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THE CARRYING DIMENSION OF A STOCHASTIC MEASURE DIFFUSION[1]

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A multiplicative stochastic measure diffusion process in R^d is the continuous analogue of an infinite particle branching Markov process in which the particles move in R^d according to a symmetric stable process of index α, $0 < \alpha \leqslant 2$. The main result of this paper is that there is a random carrying set whose Hausdorff dimension is almost surely less than or equal to α. As a corollary it follows that the corresponding random measure is singular for $d > \alpha$. The latter result is also proved by a different approach in the case $d = \alpha$.

1. Introduction. The multiplicative stochastic measure diffusion process in R^d arises as the "high density limit" of an infinite particle branching Markov process in which the particles move in R^d according to a symmetric stable process of index α, $0 < \alpha \leqslant 2$. The basic construction of the stochastic measure diffusion process, together with a study of some basic properties, is contained in Dawson [3], [4]. The main objective of this paper is to study the local structure of the resulting random measures.

We first review some basic definitions and results. Let $\mathfrak{M}(R^d)$ denote the family of Borel measures on R^d furnished with the topology of vague convergence. Let $C_K(R^d)$ denote the class of continuous real-valued functions on R^d with compact support. For a random measure on R^d, that is, an $\mathfrak{M}(R^d)$-valued random variable, the probability distribution is uniquely determined by the characteristic functional $L(\cdot)$, defined for $f \in C_K(R^d)$ by

$$L(f) \equiv \int_{\mathfrak{M}(R^d)} \exp(i\int_{R^d} f(x)\nu(dx))P(d\nu). \tag{1.1}$$

A *stochastic measure process* $\{X(t) : t \geqslant 0\}$ is an $\mathfrak{M}(R^d)$-valued stochastic process defined on a probability space $(\Omega, \mathfrak{F}, P)$. A Markov stochastic measure process with time homogeneous transition probabilities is uniquely determined by the characteristic functional of the initial distribution $X(0)$ and the characteristic functional of the probability transition function, given for $f \in C_K(R^d)$ and $\nu \in \mathfrak{M}(R^d)$ by

$$L_{t,\nu}(f) \equiv E(\exp(i\int f(x)X(t, dx))|X(0) = \nu). \tag{1.2}$$

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The *multiplicative critical stochastic measure diffusion process* in R^d is an $\mathfrak{M}(R^d)$-valued Markov process with

(1.3) $$L_{t,\nu}(f) \equiv \exp(i \textstyle\int U_t f(x) \nu(dx)),$$

where $\{U_t : t \geq 0\}$ is a semigroup of nonlinear operators on $C_K(R^d)$ which we describe below. Let G_α denote the infinitesimal generator of the Markov semigroup $\{S_t : t \geq 0\}$ of contraction operators on $C_K(R^d)$ associated with the symmetric stable process on R^d of index α, $0 < \alpha \leq 2$. Then $u(t, x) = U_t f(x)$ satisfies the nonlinear initial value problem

(1.4a) $$\frac{\partial u(t, x)}{\partial t} = G_\alpha u(t, x) + i\gamma u^2(t, x), \quad t > 0,$$

(1.4b) $$u(0, x) = f(x),$$

where γ is a given positive constant. The reader is referred to [3], [4] for the proof of existence and some basic properties of this process, some of which are summarized below. In particular, the measure diffusion process at time t is well approximated by alternating branching and diffusion processes over successive time intervals of length t/m in the following way: particles are created according to the branching mechanism defined by (1.5) and are then smeared out by the diffusion determined by the semigroup operator S_t.

We now review the basic properties of the measure diffusion process which are required in this paper.

PROPOSITION 1.1. *Let S_t and U_t be defined as above, and let $T_t : C_K(R^d) \to C_K(R^d)$ be defined by*

(1.5) $$T_t f(x) = f(x)/[1 - i\gamma t f(x)], \qquad t \geq 0.$$

Then the following hold:

(a)

$$\lim_{m\to\infty} \| U_t f - (S_{t/m} T_{t/m})^m f \| = 0 \qquad \textit{for each} \quad t > 0 \quad \textit{where} \ \| \cdot \|$$

denotes the supremum norm.

