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LONG RANGE EXCLUSION PROCESSES[1]

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Let S be a countable set and $p(x,y)$ be the transition probabilities for a discrete time Markov chain on S. Consider the motion of particles on S which obey the following rules: (a) there is always at most one particle at each site in S, (b) particles wait independent exponential times with mean one before moving, and (c) when a particle at x is to move, it moves to X_τ, where $\{X_n\}$ is the Markov chain starting at x with transition probabilities $p(x,y)$ and τ is the first time that $X_n = x$ or X_n is an unoccupied site. This process was introduced by Spitzer, and will be called a long range exclusion process because particles may travel long distances in short times. The process is well defined for finite configurations, and we will show how to use monotonicity arguments to define it for arbitrary configurations. It is shown that the configuration in which all sites are occupied may or may not be absorbing for the process. It always is if $p(x,y)$ is translation invariant on $S = Z^d$, but if $p(x,y)$ is a birth and death chain on $S = \{0, 1, 2, \cdots\}$, it is absorbing if and only if $p(x,y)$ is recurrent. For each positive function $\pi(x)$ on S such that $\pi P = \pi$, there is a product measure ν_π on $\{0, 1\}^S$ which is a natural candidate for an invariant measure for the process. When $p(x,y)$ is translation invariant on Z^d, it is probably the case that ν_π is in fact invariant if and only if π is constant. This will be verified under a mild regularity assumption, which is automatically satisfied if $d = 1$ or 2 or if the Laplace transform of $p(o,x)$ is finite in a neighborhood of the origin.

1. Introduction. In a paper [13] which has stimulated a large amount of research activity during the past decade, Spitzer proposed several models of infinite particle systems. We will begin by giving brief descriptions of three of these: the simple exclusion process, the zero range process, and the long range exclusion process. The first has been studied extensively (see the references in [11]), although important open problems remain. The second, on the other hand, has received only a moderate amount of attention ([5], [10], [15]) since Spitzer's paper appeared. These two will serve as motivation for the third model, which has not been studied so far, and which is the subject of the present work. There are inherent technical difficulties in the third process which lead to problems and behavior not encountered in the study of the first two processes. We will solve some of these problems, and will then discuss related open problems in the final section of this paper.

We will need some notation to describe the three processes. Let S be a finite or countable set, and let $p(\cdot, \cdot)$ be the transition probabilities for an irreducible discrete time Markov chain on S. For the description of the zero range process, we will need also a positive function $c(k)$ defined on the positive integers. Let $U = \{0, 1\}^S$ be the set of configurations of particles on S with at most one particle

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per site, and let $V = \{0, 1, 2, \cdots\}^S$ be the set of configurations on S with finitely many particles per site. If $\pi(\cdot)$ is a positive function on S, ν_π will be the product probability measure on U with marginals given by $\nu_\pi\{\eta : \eta(x) = 1\} = \pi(x)/(1 + \pi(x))$. By convention, ν_∞ will be the pointmass on the configuration $\eta \equiv 1$, in which all the sites are occupied. Let $\tilde{\nu}_\pi$ be the product probability measure on V with marginals given by

$$\tilde{\nu}_\pi\{\eta : \eta(x) = k\} = \gamma(x)\frac{[\pi(x)]^k}{\Pi_{j=1}^k[jc(j)]},$$

where $\gamma(x)$ is a normalizing constant which is chosen so that

$$\Sigma_{k=0}^\infty \tilde{\nu}_\pi\{\eta : \eta(x) = k\} = 1.$$

There is an implicit assumption, of course, that $\pi(\cdot)$ and $c(\cdot)$ are such that this normalizing constant exists for all x. Note that if $c(k) \equiv 1$, the marginals of $\tilde{\nu}_\pi$ are Poisson distributions, while if $kc(k) \equiv 1$, these marginals are geometric distributions.

The simple exclusion process is a continuous time Markov process on U in which particles move on S according to the following rules: a particle at x waits an exponential time with parameter one, and then chooses a site y according to the probabilities $p(x,y)$. If y is vacant at that time, the particle at x moves to y; otherwise, it stays at x. All exponential times and all choices according to $p(\cdot, \cdot)$ are mutually independent. This process gives a simple way in which an interaction can be superimposed on otherwise independent Markov chains in such a way that multiple occupancy of sites is forbidden. Under the mild assumption

$$\sup_y \Sigma_x p(x,y) < \infty, \tag{1.1}$$

a process corresponding to this description can be constructed. (This assumption will be made throughout much of this paper as well—see Section 3.) The simple exclusion process often has invariant measures which are product measures. In fact if $\pi P = \pi$ and either (a) π is constant or (b) $\pi(x)p(x,y) = \pi(y)p(y,x)$ for all $x,y \in S$, then ν_π is invariant for the process. One unfortunate property of the simple exclusion process is that $\pi P = \pi$ alone is not enough to guarantee that ν_π is invariant. For a survey of results on this process, see [11].

The zero range process is a continuous time Markov process on V in which particles move on S according to the following rules: A particle at x at time t chooses to move during the time interval $(t, t + \Delta t)$ with probability $c[\eta_t(x)]\Delta t + o(\Delta t)$. When it does choose to move, it moves to y with probability $p(x,y)$. Again, product measures are often invariant for the process, as was observed by Spitzer in [13]. In this case, however, $\pi P = \pi$ does suffice (under mild technical assumptions) to guarantee that $\tilde{\nu}_\pi$ is invariant. No extraneous symmetry or constancy conditions are required. When S is finite, this can be verified by a simple computation. In the infinite case, the verification was carried out in [10] when π is constant, and the argument there can be generalized under appropriate assumptions. Of course when $c(k) \equiv 1$, the invariance of $\tilde{\nu}_\pi$ when $\pi P = \pi$ is simply Doob's observation in [3] that certain Poisson processes are invariant for independent particle systems. For more on the independent case, see [12].

The long range exclusion process is a continuous time Markov process on U in which particles move on S according to the following rules: a particle at x waits an exponential time with parameter one. It then moves to X_τ, where $\{X_n\}$ is a Markov chain on S with transition probabilities $p(\cdot\,,\cdot)$, $X_0 = x$, $\tau = \min\{n \geq 1 : X_n = x$ or $\eta(X_n) = 0\}$, and η is the configuration of the process just before the particle at x tries to move. Thus a particle continues to search for a vacant site until it finds one, spending no time at the intervening occupied sites. This process is well defined for finite configurations, and at this point we will consider it to be defined only in the finite case. When we give a precise definition of the infinite particle system, it will be consistent with the interpretation that the particle disappears if $\tau = \infty$. For a justification of this interpretation, see Section 3.

The connection with the simple exclusion process is, of course, that the long range exclusion process provides a second natural way of superimposing an interaction upon independent particle motions in such a way that multiple occupancy of sites is excluded. One aspect of the long range exclusion process which makes it quite natural is that the sequence of sites visited by an individual particle is a random subsequence of $\{X_n\}$. This is not the case for the simple exclusion process. The relation between the long range exclusion process and the zero range process is that the former is obtained formally from the latter by taking the limit as $c(k) \to \infty$ for each $k \geq 2$, with $c(1) = 1$.

The construction of infinite particle systems has been carried out in several different ways using different techniques. The construction usually begins by writing down a formal generator, and then either using the Hille-Yosida theorem to construct the corresponding Feller semigroup [9], or solving an appropriate martingale problem [6]. In either case, the aim is to associate a uniquely defined Markov process to the given parameters—in this case the transition probabilities $p(\cdot\,,\cdot)$. In the case of the long range exclusion process, particles can move very long distances in short times when there are many particles present, and this leads to difficulties if one wishes to construct the process using the earlier techniques. In fact, the formal generator is very badly behaved (the generator applied to a function which depends on finitely many coordinates is in general unbounded), and even in nice cases, the process cannot be constructed in such a way that it will have the Feller property. For example, it will be shown in Section 4 that the process is never Feller when $p(\cdot\,,\cdot)$ corresponds to a random walk on Z^d. Thus we will use monotonicity arguments, rather than estimates, to construct the process.

This construction of the long range exclusion process, which will be denoted by η_t, will be carried out in Section 2. There it will be seen that η_t behaves rather well on a certain set $D \subset U$ of configurations, and that it immediately jumps into D

from any point not in D. This set D has the following properties:

(a) $\lim_{t\downarrow 0} P^{\eta}[\eta_t(x) \neq \eta(x)] = 0$ for each $x \in S$ if and only if $\eta \in D$;
(b) $1 \in D$, where 1 is the configuration with all sites occupied;
(c) if $\eta \leqslant \zeta \in D$ and $\zeta \neq 1$, then $\eta \in D$;
(d) $P^1[\eta_t \in D] = 1$ for all t;
(e) if $\eta \in D$, then $P^{\eta}[\eta_t \in D] = 1$, except possibly for countably many values of t; and
(f) if $\eta \notin D$, then $P^{\eta}[\eta_t \in A] = P^1[\eta_t \in A]$ for all $t > 0$ and all measurable A.

The points in D^c are reminiscent of the branch points which occur in the theory of Ray processes [4]. Unfortunately, the long range exclusion process usually fails to have the Ray property for the same reason it fails to have the Feller property. Thus the Ray theory is not available to us.

If S is finite, a simple computation given in Section 3 shows that ν_π is invariant for η_t whenever $\pi P = \pi$. This was observed by Waymire (private communication), and should not be surprising in view of the corresponding fact for the zero range process. In this respect, the long range exclusion process is better behaved than the simple exclusion process. An obvious fact when S is finite is that ν_∞ is invariant as well. The main problems we will study in this paper are to determine the extent to which these facts from the finite case carry over to the infinite case. We wish to determine when $\pi P = \pi$ implies $\nu_\pi \in \mathcal{I}$, and when $\nu_\infty \in \mathcal{I}$. One of the interesting features of this study is that they are often, but not always, the case. This situation is quite different from that encountered in other infinite particle systems which have been studied so far. For these other systems, it has essentially always been the case that once one found a class of measures which were invariant when S was finite, they also turned out to be invariant when S was infinite, at least under mild technical assumptions. The Feller property was important in making the transition from finite S to infinite S, and as has been observed, the Feller property usually fails for the long range exclusion process. Thus it should perhaps not be too surprising that the behavior of the long range exclusion process is different in this respect from the behavior of other infinite particle systems. In view of our interest in the question of invariant measures, it is fortunately true in general, as will be seen in Section 3, that D is large enough so that $\mu(D) = 1$ whenever μ is invariant, and also whenever $\mu = \nu_\pi$ for some π for which $\pi P = \pi$ (even if ν_π is not invariant).

We have not been able to answer our two basic questions in complete generality. We have, therefore, concentrated on two classes of nice cases in which fairly complete answers can be obtained, and in which the answers are not always affirmative. In order to state some of the results, recall that if μ_1 and μ_2 are two probability measures on U, we say that $\mu_1 \leqslant \mu_2$ if $\int f\,d\mu_1 \leqslant \int f\,d\mu_2$ for all monotone continuous functions f on U.

THEOREM 1.2. *If $\pi P = \pi$, then $\nu_\pi S(t)$ is a nonincreasing function of t, where $\nu_\pi S(t)$ is the distribution at time t of the process with initial distribution ν_π.*

The following theorem is probably true without further assumptions, and, in fact, no further assumptions are needed for any of the conclusions, except for the statement that ν_π is not invariant when π is not constant. For the proof of that statement, we will need to impose a mild regularity assumption, which is automatically satisfied if $d = 1$ or 2 or if the Laplace transform of $p(0, x)$ is finite in a neighborhood of the origin.

THEOREM 1.3. *Suppose $S = Z^d$ and $p(x,y) = p(0,y - x)$. If $\pi P = \pi$, then ν_π is invariant if and only if π is constant. Furthermore, ν_∞ is always invariant.*

THEOREM 1.4. *Suppose $S = \{0, 1, \cdots\}$ and $p(x,y) = 0$ for $|y - x| \geq 2$. Then ν_∞ is invariant if and only if $p(\cdot\,,\cdot)$ is recurrent. In the transient case, ν_π is not invariant for any π satisfying $\pi P = \pi$.*

Theorem 1.2 is proved in Section 3, Theorem 1.3 in Section 4, and Theorem 1.4 in Section 5. In cases when ν_π is not invariant for some π satisfying $\pi P = \pi$, it is of interest to determine the limit as t tends to ∞ of $\nu_\pi S(t)$. This is done in one class of cases by the following theorem, which is proved in Section 4. Of course the existence of the limit follows from Theorem 1.2.

THEOREM 1.5. *Suppose $S = Z^1$, $p(x,y) = p(0,y - x)$, and $\Sigma_x |x| p(0, x) < \infty$. Then*

$$\lim_{t\to\infty} \nu_\pi S(t) = \nu_{\bar{\pi}}$$

whenever $\pi P = \pi$, where $\bar{\pi}$ is the constant $\bar{\pi} = \inf_x \pi(x)$.

