

MARKOV CHAINS IN RANDOM ENVIRONMENTS: THE CASE OF MARKOVIAN ENVIRONMENTS

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A formulation of a Markov chain in a random environment is given, generalizing special cases such as branching processes, queues, birth and death chains and random walks in random environments. It is assumed that the environmental process is Markovian, each environment corresponding to a particular law of evolution on a countable state space \mathcal{X} . It is then shown that there is a natural three way classification of states of \mathcal{X} . One of the three types of states is irregular in nature, and conditions are found under which no such states exist.

1. Introduction. Consider a family of transition probabilities $\{P(\theta), \theta \in \Theta\}$ on a state space $(\mathcal{X}, \mathcal{A})$ and a stochastic sequence X_0, X_1, X_2, \dots taking values in \mathcal{X} and satisfying

$$P(X_{n+1} \in A | X_0, \dots, X_n) = P(\theta_n : X_n, A) \text{ a.s.}$$

for each n and $A \in \mathcal{A}$. If the sequence $\theta_0, \theta_1, \theta_2, \dots$ is fixed this is simply the formulation of a nontime homogeneous Markov chain. Suppose, however, that the θ_n 's are the realization of a stochastic sequence $\xi_0, \xi_1, \xi_2, \dots$ taking values in Θ and that, conditioning on the full ξ_n sequence,

$$(1.1) \quad P(X_{n+1} \in A | \xi_0, \xi_1, \xi_2, \dots; X_0, \dots, X_n) = P(\xi_n : X_n, A) \text{ a.s.}$$

for each n and $A \in \mathcal{A}$. Then we say that X_0, X_1, X_2, \dots is a *Markov chain in a random environment*. The ξ_n 's are called the *environmental process* (or the control process in some contexts).

In this study we assume $\{\xi_n\}$ is a time-homogeneous Markov chain on Θ and take \mathcal{X} to be countable with discrete σ -field. The probability of going from x to y in one step in the θ th environment is denoted $P(\theta : x, y)$, while the n -step transition probability for $\{\xi_n\}$ on (Θ, \mathcal{T}) is $K^{(n)}(\theta, \Gamma)$. In this case $\{\xi_n, X_n\}$ is a Markov chain with one-step transition probability on $\Theta \times \mathcal{X}$ determined by

$$(1.2) \quad P(\theta, x; \Gamma \times \{y\}) = K(\theta, \Gamma)P(\theta : x, y)$$

and we call $\{\xi_n, X_n\}$ the *bichain*. It is important to note that (1.1) together with the Markovian assumption on $\{\xi_n\}$ implies (1.2), and, conversely, a bichain satisfying (1.2) must satisfy (1.1). Also note that (1.2) determines a proper subclass of two dimensional Markov chains whose first component is Markovian. In particular, the subsequence $\{(\xi_{nd}, X_{nd}), n = 0, 1, 2, \dots\}$ fails to satisfy (1.2) in general, hence does not determine a Markov chain in a random environment.

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In Section 2 we take Θ countable, so the bichain moves on countable state space $\Theta \times \mathcal{X}$. Even though \mathcal{X} is countable, however, the environments on \mathcal{X} may vary continuously so in Section 3 we allow Θ to be a general state space.

The purpose of this study is to consider some aspects of the problem of classification of states for a Markov chain in a random environment. Since

$$[X_n = x] = [(\xi_n, X_n) \in \Theta \times \{x\}],$$

states of \mathcal{X} correspond to cylinder sets for the bichain. A natural classification of state sets is given by Doeblin for a Markov chain $\{Y_n\}$ on a general state space $(\mathcal{Y}, \mathfrak{B})$: $B \in \mathfrak{B}$ is *inessential* if $P_y[Y_n \in B \text{ i.o.}] = 0$ for every y (where P_y is the distribution for the chain started at y and i.o. stands for “infinitely often”). Otherwise B is *essential*. If an essential B is contained in a countable union of inessential sets then B is *improperly essential*, and otherwise B is *properly essential*. Based on this trichotomy we define $x \in \mathcal{X}$ to be *inessential*, *improperly essential* or *properly essential* according as $\Theta \times \{x\}$ has this property for the bichain.

