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RENORMALIZING THE 3-DIMENSIONAL VOTER MODEL

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It is shown that a discrete time voter model in equilibrium on $\mathbb{Z}_3$ approaches the 0-mass free field of 3-dimensional Euclidean field theory under appropriate renormalization. This result is of interest because the strong correlation between distant sites gives rise to the renormalization exponent $-\frac{5}{2}$ instead of the usual $-\frac{3}{2}$. Dawson, Ivanoff, and Spitzer have examined models on $\mathbb{R}_3$ which exhibit precisely the same limit. Because the process we consider lives on a lattice, our method of proof is necessarily quite different from theirs. In particular, we make use of a "duality" between voter models and coalescing random walks which has been exploited effectively by Holley and Liggett.

1. Introduction. Based on recent developments in mathematical physics (cf. [11], [14], [22]), Sinai [16], Dobrushin [3], [4] and others have begun to investigate the macroscopic structure of strongly dependent random fields. If $\xi = (\xi(\mathbf{i}))_{\mathbf{i}\in\mathbb{Z}_d}$ is a real-valued random field on the d-dimensional integer lattice, Sinai introduces the renormalized fields $D_k^\alpha\xi$, $\alpha \geqslant 1$, $k = 1, 2, \cdots,$ given by

$$(D_k^\alpha\xi)(\mathbf{i}) = k^{-(\alpha d/2)}\Sigma_{k\mathbf{i}\leqslant\mathbf{j}<k(\mathbf{i}+\mathbf{1})}[\xi(\mathbf{j}) - E\xi(\mathbf{j})].$$

($\mathbf{1} = (1, 1, \cdots, 1) \in \mathbb{Z}_d$.) ξ is called *self-similar of order* α if $D_k^\alpha\xi =_d \xi$ ($=_d$ and $\rightarrow_d$ will denote equality and convergence in distribution respectively) for all k. Similarly, if F is a generalized random field ([3], [4], [5]) on d-dimensional Euclidean space $\mathbb{R}_d$, defined over the Schwartz space $\mathbb{S}$ of rapidly decreasing functions, then Dobrushin considers the renormalized fields $R_r^\alpha F$, $\alpha \geqslant 1$, $r > 0$, where

$$(R_r^\alpha F)(\varphi) = r^{-(\alpha d/2)}[F(\varphi_r) - EF(\varphi_r)], \qquad \varphi \in \mathbb{S}.$$

Here $\varphi_r(\mathbf{x}) = \varphi(\mathbf{x}/r)$. In this setting F is called *self-similar of order* α if $R_r^\alpha F(\varphi) =_d F(\varphi)$ for every $\varphi \in \mathbb{S}$ (i.e., $R_r^\alpha F =_d F$).

If ξ is any translation invariant field on $\mathbb{Z}_d$, if ξ_k is defined by $\xi_k = D_k^\alpha\xi$, and if $\xi_k \rightarrow_d \xi_\infty$ as $k \rightarrow \infty$, then ξ_∞ will be a translation invariant self-similar random field of order α. When ξ has sufficiently weak correlations, then taking $\alpha = 1$ one obtains a limiting field ξ_∞ with independent Gaussian values at each site. In cases of strong correlation, on the other hand, one must take $\alpha > 1$ to obtain a limit. This is the situation of interest, when the self-similar random field ξ_∞ describes a nontrivial macroscopic dependency structure. Analogous remarks apply to

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Dobrushin's setting. Thus, if F has weak correlations and $F_r = R_r^1 F$, then we expect

$$F_r \to_d C\Phi = \text{a constant multiple of white noise as } r \to \infty.$$

Φ is the Gaussian self-similar generalized random field with covariance functional

$$E[\Phi(\varphi)\Phi(\psi)] = \int_{\mathbb{R}_d} \varphi(\mathbf{x})\psi(\mathbf{x})\, d\mathbf{x}.$$

For strongly dependent F one must take $\alpha > 1$, and more interesting self-similar generalized random fields arise. Important examples of such are the isotropic Gaussian self-similar generalized random fields with covariance functionals

$$(1) \qquad E[F(\varphi)F(\psi)] = C\int_{\mathbb{R}_d}\int_{\mathbb{R}_d} \frac{\varphi(\mathbf{x})\psi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\kappa}}\, d\mathbf{x}\, d\mathbf{y}$$

for appropriate (dimension-dependent) parameter values κ (cf. [4]). We shall be concerned almost exclusively with the case $d = 3$, $\kappa = 1$. This well-known field, to be denoted here as Ψ when $C = 1$, is called the "0-mass free field" in 3-dimensional Euclidean field theory. Recently, Dawson and Ivanoff [2], and Spitzer [18] have independently investigated the renormalization of strongly correlated equilibrium fields for certain Markovian time evolutions of infinitely many particles on $\mathbb{R}_d$. They have proved that critical branching Brownian motions in dimension $d \geqslant 3$, as well as certain critical branching random walks, have renormalization limits of the form (1) for some of the possible parameter values κ. In particular, they obtain a limiting field $C\Psi$ from critical branching Brownian motion on $\mathbb{R}_3$. Dawson [1] has proved related results for some generalized Markov processes. See also the recent paper by Holley and Stroock [10].

Our main objective in this paper is to exhibit an example similar to those of [1], [2], and [18], but derived from an infinite particle system on $\mathbb{Z}_3$: viz., a discrete time version of a Holley-Liggett voter model [8]. ("Voter models" were previously studied by Clifford and Sudbury [0].) Let $\Xi = \{0, 1\}^{\mathbb{Z}_3}$ = the space of configurations $\xi = (\xi(\mathbf{i}))_{\mathbf{i}\in\mathbb{Z}_3}$ of 0's and 1's on $\mathbb{Z}_3$. The systems we propose to study are discrete time Ξ-valued Markov processes (ξ_n); the simplest example is described as follows. At each time $n = 0, 1, \cdots$, the site $\mathbf{i} \in \mathbb{Z}_3$ is said to be occupied by a particle if $\xi_n(\mathbf{i}) = 1$, unoccupied otherwise. Given the configuration ξ_n at time n, site $\mathbf{i}$ is occupied at time $n + 1$ with probability $p_{\mathbf{i}}(\xi_n)$, independently of other sites and the past history of the process, where

$$p_{\mathbf{i}}(\xi) = \frac{1}{6}\Sigma_{\mathbf{j}\,:\,|\mathbf{j}-\mathbf{i}|=1}\xi(\mathbf{j}).$$

Let ν_λ, $0 \leqslant \lambda \leqslant 1$, be the Bernoulli product measure such that $\nu_\lambda\{\xi(\mathbf{i}) = 1\} = \lambda$ for all $\mathbf{i}$. The methods of Holley and Liggett show that if (ξ_n) has initial distribution ν_λ, and if $\nu_\lambda P^n$ denotes the measure governing the state ξ_n of the process at time n, then $\nu_\lambda P^n \Rightarrow \mu_\lambda$ as $n \to \infty$. ($\Rightarrow$ denotes weak convergence.) μ_λ is an extreme equilibrium measure for (ξ_n), is (of course) translation invariant, and is mixing with respect to shifts in $\mathbb{Z}_3$. Now let ξ be a μ_λ-distributed random field, and define a corresponding generalized random field F_λ on $\mathbb{R}_d$ by

$$F_\lambda(\varphi) = \Sigma_{\mathbf{i} \in \mathbb{Z}_3} \xi(\mathbf{i}) \varphi(\mathbf{i}) \qquad \varphi \in \mathcal{S}.$$

The main result of this paper asserts that for $0 < \lambda < 1$,

$$R_r^{5/3} F_\lambda \to_d C_\lambda \Psi \qquad \text{as} \quad r \to \infty,$$

where C_λ is a positive constant depending only on λ. We feel the result to be of interest because, in contrast to the examples of [1], [2], and [18], μ_λ is a measure on lattice configurations. As a result, the independence properties (e.g., infinite divisibility) enjoyed by the models they study have no counterpart for us. Rather, interference is inevitable in the lattice setting (cf. the last paragraph of [2]). Fortunately, a "duality equation" alleviates this problem to a large extent. Since (ξ_n) lives on the lattice, we also obtain a "discretized" renormalization theorem for Sinai's setting:

$$D_k^{5/3} \xi \to_d C_\lambda \xi_\infty \qquad \text{as} \quad k \to \infty,$$

where ξ_∞ is a Gaussian self-similar random field on $\mathbb{Z}_3$ whose covariances will be given later.

