alpha x) \leqslant q(\beta x)$ if $|\alpha| \leqslant |\beta|$, for every $\alpha, \beta \in R$ and every $x \in E$.

The following lemma is a version of the classical Lévy's inequality; it can be proved by a slight modification of the proof given in [9], Chapter 2.

LEMMA 2.1. *Let $X_1, \ldots, X_n$ be E-valued symmetric and independent random variables and let q be a measurable seminorm. Then for every $\epsilon > 0$ we have*

$$P\{\max_{1 \leqslant j \leqslant n} q(\textstyle\sum_{i=1}^{j} X_i) > \epsilon\} \leqslant 2P\{q((1/2) \textstyle\sum_{i=1}^{n} X_i) > \epsilon/2\}.$$

The next lemma is a version of Hoffmann-Jørgensen's inequality [8].

LEMMA 2.2. *Let $X_1, \ldots, X_n$ and q be as in Lemma 2.1. Then for every $s, t > 0$ the following holds:*

$$P\{q(\textstyle\sum_{i=1}^{n} X_i) > 2t + s\} \leqslant P\{\max_{1 \leqslant i \leqslant n} q(X_i) > s\} + (2P\{q((1/2) \textstyle\sum_{i=1}^{n} X_i) > t/2\})^2.$$

We will also use the following inequalities exploited by de Acosta in [1], [2].

LEMMA 2.3. *Let X, Y be independent E-valued random variables and q be a measurable seminorm. Then for every $s > 0$ and every $0 < \epsilon < 1$ we have:*

(a) $$P\{q(X + Y) > s\} \geqslant P\{q(X) > (1 + \epsilon)s\} \cdot P\{q(Y) \leqslant \epsilon \cdot s\} + P\{q(Y) > (1 + \epsilon)s\} \cdot P\{q(X) \leqslant \epsilon \cdot s\};$$

(b) $$P\{q(X + Y) > s\} \leqslant P\{q(X) > (1 - \epsilon)s\} + P\{q(Y) > (1 - \epsilon)s\} + P\{q(X) > \epsilon \cdot s\} \cdot P\{q(Y) > \epsilon \cdot s\}.$$

Now, let $(\mu_t)_{t>0}$ be a family of probability measures on $(E, B) \cdot (\mu_t)_{t>0}$ will be called a (convolution) semigroup if

$$\mu_t * \mu_s = \mu_{t+s}$$

for every $t, s > 0$. Let q be a measurable seminorm on (E, B). A semigroup $(\mu_t)_{t>0}$ will be called q-continuous if for every $s > 0$

$$\lim_{t \to 0+} \mu_t\{x: q(x) > s\} = 0.$$

The next lemma is an immediate consequence of Lemma 2.3.

LEMMA 2.4. *Let q be a measurable seminorm and $(\mu_t)_{t>0}$ be a q-continuous semigroup of probability measures. Then for every $\mu > 0$*

$$\lim_{t \to 0+} \mu_{t+u}\{x: q(x) > s\} = \mu_u\{x: q(x) > s\}$$

for every s at which $\mu_u\{x: q(x) > s\}$ is continuous. In other words $\mu_{t+u} \circ q^{-1}$ converges weakly to $\mu_u \circ q^{-1}$, as $t \to 0+$.