(b)

(1.6) $$T_t f(x) = \int_{0+}^{\infty} [\exp(i f(x) y) - 1] \mu_t(y)\, dy,$$

where

(1.7) $$\mu_t(y) = (\gamma t)^{-2} \exp(-y/\gamma t), \qquad y \geq 0$$
$$= 0, \qquad y < 0.$$

(c) *If ν is a nonatomic measure on R^d, then*

(1.8) $$L_{t,\nu}^T(f) \equiv \exp(i \textstyle\int T_t f(x) \nu(dx))$$

is the characteristic functional of a compound Poisson random field with Lévy-Khintchine-Kingman representation (1.6).

The reader is referred to [3] for the proof of (a) and to [4] for the proof of (b) and (c).

The basic property of the symmetric stable semigroup that we require is the "scaling property," that is, for $r > 0$,

$$S_t f(0) = S_{rt} f_{\alpha, r}(0) \tag{1.9}$$

where $f_{\alpha,r}(u) \equiv f(r^{-1/\alpha}u)$. In other words, if $\{Z_\alpha(t) : t \geqslant 0\}$ denotes the symmetric stable process with index α, then $r^{-1/\alpha}(Z_\alpha(rt) - Z_\alpha(0))$ has the same law as $(Z_\alpha(t) - Z_\alpha(0))$.

Given a Borel set $E \subset R^d$ and $\beta > 0$, $\delta > 0$, let

$$\wedge_\delta^\beta(E) \equiv \inf_{\mathbb{S}} \Sigma_i (d(S_i))^\beta$$

where $d(S_i)$ is the diameter of the set S_i and $\mathbb{S} \equiv \{\{S_i\} : E \subset \cup S_i, d(S_i) < \delta$ for each $i\}$. Then the *Hausdorff β-measure* of E is defined by

$$\wedge^\beta(E) \equiv \lim_{\delta \to 0} \wedge_\delta^\beta(E). \tag{1.10}$$

The *Hausdorff dimension* of E is defined by

$$\dim E \equiv \inf\{\beta > 0 : \wedge^\beta(E) = 0\} = \sup\{\beta > 0 : \wedge^\beta(E) = \infty\}. \tag{1.11}$$

Note that $0 \leqslant \dim E \leqslant d$, and if E has positive Lebesgue measure, then $\dim E = d$.

In this paper, we demonstrate the existence of a random carrying set of Hausdorff dimension α for the stochastic measure diffusion process. (A similar problem, that of determining the Hausdorff dimension of a carrying set of a random measure arising from a "curdling" process, has been posed by Mandelbrot [6].) It follows from our result that the corresponding random measure is singular if the dimension d is greater than the index α of the symmetric stable diffusion process. Finally, we prove the singularity of the random measure in the case $d = \alpha$ by rescaling in both space and time and using the fact proved in [4] that for the critical measure diffusion in the recurrent case, the measure of a compact set approaches zero in probability as t becomes infinite.

2. Statement of the results.

The main result is given by the following theorem:

THEOREM 2.1. *Let $\{X(t) : t \geqslant 0\}$ denote the multiplicative stochastic measure diffusion process in R^d defined by the characteristic functional* (1.3) *whose spatial diffusion corresponds to a symmetric stable process with index α, $0 < \alpha \leqslant 2$. Then for fixed $t > 0$, there exists a random set B such that*

$$X(t, \omega, C \cap B(\omega)) = X(t, \omega, C) \tag{2.1}$$

for every compact set C and almost every ω, and

$$\dim B(\omega) \leqslant \alpha \quad \textit{for every} \quad \omega. \tag{2.2}$$

REMARK 2.2. Note that since this is a local problem, we need only construct $B \cap V$ where V is a unit cube in R^d. Furthermore, without loss of generality, we can assume that $X(0, V) = 1$.