Section 2 introduces a more general class of discrete time Ξ-valued processes: the local homogeneous proximity processes of [19], [8], [9]. In this context we exhibit some examples where renormalized equilibria converge to white noise (= total independence). Section 3 is devoted to renormalizing the 3-dimensional voter model. A certain familiarity with duality theory for Ξ-valued processes ([19], [8], [9], [6], [7]), generalized random fields ([3], [4], [5]) and the method of semi-invariants and Ursell functions ([12], [13], [15]) will be assumed. Some concluding remarks are made in Section 4.

2. Renormalization of weakly correlated proximity processes. Let $\Xi = \{0, 1\}^{\mathbb{Z}_d}$. We discuss renormalization of equilibria for some Ξ-valued Markov processes (ξ_n): namely, some discrete-time proximity processes of [19], [20], [8]. For each finite subset B of $\mathbb{Z}_d$ ($\varnothing$ included) (the letters A and B will always denote finite subsets of $\mathbb{Z}_d$), set $\chi_B(\xi) = 1$ if $\xi(\mathbf{i}) = 1$ for all $\mathbf{i} \in B$, 0 otherwise. Put $\chi_\varnothing \equiv 1$. Let (p_B) be a (possibly substochastic) probability density on finite subsets of $\mathbb{Z}_d$. Define $(\xi_n; n = 0, 1, \cdots)$ by means of a one-step transition function of the form

$$p(\xi, \cdot) = \Pi_{\mathbf{i} \in \mathbb{Z}_d} p_\mathbf{i}(\xi, \cdot) \qquad \text{(product measure)}, \tag{2}$$

where $p_\mathbf{i}(\xi) = p_\mathbf{i}(\xi, \{1\})$ is of the form

$$p_\mathbf{i}(\xi) = p_\varnothing + \Sigma_{B \neq \varnothing} p_B \chi_{\mathbf{i}+B}(\xi): \tag{3}$$

($\mathbf{i} + B$ is B translated by the vector $\mathbf{i}$.) Thus (ξ_n) is *homogeneous*. We will also assume that (ξ_n) is *local*, i.e., $p_B > 0$ for only finitely many B. Say that (ξ_n) is *deficient* if $\Sigma_B p_B < 1$. Let (P^n) be the semigroup for (ξ_n); the process is called *ergodic* if there is a (necessarily invariant) measure μ such that $\nu P^n \Rightarrow \mu$ as $n \to \infty$

for all initial ν. Vasershtein and Leontovich [19] proved that deficient proximity processes are ergodic.

Given a measure μ on Ξ, let ξ be μ-distributed. Define the *correlation functions* $\rho(A)$ of ξ (or μ) by

$$\rho(A) = \mu\{\xi(\mathbf{i}) = 1 \quad \text{for all} \quad \mathbf{i} \in A\}.$$

Say that ξ (or μ) has *exponentially decreasing correlations* if

$$|\rho(A \cup B) - \rho(A)\rho(B)| \leqslant C_1 e^{-C_2 d(A, B)},$$

where C_1 and C_2 are constants depending only on $|A \cup B|$ (the cardinality of $A \cup B$), and $d(A, B) = \min_{\mathbf{i} \in A, \mathbf{j} \in B} |\mathbf{j} - \mathbf{i}|$ is the distance between A and B.

Define a generalized random field F_μ on $\mathbb{R}_d$ by

$$F_\mu(\varphi) = \Sigma_{\mathbf{i} \in \mathbb{Z}_d} \xi(\mathbf{i}) \varphi(\mathbf{i}). \tag{4}$$

Malyšev [15] has proved that if ξ has exponentially decreasing correlations, then

$$D_k^1 \xi \rightarrow_d C_\mu \xi_\infty \quad \text{as} \quad k \rightarrow \infty, \tag{5}$$

and

$$R_r^1 F_\mu \rightarrow_d C_\mu \Phi \quad \text{as} \quad r \rightarrow \infty. \tag{6}$$

Here ξ_∞ has independent standard normal values at each lattice site, Φ is white noise, and

$$C_\mu = \Sigma_{\mathbf{i} \in \mathbb{Z}_d} [\rho\{\mathbf{0}, \mathbf{i}\} - \rho\{\mathbf{0}\}\rho\{\mathbf{i}\}].$$

We now show how to apply Malyšev's theorem to some weakly correlated proximity processes.

THEOREM 1. *Let (ξ_n) be a deficient local proximity process with transition function p given by* (2) *and* (3). *Let μ be its unique invariant measure. Then μ has exponentially decreasing correlations. Thus, if ξ is μ-distributed and F_μ is defined by* (4), *then* (5) *and* (6) *hold.*

PROOF. In the manner of [6], construct a *dual process* $(\hat{\xi}_n)$ with state space $\hat{\Xi} = \{\text{finite subsets of } \mathbb{Z}_d\} \cup \{\Delta\}$ $(\chi_\Delta \equiv 0)$ such that

$$E_\xi[\chi_{\hat{\xi}}(\xi_n)] = \hat{E}_{\hat{\xi}}[\chi_{\hat{\xi}_n}(\xi)] \quad \text{for all} \quad \xi \in \Xi, \hat{\xi} \in \hat{\Xi}. \tag{7}$$

Since (ξ_n) is deficient, each site occupied by the dual sends the entire system to Δ with probability at least $\varepsilon = 1 - \Sigma_B p_B > 0$ during each unit of time. Therefore, since $\varnothing$ and Δ are traps for $(\hat{\xi}_n)$, $\hat{\xi}_\infty = \lim_{n \to \infty} \hat{\xi}_n \in \{\varnothing, \Delta\}$ $\hat{P}_{\hat{\xi}}$-a.s. for any initial $\hat{\xi}$. Letting $n \rightarrow \infty$ in (7), the correlation functions $\rho(A)$ for the equilibrium μ satisfy

$$\rho(A) = \hat{P}_A(\hat{\xi}_\infty = \varnothing). \tag{8}$$

Define $\tau_{\{\varnothing, \Delta\}}$ = the hitting time for $\{\varnothing, \Delta\}$; then $\hat{P}_A(\tau_{\{\varnothing, \Delta\}} > n) \leqslant (1 - \varepsilon)^n$ for all $A \neq \varnothing$. Now fix nonempty sets A and B, and construct independent processes $(\hat{\xi}_n^A)$, $(\hat{\xi}_n^B)$ on a joint probability space, distributed according to $\hat{P}_A$ and $\hat{P}_B$ respectively.