2. Construction of the process. For finite configurations, the process η_t was defined in the introduction. Let $S(t)f(\eta) = E^\eta f(\eta_t)$ be the corresponding semigroup, defined for finite η only. The extension of this definition to infinite configurations will be based on the intrinsic monotonicity of the process, which will now be described. Suppose η and ζ are finite configurations for which $\eta \leq \zeta$. A bivariate Markov chain (η_t, ζ_t) can then be constructed which satisfies (a) $S(t)f(\eta) = Ef(\eta_t)$; (b) $S(t)f(\zeta) = Ef(\zeta_t)$; (c) $\eta_o = \eta$; (d) $\zeta_o = \zeta$; and (e) $\eta_t \leq \zeta_t$ for all t. This is an example of the coupling technique which has been used so fruitfully in the field of infinite particle systems. Property (e) of this coupling implies that $S(t)f(\eta)$ is monotone in η whenever $f(\eta)$ is.

Let $\mathfrak{M}$ be the set of all bounded functions f on U which satisfy

(a) $\quad \eta \leq \zeta$ implies $f(\eta) \leq f(\zeta)$; and

(b) $\quad f(\zeta) = \lim_{\eta\uparrow\zeta, |\eta|<\infty} f(\eta),$

where $|\eta| = \Sigma_x \eta(x)$. For $f \in \mathfrak{M}$, $S(t)f(\zeta)$ is then defined for infinite ζ by

$$S(t)f(\zeta) = \lim_{\eta\uparrow\zeta, |\eta|<\infty} S(t)f(\eta). \tag{2.1}$$

$S(t)$ maps $\mathfrak{M}$ into $\mathfrak{M}$ by definition, and $S(t)$ satisfies the semigroup property on $\mathfrak{M}$. To verify the semigroup property, note that $S(t_1 + t_2)f(\eta) = S(t_1)S(t_2)f(\eta)$

for finite η, and this equality is preserved under the limit $\eta \uparrow \zeta$. $\mathfrak{M}$ is a probability determining class, so the transition probabilities $P^\eta[\eta_t \in d\zeta]$ for the infinite system are uniquely determined by

$$\int f(\zeta) P^\eta[\eta_t \in d\zeta] = E^\eta f(\eta_t) = S(t)f(\eta)$$

for $f \in \mathfrak{M}$. The above expression can then be used to extend the definition of $S(t)f$ to any function f which is bounded and Borel measurable. In general, the semigroup $S(t)$ is not strongly continuous. However, it does have the following desirable properties:

LEMMA 2.2. *(a) $Tf(\eta) = \lim_{t\downarrow 0} S(t)f(\eta)$ exists for $f \in C(U)$ and $\eta \in U$. (b) $Tf(\eta) \geqslant f(\eta)$ for $f \in \mathfrak{M} \cap C(U)$; and (c) $Tf(1) = f(1)$ for $f \in C(U)$.*

PROOF. Let $f_R(\eta) = \Pi_{u \in R}\eta(u)$, where R is a finite subset of S. Since each particle waits an exponential time with parameter one before it attempts to move,

$$E^\eta[f_R(\eta_{t+s}) | \eta_t] \geqslant e^{-|R|s}$$

on $\{f_R(\eta_t) = 1\}$ for $|\eta| < \infty$. Therefore,

$$S(t+s)f_R(\eta) \geqslant e^{-|R|s} S(t) f_R(\eta) \tag{2.3}$$

for $|\eta| < \infty$, and hence for all η by (2.1). By multiplying both sides of this relation by $\exp[|R|(t+s)]$, it follows that $S(t)f_R(\eta)\exp[|R|t]$ is nondecreasing in t for each η, so that

$$\lim_{t\downarrow 0} S(t) f_R(\eta) e^{|R|t} = \lim_{t \downarrow 0} S(t) f_R(\eta)$$

exists. Part (a) then follows from this, together with the facts that $S(t)$ is a contraction for each t, and that each function in $C(U)$ is the uniform limit of finite linear combinations of functions of the form f_R. For (b), note that $Tf(\eta) = f(\eta)$ for $|\eta| < \infty$, and Tf is monotone for $f \in \mathfrak{M} \cap C(U)$. Thus

$$Tf(\zeta) \geqslant \lim_{\eta\uparrow\zeta, |\eta|<\infty} Tf(\eta) = \lim_{\eta\uparrow\zeta,|\eta|<\infty} f(\eta) = f(\zeta)$$

for $f \in \mathfrak{M} \cap C(U)$. For the final part, take $f \in \mathfrak{M} \cap C(U)$ and note that

$$S(t)f(1) \leqslant \max_\eta f(\eta) = f(1),$$

so that $Tf(1) \leqslant f(1)$. By part (b), it then follows that $Tf(1) = f(1)$. Since T is a linear contraction, this gives $Tf(1) = f(1)$ for $f \in C(U)$.

The remainder of this section is devoted to the study of the set of initial configurations for which the process is normal (i.e., the set of η for which $Tf(\eta) = f(\eta)$ for all $f \in C(U)$) and to the behavior of the process when it starts off of that set. For $\eta \in U$ and $x, y \in S$, let

$$q(x,y,\eta) = E^x\left[\Pi_{n=1}^{\sigma_y - 1}\eta(X_n), \sigma_y \leqslant \sigma_x, \sigma_y < \infty\right], \tag{2.4}$$

where $\sigma_y = \min\{n \geqslant 1 : X_n = y\}$ is the hitting time of y for the chain $\{X_n\}$. This is the rate at which a particle at x will go to y if the configuration at the time is η.

Note that for fixed x and y, $q(x,y,\eta)$ is a monotone continuous function of η. Let

$$D = \{\eta \in U: Tf(\eta) = f(\eta) \text{ for all } f \in C(U)\}$$
$$= \{\eta \in U: Tf_x(\eta) = f_x(\eta) \text{ for all } x \in S\}$$

where $f_x(\eta) = \eta(x)$. The fact that these two sets are equal follows from part (b) of Lemma 2.2, while $1 \in D$ comes from part (c). The next lemma is needed to show that $D \setminus \{1\}$ is a monotone set.

LEMMA 2.5. *Suppose* $\eta \leqslant \zeta$, $\eta(x) = 0$, $\zeta(x) = 1$, *and* $\zeta(y) = 0$. *Then*

$$q(x,y,\zeta)\lim_{t\downarrow 0} P^\eta[\eta_t(x) = 1] \leqslant \lim_{t\downarrow 0} P^\zeta[\zeta_t(y) = 1].$$

PROOF. Note that these limits exist by Lemma 2.2. Assume for now that η and ζ are finite configurations, and let (η_t, ζ_t) be the coupled process starting at (η, ζ) which is described at the beginning of this section. Let $\tau = \inf\{t > 0: \eta_t(x) = 1\}$, and let γ be any configuration such that $\gamma(x) = 1$ and $\gamma \leqslant \zeta$. Then for $t > 0$,

$$P[\zeta_t(y) = 1] \geqslant q(x,y,\gamma)e^{-t} \times \Sigma_{u:\gamma(u)=0} P[\tau \leqslant t, \zeta_{\tau-} \geqslant \gamma, \eta_\tau(u) = 0, \eta_{\tau-}(u) = 1]$$

and

$$p[\tau \leqslant t] \leqslant \Sigma_{u:\gamma(u)=0} P[\tau \leqslant t, \zeta_{\tau-} \geqslant \gamma, \eta_\tau(u) = 0, \eta_{\tau-}(u) = 1]$$
$$+ P[\zeta_s(u) = 0 \text{ for some } s \leqslant t \text{ and some } u \text{ so that } \gamma(u) = 1]$$
$$+ P[\eta_s(u) = 0, \eta_{s-}(u) = 1 \text{ for some } s \leqslant t \text{ and some } u \text{ so that } \gamma(u) = 1].$$

Each of the last two terms in the last expression is bounded above by $|\gamma|(1 - e^{-t})$, so that

$$P^\zeta[\zeta_\tau(y) = 1] \geqslant q(x,y,\gamma)e^{-t}\{P^\eta[\eta_t(x) = 1] - 2|\gamma|(1 - e^{-t})\}.$$

For fixed γ, this relation extends to infinite η and ζ by (2.1). The desired result follows by taking limits first as $t\downarrow 0$ and then as $\gamma\uparrow\zeta$.

THEOREM 2.6. *If* $\eta \leqslant \zeta$, $\zeta \in D$ *and* $\zeta \neq 1$, *then* $\eta \in D$.

PROOF. It suffices to show that $\lim_{t\downarrow 0} P^\eta[\eta_t(x) \neq \eta(x)] = 0$ for all $x \in S$. If $\eta(x) = 1$, this is an immediate consequence of (b) of Lemma 2.2. If $\eta(x) = 0$, we need to show that $\lim_{t\downarrow 0} P^\eta[\eta_t(x) = 1] = 0$. Suppose $\zeta(x) = 0$. Then $\lim_{t\downarrow 0} P^\zeta[\zeta_t(x) = 1] = 0$ since $\zeta \in D$. Since $\eta \leqslant \zeta$, $P^\eta[\eta_t(x) = 1] \leqslant P^\zeta[\zeta_t(x) = 1]$, so the result follows. On the other hand, if $\zeta(x) = 1$, then there is a y such that $\zeta(y) = 0$ and $q(x,y,\zeta) > 0$ since $p(\cdot\,,\cdot)$ is irreducible and $\zeta \neq 1$. Using Lemma 2.5 and the fact that $\zeta \in D$ then gives $\lim_{t\downarrow 0} P^\eta[\eta_t(x) = 1] = 0$.

For fixed $\eta \in U$, the map $f \to Tf(\eta)$ defines a bounded linear functional on $C(U)$, so there is a probability measure $\mu(\eta, d\zeta)$ on U for which

$$Tf(\eta) = \int f(\zeta)\mu(\eta, d\zeta)$$

for all $f \in C(U)$. This formula defines Tf for bounded Borel measurable functions f as well.

LEMMA 2.7. *$\mu(\eta, D) = 1$ for all $\eta \in U$.*

PROOF. Take $f \in \mathfrak{M} \cap C(U)$. Then $S(t)f$ is lower semicontinuous by (2.1). Let $f_n \in C(U)$ increase to $S(t)f$. Then

$$TS(t)f = \lim_n Tf_n = \lim_n \lim_{s\downarrow 0} S(s)f_n \leq \liminf_{s\downarrow 0} S(s+t)f, \tag{2.8}$$

where the first equality comes from the monotone convergence theorem. Therefore by the bounded convergence theorem,

$$T^2 f = \lim_{t\downarrow 0} TS(t)f \leq \lim_{s,t\downarrow 0} S(s+t)f = Tf.$$

But then

$$\int [Tf(\zeta) - f(\zeta)]\mu(\eta, d\zeta) = T^2 f(\eta) - Tf(\eta) \leq 0.$$

Since $Tf \geq f$ by (b) of Lemma 2.2, it follows that $\mu(\eta, \{\zeta : Tf(\zeta) \neq f(\zeta)\}) = 0$. Applying this to $f = f_x$ for each $x \in S$ gives $\mu(\eta, D) = 1$.

COROLLARY 2.9. *For any bounded Borel measurable f,*

$$\begin{aligned} Tf(\eta) &= f(\eta) \quad &\text{if } \eta \in D \\ &= f(1) \quad &\text{if } \eta \notin D. \end{aligned}$$

PROOF. $\mu(\eta, \cdot)$ concentrates on D by Lemma 2.7, and it concentrates on $\{\zeta \in U : \zeta \geq \eta\}$ by (b) of Lemma 2.2. Therefore, if $\eta \notin D$, $\mu(\eta, \cdot)$ concentrates on

$$D \cap \{\zeta \in U : \zeta \geq \eta\},$$

which is $\{1\}$ by Theorem 2.6. Thus $Tf(\eta) = f(1)$ if $\eta \notin D$. On the other hand, $Tf(\eta) = f(\eta)$ for $\eta \in D$ by the definition of D.

LEMMA 2.10. *$S(t)f(1)$ is continuous in t for $f \in C(U)$.*

PROOF. It suffices to prove this for f of the form f_R for a finite subset R of S. By (2.3), $S(t)f_R(1)\exp(|R|t)$ is nondecreasing in t. On the other hand, $S(t)f_R \in \mathfrak{M}$ implies $S(t)f_R(\eta) \leq S(t)f_R(1)$, so that

$$S(t+s)f_R(1) = S(s)S(t)f_R(1) \leq S(t)f_R(1). \tag{2.11}$$

Therefore, $S(t)f_R(1)$ is continuous in t.