REMARK 2.3. If the Borel set $B(\omega)$ has positive Lebesgue measure, then $\dim B(\omega) = d$. Combining this with (2.2) we get the following corollary:

COROLLARY 2.4. *The random measure in R^d characterized by equation* (1.3) *with spatial diffusion governed by a symmetric stable process of index α, $0 < \alpha \leq 2$, is singular if $d > \alpha$.*

Using a different approach we obtain the following extension of this corollary:

THEOREM 2.5. *The random measure on R^2 characterized by equation* (1.3) *with spatial diffusion governed by two-dimensional Brownian motion and the random measure on R^1 characterized by* (1.3) *with spatial diffusion governed by the one-dimensional symmetric Cauchy process are almost surely singular measure-valued.*

3. Proof of Theorem 2.1. Consider a unit cube $V \subset R^d$ which for each $n \geq 1$ is subdivided into $2^{k_n d}$ equal subcubes of volume $2^{-k_n d}$, where $\{k_n, n \geq 1\}$ is an increasing sequence of nonnegative integers. The ratio of the diameter of the fixed cube V to that of the subcubes is $\Gamma_n = 2^{k_n}$.

Consider the set B obtained as follows:

$$B_0 = V$$

$$B_n \subset B_{n-1}, \qquad n \geq 1$$

$$B_n \text{ is a union of } N_n \text{ subcubes of volume } (\Gamma_n)^{-d}$$

$$B = \bigcap_{n=0}^{\infty} B_n. \tag{3.1}$$

Then B is a generalized Cantor set, and, similar to the derivation of the Hausdorff dimension of the Cantor set (see, e.g., Billingsley [2], pages 141–143), the Hausdorff dimension of B can be shown to be

$$\dim B = \liminf_{n\to\infty}[\log N_n/\log \Gamma_n]. \tag{3.2}$$

We now proceed to consider a probabilistic analogue to this construction. Let X be a random measure on V. Given $\varepsilon > 0$, let

$$N_n^\varepsilon(X) = \min\{n : \Sigma_{i=1}^n X(v_i) \geq X(V) - \varepsilon\} \tag{3.3}$$

and

$$K_n^\varepsilon \equiv \bigcup_{i=1}^{N_n^\varepsilon(X)} v_i, \tag{3.4}$$

where $\{v_i : i = 1, \cdots, N_n^\varepsilon(X)\}$ is a cover consisting of the given subcubes of volume $2^{-k_n d}$ achieving the minimum in (3.3).

LEMMA 3.1. *Assume that*

$$P\left(\frac{\log N_n^{\varepsilon_n}(X)}{\log \Gamma_n} \leq D(1 + \eta_n)\right) \geq 1 - \varepsilon_n' \tag{3.5}$$

where $\varepsilon_n \downarrow 0$, $\eta_n \downarrow 0$, and $\varepsilon_n' \downarrow 0$ as $n \to \infty$.

Then there exists a random set $B(\omega)$ *such that*

(3.6) $$X(\omega, B(\omega)) = X(\omega, V) \quad \text{a.e. } \omega,$$

(3.7) $$\dim B(\omega) \leqslant D \quad \text{a.e. } \omega.$$

PROOF. Let

(3.8) $$\Phi_n \equiv \left\{\omega : \frac{\log N_n^{\varepsilon_n}(X(\omega))}{\log \Gamma_n} \leqslant D(1 + \eta_n)\right\}$$

and

(3.9) $$K_n^{\varepsilon_n}(\omega) \equiv \bigcup_{i=1}^{N_n^{\varepsilon_n}(X(\omega))} v_i \quad \text{if} \quad \omega \in \Phi_n$$
$$\equiv \varnothing \quad \text{if} \quad \omega \notin \Phi_n.$$

Note that if necessary we can take a subsequence of $\{K_n\}$. Hence, without loss of generality, we can assume that $\Sigma\varepsilon_n' < \infty$. We now show that

(3.10) $$B(\omega) \equiv \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} K_n^{\varepsilon_n}(\omega)$$

satisfies the conditions stated in the lemma.

