The Annals of Probability
1979, Vol. 7, No. 3, 453–460

MAXIMA AND MINIMA OF STATIONARY SEQUENCES[1]

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We show that the asymptotic behavior of the normalized maxima of a stationary sequence satisfying a weak distributional mixing and bivariate condition is completely determined by the marginal distribution of the process. Sufficient conditions are given in order for the maxima and minima to be asymptotically independent. An example of a 1-dependent sequence where the maxima and minima are not asymptotically independent is also provided.

1. Introduction. Let $\{X_n : n = 1, 2, \cdots\}$ be a strictly stationary sequence with F and $F_{i_1, \cdots, i_p}(\cdot, \cdots, \cdot)$ denoting the marginal distribution function (df) and joint df of $X_{i_1}, \cdots, X_{i_p}$ respectively. Let $\{u_n\}$ be a sequence of real numbers. We shall say (cf. [2]) that the condition $D(u_n)$ is satisfied by the sequence $\{X_n\}$ if for any n, and any choice of integers $i_1 < \cdots < i_p < j_1 < \cdots < j_q, j_1 - i_p > l$, we have

$$|F_{i_1, \cdots, i_p, j_1, \cdots, j_q}(u_n, \cdots, u_n) - F_{i_1, \cdots, i_p}(u_n, \cdots, u_n)F_{j_1, \cdots, j_q}(u_n, \cdots, u_n)| \leqslant \alpha_{n, l},$$

where $\alpha_{n, l}$ is nonincreasing in l and $\lim_{n\to\infty}\alpha_{n, l_n} = 0$ for some sequence $l_n \to \infty$ with $l_n/n \to 0$. The condition $D(u_n)$ is potentially much weaker and easier to verify than strong mixing, for there are fewer pairs of events to consider and these events are of a particular form.

Let $M_n = \max\{X_1, \cdots, X_n\}$ and $u_n = u_n(x) = x/a_n + b_n$ for some constants $a_n > 0$ and b_n. Gnedenko [1] showed that if $\{X_n\}$ is an independent and identically distributed (i.i.d.) sequence then there are only three possible nondegenerate limiting distributions for the normalized maxima. Loynes [3] weakened the i.i.d. assumption in Gnedenko's trinity theorem to strong mixing and stationarity. Leadbetter [2] extended Loynes result to stationary sequences satisfying $D(u_n)$ for all x. Furthermore, Loynes [3] and O'Brien [4] showed that if a stationary strong mixing sequence satisfied a particular bivariate condition R_1, then $P(a_n(M_n - b_n) \leqslant x) \to G(x)$ if and only if $P(a_n(\hat{M}_n - b_n) \leqslant x) \to G(x)$, where $G(x)$ is a nondegenerate df and $\hat{M}_n$ is the maximum of n i.i.d. random variables with marginal df F. The condition R_1 was developed from the one used by Watson [5] for the m-dependent situation. Section 1 extends this result to a stationary sequence satisfying $D(u_n)$ and the following bivariate condition for all x. The condition $D'(u_n)$ is said to hold if $\lim\sup_n n\sum_{j=1}^{n-1}P(X_1 > u_{nk}, X_{j+1} > u_{nk}) = o(1/k)$ as $k \to \infty$.

Received December 9, 1977.

[1]This research was supported in part by the Office of Naval Research under Contract N00014–75–C–0428.

AMS 1970 *subject classifications.* Primary 60F05; secondary 60G10.

Key words and phrases. Nondegenerate limiting distributions, maximum and minimum value, stationary sequence, distributional mixing, m-dependence, stationary Gaussian sequence.

In Section 2 we prove, under a suitable mixing and bivariate condition, that the normalized maxima and minima are asymptotically independent. Section 3 contains an application to a stationary Gaussian sequence with covariance function r_n. If $r_n \log n \to 0$ or $\Sigma|r_n|^p < \infty$ for some $p > 1$, then the normalized maxima and minima are asymptotically independent. We also give an example of a 1-dependent sequence where the normalized maxima and minima marginally behave as though the sequence was i.i.d., yet jointly they are not asymptotically independent.

2. Limiting results. The aim of this section is to prove that if $P(M_n \leqslant u_n) \to e^{-\tau}$ as $n \to \infty$, then $P(\hat{M}_n \leqslant u_n) \to e^{-\tau}$.