THEOREM 2.12. *$P^1[\eta_t \in D] = 1$ for $t \geq 0$.*

PROOF. Take $x \in S$. By (b) of Lemma 2.2,

$$S(t)f_x(1) \leq S(t)Tf_x(1). \tag{2.13}$$

Setting $R = \{x\}$ in (2.11) gives

$$S(t)S(s)f_x(1) \leq S(t)f_x(1). \tag{2.14}$$

The left side of (2.14) converges to the right side of (2.13) as $s\downarrow 0$ by the bounded convergence theorem, so it follows that $S(t)f_x(1) = S(t)Tf_x(1)$. Using (b) of Lemma 2.2 again, one sees that $P^1[\eta_t \in d\zeta]$ concentrates on $\{\zeta : f_x(\zeta) = Tf_x(\zeta)\}$. Since this is true for each $x \in S$, it follows that $P^1[\eta_t \in D] = 1$.

THEOREM 2.15. *$P^\eta[\eta_t \in d\zeta] = P^1[\eta_t \in d\zeta]$ for $\eta \notin D$ and $t > 0$.*

PROOF. It suffices to prove that $S(t)f_R(\eta) = S(t)f_R(1)$ for $\eta \notin D$, $t > 0$, and all finite $R \subset S$. Now,

$$\begin{aligned} S(t)f_R(1) &= TS(t)f_R(\eta) \leqslant \liminf_{s\downarrow 0} S(t+s)f_R(\eta) \\ &\leqslant \limsup_{s\downarrow 0} S(t+s)f_R(\eta) \leqslant \limsup_{s\downarrow 0} S(t+s)f_R(1) = S(t)f_R(1). \end{aligned}$$

The first equality comes from Corollary 2.9, the first inequality from (2.8), the last inequality from the fact that $S(t+s)f_R \in \mathfrak{M}$, and the last equality from Lemma 2.10. Thus

$$\lim_{s\downarrow 0} S(t+s)f_R(\eta) = S(t)f_R(1). \tag{2.16}$$

By (2.3), $S(t)f_R(\eta)\exp(|R|t)$ is nondecreasing in t. It follows from this, from (2.16), and from Lemma 2.10 that $S(t)f_R(\eta) = S(t)f_R(1)$ for $t > 0$ and $\eta \notin D$.

THEOREM 2.17. *For each $\eta \in U$, $P^\eta[\eta_t \in D] = 1$ except possibly for countably many values of t.*

PROOF. For $0 < s < t$, the Markov property gives

$$P^\eta[\eta_s \in D^c \text{ and } \eta_t \in D^c] = \int_{D^c} P^\eta[\eta_s \in d\zeta]P^\zeta[\eta_{t-s} \in D^c].$$

The right side of this is zero by Theorems 2.12 and 2.15. Therefore, $\Sigma_t P^\eta[\eta_t \in D^c] \leqslant 1$, from which the result follows.

3. General results. In this section, we gather together those results which we can prove in essentially complete generality. Many of these will be applied to the random walk and birth and death cases of Sections 4 and 5. Let $\mathfrak{I}$ be the set of invariant probability measures for the process:

$$\begin{aligned} \mathfrak{I} &= \{\mu : \mu S(t) = \mu \text{ for all } t > 0\} \\ &= \{\mu : \textstyle\int S(t)f\,d\mu = \int f\,d\mu \text{ for all } t > 0 \text{ and } f \in C(U)\}. \end{aligned}$$

THEOREM 3.1. *Suppose $\pi P = \pi$ and $\nu_{c\pi} \in \mathfrak{I}$ for all $c > 0$. Then $\nu_\infty \in \mathfrak{I}$.*

PROOF. $\nu_\infty \geqslant \nu_{c\pi}$, so $\nu_\infty S(t) \geqslant \nu_{c\pi}S(t)$ since $S(t)$ maps $\mathfrak{M}$ into $\mathfrak{M}$. Since $\nu_{c\pi} \in \mathfrak{I}$, it follows that $\nu_\infty S(t) \geqslant \nu_{c\pi}$. Letting c tend to ∞ gives $\nu_\infty S(t) = \nu_\infty$.

THEOREM 3.2. *Suppose $p(\cdot,\cdot)$ is transient, and η is a configuration for which*

$$\Sigma_x \eta(x)P^x(\sigma_y < \infty) < \infty \tag{3.3}$$

for some, and hence every $y \in S$. Then

$$\lim_{t\to\infty} P^\eta[\eta_t(y) = 1] = 0$$

for every $y \in S$.

PROOF. First note that condition (3.3) is independent of y, since $P^z(\sigma_y < \infty) > 0$ and

$$P^x(\sigma_z < \infty)P^z(\sigma_y < \infty) \leqslant P^x(\sigma_y < \infty).$$

For finite configurations, a particle initially at x visits a subsequence of $\{X_n\}$, where $\{X_n\}$ is a realization of the discrete time Markov chain which starts at x and has transition probabilities $p(\cdot\,,\cdot)$. Furthermore, the time spent at each site is exponential with parameter one. Thus, for finite η,

$$\int_0^\infty P^\eta[\eta_t(y) = 1]\,dt = E^\eta[\int_0^\infty \eta_t(y)\,dt] \leqslant \frac{\Sigma_x \eta(x) P^x(\sigma_y < \infty)}{P^y(\sigma_y = \infty)},$$

since the expected number of visits to y for the chain starting at y is $[P^y(\sigma_y = \infty)]^{-1}$. By the monotone convergence theorem and (2.1), this is true for infinite η as well. By (3.3),

$$\int_0^\infty P^\eta[\eta_t(y) = 1]\,dt < \infty.$$

By (2.3) with $R = \{y\}$, $e^t P^\eta[\eta_t(y) = 1]$ is nondecreasing in t. Thus

$$P^\eta[\eta_t(y) = 1]\int_t^\infty e^{-(s-t)}\,ds \leqslant \int_t^\infty P^\eta[\eta_s(y) = 1]\,ds,$$

so that $\lim_{t\to\infty} P^\eta[\eta_t(y) = 1] = 0$.

COROLLARY 3.4. *Suppose that* $p(\cdot\,,\cdot)$ *is transient.*

(a) *If* $\Sigma_x P^x(\sigma_y < \infty) < \infty$ *for some* y, *then* $\nu_\infty S(t) \to \nu_0$ *as* $t \to \infty$.

(b) *If* $\Sigma_x \pi(x)/(1 + \pi(x))P^x(\sigma_y < \infty) < \infty$ *for some* y, *then* $\nu_\pi S(t) \to \nu_0$ *as* $t \to \infty$.

Here ν_0 is the pointmass on the configuration in which all sites are vacant.

PROOF. The first statement is just Theorem 3.2 when $\eta \equiv 1$. The second statement follows from Theorem 3.2 and the dominated convergence theorem, since ν_π concentrates on the set of configurations which satisfy (3.3) and

$$\nu_\pi S(t)\{\eta : \eta(y) = 1\} = \int P^\eta[\eta_t(y) = 1]\nu_\pi(d\eta).$$

The following examples illustrate the use of Corollary 3.4, and show that ν_π and ν_∞ are not necessarily invariant.

EXAMPLE 3.5. (a) $S = \{0, 1, 2, \cdots\}$, $p(x, x+1) = p$ for $x \geqslant 0$, $p(x, x-1) = q$ for $x \geqslant 1$, and $p(0,0) = q$, where $p + q = 1$ and $\frac{1}{2} < p < 1$. Then $P^x(\sigma_0 < \infty) = (q/p)^x$ for $x > 0$, so $\nu_\infty S(t) \to \nu_0$ as $t \to \infty$ by the first part of Corollary 3.4.

(b) $S = Z^1$, $p(x, x+1) = p$, and $p(x, x-1) = q$ where $p + q = 1$ and $\frac{1}{2} < p < 1$. Then $P^x(\sigma_0 < \infty) = 1$ for $x < 0$ and $P^x(\sigma_0 < \infty) = (q/p)^x$ for $x > 0$. Let $\pi(x) = (p/q)^x$. Then $\pi P = \pi$, and $\Sigma_x \pi(x)/(1 + \pi(x))P^x(\sigma_0 < \infty) < \infty$. Therefore $\nu_\pi S(t) \to \nu_0$ as $t \to \infty$ by the second part of Corollary 3.4. As will be seen in the next section, $\nu_\infty \in \mathcal{I}$ in this case.

Corollary 3.4 is not sufficiently strong to determine exactly when $\nu_\infty \in \mathcal{I}$ in the birth and death case which is treated in Section 5. We will need the following stronger result.

THEOREM 3.6. *Suppose $p(\cdot\,,\cdot)$ is transient and*

$$\Sigma_x P^x(\sigma_y < \sigma_x) < \infty \tag{3.7}$$

for some, and hence every $y \in S$. Then $\nu_\infty \notin \mathcal{I}$.

PROOF. First note that condition (3.7) is independent of y since

$$P^x(\sigma_y < \sigma_x)P^y(\sigma_z < \sigma_x) \leqslant P^x(\sigma_z < \sigma_x),$$

and for each y and z, there is a finite set $R \subset S$ so that $P^y(\sigma_z < \sigma_x)$ is bounded away from zero on the complement of R. To prove the theorem, fix a $y \in S$ and let $\tau = \inf\{t > 0 | \eta_t(y) = 1\}$ and $c = \Sigma_x P^x(\sigma_y < \sigma_x)$. If η is any finite configuration for which $\eta(y) = 0$, then

$$\lim_{s\downarrow 0}\frac{P^\eta[\tau < s]}{s} = \Sigma_x \eta(x)q(x,y,\eta) \leqslant c.$$

On the set $\{\tau > t\}$, $\eta_t(y) = 0$. Hence

$$\lim_{s\downarrow 0}\frac{P^\eta[\tau < t + s|\tau > t]}{s} \leqslant c$$

by the Markov property and the bounded convergence theorem. The required boundedness comes from the inequality $P^\eta[\tau < s] < s\Sigma_x\eta(x)$ for $\eta(y) = 0$. Therefore $P^\eta[\tau \geqslant t] \geqslant e^{-ct}$ for any finite configuration η for which $\eta(y) = 0$. Now let η be a finite configuration for which $\eta(y) = 1$. Then

$$P^\eta[\eta_t(y) = 0] \geqslant (1 - e^{-t})P^y[\sigma_y = \infty]e^{-ct}, \tag{3.8}$$

since one way for y to be vacant at time t is for the particle at y to move by time t to some point other than y itself, and for no other particle to move to y in the remaining time before time t. Since the right-hand side of (3.8) is positive and independent of η, it follows from (2.1) that

$$\nu_\infty S(t)\{\eta : \eta(y) = 0\} = P^1[\eta_t(y) = 0] > 0,$$

so that $\nu_\infty \notin \mathcal{I}$.

In several places, it will be useful to approximate the semigroup $S(t)$ from below by the semigroups $T_k(t)$ which correspond to processes in which particles move like they do in the long range exclusion process, except that only k attempts to find a vacant site are permitted. If no vacant site is found after k attempts, the particle simply disappears. The fact that $T_k(t)$ converges to $S(t)$ as $k \to \infty$, which will be proved in the next theorem, justifies the assertion in the introduction that particles in the long range exclusion process which search infinitely many times for a vacant site just disappear. For the remainder of the paper, it will be assumed that $p(\cdot\,,\cdot)$ satisfies (1.1). This condition is, of course, automatically satisfied in the cases considered in Sections 4 and 5. By Theorem 2.8 of [9], this assumption guarantees that $T_k(t)$ can be defined as the strongly continuous semigroup on $C(U)$ whose generator Ω_k, when restricted to functions f which depend on finitely many coordinates, is given by

$$\Omega_k f(\eta) = \Sigma_{\eta(x)=1,\eta(y)=0} q_k(x,y,\eta)[f(\eta_{xy}) - f(y)] + \Sigma_{\eta(x)=1}\delta_k(x,\eta)[f(\eta_x) - f(\eta)].$$

In this expression, $\eta_x(x) = 1 - \eta(x)$, $\eta_x(u) = \eta(u)$ for $u \neq x$, $\eta_{xy}(x) = \eta(y)$, $\eta_{xy}(y) = \eta(x)$, $\eta_{xy}(u) = \eta(u)$ for $u \neq x,y$,

$$q_k(x,y,\eta) = E^x[\Pi_{n=1}^{\sigma_y - 1}\eta(X_n), \sigma_y \leqslant \sigma_x, \sigma_y \leqslant k],$$

and

$$\delta_k(x,\eta) = 1 - \Sigma_{\eta(y)=0} q_k(x,y,\eta) - q_k(x,x,\eta) = E^x[\Pi_{n=1}^k \eta(X_n), \sigma_x > k].$$

The interpretation is that $q_k(x,y,\eta)$ is the rate at which the particle at x will go to y, and $\delta_k(x,\eta)$ is the rate at which it will disappear, if η is the configuration of the system at that time.