Since by hypothesis $P(\Phi_n) \geqslant 1 - \varepsilon_n'$, and $\Sigma\varepsilon_n' < \infty$, then $P(\cup_{k=1}^{\infty} \cap_{n=k}^{\infty} \Phi_n) = 1$ by the Borel-Cantelli lemma.

If $\omega \in \cap_{n=k}^{\infty}\Phi_n$, then

$$X(\omega, \cap_{n=k}^{\infty} K_n^{\varepsilon_n}(\omega)) \geqslant X(V) - \Sigma_{n=k}^{\infty}\varepsilon_n'.$$

Hence if $\omega \in \cup_{k=1}^{\infty} \cap_{n=k}^{\infty}\Phi_n$, then

$$X(\omega, \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} K_n^{\varepsilon_n}(\omega)) \geqslant X(V),$$

that is,

$$X(\omega, B(\omega)) \geqslant X(V).$$

Since, trivially, $X(V) \geqslant X(\omega, B(\omega))$, we see that $B(\omega)$ satisfies (3.6).

If $\omega \in \cap_{n=k}^{\infty}\Phi_n$, then

$$\frac{\log N_n(B(\omega))}{\log \Gamma_n} \leqslant D(1 + \eta_n) \quad \text{for all} \quad n \geqslant k,$$

and hence by (3.2),

$$\dim B(\omega) = \liminf_{n\to\infty} \frac{\log N_n(B(\omega))}{\log \Gamma_n} \leqslant D,$$

and the proof of the lemma is complete.

In order to establish an estimate of the type (3.5) for the multiplicative stochastic measure diffusion $X(t)$ at time t, we exploit the approximation given by Proposition 1.1(a). Let P_ν and $\{P_m, m \geqslant 1\}$ denote the probability measures on $\mathfrak{M}(R^d)$ with characteristic functionals $L_{t,\nu}(\cdot)$ and

(3.11) $$L_m(f) = \exp\left(i\textstyle\int(S_{t/m}T_{t/m})^m f(x)\nu(dx)\right), \qquad f \in C_K(R^d),$$

respectively. Then Proposition 1.1(a) implies that the sequence $\{P_m, m \geq 1\}$ converges weakly to P_ν as $m \to \infty$. The significance of this is that $X(t)$ can be approximated by alternating two processes over successive intervals of length (t/m). The process corresponding to $S_{t/m}$ is a deterministic diffusion of the measure according to the symmetric stable semigroup, thus yielding an absolutely continuous measure. On the other hand, the process corresponding to $T_{t/m}$ takes a nonatomic initial measure $\tilde{\nu} \in \mathfrak{M}(R^d)$ into a compound Poisson random measure with characteristic functional

$$L_{\tilde{\nu}}^T(f) = \exp\left(i \int T_{t/m} f(x) \tilde{\nu}(dx)\right). \tag{3.12}$$

In view of (1.6), the total Poisson intensity is $(\gamma t/m)^{-1}\tilde{\nu}$ and the mass distribution of the particles is negative exponential with mean $\gamma t/m$. Thus, the approximation given by Proposition 1.1(a) consists of alternately "creating particles" whose masses are exponentially distributed with mean $\gamma t/m$ and then "smearing them out" by the deterministic diffusion operator corresponding to $S_{t/m}$. In view of the scaling property (1.9) of the symmetric stable semigroup, the smeared out particle tends to be concentrated in a region whose diameter is of the order of $(t/m)^{1/\alpha}$.

The basic idea of the proof is to show that at a *given scale* the picture suggested by this approximation is in fact correct. Thus we will show that the measure diffusion random measure can be viewed as a hierarchy of smeared clusters at different scales. The nth scale is obtained by subdividing V into Γ_n^d equal subcubes of volume Γ_n^{-d} as above.