Next, let us recall that a probability measure is said to be stable of index p, $0 < p \leqslant 2$, if for every independent E-valued random variables X, Y with distributions μ and every $\alpha, \beta > 0$, $\alpha X + \beta Y$ has the same distribution as $\gamma X + z$, where $\gamma(\alpha, \beta) = (\alpha^p + \beta^p)^{1/p}$ and z is an element of E. μ is called strictly stable if z can be taken 0 for every $\alpha, \beta > 0$. It is obvious that if μ is stable of index p then the symmetrization of $\mu(=\mu*\bar{\mu})$ is strictly stable (and symmetric) of the same index. It is also evident that if μ is strictly stable of index p and μ_t is the distribution of $t^{1/p}X$ (μ = distribution of X) then $(\mu_t)_{t>0}$ is a semigroup of probability measures such that $\mu_1 = \mu$. Moreover, it can be easily seen that this semigroup is q-continuous if and only if the mapping $\alpha \to q(\alpha x)$ is continuous at 0 for μ_1-almost all $x \in E$. On the other hand, if (E, B) is a topological vector space with Borel σ-field B and q is continuous then every continuous semigroup $(\mu_t)_{t>0}$ (i.e., such that $\mu_t \Rightarrow \delta_0$ as $t \to 0+$) is q-continuous. For general information about stable measures we refer to the paper of Dudley and Kanter [5].

3. The upper bound. In this section we establish that

$$\limsup_{t\to 0+} (1/t)\mu_t\{x\colon q(x) > s\} < \infty,$$

for every q-continuous semigroup $(\mu_t)_{t>0}$.

LEMMA 3.1. *Let q be a measurable seminorm and let $(\mu_t)_{t>0}$ be a q-continuous semigroup of probability measures. For every $s > 0$ we have*

$$\limsup_{t\to 0+} (1/t)\mu_t\{x\colon q(x) > s\} < \infty.$$

PROOF. Assume that μ_t, $t > 0$ are symmetric. Let s be a fixed positive number. Suppose additionally that $\mu_1\{x\colon q(x) > s/4\} < 1/2$. Without loss of generality we can assume that $s/4$ is a continuity point of $F_1(\cdot) = \mu_1\{x\colon q(x) > \cdot\}$. Let $t_n \to 0+$. Put $k_n = [1/t_n] + 1$ and let $X_1^{(n)}, \ldots, X_{k_n}^{(n)}$ be symmetric, independent and identically distributed random variables such that $\sum_{i=1}^{k_n} X_i^{(n)}$ has the distribution $\mu_{k_n t_n}$. Using the standard inequality

$$\max_{1\le i\le n} q(X_i^{(n)}) \le 2 \max_{1\le i\le n} q\left(\sum_{j=1}^{i} X_j^{(n)}\right)$$

and Lemma 2.1 we obtain

$$P\{\max_{1\le i\le k_n} q(X_i^{(n)}) > s\} \le 2P\{q((1/2)\sum_{i=1}^{k_n} X_i^{(n)}) > s/4\} \le 2P\{q(\sum_{i=1}^{k_n} X_i^{(n)}) > s/4\}.$$

We have thus obtained

$$(\mu_{t_n}\{x\colon q(x) \le s\})^{k_n} \ge 1 - 2\mu_{k_n t_n}\{x\colon q(x) > s/4\}.$$

The above inequality can be rewritten in the form

$$k_n \cdot \mu_{t_n}\{x\colon q(x) > s\} \le \frac{1-\alpha_n^{1/k_n}}{1/k_n},$$

where $\alpha_n = 1 - 2\mu_{k_n t_n}\{x\colon q(x) > s/4\}$. Since $k_n t_n \to 1+$ the application of Lemma 2.4 yields the desired conclusion.

Next, observe that for every positive s there exists $t_0 > 0$ such that $\mu_{t_0}\{x\colon q(x) > s/4\} < 1/2$ (by the q-continuity of $(\mu_t)_{t>0}$). Write $\nu_t = \mu_{t_0}$. Then $(\nu_t)_{t>0}$ is a symmetric q-continuous semigroup such that

$$\nu_1\{x\colon q(x) > s/4\} = \mu_{t_0}\{x\colon q(x) > s/4\} < 1/2.$$

Applying to the semigroup $(\nu_t)_{t>0}$ the result just obtained, we get the desired conclusion. The rest of the proof follows from the standard method of symmetrization and is left to the reader.