LEMMA 2.1. *Suppose that $D(u_n)$, $D'(u_n)$ hold for the stationary sequence $\{X_n\}$ where $\{u_n\}$ is a nondecreasing sequence of real numbers. If $P(M_n \leqslant u_n) \to e^{-\tau}$ as $n \to \infty$ then $P(\hat{M}_n \leqslant u_n) \to e^{-\tau}$, $0 < \tau < \infty$.*

PROOF. By the $D(u_n)$ assumption, we have

$$P^k(M_n \leqslant u_{nk}) - P(M_{nk} \leqslant u_{nk}) \to 0, \tag{2.1}$$

for every fixed integer k (cf. Leadbetter [2]). Using some simple estimates and stationarity, we have

$$\begin{aligned} 1 - n(1 - F(u_{nk})) &\leqslant P(M_n \leqslant u_{nk}) \\ &\leqslant 1 - n(1 - F(u_{nk})) + n\Sigma_{j=1}^{n-1}P(X_1 > u_{nk}, X_{j+1} > u_{nk}), \end{aligned}$$

so that

$$\begin{aligned} (1 - n(1 - F(u_{nk})))^k &\leqslant P^k(M_n \leqslant u_{nk}) \\ &\leqslant \left[1 - n(1 - F(u_{nk})) + n\Sigma_{j=1}^{n-1}P(X_1 > u_{nk}, X_{j+1} > u_{nk})\right]^k. \end{aligned} \tag{2.2}$$

Define $\varphi(k) = \lim\sup_n n(1 - F(u_{nk}))$ and $\Psi(k) = \lim\inf_n n(1 - F(u_{nk}))$. We now show that

$$\tau \leqslant \lim\inf_k k\Psi(k) \leqslant \lim\sup_k k\varphi(k) \leqslant \tau. \tag{2.3}$$

Suppose $\lim\sup_k k\varphi(k) > \tau$. Then there exists an $\varepsilon > 0$ and an increasing sequence of positive integers k_j such that $k_j\varphi(k_j) > \tau + \varepsilon$ for all j. Choose $\delta > 0$ such that $\delta < \varepsilon$. Now, by the $D'(u_n)$ assumption, pick a k_j so large that $k_j > \tau + \varepsilon$ and

$$k_j \lim\sup_n n\Sigma_{j=1}^{n-1}P(X_1 > u_{nk_j}, X_{j+1} > u_{nk_j}) < \delta.$$

Moreover, since $\varphi(k_j) = \lim\sup_n n(1 - F(u_{nk_j}))$, there exists a subsequence n_s such that $n_s(1 - F(u_{n_s k_j})) \to \varphi(k_j)$. Taking the limit of the left-hand side of (2.2) through the subsequence n_s and making use of (2.1), the above construction, and some elementary inequalities, we obtain the following contradiction

$$e^{-\tau} > e^{-\tau-(\varepsilon-\delta)} \geqslant \left(1 - \frac{\tau + \varepsilon}{k_j} + \frac{\delta}{k_j}\right)^{k_j} \geqslant \left(1 - \frac{\varphi^{(k_j)}}{k_j}k_j + \frac{\delta}{k_j}\right)^{k_j} \geqslant e^{-\tau}.$$

The proof of the left-hand side of (2.3) can be proved in a similar manner.

Let $\{\hat{X}_n\}$ be an i.i.d. sequence with marginal df F. We then obtain

$$(2.4)\qquad (1 - n(1 - F(u_{nk})))^k \leqslant P^k(\hat{M}_n \leqslant u_{nk}) = P(\hat{M}_{nk} \leqslant u_{nk}) \leqslant \left(1 - n(1 - F(u_{nk})) + S_{nk}\right)^k,$$

where $\hat{M}_n = \max\{\hat{X}_1, \cdots, \hat{X}_n\}$ and $S_{nk} = n\sum_{j=1}^{n-1} P(\hat{X}_1 > u_{nk}, \hat{X}_{j+1} > u_{nk}) = n(n - 1)(1 - F(u_{nk}))^2$. By (2.3), $\limsup_n S_{nk} = o(1/k)$. It now follows that

$$(2.5)\qquad \left(1 - \frac{\tau}{k} + o\left(\frac{1}{k}\right)\right)^k \leqslant \liminf_n P(\hat{M}_{nk} \leqslant u_{nk}) \leqslant \limsup_n P(\hat{M}_{nk} \leqslant u_{nk}) \leqslant \left(1 - \frac{\tau}{k} + o\left(\frac{1}{k}\right)\right)^k,$$

using the fact that $\varphi(k) = \tau/k + o(1/k)$ and $\Psi(k) = \tau/k + o(1/k)$. The idea now is to replace nk by n in (2.5) and then let $k \to \infty$ to obtain the desired result.