Let

$$m_n \equiv [2\gamma t \Gamma_n^\alpha], \qquad n = 1, 2, 3, \cdots$$

where $[x]$ denotes the greatest integer less than or equal to x. By the scaling property (1.9) of the symmetric stable law there is a constant $c > 0$ such that

$$z_n \equiv P\left(|Z_\alpha(t/m_n) - Z_\alpha(0)| > n\Gamma_n^{-1}\right) < c/n^\alpha \tag{3.13}$$

for sufficiently large n.

Assume that ν is nonatomic and consider the random measure $X(t/m_n)$ (restricted to V) with characteristic functional

$$L_{t/m_n, \nu}(f) = \exp\left(i \int U_{t/m_n} f(x) \nu(dx)\right) \tag{3.14}$$

for f with $\mathrm{Spt}(f) \subset V$.

LEMMA 3.2.

(i) *The random measure $X(t/m_n)$ consists of a Poisson number W_C of clusters with total intensity $(\gamma t/m_n)^{-1}\nu(V)$.*

(ii) *The total mass Y_C of each cluster is exponentially distributed with mean $(\gamma t/m_n)$.*

PROOF. We begin by using the approximation $U_{t/m_n} \approx (S_{t/m} T_{t/m})^{m/m_n}$ where $m \gg m_n$. The result of first applying $T_{t/m}$ and then $S_{t/m}$ to the nonatomic initial measure ν is a compound Poisson random field of smeared particles. The total

Poisson intensity is $m\nu(V)/\gamma t$ and the masses of the particles are exponentially distributed with mean $\gamma t/m$. The particles are smeared out over a region whose diameter is of the order of $(t/m)^{1/\alpha}$.

To determine the structure of the random measure whose characteristic functional is given by (3.14), m/m_n iterations of this process are required. This iterative process can be viewed as a critical branching random walk as follows. Each smeared particle at the next iteration gives rise to a Poisson random number of particles whose location is displaced from that of its predecessor by the symmetric stable law associated with $S_{t/m}$. The number of offspring is a Poisson random variable whose mean is equal to $(m/\gamma t)$ times the mass of the predecessor. But since the mass of the predecessor is exponentially distributed with mean $(\gamma t/m)$, the offspring distribution is given by (letting $a = m/\gamma t$)

$$(3.15)\qquad \begin{aligned} P(N = k) &= \int_0^\infty \left(\frac{(a\lambda)^k}{k!} e^{-a\lambda} \right) \left[a e^{-a\lambda} \right] d\lambda = (k!)^{-1} \int_0^\infty \lambda^k e^{-2\lambda}\, d\lambda \\ &= 2^{-(k+1)}, \qquad\qquad k = 0, 1, 2, \cdots, \end{aligned}$$

that is, a geometric distribution with mean one. Hence, we have a critical branching random walk in which the particles have an exponentially distributed random mass.

By a "cluster" is meant the collection of descendents surviving at time (t/m_n) of one of the particles first created at time (t/m). Recall that for a critical Galton-Watson process $\{Z_n : n \geqslant 0\}$,

$$P(Z_n > 0) \sim 2/n\sigma^2$$

where $\sigma^2 \equiv \mathrm{Var}(Z_1)$, and

$$\lim_{n\to\infty} P(Z_n/n > z | Z_n > 0) = \exp(-2z/\sigma^2)$$

(cf. Athreya-Ney [1], page 19). It remains to note that for the geometric distribution in (3.15), $\sigma^2 = 2$, and thus

$$P(Z_n > 0) \sim \frac{1}{n}$$

and

$$(3.16)\qquad \lim_{n\to\infty} P(Z_n/n > z | Z_n > 0) = \exp(-z), \qquad z > 0.$$

Hence, $P(Z_{m/m_n} > 0) \sim m_n/m$, and the conditional distribution of Z_{m/m_n} conditioned on $\{Z_{m/m_n} > 0\}$ is approximately exponentially distributed with mean (m/m_n). We thus note that the total number of surviving clusters, W_C, has a Poisson distribution of total intensity $(m_n/\gamma t)\nu(V)$. Furthermore, each cluster consists of a random number of particles, the number of which is approximately exponentially distributed with mean (m/m_n). Each of the "small particles" created at time t/m has a mass which is exponentially distributed with mean $(\gamma t/m)$. We must now verify that the total mass of a cluster at time (t/m_n) is exponentially distributed with mean $(\gamma t/m_n)$.