THEOREM 3.9. *For all* $\eta \in U$,
(a) $T_k(t)f(\eta) \leqslant T_{k+1}(t)f(\eta) \leqslant S(t)f(\eta)$ *for* $f \in \mathfrak{M} \cap C(U)$; *and*
(b) $\lim_{k\to\infty} T_k(t)f(\eta) = S(t)f(\eta)$ *for* $f \in C(U)$.

PROOF. A simple coupling argument gives (a) for finite η. To extend this to infinite η, use (2.1) and the fact that $T_k(t)f \in C(U)$ for $f \in C(U)$. Part (b) is immediate for finite η since $\lim_{k\to\infty} q_k(x,y,\eta) = q(x,y,\eta)$ and $\lim_{k\to\infty}\delta_k(x,\eta) = 0$. To extend part (b) to infinite η, let $g(\eta) = \lim_{k\to\infty} T_k(t)f(\eta)$ for $f \in \mathfrak{M} \cap C(U)$, which exists by part (a). For any finite $\zeta \in U$, choose finite ζ_n so that $\zeta_n \uparrow \zeta$. Then

$$S(t)f(\zeta_n) = g(\zeta_n) \leqslant g(\zeta) \leqslant S(t)f(\zeta)$$

by part (a) and the finite version of (b). Since $S(t)f(\zeta) = \lim_{n\to\infty} S(t)f(\zeta_n)$ by (2.1), it follows that $g(\zeta) = S(t)f(\zeta)$.

If $\pi P = \pi$, E_π^x and P_π^x will denote expectations and probabilities computed relative to the reversed chain which has transition probabilities $P_\pi^x[X_1 = y] = (\pi(y)p(y,x))/\pi(x)$.

LEMMA 3.10. *Suppose* $\pi P = \pi$ *and* R *is a finite subset of* S. *Then*

$$\int \Omega_k f_R \, d\nu_\pi = -\Sigma_{x\in R}\int f_R(\eta) E_\pi^x[\Pi_{n=1}^k \eta(X_n), \sigma_x > k]\, d\nu_\pi.$$

PROOF. First note that

$$\begin{aligned}\Sigma_y[1-\eta(y)]q_k(x,y,\eta) &= \Sigma_y \Sigma_{j=1}^k E^x[\Pi_{n=1}^{j-1}\eta(X_n)[1-\eta(X_j)], j \leqslant \sigma_x, X_j = y]\\ &= \Sigma_{j=0}^{k-1} E^x[\Pi_{n=1}^j \eta(X_n), j+1 \leqslant \sigma_x] - \Sigma_{j=1}^k E^x[\Pi_{n=1}^j \eta(X_n), j \leqslant \sigma_x]\\ &= 1 - E^x[\Pi_{n=1}^{\sigma_x \wedge k}\eta(X_n)] \quad \text{and}\end{aligned}$$

$$\Sigma_y[1-\eta(y)]\pi(y)q_k(y,x,\eta) = \Sigma_y \pi(y)E^y[\Pi_{n=1}^{\sigma_x - 1}\eta(X_n), 1 < \sigma_x \leqslant k] + \Sigma_y \pi(y)p(y,x)$$

$$- \Sigma_y \eta(y)\pi(y)E^y\left[\Pi_{n=1}^{\sigma_x-1}\eta(X_n), \sigma_x \leq k\right]$$
$$= \Sigma_{y,z;\, z\neq x}\pi(y)p(y,z)\eta(z)E^z\left[\Pi_{n=1}^{\sigma_x-1}\eta(X_n), \sigma_x \leq k-1\right] + \pi(x)$$
$$- \Sigma_y \eta(y)\pi(y)E^y\left[\Pi_{n=1}^{\sigma_x-1}\eta(X_n), \sigma_x \leq k\right]$$
$$= \pi(x) - \pi(x)E^x\left[\Pi_{n=1}^{\sigma_x}\eta(X_n), \sigma_x \leq k\right]$$
$$- \Sigma_{y\neq x}\eta(y)\pi(y)E^y\left[\Pi_{n=1}^{\sigma_x-1}\eta(X_n), \sigma_x = k\right].$$

Therefore,

$$\Sigma_y[1-\eta(y)]\frac{\pi(y)}{\pi(x)}q_k(y,x,\eta) - \Sigma_y[1-\eta(y)]q_k(x,y,\eta)$$
$$= E^x\left[\Pi_{n=1}^{k}\eta(X_n), \sigma_x > k\right] - E_\pi^x\left[\Pi_{n=1}^{k}\eta(X_n), \sigma_x > k\right], \tag{3.11}$$

and if $\eta(x) = 1$,

$$\delta_k(x,\eta) = E^x\left[\Pi_{n=1}^{k}\eta(X_n), \sigma_x > k\right]. \tag{3.12}$$

Now

$$\int \Sigma_{\eta(x)=1,\eta(y)=0} q_k(x,y,\eta)\left[f_R(\eta_{xy}) - f_R(\eta)\right]d\nu_\pi \tag{3.13}$$
$$= \Sigma_{x \text{ or } y\in R}\int f_R(\eta)\left\{\frac{\pi(x)}{\pi(y)}\eta(y)[1-\eta(x)] - \eta(x)[1-\eta(y)]\right\}q_k(x,y,\eta)\,d\nu_\pi$$
$$= \Sigma_{x \text{ or } y\in R}\int f_R(\eta)\eta(x)[1-\eta(y)]\left\{\frac{\pi(y)}{\pi(x)}q_k(y,x,\eta) - q_k(x,y,\eta)\right\}d\nu_\pi,$$

where the first equality is obtained by making the change of variables $\eta \to \eta_{xy}$ in the integral of the first term, and the second equality comes from interchanging x and y in the first term. Since $f_R(\eta)[1-\eta(y)] = 0$ for $y \in R$, the above summation can be carried out for $x \in R$, $y \in S$ only. Using (3.11) and the fact that $f_R(\eta) = \eta(x)f_R(\eta)$ for $x \in R$, it follows that the expression in (3.13) is equal to

$$\Sigma_{x\in R}\int f_R(\eta)\left\{E^x\left[\Pi_{n=1}^{k}\eta(X_n), \sigma_x > k\right] - E_\pi^x\left[\Pi_{n=1}^{k}\eta(X_n), \sigma_x > k\right]\right\}d\nu_\pi. \tag{3.14}$$

Similarly,

$$\int \Sigma_{\eta(x)=1}\delta_k(x,\eta)\left[f_R(\eta_x) - f_R(\eta)\right]d\nu_\pi \tag{3.15}$$
$$= -\Sigma_{x\in R}\int f_R(\eta)\delta_k(x,\eta)\,d\nu_\pi$$
$$= -\Sigma_{x\in R}\int f_R(\eta)E^x\left[\Pi_{n=1}^{k}\eta(X_n), \sigma_x > k\right]d\nu_\pi.$$

The required result is obtained by adding together (3.14) and (3.15).

If S is finite, η_t is a finite state Markov chain with generator $\lim_{k\to\infty}\Omega_k f$. The following is, therefore, an immediate consequence of Lemma 3.10.

COROLLARY 3.16. *If S is finite and $\pi P = \pi$, then $\nu_\pi \in \mathcal{I}$.*

If S is infinite and $\pi P = \pi$, let R_n be an increasing sequence of finite subsets of S

such that $S = \cup_n R_n$. Define $p_n(\cdot\,,\cdot)$ by $p_n(x,x) = 1$ if $x \notin R_n$, $p_n(x,y) = 0$ if $x \notin R_n$ and $y \neq x$, and

$$p_n(x,y) = p(x,y) + Q^{-1}\left[\Sigma_{z\notin R_n} p(x,z)\right]\left[\Sigma_{z\notin R_n}\pi(z)p(z,y)\right]$$

if $x,y \in R_n$, where

$$Q = \Sigma_{z\notin R_n;\, y\in R_n}\pi(z)p(z,y) = \Sigma_{z\in R_n;\, y\notin R_n}\pi(z)p(z,y).$$

These approximations were used in [10] when $\pi(\cdot)$ is constant, and in [15] in the general case. It is easy to check that $\Sigma_y p_n(x,y) = 1$ for all x, and that $\pi P_n = \pi$. Let $S_n(t)$ be the semigroup for the long range exclusion process corresponding to $p_n(\cdot\,,\cdot)$. This is clearly a Feller semigroup, since no transitions occur off R_n. Using this sequence of semigroups, the following monotonicity result can be obtained.

THEOREM 3.17. *Suppose* $\pi P = \pi$. *Then*
(a) $\nu_\pi S(t)\downarrow$ *in* t, *and*
(b) $\nu_\pi S(t)$ *is weakly continuous in* t *for* $t \geq 0$.

PROOF. Since $\lim_{n\to\infty} p_n(x,y) = p(x,y)$ for each $x,y \in S$, it follows that $\lim_{n\to\infty} S_n(t)f(\eta) = S(t)f(\eta)$ for finite η. Therefore, for $f \in \mathfrak{M} \cap C(U)$,

$$S(t)f(\eta) \leq \liminf_{n\to\infty} S_n\sigma(t)f(\eta)$$

for all $\eta \in U$ by (2.1) and the fact that $S_n(t)f \in \mathfrak{M}$ for all n. By Corollary 3.16 and the fact that $\pi P_n = \pi$,

$$\int S_n(t)f\,d\nu_\pi = \int f\,d\nu_\pi$$

for all n and all $f \in C(U)$. Therefore, for $f \in \mathfrak{M} \cap C(U)$,

$$\int S(t)f\,d\nu_\pi \leq \lim_{n\to\infty}\int S_n(t)f\,d\nu_\pi = \int f\,d\nu_\pi.$$

Hence $\nu_\pi S(t) \leq \nu_\pi$, and then part (a) follows from the semigroup property and the fact that $S(t)$ maps $\mathfrak{M}$ into itself. Part (b) is proved in the same way as Lemma 2.10, using part (a) to get the monotonicity of $\int S(t)f\,d\nu_\pi$ in t for $f \in C(U)$.

THEOREM 3.18. (a) *If* $\mu S(t)\downarrow$ *in* t, *then* $\mu(D) = 1$.
(b) *If* $\pi P = \pi$, *then* $\nu_\pi(D) = 1$.
(c) *If* $\mu \in \mathcal{I}$, *then* $\mu(D) = 1$.

PROOF. Parts (b) and (c) follow immediately from part (a) and Theorem 3.17. For the proof of part (a), suppose $\mu S(t)\downarrow$ in t. then $\int S(t)f\,d\mu \leq \int f\,d\mu$ for $f \in \mathfrak{M} \cap C(U)$. By parts (a) and (b) of Lemma 2.2, $\int Tf\,d\mu = \int f\,d\mu$ for such f. Therefore $\mu(D) = 1$ by the definition of D.

The following will play an important role in the rest of the paper. It is useful, for example, in determining whether or not $\nu_\pi \in \mathcal{I}$.

THEOREM 3.19. *Suppose $\pi P = \pi$, and let $\mu = \lim_{t\to\infty} \nu_\pi S(t)$, which exists by Theorem* 3.17. *Then*

(a) $\Omega_k f_x(\eta)$ *is nondecreasing in k for each* $\eta \in U$, *so*

$$\Omega f_x(\eta) = \lim_{k\to\infty} \Omega_k f_x(\eta) \text{ exists;}$$

(b) $-1 \leqslant \Omega_k f_x(\eta) < \infty$ *and* $-1 \leqslant \Omega f_x(\eta) \leqslant \infty$;

(c) $(d/dt)\int S(t) f_x \, d\nu_\pi = \int \Omega f_x \, d[\nu_\pi S(t)]$; *and*

(d) $\int \Omega f_x \, d\mu = 0$.