Now we will show a strengthened form of the above result for Gaussian measures. Let us recall that μ is said to be Gaussian (in the sense of Fernique [7]) if it is stable of index 2 and if for any independent random variables X_1, X_2 with the distribution μ,

$$X_1 + X_2 \quad \text{and} \quad X_1 - X_2$$

are independent. If (E, B) is a vector space such that B is generated by a vector space $\mathscr{F}$ of linear forms on E then this definition is equivalent to the classical one: μ is Gaussian if and only if $f(\cdot)$ is a (real) Gaussian random variable, for every $f \in \mathscr{F}$.

THEOREM 3.1. *Let μ be a Gaussian measure. Assume that q is a measurable seminorm such that $\alpha \to q(\alpha x)$ is continuous at 0 μ-a.e. Then for every $s > 0$ and every positive integer k the following holds*:

$$\lim_{t \to 0+} (1/t^k)\mu\{x: q(t^{1/2}x) > s\} = 0.$$

PROOF. We use Fernique's method [7]. It suffices to prove our theorem only for μ symmetric and strictly stable; the general case then follows easily from symmetrization.

Denote $R_s(t) = \mu\{x: q(t^{1/2}x) > s\}$. Let X, Y be independent random variables with distribution μ. Then we have

$$\begin{aligned} &R_s(t)\cdot(1 - R_{s/2}(t)) \\ &\quad = P\{q(t^{1/2}X) > s\}\cdot P\{q(t^{1/2}Y) \leq s/2\} \\ &\quad = P\{q(t^{1/2}X) > s, q(t^{1/2}Y) \leq s/2\} \\ &\quad = P\{q((t/2)^{1/2}(X + Y)) > s, q((t/2)^{1/2}(X - Y)) \leq s/2\} \\ &\quad \leq P\{|q((t/2)^{1/2}(X + Y)) - q((t/2)^{1/2}(X - Y))| > s/2\} \\ &\quad \leq P\{q((2t)^{1/2}X) > s/2, q((2t)^{1/2}Y) > s/2\} \\ &\quad = (P\{q((2t)^{1/2}X) > s/2\})^2 = (R_{s/2}(2t))^2. \end{aligned}$$

Hence,

$$R_s(t)(1 - R_{s/2}(t)) \leq (R_{s/2}(2t))^2$$

which is equivalent to

$$R_s(t) \leq (R_{s/2}(2t))^2 + R_s(t)\cdot R_{s/2}(t).$$

In view of the remark given in the final part of Section 2 we obtain

$$\limsup_{t \to 0+} (1/t^2)\mu\{x: q(t^{1/2}x) > s\} < \infty$$

as an application of Lemma 3.1. Consequently, by induction

$$\lim_{t \to 0+} (1/t^k)\mu\{x: q(t^{1/2}x) > s\} < \infty$$

for every positive integer k. The proof is complete.

Let μ be a Gaussian measure on a separable metric vector space. From Theorem 3.1 it follows that for every open neighborhood U of 0 we have

$$\lim_{t \to 0+} (1/t)\mu(t^{-1/2}U^c) = 0.$$

This can be applied (via the invariance principle [3]) in proofs of various functional limit theorems (see [4]).

4. The existence of limits. In this section we state and prove the main result of our paper.

THEOREM 4.1. *Let q be a measurable seminorm and $(\mu_t)_{t>0}$ a q-continuous semigroup of probability measures. Then there exists a nonincreasing function $\theta(s)$ defined on $(0, \infty)$ such that for every continuity point of θ we have*

$$\lim_{t \to 0+} (1/t)\mu_t\{x: q(x) > s\} = \theta(s).$$

If μ_t are Gaussian then $\theta \equiv 0$; if there exists a measurable linear functional f such that

$f(\cdot)$ *is not Gaussian* (*with respect to* μ_1) *and that for some* $\alpha \in (0, 1]$

$$q \geqslant |f|^{\alpha}$$

then $\theta \not\equiv 0$.

PROOF. The proof consists of several steps.