For a fixed integer k, choose the sequence of integers r_n such that $r_n k \leqslant n < (r_n + 1)k$. It is easily seen that

$$(2.6)\qquad \liminf_n P(\hat{M}_n \leqslant u_n) \geqslant \liminf_n P\left(\hat{M}_{(r_n+1)k} \leqslant u_{(r_n+1)k}\right) - \limsup_n P\left(u_n < \hat{M}_{(r_n+1)k} \leqslant u_{(r_n+1)k}\right).$$

However,

$$\begin{aligned}\limsup_n P\left(u_n < \hat{M}_{(r_n+1)k} \leqslant u_{(r_n+1)k}\right) &\leqslant \limsup_n P\left(u_{r_n k} < \hat{M}_{(r_n+1)k} \leqslant u_{(r_n+1)k}\right)\\ &\leqslant \limsup_n (r_n + 1)kP\left(u_{r_n k} < \hat{X}_1 \leqslant u_{(r_n+1)k}\right)\\ &= \limsup_n (r_n + 1)k\left(1 - F(u_{r_n k}) - (1 - F(u_{(r_n+1)k}))\right)\\ &\leqslant k\varphi(k) - k\Psi(k) = o(1) \to 0\end{aligned}$$

as $k \to \infty$, using the hypothesis that $\{u_n\}$ is a nondecreasing sequence and (2.3). Putting (2.5) and (2.6) together we obtain

$$(2.7)\qquad \liminf_n P(\hat{M}_n \leqslant u_n) \geqslant \left(1 - \frac{\tau}{k} + o\left(\frac{1}{k}\right)\right)^k - o(1),$$

where $o(1) \to 0$ as $k \to \infty$. A similar calculation yields

$$(2.8)\qquad \limsup_n P(\hat{M}_n \leqslant u_n) \leqslant \left(1 - \frac{\tau}{k} + o\left(\frac{1}{k}\right)\right)^k + o(1).$$

The conclusion of the lemma now follows at once by letting $k \to \infty$ in (2.7) and (2.8). □

The following technical lemma will be used in allowing us to remove the nondecreasing requirement on the $\{u_n\}$ sequence.

LEMMA 2.2. *Suppose that $D(u_n)$ holds for the stationary sequence $\{X_n\}$ and $P(M_n \leqslant u_n) \to e^{-\tau}$, $0 < \tau < \infty$. Let $\{t_n\}$ be a sequence of integers tending to ∞. Then for every fixed positive integer k,*

$$P^k\big(M_{[t_n/k]} \leqslant u_{t_n}\big) \to e^{-\tau} \qquad \text{as} \quad n \to \infty,$$

where $[s]$ denotes the greatest integer not greater than s.

PROOF. Under the hypotheses $D(u_n)$ and $P(M_n \leqslant u_n) \to e^{-\tau}$, it can easily be shown that $u_n \to x_0$ where $x_0 = \sup\{x : F(x) < 1\}$. Hence $P(X_1 > u_{t_n}) \to 0$ as $n \to \infty$.

For a fixed integer k, choose a sequence of integers r_n such that $r_n k \leqslant t_n < (r_n + 1)k$. Since $0 \leqslant P(M_{r_n k} \leqslant u_{t_n}) - P(M_{t_n} \leqslant u_{t_n}) \leqslant kP(X_1 > u_{t_n}) \to 0$ as $n \to \infty$, it is enough to show $|P^k(M_{r_n} \leqslant u_{t_n}) - P(M_{r_n k} \leqslant u_{t_n})| \to 0$ as $n \to \infty$. The proof of this fact follows from Leadbetter's proof of Lemma 2.5 in [2] replacing n by r_n and nk by t_n.