Let $\varphi_P(S)$, $\varphi_C(S)$ denote the Laplace transforms of the mean distributions of the small particles and the clusters, respectively, and let

$$\psi_m(z) \equiv E\left(z^{Z_{m/m_n}} \middle| Z_{m/m_n} > 0\right).$$

We then have

$$\varphi_C(s) = \psi_m(\varphi_P(s))$$

where

$$\varphi_P(s) = \frac{a}{a+s} \quad \text{and} \quad a = m/\gamma t.$$

From the exponential limit law (3.16), it follows that

$$\psi_m(z^{m_n/m}) \to 1/(1 - \log z)$$

uniformly on bounded z-intervals. Then

$$\begin{aligned}\psi_m(\varphi_P(s)) &= \psi_m\left(\frac{1}{1+\gamma ts/m}\right)\\ &\to 1/\left[1 - \log((\exp(\gamma ts))^{-1/m_n})\right]\\ &\text{as } m \to \infty.\end{aligned}$$

Therefore

$$\psi_m(\varphi_P(s)) \to \frac{m_n/\gamma t}{m_n/\gamma t + s} \quad \text{as } m \to \infty,$$

which is the Laplace transform of the exponential distribution with mean $(\gamma t/m_n)$. Thus the proof of the lemma is complete.

LEMMA 3.3.

(i) *Let B_x^n denote a sphere of radius $n\Gamma_n^{-1}$ centered at the location of the ancestral small particle at t/m for each cluster. Then, given a constant $\kappa > 0$, there exist constants κ_1 and κ_2 such that*

$$(3.17) \qquad P_\nu\left(X\left(t/m_n, \left(\cup_{i=1}^{W_C} B_{x_i}^n\right)^C\right) > \kappa/n^\beta\right) \leqslant \kappa_1 n^{2\beta-2\alpha} + \kappa_2 n^{2\beta-\alpha}/m_n.$$

(ii) *For $\beta < \alpha$ and sufficiently large n,*

$$(3.18) \qquad P_\nu\left(\frac{\log N_n^{\kappa/n^\beta}(X(t/m_n))}{\log \Gamma_n} > \alpha + \frac{\log(2^{d+1}n^{d+2})}{\log \Gamma_n}\right) \leqslant \frac{\kappa_3(\nu(V))^2}{n^{2(\alpha-\beta)}}.$$

PROOF. We first note that the cluster has an exponentially distributed total mass and that its "expected" spatial distribution is given by $p_\alpha(t/m_n, x, \cdot)$, where $p_\alpha(t, x, \cdot)$ denotes the probability transition density of the symmetric stable process. Additionally, by (3.13),

$$z_n = \int_{(B_x^n)^C} p_\alpha(t/m_n, x, y)\,dy < c/n^\alpha.$$

Let the random variable X_i, $i = 1, \cdots, W_C$, denote that portion of the mass of the

ith cluster which lies outside $B^n_{x_i}$, $i = 1, \cdots, W_C$. Then

$$X\left(t/m_n, \cup_{i=1}^{W_C}(B^n_{x_i})^C\right) \leq \Sigma_{i=1}^{W_C} X_i$$

and

$$E\left(X\left(t/m_n, \cup_{i=1}^{W_C}(B^n_{x_i})^C\right)\right) \leq z_n. \tag{3.19}$$