Step 1. Let s be a fixed positive number. We prove that either for every positive integer k

$$\lim_{t\to 0+} (1/t^k)\mu_t\{x: q(x) > u\} = 0, \qquad s < u,$$

or

$$\liminf_{t\to 0+} (1/t)\mu_t\{x: q(x) > s\} > 0.$$

As usual we prove this fact only for $(\mu_t)_{t>0}$ symmetric.

(a) Suppose that there exists $t > 0$ and $r > 0$ such that

$$\mu_r\{x: q(x) > 2t + s\} - 4\mu_r^2\{x: q(x/2) > t/2\} \geqslant c > 0.$$

Let $X_1^{(n)}, \ldots, X_n^{(n)}$ be independent, symmetric and identically distributed E-valued random variables such that $\sum_{i=1}^n X_i^{(n)}$ has the distribution μ_1. Using Lemma 2.2 we obtain as in the proof of Lemma 3.1

$$\begin{aligned} 1 - (\mu_{r/n}\{x: q(x) > s\})^n &= P\{\max_{1\leqslant j\leqslant n} q(X_j^{(n)}) > s\} \\ &\geqslant P\{q(\textstyle\sum_{j=1}^n X_j^{(n)}) > 2t + s\} \\ &\quad - 4P^2\{q((1/2)\textstyle\sum_{j=1}^n X_j^{(n)}) > t/2\} \\ &= \mu_r\{x: q(x) > 2t + s\} \\ &\quad - 4\mu_r^2\{x: q(x/2) > t/2\}. \end{aligned}$$

We thus have

$$(n/r)\mu_{r/n}\{x: q(x) > s\} \geqslant \frac{1}{r} \cdot \frac{1 - (1-c)^{1/n}}{1/n}$$

for every n. This inequality gives

$$\liminf_n (n/r)\mu_{r/n}\{x: q(x) > s\} \geqslant \frac{1}{r} \cdot \ln \frac{1}{1-c} > 0.$$

(b) If (a) does not hold then for every $t > 0$ and every $r > 0$ we have

$$\mu_r\{x: q(x) > 2t + s\} \leqslant 4\mu_r^2\{x: q(x/2) > t/2\} \leqslant 4\mu_r^2\{x: q(x) > t/2\}.$$

Since $\limsup_{r\to 0+} (1/r)\mu_r\{x: q(x) > u\} < \infty$ for every $u > 0$ by Lemma 3.1, we obtain

$$\limsup_{r\to 0+} (1/r^2)\mu_r\{x: q(x) > u\} < \infty$$

for every $u > s$. The result now follow by induction.

Step 2. Now, we use the idea of de Acosta [2]. Consider the following family of measures defined on $(0, \infty)$

$$\lambda_t(B) = (1/t)\mu_t\{x: q(x) \in B\}, \qquad t \in (0, 1].$$

Observe that for fixed $\alpha > 0$ the family $(\lambda_t)_{t\in(0,1]}$ restricted to $I_\alpha = (\alpha, \infty)$ is weakly conditionally compact. For, by Lemma 3.1 $\lambda_t(\alpha, \infty) = (1/t)\mu_t\{x: q(x) > \alpha\} \leqslant c_\alpha < \infty$ whence $\{\lambda_t \mid I_\alpha\}_{t\in(0,1]}$ is bounded in norm. Moreover, if s_0 is sufficiently large then $\mu_1\{x: q(x) > s_0/4\} < 1/2$. Hence, for $s > s_0$ we have

$$\limsup_{t\to 0+} (1/t)\mu_t\{x: q(x) > s\} \leqslant -\ln(1 - 2\mu_1\{x: q(x) > s/4\}).$$

Therefore for every $\epsilon > 0$ the right-hand side of the above inequality is less than ϵ for sufficiently large s, so that $\{\lambda_t \mid I_\alpha\}_{t\in(0,1]}$ is uniformly tight.

By Prohorov's theorem $\{\lambda_t \mid I_\alpha\}_{t\in(0,1]}$ is weakly conditionally compact. Hence, for every sequence $t_n \to 0+$ we can choose a subsequence t_{n_k} such that $\lambda_{t_{n_k}} \mid I_\alpha$ converges weakly, for every $\alpha > 0$.