These duals can be interpreted as branching particle models, as in [8] or [6] for example. Define $(\hat{\xi}_n^{AB})$ as the (pointwise) "union evolution" of $(\hat{\xi}_n^A)$ and $(\hat{\xi}_n^B)$ endowed with a collision rule: whenever a particle from $(\hat{\xi}_n^A)$ collides with one from $(\hat{\xi}_n^B)$, then the former survives and the latter disappears. The extant particles of $(\hat{\xi}_n^{AB})$ will be $\hat{P}_{A\cup B}$-distributed. Denote by $\bar{P}$ the joint law governing $(\hat{\xi}_n^A, \hat{\xi}_n^B, \hat{\xi}_n^{AB}; n = 0, 1, \cdots)$. Adopt the convention $\hat{\xi} \subset \Delta$ for all $\hat{\xi} \in \hat{\Xi}$, and set $\tau_{AB} = \min\{n : \hat{\xi}_n^A \cap \hat{\xi}_n^B \neq \varnothing, \hat{\xi}_n^A \cup \hat{\xi}_n^B \neq \Delta\}$ ($= \infty$ if no such n exists). Then $\hat{\xi}_\infty^{AB} = \hat{\xi}_\infty^A \cup \hat{\xi}_\infty^B$ on $\{\tau_{AB} = \infty\}$. Hence, by (8) and the construction,

$$\begin{aligned}\rho(A \cup B) - \rho(A)\rho(B) &= \bar{E}\left(\chi_{\hat{\xi}_\infty^{A\cup B}} - \chi_{\hat{\xi}_\infty^A} \cdot \chi_{\hat{\xi}_\infty^B}\right)\\ &= \bar{E}\left(\chi_{\hat{\xi}_\infty^{A\cup B}} - \chi_{\hat{\xi}_\infty^A \cup \hat{\xi}_\infty^B}\right)\\ &\leqslant \bar{P}(\tau_{AB} < \infty).\end{aligned}$$

Let L be the maximal displacement of an offspring of a particle of the dual in one time unit; $L < \infty$ because (ξ_n) is local. Then clearly $\tau_{AB} \geqslant (d(A, B)/2L)\bar{P}$-a.s., while $\bar{P}((\hat{\xi}_n^A)$ hits $\{\varnothing, \Delta\}$ by time $d(A, B)/2L) \geqslant 1 - (1 - \varepsilon)^{[d(A, B)/2L]}$. ($[x]$ denotes greatest integer $\leqslant x$.) The theorem is trivial when $\varepsilon = 1$. For $0 < \varepsilon < 1$ we conclude that

$$0 \leqslant \rho(A \cup B) - \rho(A)\rho(B) \leqslant \frac{1}{1 - \varepsilon}(1 - \varepsilon)^{d(A, B)/2L},$$

whence μ has exponentially decreasing correlations.

A more delicate situation arises in the case of the Stavskaya systems [21]. For prescribed $i, j \in \mathbb{Z} : i < j$, these are a one parameter family of proximity processes (ξ_n^θ), $\theta \in [0, 1]$, such that $p_\varnothing = \theta$ and $p_{\{i, i+1, \cdots, j\}} = 1 - \theta$. Each such family has a critical value θ^* (depending on i and j), strictly between 0 and 1, such that (ξ_n^θ) is ergodic if $\theta > \theta^*$ but not if $\theta < \theta^*$. In particular, if ν_0 and ν_1 are the delta measures at "all 0's" and "all 1's" respectively, and if (P_θ^n) is the semigroup for (ξ_n^θ), then $\nu_0 P_\theta^n \Rightarrow \nu_1$ when $\theta > \theta^*$, while $\nu_0 P_\theta^n \Rightarrow \mu_\theta \neq \nu_1$ when $\theta < \theta^*$. Proofs of these assertions may be found in [21]. By combining Malyšev's theorem with a result of Vasil'ev one obtains a renormalization result for small parameter values.

THEOREM 2. *Let (ξ_n^θ) be a nonergodic Stavskaya system for some fixed $i < j$, so that $\nu_0 P_\theta^n \Rightarrow \mu_\theta \neq \nu_1$. If ξ is μ_θ-distributed and F_θ is defined as in* (4), *then for θ sufficiently small,* (5) *and* (6) *hold.*

PROOF. Use the "method of contours" to prove that μ_θ has exponentially decreasing correlations when θ is small enough (cf. Theorem 4 of [20]).

We note that the behavior of Stavskaya systems near $\theta = \theta^*$ is not known. If (ξ_n) is nonergodic at $\theta = \theta^*$, then it is quite conceivable that μ_{θ^*} has strong correlations.

3. Renormalizing the voter model on $\mathbb{Z}_3$. Throughout this section we study 3-dimensional discrete-time voter models ([8]): i.e., the proximity processes (ξ_n) with $d = 3$, and $\Sigma_{i\in\mathbb{Z}_3} p_{\{i\}} = 1$. Abbreviate $p_i = p_{\{i\}}$. We assume that $p_i > 0$ for only finitely many $\mathbf{i}$, so that (ξ_n) is *local*. In addition, we restrict attention to *nondegenerate isotropic* voter models. Letting $\mathbf{Z} = (\mathbf{Z}_1, \cdots, \mathbf{Z}_d)$ be $(p_\mathbf{i})$-distributed, the assumptions here are

(i) the group generated by $\{\mathbf{i} : p_\mathbf{i} > 0\}$ is d-dimensional,

(ii) $E(\mathbf{Z}_k) = 0 \quad 1 \leqslant k \leqslant d$,

(iii) $E(\mathbf{Z}_k \mathbf{Z}_l) = 0$, $E(\mathbf{Z}_k^2) = E(\mathbf{Z}_l^2) \quad 1 \leqslant k < l \leqslant d$. (We remark that ((ii) and (iii)) implies ((i) or $p_0 = 1$).) The model mentioned in Section 1 is nondegenerate isotropic, with

$$p_\mathbf{i} = \frac{1}{6}, \quad |\mathbf{i}| = 1$$
$$= 0 \quad \text{otherwise.}$$

Let $\hat{\Xi} = \{\text{finite subsets of } \mathbb{Z}_3\}$. Voter models on $\mathbb{Z}_3$ have dual processes $(\hat{\xi}_n)$ with state space $\hat{\Xi}$; the duals are *coalescing random walks*. Thus $(\hat{\xi}_n)$ is comprised of a finite number of random walks, each with displacement density $(p_\mathbf{i})$, and evolving independently except for the following collision rule: whenever two or more walks attempt to occupy the same site at the same time, they merge into one. The processes (ξ_n) and $(\hat{\xi}_n)$ are related by (7). For $0 \leqslant \lambda \leqslant 1$, let ν_λ be the product measure on Ξ with $\nu_\lambda\{\xi(\mathbf{i}) = 1\} = \lambda$ for all $\mathbf{i}$, and let $\rho_\lambda^n(A)$ be the correlation functions for $\nu_\lambda P^n$. In (7), set $\hat{\xi} = A$ and integrate with respect to ν_λ to get

$$\rho_\lambda^n(A) = \hat{E}_A\left(\lambda^{|\hat{\xi}_n|}\right), \qquad A \in \hat{\Xi}, n = 0, 1, \cdots,$$

the discrete time analogue of (5.9) in [8]. Since $|\hat{\xi}_n|$ is nonincreasing, $\mathrm{N} = \lim_{n\to\infty}|\hat{\xi}_n|$ exists $\hat{P}_A$-a.s. for each A. Letting $n \to \infty$, it follows that there is a measure μ_λ such that

$$\nu_\lambda P^n \Rightarrow \mu_\lambda \qquad \text{as} \quad n \to \infty,$$

and whose correlations satisfy

$$\rho_\lambda(A) = \hat{E}_A(\lambda^{\mathrm{N}}) \qquad A \in \hat{\Xi}. \tag{9}$$

Equation (9) will be the basic duality relation for our purposes. It implies immediately that $\rho_\lambda\{x\} = \lambda$ for all x. In particular, $\mu_0 = \nu_0$ and $\mu_1 = \nu_1$ ("all 0's" and "all 1's" are traps for any voter model). Holley and Liggett proved that each μ_λ is Birkhoff ergodic and extreme in the class of invariant measures for (ξ_n). Of course, it inherits translation invariance from ν_λ and homogeneity of the voter model. Results on convergence to μ_λ from initial measures other than ν_λ may also be found in [8]. Finally, we note that the situation is quite different for local homogeneous voter models on $\mathbb{Z}_1$ or $\mathbb{Z}_2$: there $\mu_\lambda = \lambda\nu_1 + (1 - \lambda)\nu_0$.

