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AN INDEPENDENCE PROPERTY OF BROWNIAN MOTION WITH DRIFT

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For d-dimensional Brownian motion starting at 0 and having constant drift it is shown that the interarrival times between successive concentric spheres and the hitting place on the outermost sphere are independent. This generalizes results of Kent and Stern. A discrete analogue extending Samuels' theorem is indicated.

Let $W(t)$ be a standard Brownian motion in R^d, $d \geqslant 1$, starting at the origin, $W(0) = 0$. Let v be a fixed vector in R^d and form the process $X(t) = W(t) + tv$, a Brownian motion with drift v; the main interest, of course, attaches to the case $v \neq 0$.

For $X(t)$ consider the hitting times $T_1, T_2, \cdots, T_k$ of concentric spheres $|x| = r_j$, $r_1 < r_2 < \cdots < r_k$. We shall prove the following result.

(1) THEOREM. *The interarrival times* $\tau_j = T_j - T_{j-1}$, $j = 1, 2, \cdots, k$, $T_0 = 0$, *and the final hitting place* $X(T_k)$ *are mutually independent.*

In dimension one and with $k = 1$ this was proved by Stern (1977), as the continuous analogue of a result of Samuels (1975) for the unsymmetric Bernoulli process. (In (2) below we indicate a generalization of the latter.) Kent (1978) proved (1) for $k = 1$ and general d, and also for $k = 2$ but without reference to $X(T_2)$. The counter-intuitiveness of all these results is striking, for one feels that there should be inferences from, e.g., "T_1 is small" to "$X(T_1)$ tends to be near $r_1 v/|v|$ on $|x| = r_1$" or to "$T_2 - T_1$ is also small."

Before proceeding to the proof of (1) we state its discrete companion.

(2) THEOREM. *Let* $\{S_n\}$ *be a random walk starting at* 0, *with independent steps* ± 1 *and probabilities* p, q. *Let* T_n *be the first hitting time for the set* $\{-n, n\}$. *Then for any* k *the interarrival times* $\tau_j = T_j - T_{j-1}$, $j = 1, 2, \cdots, k$, $T_0 = 0$, *and the position* S_{T_k} *are mutually independent.*

PROOF OF (1). We use the Randon-Nikodym (RN) method which is suggested in the concluding remarks of Kent (1978). As will be seen, this reduces the problem to the drift-free case, where the result is trivial by spherical symmetry.

Since the RN derivative of X with respect to W at time t and place $x \in R^d$ is $\exp\{-|v|^2 t/2 + v \cdot x\}$ we have (cf. Freedman (1971)) the following statement.

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(3) Let T be a stopping time and V a path-functional which is measurable $\mathcal{F}_T$. Then $E_v(V) = E_0(V \exp\{-|v|^2T/2 + v \cdot X_T\})$, where the subscripts v, 0 refer to the drift-rates assumed.

Now take $T = T_k$, $V = \exp\{u \cdot X_T - \sum_{j=1}^k s_j\tau_j\}$, where the s_j are positive variables and u is an arbitrary vector of R^d. Then $E_v(V)$ determines the joint distribution of $\tau_1, \cdots, \tau_k, X_{T_k}$. Applying (3) we obtain

$$E_v(V) = E_0\big(\exp\{-\Sigma\big(s_j + |v|^2/2\big)\tau_j + (v + u) \cdot X_T\}\big),$$

since $T = \Sigma\tau_j$. Then, since (1) holds for zero drift, the right member is the product of individual functions of the s_j and of u. This completes the proof of independence when there is nonzero drift.

The proof of (2) is even simpler; now the RN factor is $(4pq)^{T/2}(p/q)^{S_T/2}$.

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