Step 3. Let $s_n \to 0+$. Put $k_n = [1/s_n]$. Fix $t > 0$. We prove that if $\lambda_{ts_n} \mid I_\alpha$ converges weakly as $n \to \infty$, for every $\alpha > 0$, then for every $s > 0$, a continuity point of the limit distribution, we have

$$\lim_n |\lambda_{t/k_n}(s, \infty) - \lambda_{ts_n}(s, \infty)| = 0.$$

Indeed, by virtue of part (a) of Lemma 2.3 we have

$$\begin{aligned}\lambda_{t/k_n}(s, \infty) &= (k_n/t)\mu_{t/k_n}\{x: q(x) > s\}\\ &= (k_n/t)\mu_{ts_n} * \mu_{t(1/k_n - s_n)}\{x: q(x) > s\}\\ &\geq k_n s_n \lambda_{ts_n}((1+\epsilon)s, \infty)\cdot\mu_{t(1/k_n-s_n)}\{x: q(x) \leq \epsilon s\}\\ &\quad + k_n(1/k_n - s_n)\cdot\lambda_{t(1/k_n-s_n)}((1+\epsilon)s, \infty)\cdot\mu_{ts_n}\{x: q(x) \leq \epsilon s\}\\ &\geq k_n s_n\cdot\lambda_{ts_n}((1+\epsilon)s, \infty)\cdot\mu_{t(1/k_n-s_n)}\{x: q(x) \leq \epsilon s\}\end{aligned}$$

and the last term of this inequality converges to

$$\theta((1+\epsilon)s) = \lim_n \lambda_{ts_n}((1+\epsilon)s, \infty),$$

whenever $(1+\epsilon)s$ is a continuity point of θ. So, we have

$$\liminf_n \lambda_{t/k_n}(s, \infty) \geq \theta(s).$$

Analogously, using part (b) of Lemma 2.3 we obtain

$$\begin{aligned}\lambda_{t/k_n}(s, \infty) &\leq k_n s_n\cdot\lambda_{ts_n}((1-\epsilon)s, \infty)\\ &\quad + k_n(1/k_n - s_n)\lambda_{t(1/k_n-s_n)}((1-\epsilon)s, \infty)\\ &\quad + k_n s_n \lambda_{ts_n}(\epsilon s, \infty)\cdot\mu_{t(1/k_n-s_n)}\{x: q(x) > \epsilon\cdot s\}.\end{aligned}$$

By similar arguments we infer from this inequality that

$$\limsup_n \lambda_{t/k_n}(s, \infty) \leq \theta(s),$$

which, together with the previous inequality completes the proof of this step.

Step 4. Assume that μ_t, $t > 0$, are symmetric. We show that there exists a right-continuous, nonincreasing function θ defined on $(0, \infty)$ such that

$$\lim_{t\to 0+} (1/t)\mu_t\{x: q(x) > s\} = \theta(s)$$

whenever θ is continuous at s $(s > 0)$. Let $t_k \to 0+$ be such that $\lambda_{t_k} \mid I_\alpha$ converges weakly for every $\alpha > 0$. Let θ be the right-continuous function defined by

$$\theta(s) = \lim_k \lambda_{t_k}(s, \infty),$$

where s is a continuity point of the limit distribution. Now, by virtue of Step 1, it is obvious that without loss of generality we can assume $\theta(s) > 0$.

Assume, further, that s is a continuity point of θ. If $t > 0$ is sufficiently small then $\theta(2t + s) > 0$. Therefore for sufficiently large k we have

$$\mu_{t_k}\{x: q(x) > 2t + s\} - 4\mu_{t_k}^2\{x: q(x) > t/2\} = c(k)$$

by virtue of Step 1 and Lemma 3.1. Fix such a t_k. Let s_n be another sequence, $s_n \to 0+$. Write $k_n = [1/s_n]$. Let $X_1^{(n)}, \ldots, X_{k_n}^{(n)}$ be, as before, independent, symmetric and identically distributed E-valued random variables such that $\sum_{i=1}^{k_n} X_i^{(n)}$ has the distribution μ_{t_k}.