(All of these preliminary results were derived in [8] for continuous time voter models; they translate to our setting in a straightforward manner.)

As in [8], the underlying random walks without interference will be central objects of study for us. Let $\mathbf{X}_n^\mathbf{i}$ denote the walk which starts at $\mathbf{i} \in \mathbb{Z}_3$ and makes transitions according to $(p_{\mathbf{i}})$. The family of processes $\{(\mathbf{X}_n^{\mathbf{i}}); \mathbf{i} \in \mathbb{Z}_3\}$ can be constructed simultaneously on a joint probability space. Then $(\hat{\xi}_n)$ can be represented on this space in various ways, since, for example, particle coalescence can always be considered as survival of one particle and extinction of the other. We will make extensive use of such representations, omitting the routine details. Also as in [8], the difference random walk $(\mathbf{Y}_n) = (\mathbf{X}_n' - \mathbf{X}_n'')$ formed from two independent copies of $(\mathbf{X}_n)$ plays a key role. Denote the displacement density of $(\mathbf{Y}_n)$ by $(q_{\mathbf{i}})$; thus $q_{\mathbf{i}} = \Sigma_{\mathbf{j}} p_{\mathbf{j}} p_{\mathbf{j}-\mathbf{i}}$. We are now prepared to state and prove our main result.

THEOREM 3. *Let (ξ_n) be a nondegenerate isotropic (local homogeneous discrete time) voter model on $\mathbb{Z}_3$, determined by the density $(p_{\mathbf{i}})$. Fix $\lambda \in (0, 1)$, let ξ be μ_λ-distributed, and define a generalized random field F_λ on $\mathbb{R}_3$ by*

$$F_\lambda(\varphi) = \Sigma_{\mathbf{i} \in \mathbb{Z}_3} \xi(\mathbf{i}) \varphi(\mathbf{i}) \qquad \varphi \in \mathbb{S}.$$

Put $F_{\lambda, r} = R_r^{5/3} F_\lambda$, $r > 0$. Then there is a constant $C_\lambda > 0$ such that

$$F_{\lambda, r} \to_d C_\lambda \Psi \qquad \text{as} \quad r \to \infty,$$

where Ψ is the Gaussian self-similar generalized random field with covariance functional

$$E[\Psi(\varphi)\Psi(\psi)] = B(\varphi, \psi) = \int_{\mathbb{R}_3} \int_{\mathbb{R}_3} \frac{\varphi(\mathbf{x})\psi(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, d\mathbf{x} \, d\mathbf{y}.$$

With $(\mathbf{Y}_n)$ and $(q_{\mathbf{i}})$ defined as above, let

$$\gamma = \Pr\{\mathbf{Y}_n \neq 0 \quad \text{for all} \quad n \geqslant 1 | \mathbf{Y}_0 = \mathbf{0}\},$$

$$m_2 = \Sigma_{\mathbf{i} \in \mathbb{Z}_3} |\mathbf{i}|^2 p_{\mathbf{i}},$$

and let $\mathcal{G}$ be the group generated by $\{\mathbf{i} : q_{\mathbf{i}} > 0\}$. If $\mathcal{G} = \mathbb{Z}_3$, then

$$C_\lambda = \left[\frac{3\lambda(1 - \lambda)\gamma}{4\pi m_2} \right]^{\frac{1}{2}}.$$

PROOF. The desired limiting generalized random field has characteristic functional

$$E[e^{iC_\lambda \Psi(\varphi)}] = L(\varphi) = \exp\left\{ -\frac{C_\lambda^2}{2} B(\varphi, \varphi) \right\}.$$

Writing $L_r(\varphi) = E[e^{iF_{\lambda, r}(\varphi)}]$, it suffices to show that $\lim_{r \to \infty} L_r(\varphi) = L(\varphi)$ for all $\varphi \in \mathbb{S}$. To establish this, we prove directly, for each φ, that

$$(10) \qquad F_{\lambda, r}(\varphi) \to_d C_\lambda \Psi(\varphi) \qquad \text{as} \quad r \to \infty.$$

Let $M_r^m(\varphi)$ denote the mth moment of $F_{\lambda, r}(\varphi)$. Also, let $S_r^m(\varphi)$ denote the mth semi-invariant of $F_{\lambda, r}(\varphi)$, i.e., $S_r^m(\varphi)$, $m = 1, 2, \cdots$, make up the Taylor series coefficients of

$$\log E[\exp[-\mu \cdot F_{\lambda, r}(\varphi)]] = \Sigma_{m=1}^{\infty} \frac{S_r^m(\varphi)}{m!} \mu^m$$

(cf. [12], [13], [15]). By the method of moments, (10) follows once we show that

$$\lim_{r\to\infty} M_r^m(\varphi) = 0 \qquad m \text{ odd}$$

$$= [1 \cdot 3 \cdot \cdots \cdot (2m-1)]C_\lambda^m[B(\varphi,\varphi)]^{m/2} \qquad m \text{ even}.$$

Equivalently, by the method of semi-invariants, it suffices to show for each $\varphi \in \mathcal{S}$,

$$\lim_{r\to\infty} S_r^2(\varphi) = C_\lambda^2 B(\varphi,\varphi) \tag{11}$$

and

$$\lim_{r\to\infty} S_r^m(\varphi) = 0 \qquad \text{for all} \quad m \geqslant 3. \tag{12}$$

(Of course, $S_r^1(\varphi) = M_r^1(\varphi) = 0$ for all r.) We will demonstrate (11) and (12) with the aid of the Ursell functions $u(\mathbf{i}_1, \cdots, \mathbf{i}_m)$ (= truncated correlation functions) of $(\mathbf{i}_1, \cdots, \mathbf{i}_m) \in (\mathbb{Z}_3)^m$ with respect to μ_λ. Given a partition $\pi = (\pi_1, \cdots, \pi_s)$ of $\{1, \cdots, m\}$, where all π_t, $1 \leqslant t \leqslant s$, are nontrivial, if we write $A_t = \{\mathbf{i}_l : l \in \pi_t\}$, then $u(\mathbf{i}_1, \cdots, \mathbf{i}_m)$ is defined as

$$\Sigma_{s=1}^m (-1)^{s-1}(s-1)!\Sigma_{\pi=(\pi_1,\cdots,\pi_s)}\rho(A_1)\cdot \cdots \cdot \rho(A_s), \tag{13}$$

where $\rho(A_t)$ is defined in (9). A combinatorial argument shows that

$$S_r^m(\varphi) = r^{-\frac{5}{2}m}\Sigma_{\mathbf{i}_1,\cdots,\mathbf{i}_m\in\mathbb{Z}_3} u(\mathbf{i}_1,\cdots,\mathbf{i}_m)\varphi\left(\frac{\mathbf{i}_1}{r}\right)\cdot \cdots \cdot \varphi\left(\frac{\mathbf{i}_m}{r}\right) \tag{14}$$

(cf. [12], [13], [15]), a representation we will employ throughout the proof. For convenience, we divide the proofs of (11) and (12) into three propositions.

