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MULTIVARIATE DISTRIBUTIONS WITH INCREASING HAZARD RATE AVERAGE[1]

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Several conditions are considered that extend to a multivariate setting the univariate concept of an increasing hazard rate average. The relationships between the various conditions are established. In particular it is shown that if for some independent random variables $X_1, \cdots, X_k$ with increasing hazard rate average and some coherent life functions $\tau_1, \cdots, \tau_n$ of order k, $T_i = \tau_i(X_1, \cdots, X_k)$, then the joint survival function $\bar{F}(\mathbf{t}) = P(T_1 > t_1, \cdots, T_n > t_n)$ has the property that $\alpha^{-1} \log \bar{F}(\alpha \mathbf{t})$ is decreasing in $\alpha > 0$ whenever each $t_i \geqslant 0$. Various other properties of the multivariate conditions are given.

The conditions can all be stated in terms of inequalities in which equality implies that the one dimensional marginal distributions are exponential. For most of the conditions, the form of the multivariate exponential distributions that satisfy the equality is exhibited.

1. Introduction. This paper is concerned with several conditions that extend the univariate concept of "increasing hazard rate average" to the multivariate case. A univariate distribution F (or survival function $\bar{F} = 1 - F$, or corresponding random variable) is said to have an *increasing hazard rate average* (IHRA) if $F(t) = 0$ for all $t < 0$ and if

$$\left[\bar{F}(t)\right]^{1/t} \qquad \text{is decreasing in } t > 0.$$

This condition is important in reliability theory because of the following facts (Birnbaum, Esary and Marshall (1966)). Coherent systems have IHRA life distributions whenever their components have *independent* life lengths with IHRA distributions (in particular, with exponential distributions). Consequently the class of IHRA distributions is said to be "closed under the formation of coherent systems"; it is the smallest class of distributions which is closed in this sense, is closed under limits in distribution, and also contains the exponential distributions.

The assumption that the component life lengths are independent is often questionable at best, so it is important to understand multivariate extensions of the IHRA property. Direct multivariate analogs of the univariate IHRA definition are considered, but these lack intuitive appeal as does even the univariate definition. A more clearly meaningful approach is by means of characterizations like that of Birnbaum, Esary and Marshall (1966), but modified to allow for dependence. Also,

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models which account for dependence are considered. These three approaches lead to various multivariate concepts. Relationships between them are established, and related families of multivariate exponential distributions are identified.

Numerous other multivariate extensions of the univariate IHRA property are possible, and may be appropriate in some situations. However, the multivariate properties considered here are obvious ones that deserve scrutiny.

In the univariate case, the IHRA property is strictly weaker than the more intuitive increasing hazard rate (IHR) property. We do not attempt here to relate multivariate IHRA conditions to corresponding multivariate IHR properties because we believe that attempts to preserve pleasant mathematical relationships will not necessarily help in finding physically meaningful definitions. Instead, we have obtained multivariate IHRA conditions primarily by concentrating upon generalizations of the reliability motivations which explain why the univariate IHRA concept is important. Similarly, we can see no reason to be concerned with the dual concept of "decreasing hazard rate average" because we know of no reason why this concept is of practical interest even in the univariate case.

Most of the terminology used in this paper is defined by Barlow and Proschan (1975). The concept of a "coherent life function" is discussed by Esary and Marshall (1970).

2. Analogs of the univariate definition. The univariate distribution F is IHRA if $F(t) = 0$ for all $t < 0$ and if the hazard function $R = -\log \bar{F}$ satisfies

(i) $R(t)/t$ is increasing in $t > 0$,

or equivalently, if

(ii) $R(\alpha t) \leqslant \alpha R(t)$ for all $\alpha \in [0, 1]$ and $t \geqslant 0$.

When R is differentiable (so that F has density $f = F'$ and hazard rate $r = f/\bar{F}$), condition (i) can be rewritten in the form

(iii) $R(t) \leqslant tr(t)$ for all $t \geqslant 0$.

In the multivariate case, the *joint survival function* of random variables $T_1, \cdots, T_n$ is defined by $\bar{F}(\mathbf{t}) = P(T_1 > t_1, \ldots, T_n > t_n)$, and the hazard function is defined by $R(\mathbf{t}) = -\log \bar{F}(\mathbf{t})$. Conditions (i) and (ii) have the obvious analogs

(i′) $R(\alpha\mathbf{t})/\alpha$ is increasing in $\alpha > 0$ whenever each $t_i \geqslant 0$,

(ii′) $R(\alpha\mathbf{t}) \leqslant \alpha R(\mathbf{t})$ for all $\alpha \in [0, 1]$ whenever each $t_i \geqslant 0$.

A multivariate analog of the hazard rate is the hazard gradient $r(\mathbf{t}) = \nabla R(\mathbf{t})$; in terms of this, (iii) has the analog

(iii′) $R(\mathbf{t}) \leqslant \mathbf{t} \cdot r(\mathbf{t})$ whenever each $t_i \geqslant 0$.