Next, analogously as in Step 1 we obtain that

$$1 - (1 - \mu_{t_k/k_n}\{x:q(x) > s\})^{k_n} \geqslant c(k).$$

Assume that $\lambda_{t_k s_n} \mid I_\alpha$ converges weakly as $n \to \infty$, for every $\alpha > 0$, and that s is a continuity point of the limit distribution. Define

$$\beta(n) = \mu_{t_k s_n}\{x:q(x) \leqslant s\}, \qquad c = \limsup_n \beta(n)^{k_n}.$$

Observe that

$$c = \limsup_n (1 - \mu_{t_k s_n}\{x:q(x) > s\})^{k_n} > 0$$

since $\liminf_{t \to 0+} (1/t)\mu_t\{x:q(x) > s\} > 0$, by assumption. Furthermore, using Step 3 with $t = t_k$ we obtain

$$c = \limsup_n (1 - s_n t_k \lambda_{t_k s_n}(s, \infty))^{k_n} = e^{-t_k\theta'(s)}$$
$$= \lim_n \left(1 - \frac{t_k \lambda_{t_k/k_n}(s, \infty)}{k_n}\right)^{k_n} \leqslant 1 - c(k)$$

where $\theta'(s)$ is determined by the limit distribution of $\lambda_{t_k s_n}$. Hence, we have

$$\begin{aligned} \liminf_n k_n \mu_{t_k s_n}\{x:q(x) > s\} &= \liminf_n k_n(1 - \beta(n)) \\ &= \liminf_n k_n(1 - \beta(n)^{k_n})^{1/k_n} \\ &= \ln(1/c) \geqslant \ln \frac{1}{1 - c(k)}. \end{aligned}$$

We have thus obtained

$$\liminf_n (1/s_n)\mu_{t_k s_n}\{x:q(x) > s\} \geqslant \ln \frac{1}{1 - c(k)}.$$

From this inequality it follows that for every s, a point of continuity of θ,

$$\lim_{t \to 0+} \lambda_t(s, \infty) = \theta(s).$$

For, if t_k' and θ' are such that θ' is determined by the limit distribution of $\lambda_{t_k'}$ then taking $s_n^{(k)} = t_n'/t_k$ we obtain

$$(1/s_n^{(k)})\mu_{t_k s_n^{(k)}} = (1/s_n^{(k)})\mu_{t_n'} = t_k(1/t_n')\mu_{t_n'} = t_k\lambda_{t_n'}.$$

Hence, if s is a continuity point of θ' and θ then the previous inequality implies

$$\theta'(s) \geqslant \ln(1 - c(k))^{-1/t_k}.$$

Since

$$(1/t_k)c(k) = \lambda_{t_k}(2t + s, \infty) - 4t_k \cdot \lambda_{t_k}^2(t/2, \infty) \to \theta(2t + s)$$

as $k \to \infty$ (for $2t + s$ being a continuity point of θ) we obtain

$$\ln(1 - c(k))^{-1/t_k} = \ln\left(1 - \frac{1/t_k c(k)}{1/t_k}\right)^{-1/t_k} \to \theta(2t + s)$$

as $k \to \infty$. So, we have finally obtained

$$\theta'(s) \geqslant \theta(s),$$

which, by virtue of the same properties of θ and θ', completes the proof of this step.

Step 5. The rest of the proof is standard and follows from symmetrization. It will be sketched only for the sake of completeness.