LEMMA 2.3. *Suppose that $D(u_n)$ and $D'(u_n)$ hold for the stationary sequence $\{X_n\}$. Then $P(\hat{M}_n \leqslant u_n) \to e^{-\tau}$ if $P(M_n \leqslant u_n) \to e^{-\tau}$, $0 < \tau < \infty$.*

PROOF. By the remark made at the beginning of the proof of Lemma 2.2, $u_n \to x_0$. Also, since $e^{-\tau} < 1$, we have that $u_n < x_0$ for all sufficiently large n. Hence we shall assume, without loss of generality, that $u_n < x_0$ for all n.

Define $v_n = \max_{1 \leqslant j \leqslant n}\{u_j\}$. Then $v_n = u_{s(n)} = v_{s(n)}$, $s(n) \leqslant n$, $s(n) \to \infty$, and v_n is a nondecreasing sequence. Since $v_n \geqslant u_n$ for all n, condition $D'(v_n)$ is obviously satisfied and the condition $D(v_n)$ can easily be verified. Moreover,

$$\begin{aligned} e^{-\tau} &= \liminf_n P(M_n \leqslant u_n) \leqslant \liminf_n P(M_n \leqslant v_n) \leqslant \limsup_n P(M_n \leqslant v_n) \\ &\leqslant \limsup_n P(M_{s(n)} \leqslant v_n) = \limsup_n P(M_{s(n)} \leqslant u_{s(n)}) = e^{-\tau}. \end{aligned}$$

Hence $P(M_n \leqslant v_n) \to e^{-\tau}$ and thus, by Lemma 2.1, we have $P(\hat{M}_n \leqslant v_n) = (1 - (1 - F(v_n)))^n \to e^{-\tau}$. It should be noted that $P(\hat{M}_n \leqslant v_n) \to e^{-\tau}$ if and only if

$$1 - F(v_n) = \tau/n + o(1/n). \tag{2.9}$$

Now let $t_n = \max_j\{j < n : v_j < u_n\}$. Then $u_{t_n} \leqslant v_{t_n} < u_n \leqslant v_{t_n+1} = u_{t_n+1}$ and $t_n + 1 \leqslant n$. It is enough to show $\lim_{n\to\infty}(n/(t_n + 1)) = 1$ for we have

$$n(1 - F(v_n)) \leqslant n(1 - F(u_n)) \leqslant n\big(1 - F(v_{t_n})\big) = \frac{n}{t_n} t_n\big(1 - F(v_{t_n})\big). \tag{2.10}$$

Upon letting $n \to \infty$ and using (2.9), we have the outside terms of (2.10) going to τ. Thus $1 - F(u_n) = \tau/n + o(1/n)$, giving us the desired result.

Suppose $\limsup_n(n/(t_n + 1)) > 1$. Then there exists a positive integer k and a subsequence n_j such that $(t_{n_j} + 1)/k < n_j/(k + 1)$ for all j. By construction, one has

$$\begin{aligned} P\big(M_{[n_j/(k+1)]} \leqslant u_{n_j}\big) &\leqslant P\big(M_{[n_j/(k+1)]} \leqslant u_{t_{n_j}+1}\big) \\ &\leqslant P\big(M_{[(t_{n_j}+1)/k]} \leqslant u_{t_{n_j}+1}\big). \end{aligned}$$

Taking the limit as $j \to \infty$ and invoking Lemma 2.2, one obtains the contradiction $e^{-\tau/k} \geqslant e^{-\tau/k+1}$. Since $t_n + 1 \leqslant n$, it must be that $\lim_{n\to\infty}(n/(t_n + 1)) = 1$, completing the proof.

THEOREM 2.4. *Suppose that $D(u_n)$, $D'(u_n)$ are satisfied by the stationary sequence $\{X_n\}$ for all x, where $u_n = (x/a_n) + b_n$ for some constants $a_n > 0$ and b_n. Then, with the established notation,*

$$P(a_n(M_n - b_n) \leqslant x) \to G(x)$$

if and only if

$$P(a_n(\hat{M}_n - b_n) \leqslant x) \to G(x),$$

for any nondegenerate df G (*hence one of the three extreme value distributions*).