It is clear that inessential states for a Markov chain in a random environment correspond closely to the notion of transient states for a Markov chain. Also, if $\Theta \times \mathcal{X}$ is indecomposable (that is, does not contain two disjoint stochastically closed sets) and if x is properly essential, then it is known (cf. Section 1.8 of Orey (1971)) that there is a stochastically closed set $C \subset \Theta \times \mathcal{X}$ such that $\Theta \times \mathcal{X} - C$ is not properly essential (possibly empty) and for any starting distribution ψ on C ,

$$P_\psi[X_n = x \text{ i.o.}] = P_\psi[(\xi_n, X_n) \in \Theta \times \{x\} \text{ i.o.}] = 1,$$

so properly essential states for Markov chains in random environments are similar to recurrent states for a Markov chain.

Improperly essential states, on the other hand, present various anomalies. It would be nice if we knew \mathcal{X} had no such states. In this case we say that the bichain is *proper*, otherwise *improper*. Most of what follows consists of an attempt to provide usable conditions under which the bichain is proper. Of course, a sufficient condition is that $\Theta \times \mathcal{X}$ contain no improperly essential subset, in other words that $\Theta \times \mathcal{X}$ is a “final set” in the sense of Doeblin. It is known that this is equivalent to the assumption that the chain on $\Theta \times \mathcal{X}$ is φ -recurrent (that is, “recurrent in the sense of Harris”). The requirement that the bichain be proper, however, is considerably weaker and allows for “transient” behavior of the \mathcal{X} component.

To see how knowledge that the bichain is proper may be useful, consider the case that $\mathcal{X} = \{0, 1, 2, \dots\}$ and where $P(\theta : 0, 0) = 1$ for every θ , while for the bichain $P_{(\theta, x)}[\text{some } X_n = 0] > 0$ for every (θ, x) . It follows easily that no positive x can be properly essential. Provided the bichain is proper, it then follows that positive x are inessential and consequently that

$$P[X_n \rightarrow 0 \text{ or } \infty] = 1$$

whatever the starting distribution for the bichain. This kind of stability has been of primary importance in several of the special theories for types of Markov chains in

random environments, e.g., see Tanny (1977) for a strong form of this result for branching processes in random environments.

2. Countable Θ . *In this section the environmental process is taken to be an irreducible, recurrent Markov chain on a countable state space Θ . Then the bichain also moves on a countable state space $\Theta \times \mathcal{X}$. For a given $x \in \mathcal{X}$, if (θ, x) is recurrent for any $\theta \in \Theta$, then x must be properly essential, while, if (θ, x) is transient for every $\theta \in \Theta$, then x may be either inessential or improperly essential. We will consider several conditions under which the evolution is proper, eliminating the possibility of improperly essential states.*

Since the environmental chain is recurrent, every $\theta \in \Theta$ will be visited infinitely often with probability one. Suppose the chain is started at θ , let τ_n denote the time of the n th return to θ and $\tau_0 = 0$. Then the sequence $X_{\tau_0}, X_{\tau_1}, X_{\tau_2}, \dots$ is a time homogeneous Markov chain on \mathcal{X} and we denote its one step transition probability by $R_\theta(x, y)$. In effect we obtain this chain by looking at the bichain during its visits to $\{\theta\} \times \mathcal{X}$. We will find these chains useful in several ways.