It is easily verified that these conditions are equivalent, although (iii′) requires R to be differentiable. This motivates our consideration of

CONDITION A. Either (i′), (ii′) or (iii′) is satisfied.

Other multivariate analogs of the univariate definition have been considered by Buchanan and Singpurwalla (1977).

3. Additional conditions. In dealing with large systems, it is common practice to determine the life distribution of various subsystems and then to combine such partial results as successively larger subsystems are studied. An easy consequence of the characterization of Birnbaum, Esary and Marshall (1966) which has a direct bearing on such a procedure is the following:

(iv) independent random variables $T_1, \cdots, T_n$ have IHRA distributions if and only if $\tau(T_1, \cdots, T_n)$ has an IHRA distribution for all coherent life functions τ.

But in the above context the assumption of independence is often invalid. This leads to consideration of

CONDITION B. The random variables $T_1, \cdots, T_n$ have a joint distribution such that $\tau(T_1, \cdots, T_n)$ has an IHRA distribution for all coherent life functions τ.

Because $\tau(T_1, \cdots, T_n) = T_i$ is a coherent life function, Condition B implies that T_i is IHRA, $i = 1, 2, \cdots, n$. But $T_1, \cdots, T_n$ need not be independent.

When making a system analysis by combining subsystem information as described above, the subsystem life lengths $T_1, \cdots, T_n$ are often dependent as a result of the subsystems having components in common. In such a circumstance,

$$T_i = \tau_i(X_1, \cdots, X_k)$$

where τ_i is the life function of the ith subsystem and $X_1, \cdots, X_k$ are component life lengths. This model for dependence leads to

CONDITION C. $T_1, \cdots, T_n$ have a representation as

$$T_i = \tau_i(X_1, \cdots, X_k), \qquad i = 1, 2, \cdots, n,$$

where $X_1, \cdots, X_k$ are independent IHRA random variables and $\tau_1, \cdots, \tau_n$ are coherent life functions of order k.

Although Condition C can be viewed as a model for dependence, it is also a natural extension of the univariate characterization (iv). Of course (iv) says that if Condition C is satisfied, each T_i has an IHRA distribution.

If each of the coherent systems in Condition C is a series system and if each X_j is exponentially distributed, then $T_1, \cdots, T_n$ have a multivariate exponential distribution of the kind introduced by Marshall and Olkin (1967). It is of some interest to modify Condition C by admitting only series systems. This case arises, e.g., if $T_1, \cdots, T_n$ are the minimal path life lengths for some coherent system with independent components.

CONDITION D. For some independent IHRA random variables $X_1, \cdots, X_k$ and nonempty subsets S_i of $\{1, 2, \cdots, k\}$,

$$T_i = \min_{j \in S_i} X_j, \qquad i = 1, 2, \ldots, n.$$

Condition D is strictly stronger than Condition C (Section 9) but the distributions satisfying Condition D can be used to generate all of those satisfying Condition C in a rather simple way (Section 7).

Condition B can be modified in the same way that D modifies C.

CONDITION E. $\min_{i \in S} T_i$ is IHRA for all nonempty subsets S of $\{1, 2, \cdots, n\}$.

Condition A is the only condition we have given that is imposed upon the survival function rather than on the corresponding random variables (the formulation of Condition D in terms of survival functions is easy but not particularly illuminating). The following condition, in terms of random variables, is equivalent to Condition A (Section 8).

CONDITION F. $T_1, \cdots, T_n$ have a joint distribution such that

$$\min_i a_i T_i \text{ is IHRA whenever each } a_i \geqslant 0.$$

4. Relationships between the various conditions. Figure 1 summarizes the relationships between Conditions A–F. No further implications can be added to this diagram. Proofs and counterexamples are given in Sections 8 and 9.

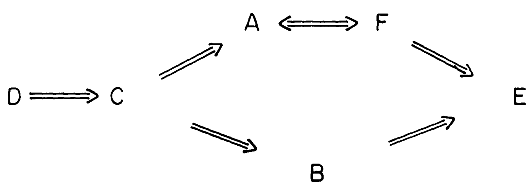

FIG. 1.

5. Related multivariate exponential distributions. Univariate exponential distributions, characterized by having constant hazard rates, play a central role in the study of univariate IHRA distributions. Here, families of multivariate exponential distributions are identified which play similar roles with respect to Conditions A–F.

CONDITION A. The cases of equality in the various defining conditions are:

$$\frac{R(\alpha \mathbf{t})}{\alpha} \text{ is independent of } a > 0 \text{ whenever each } t_i \geqslant 0;$$

or

$$R(\alpha \mathbf{t}) = \alpha R(\mathbf{t}) \text{ for all } \alpha \in [0, 1] \text{ whenever each } t_i \geqslant 0;$$

or

$$R(\mathbf{t}) = \mathbf{t} \cdot r(\mathbf{t}) \text{ whenever each } t_i \geqslant 0.$$

Because Condition A is equivalent to Condition F, the distributions which satisfy these equalities are characterized in terms of the corresponding random variables $T_1, \cdots, T_n$ as follows:

$$\min_i a_i T_i \text{ is exponentially distributed whenever each } a_i > 0.$$

This rather broad class of multivariate exponential distributions was introduced by Esary and Marshall (1974); it includes the multivariate exponential distributions of Marshall and Olkin (1967). Representations of the corresponding survival functions have been obtained by Pickands (1976). We do not claim that these distributions are the only ones satisfying Condition A that have exponential marginals.