Furthermore,

$$\begin{aligned}\operatorname{Var}\left(X\left(t/m_n, \cup_{i=1}^{W_C}(B^n_{x_i})^C\right)\right) &= E(W_C)\cdot \operatorname{Var}\left(X\left(t/m_n, (B^n_{x_i})^C\right)\right)\\ &\quad + \left[E\left(X\left(t/m_n, (B^n_{x_i})^C\right)\right)\right]^2 \cdot \operatorname{Var}(W_C)\\ &= (\gamma t/m_n)^{-1}\operatorname{Var}\left(X\left(t/m_n, (B^n_{x_i})^C\right)\right)\\ &\quad + z_n^2(\gamma t/m_n)^2(\gamma t/m_n)^{-1}.\end{aligned} \tag{3.20}$$

To determine $\operatorname{Var}(X(t/m_n, (B^n_x)^C))$, we compute the exact characteristic functional of the random measure at time t/m_n associated with a cluster with center at x. This is computed from (1.3) as

$$\begin{aligned}L_{C,x,t}(\varphi) &= \lim_{z\downarrow 0}\{[\exp(iU_t\varphi(x)z) - \exp(-z/\gamma t)]/[1 - \exp(-z/\gamma t)]\}\\ &= 1 + i\gamma t U_t\varphi(x).\end{aligned} \tag{3.21}$$

Thus,

$$\operatorname{Var}\left(X\left(t/m_n, (B^n_x)^C\right)\right) = (\gamma t/m_n)v(t/m_n) - (\gamma t/m_n)^2 z_n^2,$$

where, by [4], Equation (4.7),

$$\begin{aligned}v(t') &= \textstyle\int_0^{t'}\int\int_{(B^n_x)^C}\int_{(B^n_x)^C} p_\alpha(s, x, y)p_\alpha(t' - s, y, w)p_\alpha(t' - s, y, v)\,dw\,dv\,dy\,ds\\ &\leq t'\textstyle\int_{(B^n_x)^C} p_\alpha(t', x, w)\,dw.\end{aligned}$$

Hence $v(t/m_n) \leq (t/m_n)z_n$, so

$$\operatorname{Var}\left(X\left(t/m_n, (B^n_x)^C\right)\right) \leq \gamma(t/m_n)^2 z_n - \gamma^2(t/m_n)^2 z_n^2. \tag{3.22}$$

Therefore, from (3.20) we have

$$\operatorname{Var}\left(X\left(t/m_n, \cup_{i=1}^{W_C}(B^n_{x_i})^C\right)\right) \leq (t/m_n)z_n$$

and

$$E\left(\left[X\left(t/m_n, \cup_{i=1}^{W_C}(B^n_{x_i})^C\right)\right]^2\right) \leq z_n^2 + (t/m_n)z_n.$$

Now using Chebyshev's inequality we have

$$P\left(X\left(t/m_n, \left(\cup_{i=1}^{W_C} B^n_{x_i}\right)^C\right) > \kappa/n^\beta\right) \leq \kappa_1 n^{2\beta-2\alpha} + \kappa_2 n^{2\beta-\alpha}/m_n,$$

and the proof of (i) is complete. But from (3.17) together with

$$P(W_C > n^2 m_n/\gamma t) \leq \kappa(\nu(V))^2/n^4, \tag{3.23}$$

it follows that for $\beta < \alpha$ and sufficiently large n,

$$P_\nu\left(N_n^{\kappa/n^\beta}(X(t/m_n)) > 2^d n^{d+2} m_n/\gamma t\right) \leqslant \kappa(\nu(V))^2/n^4 + \kappa_1 n^{2\beta-2\alpha} + \kappa_2 n^{2\beta-\alpha}/m_n \leqslant \kappa_2(\nu(V))^2/n^{2(\alpha-\beta)}.$$

Hence

$$P_\nu\left(\log N_n^{\kappa/n^\beta}(X(t/m_n))/\log \Gamma_n > \alpha + \log(2^{d+1}n^{d+2})/\log \Gamma_n\right) \leqslant \kappa_3(\nu(V))^2/n^{2(\alpha-\beta)},$$

and the proof of (ii) is complete.