PROPOSITION 2.1. (a) *A state $x \in \mathcal{X}$ is recurrent for the R_θ chain if, and only if, (θ, x) is recurrent for the bichain.*

(b) *Let the environmental chain be positive recurrent. Then a state $x \in \mathcal{X}$ is positive recurrent for the R_θ chain if, and only if, (θ, x) is positive recurrent for the bichain.*

PROOF. The first assertion follows directly from definitions. For the second, notice that, if (θ, x) is positive recurrent, τ_n denotes the time of n th return of the environmental chain to θ and τ_σ is the first τ_n such that $X_{\tau_n} = x$ (for $n \geq 1$), then the mean return time to x for the R_θ chain equals

$$E_{(\theta, x)}(\sigma) \leq E_{(\theta, x)}(\tau_\sigma) < \infty$$

since τ_σ is precisely the return time of the bichain to (θ, x) . Thus positive recurrence of (θ, x) implies x is positive recurrent for R_θ . For the converse, note that $\delta_n = \tau_n - \tau_{n-1}$, $n = 1, 2, 3, \dots$, are independent and identically distributed. Since σ is a stopping time, it follows by Wald's equality that

$$E_{(\theta, x)}(\tau_\sigma) = E_{(\theta, x)}(\sum_{n=1}^{\sigma} \delta_n) = E_\theta(\tau_1) \cdot m_\theta(x)$$

where $m_\theta(x)$ denotes the mean return time to x for the R_θ chain. \square

PROPOSITION 2.2. *If \mathcal{X} is finite then the evolution is proper and \mathcal{X} has at least one properly essential state.*

PROOF. For each θ , the R_θ chain must have at least one recurrent state x since \mathcal{X} is finite. Then (θ, x) is recurrent and x is properly essential. Moreover, the set of recurrent states for the R_θ chain is closed and there are only a finite number of transient states, so this chain must eventually leave the transient states and not return. At this point the bichain will be in its set of recurrent states, which is also closed. Since the \mathcal{X} component of every such state must be properly essential, it

follows that any $x \in \mathfrak{X}$ that is not properly essential is visited only finitely often with probability one, hence is inessential. \square

PROPOSITION 2.3. *If Θ is finite then the evolution is proper.*

This is immediate since every $\Theta \times \{x\}$ is a finite set in this case and so cannot be improperly essential.

PROPOSITION 2.4. *If $\inf_{\theta} K(\theta, \theta^*) > 0$ for some θ^* , then the evolution is proper.*

PROOF. Let $K(\theta, \theta^*) \geq \varepsilon > 0$ for all θ . Then, for each $x, y \in \mathfrak{X}$, and $\theta \in \Theta$,

$$\begin{aligned} P_{(\theta,y)}[(\xi_n, X_n) = (\theta^*, x)] &= E_{(\theta,y)}(K(\xi_{n-1}, \theta^*)P(\xi_{n-1}: X_{n-1}, x)) \\ &\geq \varepsilon P_{(\theta,y)}[X_n = x]. \end{aligned}$$

Now, if (θ^*, x) is transient, then

$$\sum_{\theta} P_{(\theta,y)}[X_n = x] \leq \frac{1}{\varepsilon} \sum_{\theta} P_{(\theta,y)}[(\xi_n, X_n) = (\theta^*, x)] < \infty$$

and it follows by Cantelli's lemma that x is inessential. Of course, if (θ^*, x) is recurrent then x is properly essential, so only these two possibilities exist. \square

An important special case of Proposition 2.4 is when the ξ_n 's are independent. In this case the X_n sequence is Markovian and the correspondence of "transient" to "inessential" and "recurrent" to "properly essential" holds.

This last result naturally leads to the conjecture that it would be sufficient for $K^{(n)}(\theta, \theta^*) \geq \varepsilon > 0$ for all θ and some n and θ^* . This is not the case, as shown by the following example:

Let $\Theta = \mathfrak{X} = \{0, 1, 2, \dots\}$. For $n = 0, 1, 2, \dots$ let

$$K(0, 2n) = q_{2n} > 0$$

where $\sum_{n=0}^{\infty} q_{2n} = 1$ and, for $n \geq 1$, let

$$K(2n, 2n - 1) = K(2n - 1, 0) = 1.$$

Also let

$$P(2n: 0) = 1 \text{ if } 0 \leq x < 2n$$

$$P(2n: x, x) = 1 \text{ if } x \geq 2n$$

$$P(2n - 1: 0, 2n) = 1 \text{ and } P(2n - 1: x, x) = 1 \text{ if } x \neq 0.$$

In this case $K^{(2)}(\theta, 0) \geq q_0^2 > 0$ for all θ .