Observe that from part (a) of Lemma 2.3 we have

$$(1/t)\mu_t * \bar{\mu}_t\{x : q(x) > (1+\epsilon)^{-1}s\} \geq (2/t)\mu_t\{x : q(x) > s\}\mu_t\left\{x : q(x) \leq \frac{\epsilon}{1+\epsilon}s\right\},$$

which implies that if θ is determined by the limit distribution of $(1/t)\mu_t * \bar{\mu}_t$ then

$$\limsup_{t\to 0+} (1/t)\mu_t\{x : q(x) > s\} \leq (1/2)\theta(s),$$

whenever θ is continuous at s. On the other hand, by part (b) of Lemma 2.3 we obtain

$$(1/t)\mu_t * \bar{\mu}_t\{x : q(x) > (1-\epsilon)^{-1} \cdot s\} \leq (2/t)\mu_t\{x : q(x) > s\} + (1/t)\mu_t^2\left\{x : q(x) > \frac{\epsilon}{1-\epsilon}s\right\}$$

and, analogously,

$$\liminf_{t\to 0+} (1/t)\mu_t\{x : q(x) > s\} \geq (1/2)\theta(s),$$

if θ is continuous at s. This ends the proof of this step.

Step 6. By virtue of Step 1, we have two possibilities: either for every $k \geq 1$ and every $s > 0$

$$\lim_{t\to 0+} (1/t^k)\mu_t\{x : q(x) > s\} = 0$$

or there exists an $s > 0$ such that

$$0 < \liminf_{t\to 0+} (1/t)\mu_t\{x : q(x) > s\} \leq \limsup_{t\to 0+} (1/t)\mu_t\{x : q(x) > s\} < \infty.$$

If there exists a measurable linear functional f such that $f(\cdot)$ is not Gaussian and

$$q(\cdot) \geq |f(\cdot)|^\alpha$$

for some $\alpha \in (0, 1]$, then $\nu_t = \mu_t \circ f^{-1}$ is a continuous semigroup of probability measures on the real line and ν_1 is not Gaussian. By [6], Problem 17, page 597, we infer that there exists a positive number s such that

$$\liminf_{t\to 0+} (1/t)\nu_t\{y : |y| > s\} > 0.$$

Thus, in this case

$$\liminf_{t\to 0+} (1/t)\mu_t\{x : q(x) > s\} > 0.$$

If μ_t, $t > 0$ are Gaussian then their symmetrizations, $\mu_t * \bar{\mu}_t$ are also Gaussian. Since $\mu_1 * \bar{\mu}_1$ is symmetric and Gaussian, there exists the unique symmetric Gaussian root of $\mu_1 * \bar{\mu}_1$ of order 2^n. Since $\gamma_{1/2^n}(A) = \mu_1 * \bar{\mu}_1\{x : (1/2^n)x \in A\}$ is such a 2^nth root, we infer that $\gamma_{1/2^n} = \mu_{1/2^n} * \bar{\mu}_{1/2^n}$. From Theorem 3.1 it follows that

$$2^n\mu_{1/2^n} * \bar{\mu}_{1/2^n}\{x : q(x) > s\} \to 0.$$

Hence, by Step 1 we obtain that

$$\lim_{t\to 0+} (1/t)\mu_t * \bar{\mu}_t\{x : q(x) > s\} = 0.$$

By symmetrization we obtain

$$\lim_{t\to 0+} (1/t^k)\mu_t\{x : q(x) > s\} = 0.$$

The proof of the theorem is complete.

We derive the following corollaries.

COROLLARY 4.1. *Let (E, B) be a separable normed vector space with the Borel σ-field B. Then, given a continuous semigroup $(\mu_t)_{t>0}$ of probability measures there is a nonincreasing right-continuous function θ defined on $(0, \infty)$ such that*

$$(1/t)\mu_t\{x : \|x\| > s\} \to \theta(s)$$

for every $s > 0$ at which θ is continuous. μ_1 is Gaussian if and only if $\theta \equiv 0$.