PROOF. By the definition of Ω_k,

$$(3.20) \quad \Omega_k f_x(\eta) = [1 - \eta(x)] \Sigma_{\eta(y)=1} q_k(y, x, \eta) - \eta(x)[1 - q_k(x, x, \eta)].$$

Since $q_k(y, x, \eta)$ is nondecreasing in k,

$$(3.21) \quad \begin{aligned} \Omega f_x(\eta) &= \lim_{k\to\infty} \Omega_k f_x(\eta) \\ &= [1 - \eta(x)] \Sigma_{\eta(y)=1} q(y, x, \eta) - \eta(x)[1 - q(x, x, \eta)], \end{aligned}$$

which proves (a) and (b). For part (c), note that

$$\int \Omega_k f_x \, d\nu_\pi = \lim_{t\downarrow 0} \int \frac{T_k(t) f_x - f_x}{t} \, d\nu_\pi \leqslant \liminf_{t\downarrow 0} \int \frac{S(t) f_x - f_x}{t} \, d\nu_\pi \leqslant 0$$

by parts (a) of Theorems 3.9 and 3.17. Therefore $\int \Omega f_x \, d\nu_\pi \leqslant 0$ by parts (a) and (b) and the monotone convergence theorem. By (3.21), it follows that

$$\int [1 - \eta(x)] \Sigma_{\eta(y)=1} q(y, x, \eta) \, d\nu_\pi \leqslant 1.$$

Since $q(y, x, \eta)$ does not depend on the coordinate $\eta(x)$ and ν_π is a product measure,

$$(3.22) \qquad \int \left[\Sigma_{\eta(y)=1} q(y, x, \eta) \right] d\nu_\pi \leqslant 1 + \pi(x) < \infty.$$

The integrand in (3.22) is a monotone function of η for each x, so

$$(3.23) \qquad \int \left[\Sigma_{\eta(y)=1} q(y, x, \eta) \right] d[\nu_\pi S(t)] < \infty$$

for each t by part (a) of Theorem 3.17. Suppose now that η_k is a sequence in U such that $\eta_k \uparrow \eta$. Then for k large enough that $\eta_k(x) = \eta(x)$, $\Omega_j f_x(\eta_k)$ is increasing in k as well as in j. Thus

$$\lim_{k\to\infty} \Omega_k f_x(\eta_k) = \Omega f_x(\eta).$$

By Theorem 3.9, $\nu_\pi T_k(t) \leqslant \nu_\pi T_{k+1}(t) \leqslant \nu_\pi S(t)$ and $\nu_\pi T_k(t) \to \nu_\pi S(t)$ as $k \to \infty$, so one can define random elements η_k and η of U on the same probability space so that η_k has distribution $\nu_\pi T_k(t)$, η has distribution $\nu_\pi S(t)$ and $\eta_k \uparrow \eta$ a.s. Then

$$\int \Omega_k f_x \, d[\nu_\pi T_k(t)] = E\Omega_k f_x(\eta_k) \to E\Omega f_x(\eta) = \int \Omega f_x \, d[\nu_\pi S(t)]$$

by the dominated convergence theorem, where the domination comes from (3.23). Since $T_k(t)$ is a Feller process with generator Ω_k, it follows that

$$(3.24) \qquad \begin{aligned} \lim_{k\to\infty} \frac{d}{dt} \int T_k(t) f_x \, d\nu_\pi &= \lim_{k\to\infty} \int \Omega_k f_x \, d[\nu_\pi T_k(t)] \\ &= \int \Omega f_x \, d\nu_\pi S(t). \end{aligned}$$

Note that $\eta_k \uparrow \eta$, or $\eta_k \downarrow \eta$ and $\Omega f_x(\eta_k) < \infty$ imply that

$$\Omega f_x(\eta_k) \to \Omega f_x(\eta). \tag{3.25}$$

Therefore, $\int \Omega f_x \, d\nu_\pi S(t)$ is a continuous function of t by Theorem 3.17 and (3.23). Since $\int T_k(t) f_x \, d\nu_\pi \to \int S(t) f_x \, d\nu_\pi$ by Theorem 3.9, it follows from (3.24) that $\int S(t) f_x \, d\nu_\pi$ is a continuously differentiable function of t with derivative $\int \Omega f_x \, d\nu_\pi S(t)$. This proves part (c). Now $\int S(t) f_x \, d\nu_\pi$ is a bounded monotone function of t, so that

$$\lim_{t\to\infty} \frac{d}{dt} \int S(t) f_x \, d\nu_\pi = 0.$$

Part (d) follows from this and (3.25).

Put $\tau_1 = 0$, and for $k \geqslant 2$, define $\tau_k = \min\{n \geqslant 1 : \text{number of distinct points in } \{X_o, \cdots, X_n\} \text{ is } k\}$.

COROLLARY 3.26. *Suppose $\pi P = \pi$ and*

$$P_\pi^x\left[\sigma_x = \infty, \Sigma_{n=1}^\infty \frac{1}{\pi(X_{\tau_n})} < \infty\right] > 0 \tag{3.27}$$

for some $x \in S$. Then $\nu_\pi \notin \mathcal{I}$.

PROOF. By part (d) of Theorem 3.19, it suffices to show that $\int \Omega f_x \, d\nu_\pi \neq 0$. By Lemma 3.10, parts (a) and (b) of Theorem 3.19, and the monotone convergence theorem,

$$\int \Omega f_x \, d\nu_\pi = -\int E_\pi^x\left[\Pi_{n=0}^\infty \eta(X_n), \sigma_x = \infty\right] d\nu_\pi.$$

Since $\{X_{\tau_n}, n \geqslant 1\}$ is the set of distinct points visited by the chain $\{X_n, n \geqslant 0\}$,

$$\int \Omega f_x \, d\nu_\pi = -E_\pi^x\left[\Pi_{n=1}^\infty \frac{\pi(X_{\tau_n})}{1 + \pi(X_{\tau_n})}, \sigma_x = \infty\right],$$

which is strictly negative by (3.27).

Finally, we will show that, in many cases, D is a proper subset of U and η_t does not have the Feller property, which means that $S(t)$ does not map $C(U)$ into $C(U)$. The assumptions of the following theorem will be verified in Section 4 when $p(\cdot, \cdot)$ is an arbitrary random walk on Z^d, and in Section 5 when $p(\cdot, \cdot)$ corresponds to a recurrent birth and death chain on $\{0, 1, 2 \cdots\}$ for which $\inf_x p(x, x+1) > 0$. Note that assumption (3.29) below implies (3.30) when $p(\cdot, \cdot)$ is transient by Theorem 3.6.

THEOREM 3.28. *Suppose*

$$\nu_\infty \in \mathcal{I}, \tag{3.29}$$

and

$$\Sigma_y P^y(\sigma_x < \sigma_y) = \infty \tag{3.30}$$

for some, and hence every $x \in S$. *Then*

$$0 < \Sigma_x[1 - \eta(x)] < \infty \text{ implies } \eta \notin D, \text{ and} \tag{3.31}$$

$$\eta_t \text{ does not have the Feller property.} \tag{3.32}$$

PROOF. Conclusion (3.31) implies (3.32) very simply. To see this, let η_n be a sequence in U satisfying $0 < \Sigma_x[1 - \eta_n(x)] < \infty$ and $\eta_n(x) \to 0$ for each x. Then $P^{\eta_n}[\eta_t(x) = 1] = P^1[\eta_t(x) = 1] = 1$ for $t > 0$ by Theorem 2.15 and assumption (3.29). On the other hand, $P^0[\eta_t(x) = 1] = 0$ by definition, where 0 is the empty configuration. In order to prove (3.31), it suffices to prove it for η such that $\Sigma_x[1 - \eta(x)] = 1$. To see this, we argue by induction. Suppose we know that $\eta \notin D$ whenever $0 < \Sigma_x[1 - \eta(x)] < n$, and let η satisfy $\Sigma_x[1 - \eta(x)] = n$. Let u be a site for which $\eta(u) = 0$, and note that, by a coupling argument,

$$\Sigma_x\big[P^{\eta}[\eta_t(x) = 0] - P^{\eta_u}[\eta_t(x) = 0]\big] \leqslant 1 \tag{3.33}$$

for $t > 0$. Since $\Sigma_x[1 - \eta_u(x)] = n - 1$, the induction assumption gives $\eta_u \notin D$. But then Theorem 2.15 and (3.29) imply that $P^{\eta_u}[\eta_t(x) = 0] = 0$. Thus $P^{\eta}[\eta_t \in d\zeta]$ concentrates on $\{\zeta \in U : \Sigma_x[1 - \zeta(x)] \leqslant 1\}$, so that $P^{\eta}[\eta_t \equiv 1] = 1$ follows again from the induction assumption, Theorem 2.15, (3.29), and the Markov property. This implies that $\eta \notin D$, so that the induction step is complete. It remains to prove that $\eta \notin D$ when $\Sigma_x[1 - \eta(x)] = 1$. To do this, fix $x \in S$ and assume $1_x \in D$, from which we will deduce a contradiction. Let R be a finite subset of S such that $x \in R$, and let η be a finite configuration which satisfies $\eta = 1$ on R. Define $\tau = \inf\{t > 0 : \eta_t(x) = 1\}$ and let σ_{R^c} be the hitting time of R^c for the chain $\{X_n\}$. Then

$$e^t P^{\eta_x}[\eta_t(x) = 1] \geqslant P^{\eta_x}[\tau \leqslant t], \tag{3.34}$$

which is bounded below by

$$\Sigma_{y \in R} P^y[\sigma_x < \sigma_{R^c} \wedge \sigma_y] \int_0^t e^{-s} P^{\eta_x}[\eta_s(u) = 1 \text{ for } u \in R \setminus \{x\}]\, ds - |R|^2(1 - e^{-t})^2. \tag{3.35}$$

By (2.1), the expression in (3.35) with η_x replaced by 1_x is a lower bound for $e^t P^{1_x}[\eta_t(x) = 1]$. By (3.29) and (3.33), $P^{1_x}[\eta_s(u) = 1 \text{ for } u \in R \setminus \{x\}, \eta_s(x) = 0] = P^{1_x}[\eta_s(x) = 0]$, while the assumption that $1_x \in D$ gives $\lim_{s \downarrow 0} P^{1_x}[\eta_s(x) = 0] = 1$. Putting these facts together, it follows that

$$\liminf_{t \downarrow 0} \frac{P^{1_x}[\eta_t(x) = 1]}{t} \geqslant \Sigma_{y \in R} P^y[\sigma_x < \sigma_{R^c} \wedge \sigma_y].$$

Letting $R \uparrow S$ and using (3.30), this gives

$$\lim_{t \downarrow 0} \frac{P^{1_x}[\eta_t(x) = 1]}{t} = \infty. \tag{3.36}$$

$P^{\eta}[\tau \geqslant t]$ is monotone decreasing in η for $|\eta| < \infty$, so we may define the function

$h(t) = \lim_{\eta \uparrow 1_x, |\eta| < \infty} P^{\eta}[\tau \geq t]$. Using assumption (3.29) again, an argument similar to that above implies that $h(t + s) = h(t)h(s)$ for $s > 0$ and $t > 0$. By (3.36), $\lim_{t \downarrow 0}(1 - h(t))/t = \infty$. Therefore $h(t) = 0$ for all $t > 0$, and hence by (3.34), $\lim_{t \downarrow 0} P^{1_x}[\eta_t(x) = 1] = 1$. This implies that $1_x \in D^c$, which gives the desired contradiction.

4. The random walk case. Throughout this section, we will assume that $S = Z^d$, the d-dimensional integers, and that $p(x,y) = p(0,y - x)$. In this context, we will be able to determine under a mild assumption exactly when $\nu_\pi \in \mathcal{I}$ for $\pi P = \pi$, and will be able to prove that ν_∞ is always in $\mathcal{I}$. Furthermore, we will evaluate $\lim_{t \to \infty} \nu_\pi S(t)$ for a class of cases for which $\nu_\pi \notin \mathcal{I}$. Let $\mathbb{S}$ be the set of probability measures on U which are translation invariant. Of course, $\mu \in \mathbb{S}$ implies $\mu S(t) \in \mathbb{S}$ for all t.

LEMMA 4.1. *Suppose* $\mu \in \mathbb{S}$ *and* $\mu\{1\} = 0$. *Then*
(a) $E^x \mu\{\eta : \eta(X_n) = 1$ *for all* $n\} = 0$, *and*
(b) $\int \Omega f_x \, d\mu = 0$ *for all* $x \in S$.