If the bichain is started at state $(2n - 1, 0)$, it moves to state $(0, 2n)$ at time 1 and then can return to $\Theta \times \{0\}$ only at states $(\theta, 0)$ with θ odd and $\theta > 2n$. Thus the states $(2n - 1, 0)$ are all transient. Starting at $(2n, 0)$, the bichain moves to $(2n - 1, 0)$ (if $n \geq 1$) and so it cannot return to $(2n, 0)$. Thus $(2n, 0)$ is transient for $n \geq 1$. Finally, starting at $(0, 0)$, the bichain remains there only so long as the Θ chain remains at 0, and then the bichain leaves $(0, 0)$ never to return. Thus all states of

$\Theta \times \{0\}$ are transient. On the other hand, $P_{\theta,x}[\text{some } X_n = 0] = P_\theta[\text{some } \xi_n > x] = 1$ for every (θ, x) , hence $P_{\theta,x}[X_n = 0 \text{ i.o.}] = 1$ for every (θ, x) , and 0 is improperly essential.

THEOREM 2.1. *Suppose that (a) for each $x \in \mathfrak{X}$ there exists a finite set N_x such that*

$$\inf_{\theta} P(\theta : x, N_x) > 0,$$

and (b) there exists a $\theta^ \in \Theta$ and integer m such that*

$$\inf_{\theta} \max_{n < m} K^{(n)}(\theta, \theta^*) > 0.$$

Then the evolution is proper.

NOTES. The hypothesis (b) of the theorem is equivalent to assuming that the Θ chain is uniformly φ -recurrent, and we will use this condition in the general state space version of this theorem in Section 3. In the example preceding the theorem, hypothesis (b) is satisfied with $m = 2$ and hypothesis (a) holds for every $x \in \mathfrak{X}$ except $x = 0$. This one exception suffices to void the conclusion of the theorem. This theorem generalizes a result of Torrez (1979) for birth and death chains in random environments (where (a) is satisfied with $N_x = \{x - 1, x, x + 1\}$). Even with this strong form of (a), Torrez gives an example where the environmental chain is positive recurrent but not uniformly φ -recurrent ((b) fails) and the evolution is *not* proper.

PROOF. Choose the N_x so $x \in N_x$. Let $N_x^{(1)} = N_x$ and, for $n = 2, 3, \dots$,

$$N_x^{(n)} = \cup_{y \in N_{x_{n-1}}} N_y.$$

Then the $N_x^{(n)}$ are finite and increase in n for fixed x . We will show by induction that

$$\inf_{\theta_0, \dots, \theta_{n-1}} P_{(\theta_0, x)}(X_n \in N_x^{(n)} | \theta_0, \dots, \theta_{n-1}) > 0.$$

For $n = 1$ this is condition (a) of the theorem. If true for n , then for $n + 1$

$$\begin{aligned} & \inf_{\theta_0, \dots, \theta_n} P_{(\theta_0, x)}(X_{n+1} \in N_x^{(n+1)} | \theta_0, \dots, \theta_n) \\ & \geq \inf_{\theta_0, \dots, \theta_n} \left\{ \sum_{y \in N_x^{(n)}} P_{(\theta_0, x)}(X_n = y | \theta_0, \dots, \theta_{n-1}) P(\theta_n : y, N_y) \right\} \\ & \geq \min_{y \in N_x^{(n)}} \left\{ \inf_{\theta} P(\theta : y, N_y) \right\} \inf_{\theta_0, \dots, \theta_{n-1}} P_{(\theta_0, x)}(X_n \in N_x^{(n)} | \theta_0, \dots, \theta_{n-1}) \\ & > 0. \end{aligned}$$