PROOF. First assume $P(a_n(M_n - b_n) \leqslant x) \to G(x)$ for some nondegenerate df G. By the preceding lemma, it follows that $P(a_n(\hat{M}_n - b_n) \leqslant x) \to G(x)$ for all x where $0 < G(x) < 1$. Since G is continuous this readily extends to all x.

For the other direction, see Theorem 3.2 in [2]. □

In the i.i.d. case the limiting distribution of the maxima is *completely* determined by the tail behavior of the marginal df F. As a consequence of Theorem 2.4, the same is also true of a stationary sequence satisfying conditions D and D'. It should also be pointed out that by considering the sequence $\{-X_n\}$, one can obtain similar results for the minima.

3. Joint limiting distribution of the maxima and minima. In this section we give sufficient conditions in order for the normalized maxima and minima (both jointly and marginally) to behave as though the sequence was i.i.d.

Let $\{u_n\}$ and $\{v_n\}$ be two sequences of real numbers. We shall say that the condition $D(v_n, u_n)$ is satisfied if for any n and any choice of integers $i_1 < \cdots < i_p < j_1 < \cdots < j_q$, $j_1 - i_p > l$ then

$$|F_{i_1 \cdots i_p j_1 \cdots j_q}(u_n, \cdots, u_n) - F_{i_1 \cdots i_p}(u_n, \cdots, u_n)F_{j_1 \cdots j_q}(u_n, \cdots, u_n)| \leqslant \alpha_{n,l},$$

$$|P(X_{i_1} > v_n, \cdots, X_{j_q} > v_n) - P(X_{i_1} > v_n, \cdots, X_{i_p} > v_n) \\ P(X_{j_1} > v_n, \cdots, X_{j_q} > v_n)| \leqslant \alpha_{n,l},$$

and

$$|P(v_n < X_{i_1} \leqslant u_n, \cdots, v_n < X_{j_q} \leqslant u_n) - P(v_n < X_{i_1} \leqslant u_n, \cdots, v_n < X_{i_p} \leqslant u_n) \\ P(v_n < X_{j_1} \leqslant u_n, \cdots, v_n < X_{j_q} \leqslant u_n)| \leqslant \alpha_{n,l},$$

where $\alpha_{n,l}$ is nonincreasing in l and $\lim_{n\to\infty}\alpha_{n,l_n} = 0$ for some sequence $l_n \to \infty$ with $l_n/n \to 0$. In defining the mixing condition $D(v_n, u_n)$, we could have bounded the three inequalities with different functions satisfying the above criterion. However, by taking $\alpha_{n,l}$ to be the maximum of the three functions and l_n the maximum of the respective sequences, we see that the two definitions are equivalent. Again, the condition $D(v_n, u_n)$ is potentially much weaker than strong mixing.

In order to obtain the asymptotic independence of the normalized maxima and minima, we shall need the following condition. Let

$$S_{nk} = n\Sigma_{j=1}^{n-1}\big[P(X_1 > u_{nk}, X_{j+1} > u_{nk}) + P(X_1 > u_{nk}, X_{j+1} \leqslant v_{nk}) + P(X_1 \leqslant v_{nk}, X_{j+1} > u_{nk}) + P(X_1 \leqslant v_{nk}, X_{j+1} \leqslant v_{nk})\big].$$

The condition $D'(v_n, u_n)$ is said to hold if $\limsup_n S_{nk} = o(1/k)$ as $k \to \infty$.

LEMMA 3.1. *Suppose $D(v_n, u_n)$ is satisfied by the stationary sequence $\{X_n\}$. Then for every fixed positive integer k,*

$$P(M_{n_k} \leqslant u_{nk}, W_{nk} > v_{nk}) - P^k(M_n \leqslant u_{nk}, W_n > v_{nk}) \to 0 \text{ as } n \to \infty,$$

where $W_n = \min\{X_1, \cdots, X_n\}$.

The proof is omitted since it follows the same lines of argument as in the proof of Lemma 2.5 of [2] with the obvious modifications.