PROOF. By the Hewitt-Savage zero-one law,

$$P^x\left[\cup_{n=1}^{\infty}\{\eta(X_k) = 1 \text{ for all } k \geq n\}\right] = 0 \text{ or } 1$$

for all $\eta \in U$ and $x \in S$. Since the random walk is irreducible, this value does not depend on x. Let

$$\Lambda = \left\{\eta \in U : P^x\left[\cup_{n=1}^{\infty}\{\eta(X_k) = 1 \text{ for all } k \geq n\}\right] = 1\right\}.$$

Then Λ is a translation invariant set, so $\mu(\Lambda) = 0$ or 1 if μ is ergodic. Suppose $\mu(\Lambda) = 1$. Then

$$\lim_{n \to \infty} \frac{1}{n} \Sigma_{k=1}^{n} \eta(X_k) = 1$$

a.s. with respect to the product measure $\mu \times P^x$. Therefore $\lim_{n \to \infty} 1/n \Sigma_{k=1}^{n} \int E^x \eta(X_k) \, d\mu = 1$. But $\int E^x \eta(X_k) \, d\mu = E^x \mu\{\eta : \eta(X_k) = 1\} = \mu\{\eta : \eta(x) = 1\}$, so that $\mu = \nu_\infty$. It follows that $\mu(\Lambda) = 0$ for every ergodic $\mu \in \mathbb{S}$ other than ν_∞. Part (a) then follows from the fact that every element of $\mathbb{S}$ is a mixture of ergodic elements of $\mathbb{S}$. Turning to the proof of part (b), note that (3.12) and the bounded convergence theorem give

$$\lim_{k \to \infty} \int \eta(x) \delta_k(x, \eta) \, d\mu = \int \eta(x) E^x\left[\Pi_{n=1}^{\infty} \eta(X_n), \sigma_x = \infty\right] d\mu,$$

and this last term is zero by part (a). Since $\mu \in \mathbb{S}$, $\int \eta(x)[1 - \eta(y)] q_k(x,y,\eta) \, d\mu$ is a function of $y - x$, so that

$$\Sigma_y \int \eta(y)\left[1 - \eta(x)\right] q_k(y,x,\eta) \, d\mu = \Sigma_y \int \eta(x)\left[1 - \eta(y)\right] q_k(x,y,\eta) \, d\mu,$$

which equals $\int \eta(x)[1 - \delta_k(x,\eta) - q_k(x,x,\eta)] \, d\mu$ by the definition of $\delta_k(x,\eta)$. Therefore,

$$\int \Omega f_x \, d\mu = \lim_{k \to \infty} \int \Omega_k f_x \, d\mu = 0$$

by (3.20).

THEOREM 4.2. *If $\pi(\cdot)$ is constant, then $\nu_\pi \in \mathcal{I}$.*

PROOF. Since $\pi(\cdot)$ is constant, $\nu_\pi S(t) \in \mathcal{S}$ for all $t \geqslant 0$. Also $\nu_\pi S(t) \leqslant \nu_\pi$, so that $\nu_\pi S(t)\{1\} = 0$. Therefore $\int S(t) f_x \, d\nu_\pi = \int f_x \, d\nu_\pi$ for all t by Lemma 4.1 and part (c) of Theorem 3.19. Since $\nu_\pi S(t) \leqslant \nu_\pi$ and $\nu_\pi S(t)\{\eta(x) = 1\} = \nu_\pi\{\eta(x) = 1\}$, it follows that $\nu_\pi S(t) = \nu_\pi$.

The structure of the set of invariant measures for a random walk is well known [2]. In order to describe it, let $\varphi(a) = \Sigma_{x \in S} \exp[-\langle x, a \rangle] p(0, x)$ for $a \in R^d$, where $\langle x, a \rangle$ is the usual inner product in R^d, and let $T = \{a \in R^d : \varphi(a) = 1\}$. Then $\pi P = \pi$ if and only if π is of the form

$$\pi(x) = c \int_T e^{\langle x, a \rangle} \gamma(da) \tag{4.3}$$

for some constant c and some probability measure γ on T. Of course $\pi(\cdot) =$ constant is obtained by letting γ be the pointmass at $0 \in T$. For many random walks, $T \neq \{0\}$, so that there are nonconstant π's which satisfy $\pi P = \pi$.

COROLLARY 4.4. $\nu_\infty \in \mathcal{I}$.

PROOF. This is an immediate consequence of Theorem 3.1 and 4.2.

COROLLARY 4.5. (a) *$D^c \neq \phi$, and* (b) *the process η_t does not have the Feller property.*

PROOF. This follows from Theorem 3.28 and Corollary 4.4, once we have verified (3.30). Since $P^y(\sigma_x < \sigma_y)$ is a function of $y - x$ only, in order to verify (3.30) it suffices to check that $\Sigma_x P^o(\sigma_x < \sigma_0) = \infty$. Let R_n be the number of distinct points in $\{X_0, \cdots, X_n\}$. Then $\Sigma_x P^o(\sigma_x < \sigma_0) = E^o[R_{\sigma_0}]$ if R_∞ is interpreted to be $+\infty$. If $p(\cdot, \cdot)$ is transient, $R_{\sigma_0} = \infty$ with positive probability, so that $E^o[R_{\sigma_0}] = \infty$. In the recurrent case, $E^o[R_{\sigma_0}] = \infty$ is problem 11 on page 394 of [14]. It is easy to show, by comparing the chain with its marginals that it is enough to prove this in one dimension, and, of course, it suffices to consider the aperiodic case. The proof of this problem in one dimension in the aperiodic case is an application of two potential theoretic facts. The first is that $P^o(\sigma_x < \sigma_0) = [a(x) + a(-x)]^{-1}$, which follows from Theorem 2 of Section 30 of [14]. The second is Proposition 4 of Section 28 of [14], which asserts that

$$\lim_{|x| \to \infty} \frac{a(x) + a(-x)}{|x|} = 2\left[\Sigma_x x^2 p(0, x)\right]^{-1},$$

even if $\Sigma_x x^2 p(0, x) = \infty$. Here $a(x)$ is the recurrent potential kernel. These two facts imply that $|x| P^o(\sigma_x < \sigma_0)$ is bounded below, so that $\Sigma_x P^o(\sigma_x < \sigma_0) = \infty$.

It is probably true in the general random walk case that $\nu_\pi \notin \mathcal{I}$ whenever $\pi P = \pi$ and π is not constant. This would follow from Corollary 3.26 if we could verify condition (3.27) in this context. While (3.27) probably holds without further assumption, we have been unable to verify it without imposing a mild regularity condition on the random walk. In order to describe this condition, let T and φ be as in the paragraph following the proof of Theorem 4.2. For each $a \in T \setminus \{0\}$, consider all hyperplanes L through the origin in R^d such that $a/2$ is in the (relative) interior of $L \cap \{\varphi < \infty\}$. The line through 0 and a has this property, since φ is strictly convex and $\varphi(0) = \varphi(a) = 1$. Furthermore, if L_1 and L_2 are two hyperplanes with this property, then the hyperplane spanned by L_1 and L_2 also has the property. Thus there is a unique maximal hyperplane which has this property. It will be denoted by L_a. Let $\mathfrak{L}$ be the set of all distinct hyperplanes of the form L_a for some $a \in T \setminus \{0\}$.

LEMMA 4.6. *Assume that* $T \neq \{0\}$.
(a) *If* $\varphi < \infty$ *in a neighborhood of the origin, then* $\mathfrak{L} = \{R^d\}$;
(b) *if* $\{\varphi < \infty\}$ *is strictly convex, then* $\mathfrak{L} = \{R^d\}$;
(c) *if* $d = 1$, *then* $\mathfrak{L} = \{R^1\}$;
(d) *if* $d = 2$, *then* $\mathfrak{L}$ *contains at most three elements*;
(e) *if the d coordinates of the random walk are independent, then* $\mathfrak{L}$ *contains at most* $2^d - 1$ *elements.*

PROOF. Statements (a) and (b) are immediate, and (c) follows from (b). For (d), assume that $\mathfrak{L}$ contains more than three elements. Then it must contain at least three lines, say $L_{a_1}, L_{a_2}, L_{a_3}$. Since φ is strictly convex, the convex hull of $\{0, a_1, a_2, a_3\}$ is a quadrilateral. But then if a_i is the vertex opposite 0 in this quadrilateral, $L_{a_i} = R^2$, which gives a contradiction. For (e), it suffices to note that for each $a \in T \setminus \{0\}$, L_a must be a coordinate hyperplane.

THEOREM 4.7. *Suppose* π *is given by* (4.3), *and assume that for some* $L \in \mathfrak{L}$, $\gamma\{a \in T \setminus \{0\} : L_a = L\} > 0$. *Then* $v_\pi \notin \mathcal{G}$.

PROOF. For $a \in T$, abbreviate $E^x_{\pi_a}$ and $P^x_{\pi_a}$ by E^x_a and P^x_a, where $\pi_a(x) = \exp\langle x, a\rangle$. Then

$$\pi(x)P^x_\pi(\cdot) = c\textstyle\int_T \pi_a(x)P^x_a(\cdot)\gamma(da),$$

so that to verify condition (3.27) for π, it suffices to show that γ puts positive measure on the set of a's for which

$$P^0_a\left[\sigma_0 = \infty, \Sigma^\infty_{n=1}\frac{1}{\pi(X_{\tau_n})} < \infty\right] > 0.$$

Since $\Sigma^\infty_{n=1}1/\pi(X_{\tau_n}) \leqslant \Sigma^\infty_{n=1}1/\pi(X_n)$, since the events $\{\sigma_0 = \infty\}$ and $\{\Sigma^\infty_{n=1}1/\pi(X_n) < \infty\}$ are independent by the strong Markov property, and since $\{X_n\}$ relative to P^0_a is transient, it suffices to show that γ puts positive measure on the set of a's for which

$$P^0_a\left[\Sigma^\infty_{n=1}\frac{1}{\pi(X_n)} < \infty\right] > 0.$$

In order to show this, note that relative to P^0_a, $\langle X_n, a - b\rangle$ is a random walk with

Laplace transform

$$\psi(t) = E_a^0 e^{-t\langle X_1, a-b\rangle} = \varphi[(1-t)a + tb].$$

Since $\psi(0) = \varphi(a) = 1$, if $\psi(1) = \varphi(b) \leqslant 1$, it then follows by the strong law of large numbers that

$$P_a^o\left[\liminf_{n\to\infty} \frac{1}{n}\langle X_n, a-b\rangle > 0\right] = 1.$$

Let L be as in the statement of the theorem, and let $a \in T\setminus\{0\}$ be such that $L_a = L$. By definition, and by the fact that a convex function is continuous on any open set on which it is finite, there is a neighborhood N of $a/2$ such that $\varphi(b) \leqslant 1$ for all $b \in N \cap L$. Therefore, there is a neighborhood N' of a such that

$$P_a^0\left[\liminf_{n\to\infty} \frac{1}{n}\langle X_n, b\rangle > 0\right] = 1$$

for all $b \in N' \cap L$. If $\gamma(N' \cap L) > 0$, it follows from Jensen's inequality that

$$[\pi(x)]^{1/n} \geqslant [c\gamma(N' \cap L)]^{1/n} \int_{N'\cap L} e^{(1/n)\langle x, b\rangle} \frac{\gamma(db)}{\gamma(N' \cap L)}.$$

Putting these facts together, we see by Fatou's lemma that for every a such that $L_a = L$ which is in the support of the restriction of γ to L,

$$P_a^0\left[\liminf_{n\to\infty}[\pi(X_n)]^{1/n} > 1\right] = 1.$$

This gives

$$P_a^0\left[\Sigma_{n=1}^{\infty} 1/\pi(X_n) < \infty\right] = 1$$

for such a, thus concluding the proof of the theorem.

COROLLARY 4.8. *If $\mathfrak{L}$ consists of at most countably many hyperplanes, then $\nu_\pi \notin \mathcal{I}$ whenever $\pi P = \pi$ and π is not constant. In particular, this is the case whenever any of the assumptions of Lemma 4.6 are satisfied.*

When π is not constant, it is of interest to determine the limit of $\nu_\pi S(t)$ as $t \to \infty$. The existence of this limit is, of course, guaranteed by Theorem 3.17. We will determine this limit in one dimension under the assumption that the random walk has a finite mean. Our result is probably true for an arbitrary random walk in any number of dimensions, but the proof in general would probably be substantially more complicated. For the remainder of this section, we will assume then that $S = Z^1$, $\Sigma_x |x| p(0,x) < \infty$, and that there exists a nonconstant π such that $\pi P = \pi$. It follows from the existence of this π that $\Sigma_x x p(0,x) \neq 0$, so we may assume without loss of generality that $m = \Sigma_x x p(0,x) > 0$. It then follows that $\Sigma_{x<0} |x|^k p(0,x) < \infty$ for all $k \geqslant 1$, and that $\pi(x) = c_1 + c_2\lambda^x$ for some $\lambda > 1$.

LEMMA 4.9. *Let l be any real number other than an integer. Then*

(a) $\Sigma_{y<l<x}\eta(x)[1-\eta(y)]q(x,y,\eta)$ *and*

(b) $\eta(x)\Sigma_{y<l}[1-\eta(y)]q(x,y,\eta)$

are continuous functions of η.

PROOF. Since $q(x,y,\eta)$ is continuous in η for each $x,y \in S$, it suffices to show that the above series converge uniformly in η. This follows from the inequality

$$\eta(x)[1-\eta(y)]q(x,y,\eta) \leq P^x(\sigma_y < \infty),$$

and from

$$\Sigma_{y<l<x} P^x(\sigma_y < \infty) \leq \Sigma_{k=1}^{\infty} kP^o[\min_{n>0} X_n \leq -k]$$
$$\leq E^o[\min_{n>0} X_n]^2 < \infty$$

since $\Sigma_{x<0}|x|^3 p(0,x) > \infty$ (see [8]).