Moreover,

$$\begin{aligned} & \min_{n < m} \inf_{\theta_0, \dots, \theta_{n-1}} P_{(\theta_0, x)}(X_n \in N_x^{(m)} | \theta_0, \dots, \theta_{n-1}) \\ & \geq \min_{n < m} \inf_{\theta_0, \dots, \theta_{n-1}} P_{(\theta_0, x)}(X_n \in N_x^{(n)} | \theta_0, \dots, \theta_{n-1}) > 0. \end{aligned}$$

Combining this inequality with hypothesis (b) and letting τ be the first positive time that $\xi_n = \theta^*$ we have, for each $x \in \mathfrak{X}$,

$$\inf_{\theta} P_{(\theta,x)}[\xi_n = \theta^* \text{ and } X_n \in N_x^{(m)} \text{ for some } n \leq m] \tag{2.1}$$

$$= \inf_{\theta} \sum_{n=1}^m \left\{ P_{(\theta,x)}[\tau = n] \cdot E_{(\theta,x)} \left(P_{(\theta_0,x)}(X_n \in N_x^{(m)} | \xi_0, \dots, \xi_{n-1}) \right) \right\} > 0.$$

It would appear that we should have $P_{(\theta_0,x)}(X_n \in N_x^{(m)} | \xi_0, \dots, \xi_n)$ in the above relation but the n th environment, indexed by ξ_n , does not affect the distribution of X_n .

Now let

$$A_x = \{y \in N_x^{(m)} : L(\theta^*, y; \Theta \times \{x\}) > 0\}$$

(the L and Q functions are those of Doeblin—cf. Chung (1964) for definitions) and let $B_x = N_x^{(m)} - A_x$. Assume that x is not properly essential. Then, for each $y \in A_x$, (θ^*, y) leads to (θ, x) for some θ , and, since each (θ, x) is transient, (θ^*, y) is transient, too.

The relation (2.1), together with Proposition 7 of Chung (1964) implies that, for every $\theta \in \Theta$ and $z \in \mathcal{X}$,

$$Q(\theta, z; \Theta \times \{x\}, \{\theta^*\} \times N_x^{(m)}) = Q(\theta, z; \Theta \times \{x\}). \tag{2.2}$$

But if the bichain is in $\{\theta^*\} \times N_x^{(m)}$ infinitely often it must enter $\{\theta^*\} \times B_x$, since $\{\theta^*\} \times A_x$ is a finite set of transient states. Once in $\{\theta^*\} \times B_x$, the bichain cannot return to $\Theta \times \{x\}$. Thus the left-hand side of (2.2), hence the right-hand side, must equal 0. Consequently x is inessential. \square

3. General Θ . We assume throughout this section that the environmental sequence is a φ -recurrent (recurrent in the sense of Harris) Markov chain on a general state space Θ . We will establish two results corresponding to Proposition 2.2 and Theorem 2.1 of the preceding section.

PROPOSITION 3.1. *If \mathcal{X} is finite then there is at least one properly essential state in \mathcal{X} , and the evolution is proper.*

PROOF. 1. Let C be any closed set in $\Theta \times \mathcal{X}$. Let

$$\Phi = \{\theta \in \Theta : (\theta, x) \in C \text{ for some } x \in \mathcal{X}\}.$$

Then Φ is a closed set in Θ . We will show that, for some $x \in \mathcal{X}$, $(\Phi \times \{x\}) \cap C$ must be properly essential.