COROLLARY 4.2. *Let (E, B) be a measurable vector space and q be a measurable seminorm. Let μ be a stable measure of index p. Assume that $\alpha \to q(\alpha x)$ is continuous at 0. Then*

$$\lim_{t\to 0+} (1/t^p)\mu\{x:q(x/t) > s\} = \theta(s)$$

for every $s > 0$ at which θ is continuous, where θ is as in the previous corollary.

If there exists a measurable linear functional f such that $q \geqslant |f|$ and $\mu\circ f^{-1}$ is nondegenerate then $\theta \not\equiv 0$ whenever $p < 2$. If q is r-homogeneous (i.e., $q(\alpha x) = |\alpha|^r q(x)$) *then θ is continuous.*

PROOF. This is an immediate consequence of the application of our theorem to $\mu * \bar{\mu}$ and the same arguments as in the Step 5 of the proof of our theorem.

Now, we pay our attention to some nonhomogeneous seminorms.

Let Φ be a continuous nondecreasing function defined for $u \geqslant 0$ and such that $\Phi(u) = 0$ if and only if $u = 0$. Assume additionally that Φ is subadditive. Let R^∞ be the space of all sequences of reals. Let m be a nonnegative measure on integers. Put

$$\|x\|_\Phi = \sum_{i=1}^\infty \Phi(|x(i)|)m(i).$$

Let l_Φ be the set of all $x \in R^\infty$ such that $\|x\|_\Phi < \infty$. l_Φ is a linear space under usual addition and scalar multiplication, and $\|\cdot\|_\Phi$ is (in general nonhomogeneous) a seminorm on l_Φ. $(l_\Phi, \|\cdot\|_\Phi)$ is called Orlicz space (see [10]).

COROLLARY 4.3. *Let $(\mu_t)_{t>0}$ be a continuous semigroup of Borel probability measures on $(l_\Phi, \|\cdot\|_\Phi)$. Then there exists a nondecreasing right-continuous function defined on* $(0, \infty)$ *such that*

$$\lim_{t\to 0+} (1/t)\mu_t\{x:\|x\|_\Phi > s\} = \theta(s)$$

for every s at which θ is continuous. $\theta \equiv 0$ if and only if μ_1 is Gaussian.

PROOF. Recall that $(\mu_t)_{t>0}$ is said to be continuous if and only if μ_t converges weakly to 0, as $t \to 0+$. Now, it remains to prove that if $\theta \equiv 0$ then μ_1 is Gaussian. Let $x = (x_i) \in l_\Phi$. Then $\Phi(|x_i|) > s/a_i$ implies $\|x\|_\Phi > s$, where $a_i = m(i)$. Hence, if we denote $f_i(x) = x_i$ then there is an $u = u(s)$ such that

$$\{x:\|x\|_\Phi > s\} \supset \{x:\Phi(|f_i(x)|) > s/a_i\} = \{x:|f_i(x)| > u(s)\}$$

Moreover, $u(s) \to 0$ as $s \to 0$. Thus, $\theta \equiv 0$ implies that $\mu_1 \circ f_i^{-1}$ is Gaussian, for every i. Since f_i's generate the Borel σ-field in l_Φ, μ_1 is Gaussian.

Finally, we state one more application of our theorem.

EXAMPLE. Let $(p_n)_{n=1}^\infty$ be a sequence of real numbers, $0 < p_n \leqslant 1$. Given $(x_n) = x \in R^\infty$ put

$$\|x\|_{(p_n)} = \sum_{n=1}^\infty |x_n|^{p_n}.$$

Define $l_{(p_n)} = \{x \in R^\infty:\|x\|_{(p_n)} < \infty\}$. Then $l_{(p_n)}$ with the seminorm $\|\cdot\|_{(p_n)}$ is a complete metric linear space. Since

$$\|x\|_{(p_n)} = \sum_{n=1}^\infty |x_n|^{p_n} \geqslant |x_n|^{p_n}$$

we obtain that if $(\mu_t)_{t>0}$ is a continuous semigroup of measures, then $\theta \equiv 0$ if and only if μ_1 is Gaussian.

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