PROPOSITION 1. *Equation* (11) *holds.*

PROOF OF PROPOSITION 1. Using (9), we find that $u(\mathbf{i},\mathbf{i}) = \lambda(1-\lambda)$ for each $\mathbf{i}$, while for $\mathbf{i} \neq \mathbf{j}$, $u(\mathbf{i},\mathbf{j}) = \lambda(1-\lambda)\hat{P}_{\{\mathbf{i},\mathbf{j}\}}(\mathbf{N}=1) = \lambda(1-\lambda)h(\mathbf{j}-\mathbf{i})$, where $h(\mathbf{k})$ is the probability that the difference random walk $(\mathbf{Y}_n)$ ever hits $\mathbf{0}$ when it starts at $\mathbf{k}$. Since (ξ_n) is nondegenerate isotropic, so are $(\mathbf{X}_n)$ and $(\mathbf{Y}_n)$. Results in Sections 26 and 7 of [17] guarantee a constant $C > 0$ such that

$$h(\mathbf{k}) \sim \frac{C}{|\mathbf{k}|} \qquad \text{as} \quad \mathbf{k}\to\infty, \mathbf{k}\in\mathcal{G} \tag{15}$$

$$(=0 \qquad \mathbf{k}\in\mathbb{Z}_3 - \mathcal{G}),$$

with $C = 3\gamma/4\pi m_2$ if $\mathcal{G} = \mathbb{Z}_3$. Fix $\varepsilon > 0$. If $|\mathbb{Z}_3/\mathcal{G}| = K$, then for any $M \geqslant 0$,

$$r^{-5}\Sigma_{\mathbf{i},\mathbf{j}\in\mathbb{Z}_3:\,\mathbf{j}-\mathbf{i}\in\mathcal{G},\,|\mathbf{j}-\mathbf{i}|\geqslant M}\frac{1}{|\mathbf{j}-\mathbf{i}|}\varphi\left(\frac{\mathbf{i}}{r}\right)\varphi\left(\frac{\mathbf{j}}{r}\right)$$

$$= r^{-6}\Sigma_{\mathbf{i},\mathbf{j}\in\mathbb{Z}_3:\,\mathbf{j}-\mathbf{i}\in\mathcal{G},\,|\frac{\mathbf{j}}{r}-\frac{\mathbf{i}}{r}|\geqslant\frac{M}{r}}\frac{1}{\left|\frac{\mathbf{j}}{r}-\frac{\mathbf{i}}{r}\right|}\varphi\left(\frac{\mathbf{i}}{r}\right)\varphi\left(\frac{\mathbf{j}}{r}\right)$$

$$\to K^{-1}B(\varphi,\varphi) \qquad \text{as} \quad r\to\infty.$$

Setting $C_\lambda^2 = \lambda(1 - \lambda)CK^{-1}$, and using (15), we can choose M large enough that

$$\limsup_{r\to\infty}\left|r^{-5}\Sigma_{\mathbf{j}-\mathbf{i}\in\mathcal{G};\,|\mathbf{j}-\mathbf{i}|\geq M}\left(u(\mathbf{i},\mathbf{j}) - \frac{\lambda(1-\lambda)C}{|\mathbf{j}-\mathbf{i}|}\right)\varphi\left(\frac{\mathbf{i}}{r}\right)\varphi\left(\frac{\mathbf{j}}{r}\right)\right| \leq \varepsilon C_\lambda^2 B(\varphi,\varphi).$$

Also,

$$r^{-5}\Sigma_{|\mathbf{j}-\mathbf{i}|<M}u(\mathbf{i},\mathbf{j})\varphi\left(\frac{\mathbf{i}}{r}\right)\varphi\left(\frac{\mathbf{j}}{r}\right) \sim 0(r^{-2}).$$

Using (14), we conclude that

$$\limsup_{r\to\infty}|S_r^2(\varphi) - C_\lambda^2 B(\varphi,\varphi)| \leq \varepsilon C_\lambda^2 B(\varphi,\varphi).$$

Since ε is arbitrary, the proof of Proposition 1 is finished.

PROPOSITION 2. *Let $u(\mathbf{i}_1, \cdots, \mathbf{i}_m)$ be an mth order Ursell function for μ_λ. Then*

$$|u(\mathbf{i}_1, \cdots, \mathbf{i}_m)| \leq K_m \hat{P}_{\{\mathbf{i}_1,\ldots,\mathbf{i}_m\}}(\mathrm{N} = 1),$$

where K_m is a constant depending only on m.

(Note that $\mathbf{i}_l = \mathbf{i}_{l'}$ for some l, l' is allowed.)

PROOF. Construct m independent random walks $\mathbf{X}_n^l$, starting at the respective sites $\mathbf{i}_l$, with displacement density $(p_\mathbf{i})$, on a joint probability space. Let $\tilde{P}$ and $\tilde{E}$ be the probability law and expectation operator on this space. For each (nontrivial—this means that no member of π may equal $\varnothing$) partition $\pi = (\pi_1, \cdots, \pi_s)$ of $\{1, \cdots, m\}$, let $\mathbf{X}_n^\pi$ denote the process such that walks whose superscripts belong to the same π_t interact, whereas those with superscripts from distinct members of π do not. The interaction is as follows: if $\mathbf{X}_n^l$ and $\mathbf{X}_n^{l'}$ collide, then $\mathbf{X}_n^l$ survives (i.e., does not disappear) if and only if $l < l'$; interaction takes place at time 0 whenever $\mathbf{i}_l = \mathbf{i}_{l'}$. Thus $(\mathbf{X}_n^{(\{1\},\cdots,\{m\})})$ is the totally independent process (no interaction), whereas $(\mathbf{X}_n^{(\{1,\cdots,m\})})$ can be identified with the dual $\hat{\xi}_n$ starting at $\{\mathbf{i}_1, \cdots, \mathbf{i}_m\}$ in the obvious way. Intermediate $(\mathbf{X}_n^\pi)$ have intermediate collision rules, and are all to be thought of as evolving simultaneously on the same probability space governed by $\tilde{P}$. Equation (13) states that

(13) $\quad u(\mathbf{i}_1, \cdots, \mathbf{i}_m) = \Sigma_{s=1}^m(-1)^{s-1}(s-1)!\Sigma_{\pi=(\pi_1,\ldots,\pi_s)}\rho(A_1)\cdot \cdots \cdot\rho(A_s).$

For each π, let $\mathrm{N}_\pi = \lim_{n\to\infty}|\mathbf{X}_n^\pi|$. Then from (9) and the construction,

$$u(\mathbf{i}_1, \cdots, \mathbf{i}_m) = \Sigma_{s=1}^m(-1)^{s-1}(s-1)!\Sigma_\pi \tilde{E}(\lambda^{\mathrm{N}_\pi})$$
$$= \tilde{E}(\Sigma),$$

where

$$\Sigma = \Sigma_{s=1}^m(-1)^{s-1}(s-1)!\Sigma_\pi\lambda^{\mathrm{N}_\pi}.$$

Now suppose that (π', π'') is a (nontrivial) partition of $\{1, \cdots, m\}$. It is known ([12], [13], [15]) that the right side of (13) can be rearranged into a finite sum of the form

(16) $\qquad \Sigma \pm [\rho(A_1 \cup A_2) - \rho(A_1)\cdot\rho(A_2)]\rho(A_3)\cdot \cdots \cdot\rho(A_s),$

over all partitions π such that $\pi_1 \subset \pi'$ and $\pi_2 \subset \pi''$.