For any real number l other than an integer, let

$$H_l(\eta) = \Sigma_{x<l<y}\eta(x)[1-\eta(y)]q(x,y,\eta) - \Sigma_{y<l<x}\eta(x)[1-\eta(y)]q(x,y,\eta)$$
$$+ \Sigma_{x<l}\eta(x)[1 - \Sigma_y[1-\eta(y)]q(x,y,\eta) - q(x,x,\eta)].$$

The interpretation is that $H_l(\eta)$ is the net rate at which particles move across l from left to right when the configuration is η. Note that the third term corresponds to particles which start at a point to the left of l and "disappear" at $+\infty$. Of course, $H_l(\eta)$ may be $+\infty$ for some η, but it is lower semicontinuous by Lemma 4.9, and hence is bounded below. Therefore $\int H_l d\mu$ is well defined for each probability measure μ.

Let $\{\tau_k\}$ be defined as in Section 3, and let $\{\mathcal{F}_n\}$ be the natural σ-algebras associated with the random walk $\{X_n\}$. Of course, τ_k is a stopping time for each k. The σ-algebras $\{\mathcal{F}_{\tau_k \wedge \sigma_0}\}$ are defined as usual.

LEMMA 4.10. *(a)* $E^o(\tau_k) \leq ck$ *for some constant* c. *(b)* $E[X_{\tau_k\wedge\sigma_0} - X_{\tau_j\wedge\sigma_0} | \mathcal{F}_{\tau_j\wedge\sigma_0}] = mE[\tau_k \wedge \sigma_0 - \tau_j \wedge \sigma_0 | \mathcal{F}_{\tau_j\wedge\sigma_0}] \geq 0$ *for* $j < k$.

PROOF. Part (a) follows easily from the fact that starting from zero, the expected hitting time of the positive axis is finite, since τ_k is less than or equal to the time of the kth new maximum for $\{X_n\}$. Part (b) comes from part (a) and the fact that $\{X_n - nm\}$ is a martingale.

THEOREM 4.11. *Suppose* $\mu \in \mathbb{S}$, *and define*

$$\rho(T) = \mu\{\eta : \eta(x) = 1 \text{ for all } x \in T\}$$

for finite subsets T *of* S. *Then*

$$\int H_l d\mu = m\Sigma_{k=1}^{\infty} E^o[\rho(\{X_0,\ldots,X_{\tau_k}\})[\tau_{k+1}\wedge\sigma_0 - \tau_k\wedge\sigma_0]], \tag{4.12}$$

where both sides may be $+\infty$.

REMARK. An important feature of the expression on the right side is that it is an increasing function of μ. This monotonicity will be used in a crucial way in the proof of the convergence theorem. It is interesting to note that the analogous expression for the simple exclusion process is

$$\Sigma_x xp(0,x)[\rho(\{0\}) - \rho(\{0,x\})],$$

which is not monotone in μ. This is one of the few ways in which the long range exclusion process is easier to deal with than the simple exclusion process.

PROOF. If $\mu\{1\} > 0$, both sides of (4.12) are $+\infty$, so we may assume without loss of generality that $\mu\{1\} = 0$. By part (a) of Lemma 4.1, μ concentrates on the set of η for which

$$\eta(x)\{\Sigma_y[1-\eta(y)]q(x,y,\eta) + q(x,x,\eta)\} = \eta(x),$$

so that the integral of the third sum in the definition of $H_l(\eta)$ is zero. Since $\mu \in \mathbb{S}$, $\int\eta(x)[1-\eta(y)]q(x,y,\eta)\,d\mu$ is a function of $y-x$ only. Thus $\int H_l(\eta)\,d\mu$ equals the integral with respect to μ of

$$\Sigma_y y E^o\big[\Pi_{n=0}^{\sigma_y-1}\eta(X_n)[1-\eta(y)], \sigma_y < \sigma_0\big],$$

which is

$$E^o\Big[\Sigma_y y\Big[\rho\big(\{X_0,\cdots,X_{\sigma_y-1}\}\big) - \rho\big(\{X_0,\cdots,X_{\sigma_y}\}\big)\Big]1_{\{\sigma_y<\sigma_0\}}\Big].$$

The order of integration and expectation can be interchanged at will because $\Sigma_{y<0}|y|P^o(\sigma_y < \infty) < \infty$. Hence

$$\int H_l\,d\mu = E^0\Sigma_{k=1}^{\infty}X_{\tau_{k+1}\wedge\sigma_0}\big[W_k - W_{k+1}\big], \tag{4.13}$$

where $W_k = \rho(\{X_0,\cdots,X_{\tau_k}\})$. Equation (4.13) comes from the fact that with probability one, $\{\tau_k, k \geq 2\}$ is the same as the set of finite times in $\{\sigma_y, y \neq 0\}$. Note that if $\tau_{k+1} = \sigma_y$, it is not the case in general that $\tau_k = \sigma_y - 1$, but it is true that $\{X_0,\cdots,X_{\sigma_y-1}\} = \{X_0,\cdots,X_{\tau_k}\}$. Summing by parts gives

$$\int H_l\,d\mu = \lim_{N\to\infty}\Big\{\Sigma_{k=1}^{N-1}E^o\Big[\big(X_{\tau_{k+1}\wedge\sigma_0} - X_{\tau_k\wedge\sigma_0}\big)W_k\Big] - E^o\Big[X_{\tau_N\wedge\sigma_0}W_N\Big]\Big\}. \tag{4.14}$$

Since $W_k = \rho(\{X_0,\cdots,X_{\tau_k\wedge\sigma_0}\})$ on the set $\{\tau_{k+1}\wedge\sigma_0 \neq \tau_k\wedge\sigma_0\}$ and $\rho(\{X_0,\cdots X_{\tau_k\wedge\sigma_0}\}) \in \mathcal{F}_{\tau_k\wedge\sigma_0}$, it follows from Lemma 4.10 that

$$E^o\Big[\big(X_{\tau_{k+1}\wedge\sigma_0} - X_{\tau_k\wedge\sigma_0}\big)W_k\Big] = mE^o\big[(\tau_{k+1}\wedge\sigma_0 - \tau_k\wedge\sigma_0)W_k\big], \tag{4.15}$$

which is nonnegative. Now, $E^o[X_{\tau_N\wedge\sigma_0}W_N]$ is bounded below since $W_N \leq 1$ and $E^o[\min_{n\geq 0}X_n] > -\infty$. Therefore if $\int H_l\,d\mu = \infty$, it follows that the right side of (4.12) is ∞ as well, and therefore the result holds. Assume then that $\int H_l\,d\mu < \infty$. By part (a) of Lemma 4.1, $W_N \to 0$ a.s., so that

$$\begin{aligned} E^o\Big[X_{\tau_N\wedge\sigma_0}W_N\Big] &= E^o\Big[X_{\tau_N\wedge\sigma_0}\Sigma_{j=N}^{\infty}(W_j - W_{j+1})\Big] \\ &\leq E^o\Big[\Sigma_{j=N}^{\infty}X_{\tau_{j+1}\wedge\sigma_0}(W_j - W_{j+1})\Big] \\ &\quad + E^o\Big[W_N\sup_{j\geq N}\big(X_{\tau_N\wedge\sigma_0} - X_{\tau_j\wedge\sigma_0}\big)\Big]. \end{aligned} \tag{4.16}$$

Since $\int H_l\,d\mu < \infty$, it follows from (4.13) that the first term on the right side of the inequality in (4.16) tends to zero as $N \to \infty$. On the other hand, the strong Markov property for $\{X_n\}$ gives

$$E^o\Big[W_N \sup_{j \geqslant N}\big(X_{\tau_N \wedge \sigma_0} - X_{\tau_j \wedge \sigma_0}\big)\Big] \leqslant E^o(W_N)E^o|\min_{n>0} X_n|.$$

To get this, one must observe that even though $W_N \notin \mathcal{F}_{\tau_N \wedge \sigma_0}$, it is true that $W_N 1_{\{\tau_N < \sigma_0\}}$ is $\mathcal{F}_{\tau_N \wedge \sigma_0}$ measurable, and $\sup_{j \geqslant N}(X_{\tau_N \wedge \sigma_0} - X_{\tau_j \wedge \sigma_0}) = 0$ on $\{\sigma_0 < \tau_N\}$. Therefore, since $W_N \to 0$ a.s., the second term on the right side of (4.16) tends to zero as well. The desired result then follows from (4.14) and (4.15).

LEMMA 4.17. *Let* $\mu = \lim_{t\to\infty} \nu_\pi S(t)$, *where* $\pi P = \pi$. *Then* $\int H_l \, d\mu$ *is independent of* l.

PROOF. It suffices to show that $\int H_l \, d\mu = \int H_{l+1} \, d\mu$. We may assume without loss of generality that either $\int H_l \, d\mu$ or $\int H_{l+1} \, d\mu$ is finite, since otherwise they are both $+\infty$. Since at least one is finite, we may write

$$\begin{aligned}\int H_{l+1}\, d\mu - \int H_l \, d\mu &= \int \eta(u)\Sigma_y[1 - \eta(y)]q(u,y,\eta)\, d\mu \\ &\quad - \int [1 - \eta(u)]\Sigma_x \eta(x)q(x,u,\eta)\, d\mu \\ &\quad + \int \eta(u)\big[1 - \Sigma_y[1-\eta(y)]q(u,y,\eta) - q(u,u,\eta)\big]\, d\mu,\end{aligned}$$

where u is the integer between l and $l+1$. Therefore $\int H_{l+1} \, d\mu - \int H_l \, d\mu = -\lim_{k\to\infty} \int \Omega_k f_u \, d\mu = -\int \Omega f_u \, d\mu = 0$ by Theorem 3.19, which completes the proof.

LEMMA 4.18. *If* $\pi P = \pi$, *then*

$$\Sigma_{x<l} \int E^x\big[\Pi_{n=0}^{\sigma-1}\eta(X_n)\big]\, d\nu_\pi < \infty$$

where $\sigma = \min\{n \geqslant 1 : X_n > l\}$.

PROOF. Let $\gamma = \sup_{x<l}\pi(x)[1+\pi(x)]^{-1}$, which is less than 1, and let N be the number of distinct points in $\{X_0, \cdots, X_{\sigma-1}\}$.

$$\begin{aligned}\Sigma_{x<l}\int E^x\big[\Pi_{n=0}^{\sigma-1}\eta(X_n)\big]\, d\nu_\pi &\leqslant \Sigma_{x<l}E^x\gamma^N \\ &= \Sigma_{k=1}^\infty \gamma^k \Sigma_{x<l} P^x[N = k] \\ &\leqslant \Sigma_{k=1}^\infty \gamma^k \Sigma_{k<l} P^x[X_{\tau_{k+1}} > l] \\ &\leqslant \Sigma_{k=1}^\infty \gamma^k E^o|X_{\tau_{k+1}}| \\ &\leqslant [\Sigma_x |x| p(0,x)]\Sigma_{k=1}^\infty \gamma^k E(\tau_{k+1}) < \infty\end{aligned}$$

by Lemma 4.10.

THEOREM 4.19. *Under the assumptions stated just before Lemma* 4.9,

$$\lim_{t\to\infty}\nu_\pi S(t) = \nu_{\bar\pi}$$

whenever $\pi P = \pi$, *where* $\bar\pi$ *is the constant* $\inf_x \pi(x)$.

PROOF. We may assume that π is not constant, since otherwise the result is an immediate consequence of Theorem 4.2. Let $\mu = \lim_{t\to\infty}\nu_\pi S(t)$, which exists by

Theorem 3.17. Let μ_n be the translate of μ defined by

$$\mu_n\{\eta : \eta(x) = 1 \text{ for } x \in T\} = \mu\{\eta : \eta(x + n) = 1 \text{ for } x \in T\}.$$

Since $\pi(x)$ is increasing in x, ν_π is monotone under translations. Since $S(t)$ maps $\mathfrak{M}$ into itself, $\nu_\pi S(t)$ is monotone under translations for each t. Therefore $\mu_n \leq \mu_{n+1}$, and hence $\lim_{n\to-\infty}\mu_n$ and $\lim_{n\to+\infty}\mu_n$ both exist. Since $\nu_{\bar{\pi}} \leq \mu \leq \nu_\pi$, and $\lim_{x\to-\infty}\pi(x) = \bar{\pi}$, it follows that $\lim_{n\to-\infty}\mu_n = \nu_{\bar{\pi}}$. It suffices then to show that $\nu = \lim_{n\to\infty}\mu_n$ is $\nu_{\bar{\pi}}$ as well, since then by monotonicity, $\mu_n = \nu_{\bar{\pi}}$ for each n. Since ν is a limit of translates of a fixed measure, $\nu \in \mathcal{S}$. Since μ_n is a translate of μ, $\int H_{k+\frac{1}{2}}\, d\mu_n$ is a function of $n + k$ only. Therefore by Lemma 4.17, $\int H_l d\mu_n$ does not depend on n or l. The dominated convergence theorem and Lemma 4.18 give

$$\int H_l d\nu_{\bar{\pi}} = \lim_{n\to-\infty}\int H_l d\mu_n,$$

since $\mu_n \leq \nu_\pi$ for $n \leq 0$, while

$$\int H_l d\nu \leq \lim_{n\to+\infty}\int H_l d\mu_n$$

is a consequence of the lower semicontinuity of $H_l(\eta)$. Therefore $\int H_l d\nu \leq \int H_l d\nu_{\bar{\pi}}$. Hence $\nu\{\eta : \eta(0) = 1\} = \nu_{\bar{\pi}}\{\eta : \eta(0) = 1\}$ follows from $\nu, \nu_{\bar{\pi}} \in \mathcal{S}$, $\nu \geq \nu_{\bar{\pi}}$, $\int H_l d\nu_{\bar{\pi}} < \infty$ and Theorem 4.11. Using $\nu \geq \nu_{\bar{\pi}}$ again gives $\nu = \nu_{\bar{\pi}}$, thus completing the proof.