To prove the main theorem, note that we can apply Lemmas 3.2 and 3.3 when ν is replaced by the nonatomic random measure $X((m_n - 1)t/m_n)$. Since the random measure $X((m_n - 1)t/m_n)$ has finite moments (cf. [4]), Chebyshev's inequality yields

$$P_\nu\left(X\left(\frac{m_n - 1}{m_n}t, V\right) > n^{(\alpha-\beta)/2}\right) < \kappa_4/n^{(\alpha-\beta)}. \tag{3.24}$$

Then, by the Markov property, we have for $\beta < \alpha$ and sufficiently large n,

$$P_\nu\left(\log N_n^{\kappa/n^\beta}(X(t))/\log \Gamma_n > \alpha + \log(2^{d+1}n^{d+2})/\log \Gamma_n\right) < \kappa_5/n^{(\alpha-\beta)}. \tag{3.25}$$

Theorem (2.1) then immediately follows from (3.25) and Lemma (3.1).

4. Proof of Theorem 2.5. We first introduce the rescaling transformation (cf. [4], Section 5) $X(t) \to X^{(K)}(t)$ as follows:

$$\langle X^{(K)}(t), \varphi\rangle \equiv \langle X(t), \varphi_K\rangle \tag{4.1}$$

where $\varphi_K(x) \equiv \varphi(x/K)$ and $K > 0$.

In [4], Section 5, it is shown that the characteristic functional of $X^{(K)}(t)$ is given by

$$L_{t,\nu}^{(K)}(f) = \exp\left(i\int u^{(K)}(t, x)\nu(dx)\right) \tag{4.2}$$

where

$$u^{(K)}(t, x) = \Sigma_{k=1}^{\infty} K^{\alpha(k-1)} u_k(t/K^\alpha, x/K). \tag{4.3}$$

We complete the proof in the case $d = 2$; the proof in the case $d = 1$ is essentially the same. If $d = 2$, $\alpha = 2$ and ν is Lebesgue measure, then

$$\int u^{(K)}(t, x)dx = \Sigma_{k=1}^{\infty} K^{2k} \int u_k(t/K^2, y)dy. \tag{4.4}$$

Hence,

$$X^{(K)}(K^2 t)/K^2 \simeq_{\mathfrak{L}} X(t). \tag{4.5}$$

Therefore,

$$X(t, A) \simeq_{\mathfrak{L}} X\left(1, A_{1/t^{\frac{1}{2}}}\right)/(1/t) \tag{4.6}$$

where $x \in A_{1/t^{\frac{1}{2}}}$ if and only if $t^{\frac{1}{2}}x \in A$.

But from [4], Theorem 3.1, if A is compact, then

(4.7) $$X(t, A) \to 0 \quad \text{in probability as} \quad t \to \infty.$$

Hence letting $A(x)$ denote a unit cube centered at an arbitrary point $x \in R^2$,

(4.8) $$X(1, A_\varepsilon(x))/|A_\varepsilon(x)| \to 0 \quad \text{in probability as} \quad \varepsilon \to 0$$

where $|A_\varepsilon(x)|$ denotes the Lebesgue measure of $A_\varepsilon(x)$. But if $X(1, \cdot)$ has a nontrivial absolutely continuous component, then

(4.9) $$P(\{\omega : \liminf_{\varepsilon \to 0} X(1, \omega, A_\varepsilon(x))/|A_\varepsilon(x)| > 0\}) > 0$$

for a set of x of positive Lebesgue measure.

It follows that $X(1, \cdot)$ has no absolutely continuous component since otherwise (4.8) and (4.9) yield a contradiction.

REMARK. Equation (4.5) implies that $X(\cdot)$ is self-similar under the transformation $X \to X^{(K)}$, $t \to K^\alpha t$ in the case $d = \alpha$. This fact has also been noted in a recent manuscript of Holley and Stroock [5] and is implicit in recent unpublished work of Spitzer.

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