LEMMA 3.2. *If $D(v_n, u_n)$, $D'(v_n, u_n)$ hold and $P(\hat{M}_n \leqslant u_n, \hat{W}_n > v_n) \to e^{-\tau}$, where $\hat{W}_n = \min\{\hat{X}_1, \cdots, \hat{X}_n\}$, $\hat{M}_n = \max\{\hat{X}_1, \cdots, \hat{X}_n\}$, then $P(M_n \leqslant u_n, W_n > v_n) \to e^{-\tau}$, $0 \leqslant \tau < \infty$.*

PROOF. $P(\hat{M}_n \leqslant u_n, \hat{W}_n > v_n) = P(v_n < \hat{X}_1 \leqslant u_n, \cdots, v_n < \hat{X}_n \leqslant u_n) = (1 - (1 - F(u_n) + F(v_n)))^n \to e^{-\tau}$ if and only if

$$1 - F(u_n) + F(v_n) = \tau/n + o(1/n). \tag{3.1}$$

Using elementary bounds and stationarity, we obtain

$$n(1 - F(u_{nk}) + F(v_{nk})) - n\Sigma_{j=1}^{n-1}P(A_1 \cap A_{j+1}) \leqslant 1 - P(M_n \leqslant u_n, W_n > v_n) \leqslant n(1 - F(u_{nk}) + F(v_{nk})),$$

where $A_s = \{X_s > u_{nk}\} \cup \{X_s \leqslant v_{nk}\}$. Thus,

$$(1 - n(1 - F(u_{nk}) + F(v_{nk})))^k \leqslant P^k(M_n \leqslant u_{nk}, W_n > v_{nk}) \leqslant (1 - n(1 - F(u_{nk}) + F(v_{nk})) + S_{nk}))^k,$$

and now using Lemma 3.1 and (3.1) we have

$$\begin{aligned}(1 - \tau/k)^k &\leqslant \liminf_n P(M_{nk} \leqslant u_{nk}, W_{nk} > v_{nk}) \\ &\leqslant \limsup_n P(M_{nk} \leqslant u_{nk}, W_{nk} > v_{nk}) \\ &\leqslant (1 - \tau/k + o(1/k))^k.\end{aligned} \tag{3.2}$$

As in the proof of Lemma 2.1, the result follows upon replacing nk by n in (3.2) and then letting $k \to \infty$.

THEOREM 3.3. *Suppose there exist real constants $a_n > 0$, b_n, $c_n > 0$, and d_n such that $D(v_n, u_n)$, $D'(v_n, u_n)$ hold for all x and y with $v_n = -y/c_n + d_n$ and $u_n = x/a_n + b_n$. Then*

$$P(a_n(\hat{M}_n - b_n) \leqslant x, -c_n(\hat{W}_n - d_n) < y) \to G(x,y)$$

if and only if $P(a_n(M_n - b_n) \leqslant x, -c_n(W_n - d_n) < y) \to G(x,y)$ for some df G *with $G(x, \infty)$ and $G(\infty, y)$ nondegenerate distribution functions. Moreover, if this is the case, then $G(x,y) = G(x, \infty)G(\infty, y)$ and $P(a_n(M_n - b_n) \leqslant x, c_n(W_n - d_n) \leqslant y) \to G(x, \infty)(1 - G(\infty, -y))$ for all x and y.*

PROOF. Suppose $P(a_n(\hat{M}_n - b_n) \leqslant x, -c_n(\hat{W}_n - d_n) < y) \to G(x,y)$. As is well known, the maxima and minima from an i.i.d. sequence are asymptotically independent. Therefore, $G(x,y) = G(x, \infty)G(\infty, y)$. By the preceding lemma, with $\tau = -\log G(x,y)$, we have $P(a_n(M_n - b_n) \leqslant x, -c_n(W_n - d_n) < y) \to G(x,y)$ for all x, y such that $G(x, \infty)G(\infty, y) > 0$. But the continuity of $G(x, \infty)G(\infty, y)$ allows us to extend this for all x and y.

Now assume $P(a_n(M_n - b_n) \leqslant x, -c_n(W_n - d_n) < y) \to G(x,y)$. It follows, by Theorem 2.4 and the remark made at the end of Section 2, that $P(a_n(\hat{M}_n - b_n) \leqslant x) \to G(x, \infty)$ and $P(-c_n(\hat{W}_n - d_n) < y) \to G(\infty, y)$. The asymptotic independence of $\hat{M}_n$ and $\hat{W}_n$ implies $P(a_n(\hat{M}_n - b_n) \leqslant x, -c_n(\hat{W}_n - d_n) < y) \to G(x, \infty)G(\infty, y)$. Invoking the previous paragraph, we obtain