Suppose the converse. Then for each $x \in \mathcal{X}$ there exists a countable partition $\{\Phi_n^{(x)}\}$ of Φ such that $(\Phi_n^{(x)} \times \{x\}) \cap C$ is inessential for each n . Let $\{\Phi_n\}$ be a countable partition refining each of the $\{\Phi_n^{(x)}\}$, $x \in \mathcal{X}$. Then $(\Phi_n \times \{x\}) \cap C$ is inessential for each x , hence $(\Phi_n \times \mathcal{X}) \cap C$ is inessential. Starting the bichain at $(\theta, x) \in (\Phi_n \times \mathcal{X}) \cap C$, it follows that

$$K^{(n)}(\theta, \Phi_n) = P^{(n)}(\theta, x; (\Phi_n \times \mathcal{X}) \cap C) \rightarrow 0$$

as $n \rightarrow \infty$. But then each Φ_n is not properly essential, consequently $\Phi - U\Phi_n$ is not

properly essential. Since Φ is closed and the control chain is φ -recurrent this is not possible, and our claim follows *ab contrario*.

2. Taking $C = \Theta \times \mathcal{X}$, it follows from part 1 of the proof that \mathcal{X} has a nonempty subset \mathcal{X}^+ of properly essential states. Since $\mathcal{X} - \mathcal{X}^+$ is finite (possibly empty), $\Theta \times (\mathcal{X} - \mathcal{X}^+)$ is not properly essential. For each $x \in \mathcal{X} - \mathcal{X}^+$ and $\varepsilon > 0$, let

$$D_{x,\varepsilon} = \{(\theta, y) : L(\theta, y; \Theta \times (\mathcal{X} - \mathcal{X}^+)) > \varepsilon\}.$$

If, for some $\varepsilon > 0$, $D_{x,\varepsilon}$ were properly essential, then it would follow that $\Theta \times (\mathcal{X} - \mathcal{X}^+)$ would be properly essential (cf. Proposition 9 of Chung (1964)). Thus $D_{x,\varepsilon}$ is not properly essential for $\varepsilon > 0$ and $D_{x,0} = \cup_n D_{x,1/n}$ is not properly essential.

Now let

$$F = \Theta \times \mathcal{X}^+ - \cup_{x \in \mathcal{X}^+} D_{x,0}.$$

Then F is properly essential and

$$F = (\Theta \times (\mathcal{X} - \mathcal{X}^+))^c \cap (\Theta \times (\mathcal{X} - \mathcal{X}^+))^0$$

is closed (cf. Chung (1964)).

Also F^0 is either empty or closed. If it is closed, then for some x $(\Theta \times \{x\}) \cap F^0$ is properly essential by part 1. But then $x \in \mathcal{X}^+$, and, since $F \cap F^0 = \emptyset$,

$$(\Theta \times \{x\}) \cap F^0 \subset D_{x,0}$$

which is not properly essential. It follows that $F^0 = \emptyset$.

3. Let φ be a finite measure on Θ equivalent to the invariant measure π . Suppose \mathcal{X} contains N points, and for each $x \in \mathcal{X}$ choose $\Gamma_x \subset \Theta$, so that, for the bichain

$$\inf_{\theta \in \Gamma_x} L(\theta, x; F) = \varepsilon_x > 0,$$

and so $\varphi(\Gamma_x) > \varphi(\Theta)(1 - 1/2N)$. (Recall that $F^0 = \emptyset$, so this is possible.) Let $\Gamma = \cap \Gamma_x$, the intersection over all $x \in \mathcal{X}$. Then $\varphi(\Gamma) > \varphi(\Theta)/2 > 0$ so Γ is properly essential. Moreover,

$$\inf_{\theta \in \Gamma, x \in \mathcal{X}} L(\theta, x; F) \geq \min_{x \in \mathcal{X}} \varepsilon_x > 0.$$

Starting at any (θ, x) , the bichain enters $\Gamma \times \mathcal{X}$ infinitely often with probability one. But then it must enter the closed set F eventually and stay there with probability one. Since $F \subset \Theta \times \mathcal{X}^+$, it follows that $\Theta \times (\mathcal{X} - \mathcal{X}^+)$ is inessential. Thus every state of \mathcal{X} is either properly essential or inessential. \square

THEOREM 3.1. *Suppose that (a) for each $x \in \mathcal{X}$ there exists a finite set N_x such that*

$$\inf_{\theta} P_{\theta}(x, N_x) > 0,$$

and (b) the environmental chain is uniformly φ -recurrent. Then the evolution is proper.