Since the rearrangement procedure is purely combinatorial, exactly the same manipulations show that for $m \geqslant 2$, Σ can be rewritten as

$$\Sigma = \Sigma \pm \left[\lambda^{N_{\bar{\pi}}} - \lambda^{N_{\pi}}\right],$$

where $\bar{\pi} = (\pi_1 \cup \pi_2, \pi_3, \cdots, \pi_s)$, $\pi = (\pi_1, \pi_2, \pi_3, \cdots, \pi_s)$. (The lemma is trivial in case $m = 1$.) Suppose $N_{(\{1,\cdots,m\})} > 1$. Then there is some partition (π', π'') such that no process $(X_n^{l'})$ with a superscript in π' ever collides with any $(X_n^{l''})$ with a superscript in π''. Hence $N_{\bar{\pi}} = N_\pi$ for all $\bar{\pi}$, π entering into the last sum Σ, and so $\Sigma = 0$ on $\{N_{(\{1,\cdots,m\})} > 1\}$. On $\{N_{(\{1,\cdots,m\})} = 1\}$, $|\Sigma|$ is clearly bounded by

$$K_m = \Sigma_{s=1}^m (s-1)!|\{\pi : \pi = (\pi_1, \cdots, \pi_s)\}|.$$

Thus

$$|u(\mathbf{i}_1, \cdots, \mathbf{i}_m)| \leqslant K_m \tilde{P}(N_{(\{1,\cdots,m\})} = 1) = K_m \hat{P}_{\{\mathbf{i}_1,\cdots,\mathbf{i}_m\}}(N = 1),$$

completing the proof of Proposition 2.

PROPOSITION 3. *Equation* (12) *holds.*

PROOF OF PROPOSITION 3. Applying Proposition 2 to (13), it suffices to show that

$$(17) \qquad r^{-\frac{5}{2}m}\Sigma_{\mathbf{i}_1,\cdots,\mathbf{i}_m \in \mathbb{Z}_3}\hat{P}_{\{\mathbf{i}_1,\cdots,\mathbf{i}_m\}}(N = 1)\varphi\left(\frac{\mathbf{i}_1}{r}\right)\cdot \cdots \cdot \varphi\left(\frac{\mathbf{i}_m}{r}\right) \to 0$$

as $r \to \infty$, for φ nonnegative. Let us denote a nontrivial partition of $\{1, \cdots, m\}$ into two subsets π_1 and π_2 by (π_1, π_2), and abbreviate $N_{\pi_i} = N_{\{\pi_i\}}$. A simple set-theoretic argument shows that

$$\hat{P}_{\{\mathbf{i}_1,\cdots,\mathbf{i}_m\}}(N = 1) \leqslant \Sigma_{(\pi_1,\pi_2)}\hat{P}_{\{\mathbf{i}_1,\cdots,\mathbf{i}_k,\cdots,\mathbf{i}_m : k \in \pi_1\}}(N_{\pi_1} = 1)\cdot$$
$$\hat{P}_{\{\mathbf{i}_1,\cdots,\mathbf{i}_k,\cdots,\mathbf{i}_m : k \in \pi_2\}}(N_{\pi_2} = 1)\cdot s(\mathbf{i}_1, \cdots, \mathbf{i}_m; \pi_1, \pi_2),$$

where $s(\mathbf{i}_1, \cdots, \mathbf{i}_m; \pi_1, \pi_2)$ is the probability that, conditioned on $N_{\pi_1} = 1$ and $N_{\pi_2} = 1$, the two remaining particles eventually coalesce. Therefore, to demonstrate (17), we show that

$$(18) \qquad r^{-\frac{5}{2}m}\Sigma_{\mathbf{i}_1,\cdots,\mathbf{i}_m \in \mathbb{Z}_3}\hat{P}_{\{\mathbf{i}_1,\cdots,\mathbf{i}_{n-1}\}}(N_{\pi_1} = 1)\cdot \hat{P}_{\{\mathbf{i}_n,\cdots,\mathbf{i}_m\}}(N_{\pi_2} = 1)\cdot$$
$$s(\mathbf{i}_1, \cdots, \mathbf{i}_m; \pi_1, \pi_2)\cdot\varphi\left(\frac{\mathbf{i}_1}{r}\right)\cdot \cdots \cdot \varphi\left(\frac{\mathbf{i}_m}{r}\right) \to 0,$$

where $\pi_1 = \{1, \cdots, n-1\}$, $\pi_2 = \{n, n+1, \cdots, m\}$, $2 \leqslant n \leqslant m$. Also, set $n_1 = n - 1$ and $n_2 = m - n + 1$.

We now proceed to reduce the left side of (18) to the form

$$Ar^{-\frac{5}{2}m-1}\Sigma_{\mathbf{i}_1,\cdots,\mathbf{i}_m}\hat{P}_{\{\mathbf{i}_1,\cdots,\mathbf{i}_{n_1}\}}(N_{\pi_1} = 1)\cdot \hat{P}_{\{\mathbf{i}_n,\cdots,\mathbf{i}_m\}}(N_{\pi_2} = 1)\cdot\varphi\left(\frac{\mathbf{i}_1}{r}\right)\cdot \cdots \cdot \varphi\left(\frac{\mathbf{i}_m}{r}\right)$$
$$= Ar^{-1}\cdot r^{-\frac{5}{2}n_1}\Sigma_{\mathbf{i}_1,\cdots,\mathbf{i}_{n_1}}\hat{P}_{\{\mathbf{i}_1,\cdots,\mathbf{i}_{n_1}\}}(N_{\pi_1} = 1)\varphi\left(\frac{\mathbf{i}_1}{r}\right)\cdot \cdots \cdot \varphi\left(\frac{\mathbf{i}_{n_1}}{r}\right)\cdot$$
$$r^{-\frac{5}{2}n_2}\Sigma_{\mathbf{i}_n,\cdots,\mathbf{i}_m}\hat{P}_{\{\mathbf{i}_n,\cdots,\mathbf{i}_m\}}(N_{\pi_2} = 1)\varphi\left(\frac{\mathbf{i}_n}{r}\right)\cdot \cdots \cdot \varphi\left(\frac{\mathbf{i}_m}{r}\right),$$

where A is independent of r. This reduction will enable us to employ induction on m to conclude that (18), and hence (17), is valid. To effect such a reduction, we partition $(\mathbb{Z}_3)^k$ into subsets c, henceforth called *configurations*, such that the ordered k-tuples $(\mathbf{i}_1, \cdots, \mathbf{i}_k)$ and $(\mathbf{i}'_1, \cdots, \mathbf{i}'_k)$ belong to the same configuration iff there exists an $\mathbf{a} \in \mathbb{Z}_3$ such that $(\mathbf{i}_1 + \mathbf{a}, \cdots, \mathbf{i}_k + \mathbf{a}) = (\mathbf{i}'_1, \cdots, \mathbf{i}'_k)$. Denoting the partition by $\mathcal{C}_k$, and noting that $\hat{P}_{\cdot}(\mathbf{N}_{\pi_i} = 1)$ is a function of the configuration, we rewrite the left side of (18) as