$$P(a_n(M_n - b_n) \leqslant x, -c_n(W_n - d_n) < y) \to G(x, \infty)G(\infty, y),$$

for all x and y. Hence we must have $G(x,y) = G(x, \infty)G(\infty, y)$. Also, since $P(a_n(M_n - b_n) \leqslant x, c_n(W_n - d_n) \leqslant y) + P(a_n(M_n - b_n) \leqslant x, -c_n(W_n - d_n) < y) = P(a_n(M_n - b_n) \leqslant x)$, we have

$$P(a_n(M_n - b_n) \leqslant x, c_n(W_n - d_n) \leqslant y) \to G(x, \infty)(1 - G(\infty, -y)).$$

This completes the proof of the theorem.

It is worth noting that there are examples of sequences satisfying the $D(v_n, u_n)$ condition where the maxima and minima are asymptotically independent, yet marginally they do not behave as though the sequence was i.i.d. The crucial hypothesis in establishing the asymptotic independence of the maxima and minima seems to be the "cross terms" in the $D'(v_n, u_n)$ condition.

4. Examples. We exhibit two sequences illustrating the conditions of Theorem 3.3.

Let $\{X_n\}$ be a stationary Gaussian sequence with $EX_1 = 0$, $EX_1^2 = 1$, and covariance function $r_n = EX_1X_{n+1}$. It has been shown in [2] and by others that if $r_n \log n \to 0$ or $\Sigma|r_n|^p < \infty$ for some $p > 1$, then $P(a_n(M_n - b_n) \leqslant x) \to e^{-e^{-x}}$ and $P(a_n(W_n + b_n) \leqslant y) \to 1 - e^{-e^{y}}$ where $a_n = (2 \log n)^{\frac{1}{2}}$ and

$$b_n = (2 \log n)^{\frac{1}{2}} - \tfrac{1}{2}(2 \log n)^{-\frac{1}{2}}(\log \log n + \log 4\pi).$$

Let $u_n = x/a_n + b_n$ and $v_n = -y/a_n - b_n$. Using a general form of Lemma 4.2

and Lemma 4.3 in [2], one can show that $D(v_n, u_n)$, $D'(v_n, u_n)$ hold for all x providing $r_n \log n \to 0$ or $\Sigma|r_n|^p < \infty$ for some $p > 1$. Thus, under these types of restrictions on the covariance function, the maxima and minima are asymptotically independent.

The next example is a 1-dependent sequence where the normalized maxima and minima are not asymptotically independent. First, let $\{Y_n\}$ be an i.i.d. sequence with marginal df F which is symmetric about the origin and belongs to the domain of attraction of some extreme value distribution G. Hence, there exist constants $a_n > 0$, b_n such that

$$1 - F(x/a_n + b_n) = -\frac{\log G(x)}{n} + o(1/n)$$

for all x where $G(x) > 0$, and

$$F(-y/a_n - b_n) = \frac{-\log G(y)}{n} + o(1/n), \ G(y) > 0.$$

Now let $\{J_n\}$ be an i.i.d. sequence of Bernoulli trials, independent of the Y_n sequence, with $P(J_1 = 1) = \alpha > 0$ and $P(J_1 = 0) = 1 - \alpha$. Define $X_n = Y_{n+1}$ if $J_n = 1$ and $X_n = -Y_n$ if $J_n = 0$. It is clear that $\{X_n\}$ is a stationary 1-dependent sequence with marginal df F. Let $u_n = x/a_n + b_n$ and $v_n = -y/a_n - b_n$. It is easy to verify that $D'(u_n)$ is satisfied by $\{X_n\}$ and $\{-X_n\}$ so that $P(a_n(M_n - b_n) \leqslant x) \to G(x)$ and $P(-a_n(W_n + b_n) \leqslant y) \to G(y)$. However, one can show that

$$P(a_n(M_n - b_n) \leqslant x, -a_n(W_n + b_n) < y) \to G(x)^{1-2\alpha(1-\alpha)}G(y) \text{ if } y \leqslant x,$$

$$\to G(x)G(y)^{1-2\alpha(1-\alpha)} \text{ if } x < y.$$

Although the limiting marginal distribution of the maxima and minima in this example behaves as if the sequence were i.i.d., the asymptotic joint distribution does not.

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