5. The birth and death case. Throughout this section, we will assume that $S = \{0, 1, 2, \cdots\}$ and $p(x,y) = 0$ if $|y - x| \geq 2$. As is well known in this case, $\pi P = \pi$ if and only if π is a constant multiple of

$$\tilde{\pi}(x) = \Pi_{y=0}^{x-1}\frac{p(y,y+1)}{p(y+1,y)},$$

and $p(\cdot\,,\cdot)$ is recurrent if and only if $\Sigma_{x=0}^{\infty}\gamma(x) = \infty$, where

(5.1) $$\gamma(x) = \Pi_{y=0}^{x-1}\frac{p(y+1,y)}{p(y+1,y+2)} = \frac{p(0,1)}{\tilde{\pi}(x)p(x,x+1)}.$$

THEOREM 5.2. *Suppose $p(\cdot\,,\cdot)$ is transient. Then*

(a) $\nu_\infty \notin \mathcal{I}$, *and*

(b) $\nu_\pi \notin \mathcal{I}$ *for any π satisfying $\pi P = \pi$.*

PROOF.

$$\begin{aligned} P^x(\sigma_0 < \sigma_x) &= p(x,x-1)P^{x-1}(\sigma_0 < \sigma_x) \\ &= \frac{p(x,x-1)\gamma(x-1)}{\Sigma_{y=0}^{x-1}\gamma(y)} \leq \frac{\gamma(x-1)}{\gamma(0)}. \end{aligned}$$

Part (a) follows from Theorem 3.6, since $p(\cdot\,,\cdot)$ is transient. For part (b), note that $X_{\tau_n} = n - 1$ for the chain starting at 0. Therefore by Corollary 3.26, it suffices to show that $\Sigma_{x=0}^{\infty}[\pi(x)]^{-1} < \infty$. But this follows from the transience again, since $[\pi(x)]^{-1}$ is bounded above by a constant multiple of $\gamma(x)$.

In order to prove that $\nu_\infty \in \mathcal{I}$ in the recurrent case, we need to use special properties of the birth and death chain. The following is the key preliminary result.

LEMMA 5.3. *Consider the continuous time Markov chain on* $\{1, 2, \cdots\}$ *whose Q-matrix is given by*

$$q(x,y) = \frac{\gamma(y-1)}{\Sigma_{z=y}^{x-1}\gamma(z)} \quad \textit{for} \quad x > y$$
$$= 0 \quad \textit{for} \quad x < y,$$

where $\gamma(\cdot)$ *is an arbitrary positive sequence. Let* $q_t(x,y)$ *be the corresponding transition probabilities at time* t. *If* $\Sigma_x\gamma(x) = \infty$, *then* $\lim_{x\to\infty}q_t(x,y) = 0$ *for all* $t > 0$ *and* $y \geq 1$.

PROOF. Let $\mathcal{L}f(x) = \Sigma_y q(x,y)[f(y) - f(x)]$ be the generator of the Markov chain. Define $\Gamma(0) = 0$, $\Gamma(x) = \Sigma_{y=0}^{x-1}\gamma(y)$ for $x \geq 1$, and $f_o(x) = h[\Gamma(x)]$ for $x \geq 1$, where $h(s) = s^{-\frac{1}{2}}$. It suffices to prove that

$$0 \leq \mathcal{L}f_o \leq 2f_o, \tag{5.4}$$

since then

$$0 \leq \frac{d}{dt}\Sigma_y q_t(x,y)f_o(y) \leq 2\Sigma_y q_t(x,y)f_o(y),$$

which implies that

$$\Sigma_y q_t(x,y)f_o(y) \leq e^{2t}f_o(x).$$

The conclusion of the lemma then follows from $\lim_{x\to\infty}f_o(x) = 0$, which in turn is a consequence of $\Sigma_x\gamma(x) = \infty$. The first inequality in (5.4) is immediate from the monotonicity of f_o and the fact that $q(x,y) > 0$ only for $y < x$. For the second inequality, use the convexity of h to show that

$$\frac{h(s) - h[\Gamma(x)]}{\Gamma(x) - s}$$

is decreasing in s for $0 < s < \Gamma(x)$. Therefore,

$$\frac{h[\Gamma(y)] - h[\Gamma(x)]}{\Gamma(x) - \Gamma(y)} \leq \frac{h(s) - h[\Gamma(x)]}{\Gamma(x) - s}$$

for $0 < s < \Gamma(y)$ and $y < x$. Hence

$$\frac{\mathcal{L}f_o(x)}{f_o(x)} = \frac{\Sigma_{y<x}\frac{\Gamma(y) - \Gamma(y-1)}{\Gamma(x) - \Gamma(y)}\{h[\Gamma(y)] - h[\Gamma(x)]\}}{h[\Gamma(x)]}$$
$$\leq \frac{1}{h[\Gamma(x)]}\int_0^{\Gamma(x)}\frac{h(s) - h[\Gamma(x)]}{\Gamma(x) - s}\,ds.$$

But

$$\frac{1}{h(t)}\int_0^t \frac{h(s)-h(t)}{t-s} = \int_0^t \frac{ds}{s^{\frac{1}{2}}(t^{\frac{1}{2}}+s^{\frac{1}{2}})}$$
$$\leqslant t^{-\frac{1}{2}}\int_0^t \frac{ds}{s^{\frac{1}{2}}} = 2,$$

which completes the proof of (5.4).

THEOREM 5.5. *Suppose $p(\cdot\,,\cdot)$ is recurrent. Then $\nu_\infty \in \mathcal{G}$.*

PROOF. Since

$$P^y[\sigma_x < \sigma_y] \leqslant \frac{\gamma(y-1)}{\Sigma_{z=y}^{x-1}\gamma(z)}$$

for $y < x$, one can construct a bivariate process (η_t, Z_t) on $\{(\eta, z): \eta \in U, \Sigma_x \eta(x) < \infty, z \geqslant 1, \eta(x) = 1 \text{ for } x < z\}$ in such a way that η_t is the long range exclusion process and Z_t is the Markov chain defined in Lemma 5.3 using the $\gamma(\cdot)$ from (5.1). Therefore, for fixed z, if $\eta(x) = 1$ for all $x < z$, it follows that

$$P^\eta[\eta_t(x) = 0] \leqslant \Sigma_{y<x} q_t(z,y)$$

for all $x < z$ and all t. Hence

$$P^1[\eta_t(x) = 0] \leqslant \lim_{z\to\infty}\Sigma_{y<x} q_t(z,y) = 0$$

by (2.1) and Lemma 5.3, and this implies that $\nu_\infty \in \mathcal{G}$.

COROLLARY 5.6. *If $p(\cdot\,,\cdot)$ is recurrent and $\inf_x p(x, x+1) > 0$, then $D^c \neq \varnothing$ and η_t does not have th.. Feller property.*

PROOF. By Theorems 3.28 and 5.5, it suffices to show that $\Sigma_y P^y[\sigma_0 < \sigma_y] = \infty$. But

$$P^y[\sigma_o < \sigma_y] = \frac{p(y, y+1)\gamma(y)}{\Sigma_{z=0}^{y-1}\gamma(z)},$$

and

$$\Sigma_{y=1}^{\infty}\frac{\gamma(y)}{\Sigma_{z=0}^{y-1}\gamma(z)} \geqslant \int_{\gamma(0)}^{\infty}\frac{1}{x}\,dx = \infty$$

since $\Sigma_y \gamma(y) = \infty$, so the result follows.

6. Open problems. In this section we list some of the questions which have arisen naturally in this paper, but which have not been resolved:

(a) In the transient case, is (3.7) a necessary as well as a sufficient condition for $\nu_\infty \notin \mathcal{G}$?

(b) Is it always true that $\nu_\infty \in \mathcal{G}$ in the recurrent case?

(c) Is (3.27) a necessary as well as a sufficient condition for $\nu_\pi \notin \mathcal{G}$ when $\pi P = \pi$? In particular, is $\nu_\pi \in \mathcal{G}$ whenever $p(\cdot\,,\cdot)$ is recurrent and $\pi P = \pi$? Is

$\nu_\pi \in \mathcal{I}$ whenever $p(\cdot, \cdot)$ is doubly stochastic and π is constant?

(d) If $\pi P = \pi$, does $\nu_\pi \in \mathcal{I}$ imply $\nu_{c\pi} \in \mathcal{I}$ for all constants c?

(e) Does $\nu_\infty \notin \mathcal{I}$ imply that $\nu_\infty S(t) \to \nu_0$ and hence that $\mathcal{I} = \{\nu_0\}$?

(f) Is $P^\eta[\eta_t(x) = 1]$ continuous in t for $t > 0$? This would be nice to know if one is interested in path properties of the process, since it would imply that the right continuous modification of η_t (which does exist) is a version of η_t.

(g) If one replaces η_t by its right continuous modification, is $P^\eta[\eta_t \in D$ for all $t] = 1$ for $\eta \in D$?

(h) Is $\mathcal{I}$ weakly closed? Is $\lim_{t\to\infty}\mu S(t) \in \mathcal{I}$ when the limit exists? These are trivial facts for Feller processes, but as has been shown, η_t is usually not Feller.

One collection of questions deals not with η_t directly, but rather with the sequence of random variables $\{\tau_n\}$ which were introduced in Section 3. The fact that these come up naturally in the study of η_t suggests that it would be of interest to know more about them. One question which is motivated by Lemma 4.10 and the expression for $\int H_t d_\mu$ in Theorem 4.11 is the following: for a general random walk $p(\cdot, \cdot)$ how big can $E^o(\tau_k)$ be? It is easy to show using the fact that

$$\sup_x P^0[X_n = x] \leqslant \frac{c}{n^{\frac{1}{2}}}$$

for some constant c (see Theorem 1 of [7]), that

$$\sup_k \frac{E^o(\tau_k)}{k^3} < \infty.$$

The problem is to obtain better estimates for the rate of growth of $E^o(\tau_k)$. In particular, is it true that $\sup_k E^o(\tau_k)/k^2 < \infty$ for a general random walk, and that $\sup_k E^o(\tau_k)/k < \infty$ for a transient random walk? The conjecture in the general case is motivated by the fact that $E^o(\tau_k)$ is of the order k^2 for the simple symmetric random walk on Z^1, and that should, in some sense, be the worst case. Some information about $E^o(\tau_k)$ can be obtained by noting the close connection between $\{\tau_k\}$ and $\{R_n\}$, where R_n is the number of distinct points in $\{X_0, \cdots, X_n\}$. For example, an elegant theorem of Kesten, Spitzer and Whitman (Theorem 6.35 of [1]) states that

$$\lim_{n\to\infty} \frac{R_n}{n} = P^o[X_n \neq 0 \text{ for all } n] \text{ a.s.}$$

Since $R_{\tau_k} = k$, one obtains from this and Fatou's lemma that

$$\liminf_{k\to\infty} \frac{E(\tau_k)}{k} > 0$$

in the transient case, and

$$\lim_{k\to\infty} \frac{E(\tau_k)}{k} = \infty$$

in the recurrent case. It seems more difficult to obtain upper bounds on $E(\tau_k)$, since large values of τ_k correspond to small values of R_n.

Finally, is it the case for an arbitrary random walk on Z^d that $\pi P = \pi$ and π nonconstant implies that

$$P_\pi^o\left[\Sigma_{n=1}^\infty \frac{1}{\pi(X_n)} < \infty\right] > 0?$$

If so, then no regularity assumptions would be needed in Theorem 1.3 and Corollary 4.8.

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