$$(19) \qquad r^{-\frac{5}{2}m}\Sigma_{c\in\mathcal{C}_{n_1}}\Sigma_{c'\in\mathcal{C}_{n_2}}\Sigma_{\mathbf{i}_1}\Sigma_{\mathbf{i}_n}\hat{P}_c(\mathbf{N}_{\pi_1} = 1)\hat{P}_{c'}(\mathbf{N}_{\pi_2} = 1)$$
$$\cdot\, s(\mathbf{i}_1, \cdots, \mathbf{i}_m; \pi_1, \pi_2)\cdot\varphi\left(\frac{\mathbf{i}_1}{r}\right)\cdot \cdots \cdot\varphi\left(\frac{\mathbf{i}_m}{r}\right),$$

where $(\mathbf{i}_1, \cdots, \mathbf{i}_{n_1}) \in c$, $(\mathbf{i}_n, \cdots, \mathbf{i}_m) \in c'$. We also introduce the following notation:

$$f^{(c,r)}(\mathbf{i}_1) \equiv \varphi\left(\frac{\mathbf{i}_1}{r}\right)\cdot \cdots \cdot\varphi\left(\frac{\mathbf{i}_{n_1}}{r}\right),$$
$$f^{(c',r)}(-\mathbf{i}_n) \equiv \varphi\left(\frac{\mathbf{i}_n}{r}\right)\cdot \cdots \cdot\varphi\left(\frac{\mathbf{i}_m}{r}\right),$$
$$g(c) \equiv \hat{P}_c(\mathbf{N}_{\pi_1} = 1),$$
$$g(c') \equiv \hat{P}_{c'}(\mathbf{N}_{\pi_2} = 1),$$
$$\tau_i \equiv \inf\{k : |X_k^{\pi_i}| = 1\}(= \infty \qquad \text{if} \quad \mathbf{N}_{\pi_i} > 1),$$
$$k^{(c,c')}(\mathbf{x}) \equiv \hat{P}\left[\mathbf{X}^{\pi_1}_{\tau_1\vee\tau_2} - \mathbf{X}^{\pi_2}_{\tau_1\vee\tau_2} = \mathbf{x}\,|\,\tau_1 \vee \tau_2 < \infty;\ \mathbf{i}_1 = \mathbf{0}, \mathbf{i}_n = \mathbf{0}\right].$$

(Again, $\hat{P}[\cdot\,|\,\cdot\,]$ is a function of the configuration.)With $h(\mathbf{y})$ defined as in Proposition 1, note that

$$s(\mathbf{i}_1, \cdots, \mathbf{i}_m; \pi_1, \pi_2) = \Sigma_{\mathbf{x}}k^{(c,c')}(\mathbf{x})h(\mathbf{i}_1 - \mathbf{i}_n + \mathbf{x})$$

for $(\mathbf{i}_1, \cdots, \mathbf{i}_{n_1}) \in c$, $(\mathbf{i}_n, \cdots, \mathbf{i}_m) \in c'$. It follows that (19) equals

$$r^{-\frac{5}{2}m}\Sigma_c\Sigma_{c'}g(c)g(c')\Sigma_{\mathbf{i}_1}\Sigma_{\mathbf{i}_n}f^{(c,r)}(\mathbf{i}_1)f^{(c',r)}(\mathbf{i}_n)\cdot\Sigma_{\mathbf{x}}k^{(c,c')}(\mathbf{x})h(\mathbf{i}_1 + \mathbf{i}_n + \mathbf{x}).$$

Recall that $h(\mathbf{y})$ is less than $A_1/(|y| \vee 1)$ for some constant A_1. Letting $*$ denote convolution, and setting $\mathbf{y} = \mathbf{i}_1 + \mathbf{i}_n + \mathbf{x}$ and $|\mathbf{y}|_1 = |\mathbf{y}| \vee 1$, it follows that (19) is less than

$$A_1r^{-\frac{5}{2}m}\Sigma_c\Sigma_{c'}g(c)g(c')\Sigma_{\mathbf{y}}(f^{(c,r)} * f^{(c',r)} * k^{(c,c')})(\mathbf{y})/|\mathbf{y}|_1,$$

which is at most

$$(20) \qquad A_1r^{-\frac{5}{2}m-1}\Sigma_c\Sigma_{c'}g(c)g(c')\Sigma_{|\mathbf{y}|/r\leqslant 1}(f^{(c,r)} * f^{(c',r)} * k^{(c,c')})(\mathbf{y})\Big/\frac{|\mathbf{y}|_1}{r}$$
$$+A_1r^{-\frac{5}{2}m-1}\Sigma_c\Sigma_{c'}g(c)g(c')\Sigma_{|\mathbf{y}|/r>1}(f^{(c,r)} * f^{(c',r)} * k^{(c,c')})(\mathbf{y}).$$

To obtain an upper bound for the first term of (20), we introduce the further notation:

$$F^{(c,r)} \equiv \max_{\mathbf{i}_1} f^{(c,r)}(\mathbf{i}_1),$$

$$\varphi_1(\mathbf{x}) \equiv \max_{|\alpha| \leqslant 1} \varphi(\mathbf{x} + \alpha)$$

(note that φ_1 is also rapidly decreasing),

$$f_1^{(c,r)}(\mathbf{i}_1) \equiv \varphi_1\left(\frac{\mathbf{i}_1}{r}\right) \cdot \cdots \cdot \varphi_1\left(\frac{\mathbf{i}_{n_1}}{r}\right),$$

$$f_2^{(c,r)}(\mathbf{i}_1) \equiv f_1^{(c,r)}(\mathbf{i}_1) \wedge F^{(c,r)},$$

$$M_1^{(c,r)} \equiv \Sigma_{\mathbf{i}_1} f_1^{(c,r)}(\mathbf{i}_1),$$

$$M_2^{(c,r)} \equiv \Sigma_{\mathbf{i}_1} f_2^{(c,r)}(\mathbf{i}_1),$$

$$M^{(c',r)} \equiv \Sigma_{\mathbf{i}_n} f^{(c',r)}(\mathbf{i}_n),$$

$$\bar{f}_2^{(c,r)}(\mathbf{i}_1) \equiv f_2^{(c,r)}(\mathbf{i}_1) / M_2^{(c,r)},$$

$$\bar{f}^{(c',r)}(\mathbf{i}_n) \equiv f^{(c',r)}(\mathbf{i}_n) / M^{(c',r)}.$$

The first term of (20) is at most

$$A_1 r^{-\frac{5}{2}m-1} \Sigma_c \Sigma_{c'} g(c) g(c') \Sigma_{|\mathbf{y}|/r \leqslant 1} \big(f_2^{(c,r)} * f^{(c',r)} * k^{(c,c')}\big)(y) \Big/ \frac{|y|_1}{r},$$

which equals

(21)

$$A_1 r^{-\frac{5}{2}m-1} \Sigma_c \Sigma_{c'} g(c) g(c') M_{22}^{(c,r)} M^{(c',r)} \Sigma_{|y|/r \leqslant 1} \big(\bar{f}_2^{(c,r)} * \bar{f}^{(c',r)} * k^{(c,c')}\big)(\mathbf{y}) \Big/ \frac{|\mathbf{y}|_1}{r}.$$