PROOF. Let π be invariant probability measure on Θ . Choose m so $\inf_{\theta} P_{\theta}[\tau_{\Gamma} \leq m] > 0$ whenever $\pi(\Gamma) \geq \frac{1}{2}$ where τ_{Γ} is the first positive n such that $\xi_n \in \Gamma$. (This can be done. For example, cf. Lemma 1.1 of Cogburn (1975).) Choose a state x that is not properly essential (if any) and construct the set $N_x^{(m)}$ as in the proof of Theorem 2.1. Suppose $N_x^{(m)}$ has ν points. Now choose $\Phi_n \subset \Theta$ so $\cup \Phi_n = \Theta$, $\Phi_n \uparrow$ and $\Phi_n \times \{x\}$ is inessential for each n . For each $y \in \mathcal{X}$, let

$$\Psi_y = \{\theta : L(\theta, y; \Theta \times \{x\}) = 0\}.$$

If $\pi(\Theta - \Psi_y) > \frac{1}{2\nu}$, then choose $\Psi_y^* \subset \Theta - \Psi_y$ and n_y so that

$$(3.1) \quad \inf_{\theta \in \Psi_y^*} L(\theta, y; \Phi_{n_y} \times \{x\}) > 0$$

and so that $\pi(\Psi_y^*) \geq \pi(\Theta - \Psi_y) - \frac{1}{2\nu}$. This is possible by Ergoroff's lemma since

$$L(\theta, y; \Phi_n \times \{x\}) \uparrow L(\theta, y; \Theta \times \{x\}) > 0$$

as $n \uparrow \infty$ for every $\theta \in \Theta - \Psi_y$. If $\pi(\Theta - \Psi_y) \leq \frac{1}{2\nu}$, let $\Psi_y^* = \emptyset$. In either case, let $\Lambda_y = \Psi_y \cup \Psi_y^*$ and let

$$\Lambda = \cap_{y \in N_x^{(m)}} \Lambda_y.$$

Then $\pi(\Lambda) \geq \frac{1}{2}$, so, using the argument in the first part of Theorem 2.1,

$$(3.2) \quad \inf_{\theta} L(\theta, x; \Lambda \times N_x^{(m)}) > 0.$$

Now by (3.1) and since $\Phi_{n_y} \times \{x\}$ is inessential,

$$Q(\theta, y; \Psi_y^* \times \{y\}) = Q(\theta, y; \Psi_y^* \times \{y\}; \Phi_{n_y} \times \{x\}) = 0.$$

Since this holds for every θ , it follows that $\Psi_y^* \times \{y\}$ is inessential for each y . Thus

$$D = \cup_{y \in N_x^{(m)}} (\Psi_y^* \times \{y\})$$

is inessential. Note that

$$\Lambda \times N_x^{(m)} - D \subset \cup_{y \in N_x^{(m)}} (\Psi_y \times \{y\}) \subset (\Theta \times \{x\})^0.$$

Now with probability one sample sequences entering $\Lambda \times N_x^{(m)}$ infinitely often enter $\Lambda \times N_x^{(m)} - D$ and so do not enter $\Theta \times \{x\}$ infinitely often. But (3.2) implies that, for every θ, y ,

$$Q(\theta, y; \Theta \times \{x\}) = Q(\theta, y; \Theta \times \{x\}; \Lambda \times N_x^{(m)}).$$

Since the right-hand side is zero, we have established that any x that is not properly essential must be inessential. \square

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