Now, $\max_{\mathbf{y}}(\bar{f}_2^{(c,r)} * \bar{f}^{(c',r)} * k^{(c,c')})(\mathbf{y}) \leqslant \max_{\mathbf{y}} \bar{f}_2^{(c,r)}(\mathbf{y})$. Moreover, $\bar{f}_2^{(c,r)}$ was constructed so that the value $\bar{F}_2^{(c,r)} \equiv \max_{\mathbf{y}} \bar{f}_2^{(c,r)}(\mathbf{y}) = F^{(c,r)}/M_2^{(c,r)}$ is assumed at at least $(2r)^3$ different sites, and hence $\bar{F}_2^{(c,r)} \leqslant (2r)^{-3}$. Therefore, (21) is at most

$$A_1 r^{-\frac{5}{2}m-1} \Sigma_c \Sigma_{c'} g(c) g(c') M_2^{(c,r)} M^{(c',r)} \cdot (2r)^{-3} \Sigma_{|\mathbf{y}|/r \leqslant 1} 1 \Big/ \frac{|\mathbf{y}|_1}{r}$$

$$\leqslant A_2 r^{-\frac{5}{2}m-1} \Sigma_c \Sigma_{c'} g(c) g(c') M_2^{(c,r)} M^{(c',r)}$$

for some constant $A_2 > A_1$, since $\sup_r (2r)^{-3} \Sigma_{|\mathbf{y}|/r \leqslant 1} 1/(|\mathbf{y}|_1/r) < \infty$.

Now the inner sum of the second term in (20) is less than $M_2^{(c,r)}M^{(c',r)}$. With $A_3 = A_1 + A_2$, we conclude that (18) is majorized by

$$A_3 r^{-\frac{5}{2}m-1}\Sigma_c \Sigma_{c'} g(c)g(c')M_1^{(c,r)}M^{(c',r)}$$

$$= A_3 r^{-\frac{5}{2}m-1}\Sigma_{\mathbf{i}_1,\dots,\mathbf{i}_m}\hat{P}_{\{\mathbf{i}_1,\dots,\mathbf{i}_{n_1}\}}(\mathbf{N}_{\pi_1}=1)\cdot\hat{P}_{\{\mathbf{i}_n,\dots,\mathbf{i}_m\}}(\mathbf{N}_{\pi_2}=1)$$

$$\cdot\varphi_1\left(\frac{\mathbf{i}_1}{r}\right)\cdot\cdots\cdot\varphi_1\left(\frac{\mathbf{i}_{n_1}}{r}\right)\varphi\left(\frac{\mathbf{i}_n}{r}\right)\cdot\cdots\cdot\varphi\left(\frac{\mathbf{i}_m}{r}\right)$$

$$= A_3 r^{-1}\cdot r^{-\frac{5}{2}n_1}\Sigma_{\mathbf{i}_1,\dots,\mathbf{i}_{n_1}}\hat{P}_{\{\mathbf{i}_1,\dots,\mathbf{i}_{n_1}\}}(\mathbf{N}_{\pi_1}=1)\cdot\varphi_1\left(\frac{\mathbf{i}_1}{r}\right)\cdot\cdots\cdot\varphi_1\left(\frac{\mathbf{i}_{n_1}}{r}\right)$$

$$\cdot r^{-\frac{5}{2}n_2}\Sigma_{\mathbf{i}_n,\dots,\mathbf{i}_m}\hat{P}_{\{\mathbf{i}_n,\dots,\mathbf{i}_m\}}(\mathbf{N}_{\pi_2}=1)\cdot\varphi\left(\frac{\mathbf{i}_n}{r}\right)\cdot\cdots\cdot\varphi\left(\frac{\mathbf{i}_m}{r}\right).$$

If we set

$$T_r^k(\varphi) = r^{-\frac{5}{2}k}\Sigma_{\mathbf{i}_1,\dots,\mathbf{i}_k}\hat{P}_{\{\mathbf{i}_1,\dots,\mathbf{i}_k\}}(\mathbf{N}=1)\cdot\varphi\left(\frac{\mathbf{i}_1}{r}\right)\cdot\cdots\cdot\varphi\left(\frac{\mathbf{i}_k}{r}\right),$$

then it follows that the left-hand side of (17), $T_r^m(\varphi)$, is at most

$$\Sigma_{k=1}^{m-1}A_3\binom{m}{k}r^{-1}T_r^k(\varphi_1)T_r^{m-k}(\varphi).$$

Since (11) states that $T_r^2(\varphi)$ is bounded, and $r^{-\frac{1}{2}}T_r^1(\varphi)$ is clearly bounded, induction on m shows that $r^{(m-2)/2}T_r^m(\varphi)$ is bounded as $r \to \infty$. This is more than enough to demonstrate (17) as $r \to \infty$. Hence Proposition 3 and the theorem are proved.

4. Additional remarks.

(i) With (ξ_n) as in Theorem 3, let ξ_{n_0} be the field at some fixed time n_0. Because the process is local, there is a $d = d(n_0) < \infty$ such that $\xi_{n_0}|_A$ and $\xi_{n_0}|_B$ are independent whenever $d(A, B) \geq d$. Clearly ξ_{n_0} has *exponentially decreasing correlations*. Thus (5) and (6) hold at each time $n_0 < \infty$, but not in the limit. The same observation was made by Dawson and Ivanoff in [2].

(ii) In much the same way, one can prove a "Sinai-type" block renormalization theorem. Namely, if ξ is the μ_λ-distributed equilibrium for a nondegenerate isotropic (local homogeneous) 3-dimensional voter model, then

$$D_k^{5/3}\xi \to_d C_\lambda \xi_\infty \qquad \text{as} \quad k \to \infty.$$

C_λ is as in the theorem, and ξ_∞ is the isotropic Gaussian self-similar random field on $\mathbb{Z}_3$ with covariances given by

$$E[\xi_\infty(\mathbf{i})\xi_\infty(\mathbf{j})] = \int_I\int_J \frac{1}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{x}\,d\mathbf{y},$$

where I and J are the cubes of side 1 in $\mathbb{R}_3$ centered at $\mathbf{i}$ and $\mathbf{j}$ respectively. Details are left to the reader.

(iii) Analogous results can be derived for continuous time proximity processes and voter models, as formulated in [6], [7], [8], [9]. The dual of the voter model is then comprised of continuous time coalescing random walks, to which the same methods apply. In the continuous time simple random walk case, the difference walk $\mathbf{Y}_t$ is simply $\mathbf{X}_t$ with twice the jump rates. The constant γ for the corresponding voter model (ξ_t) is therefore known [17, page 103] to be $\gamma \approx .65046267$, and so C_λ can be computed to several decimal places.

(iv) Theorem 3 may be extended to local homogeneous voter models in dimension $d \geqslant 3$. The order in dimension d is $\alpha = 1 + 2/d$. Needless to say, if (ξ_n) is not isotropic, then the limiting self-similar field need not be isotropic.

(v) The methods used to prove Theorem 3 apply equally well to any voter model whose random walks are transient and in the domain of attraction of a symmetric stable law. Thus, if $h(\mathbf{k}) = 0(1/|\mathbf{k}|^\kappa)$ for some κ, a limiting field of the form (1) arises. In particular, certain one- and two-dimensional voter models lead to such fields. Of course these systems are far from local.

(vi) It would be interesting to know whether the Holley-Stroock space-time renormalization [9] leading to the generalized Ornstein-Uhlenbeck process carries over for the voter model. At the minimum, this would require an extension of our techniques: the cancellation procedure employed below (16) in Proposition 2 will no longer be valid.

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