CONDITION B. It is not difficult to show that if $T_1, \cdots, T_n$ satisfy $\tau(T_1, \cdots, T_n)$ is exponentially distributed for all coherent life functions τ, then there exists a permutation π such that

$$P\{T_{\pi(1)} \leqslant \cdots \leqslant T_{\pi(n)}\} = 1.$$

Conversely, if such a permutation exists and each T_i is exponentially distributed, then $\tau(T_1, \cdots, T_n)$ is exponentially distributed for all coherent life functions τ. In fact if T_i, T_j, $\min(T_i, T_j)$, and $\max(T_i, T_j)$ are all exponentially distributed, then it follows in a straightforward manner that either $P\{T_i \leqslant T_j\} = 1$ or $P\{T_j \leqslant T_i\} = 1$. The family of distributions discussed above under Condition A includes a number of examples for which the corresponding random variables are ordered with probability one.

The family of multivariate distributions with corresponding random variables $T_1, \cdots, T_n$ that satisfy

(v) $T_1, \cdots, T_n$ are each exponentially distributed,

(vi) $\tau(T_1, \cdots, T_n)$ has an IHRA distribution for all coherent life functions τ,

is extraordinarily rich. In fact, Esary and Marshall (1974) show that if

(vii) $\min_{i \in S} T_i$ has an exponential distribution for all $S \subset \{1, 2, \cdots, n\}$

then (v) and (vi) are satisfied. The family of joint distributions satisfying (vii) includes those discussed above under Condition A and many others.

CONDITION C. If $T_1, \cdots, T_n$ satisfy Condition C and are exponentially distributed, then we conjecture that $T_1, \cdots, T_n$ have the multivariate exponential distribution of Marshall and Olkin (1967). This conclusion can be obtained using Theorem 4.3 of Esary, Marshall and Proschan (1970) if the additional assumption is made that there is some point interior to the support of each X_j.

CONDITION D. Since the minimum of independent IHRA random variables is exponentially distributed only if the random variables are all exponentially distributed, it follows that if $T_1, \cdots, T_n$ satisfy Condition D and each T_i is exponentially distributed, then $T_1, \cdots, T_n$ must have the multivariate exponential distribution of Marshall and Olkin (1967).

CONDITION E. The large class of joint distributions for which $\min_{i \in S} T_i$ is exponentially distributed for all nonempty subsets S of $\{1, 2, \cdots, n\}$ has been discussed by Esary and Marshall (1974).

6. Two important properties of the conditions. The following properties are reasonable requirements for any condition that might be used as a definition of "multivariate IHRA":

(P1) $T_1, \cdots, T_n$ satisfy Condition* $\Rightarrow$ each nonempty subset of $T_1, \cdots, T_n$ satisfies Condition*,

(P2) $S_1, \cdots, S_n$ satisfy Condition*, $T_1, \cdots, T_m$ satisfy Condition*, and $(S_1, \cdots, S_n)$, $(T_1, \cdots, T_m)$ are independent $\Rightarrow S_1, \cdots, S_n, T_1, \cdots, T_m$ satisfy Condition*.

All of the Conditions A–F satisfy P1 as can be easily verified. Conditions A and C–F also satisfy P2. Whether or not Condition B satisfies P2 is unknown.

7. Additional properties of the conditions. Here some miscellaneous properties of Conditions A–F are mentioned.

Association. The random variables $T_1, \cdots, T_n$ of Conditions C and D are generated as increasing functions of independent random variables and as such, they are associated (see Esary, Proschan and Walkup (1967)). On the other hand, if U is uniformly distributed on [0, 1] and $V = 1 - U$, then the distribution of (U, V) satisfies Conditions A, B, E and F. Since this distribution has correlation -1, these conditions do not imply association or any other notion of positive dependence.

Absolute continuity. The strongest condition we have introduced is Condition D. If $T_1, \cdots, T_n$ satisfy this condition and are jointly absolutely continuous, then $T_1, \cdots, T_n$ are independent. For suppose that $T_1 = \min_{j \in S_1} X_j$ and $T_2 = \min_{j \in S_2} X_j$ are jointly absolutely continuous. Then $P(T_1 = T_2) = 0$ and one can take S_1, S_2 to be disjoint. Consequently T_1 and T_2 are independent. The general result now follows because under Condition D, pairwise independence implies independence.

There do exist distributions satisfying Condition C which are absolutely continuous. For example, suppose X_i has the absolutely continuous distribution F_i, $i = 1, 2, 3, 4$. Let T_1 and T_2 be the life lengths of the coherent systems of Figure 2.

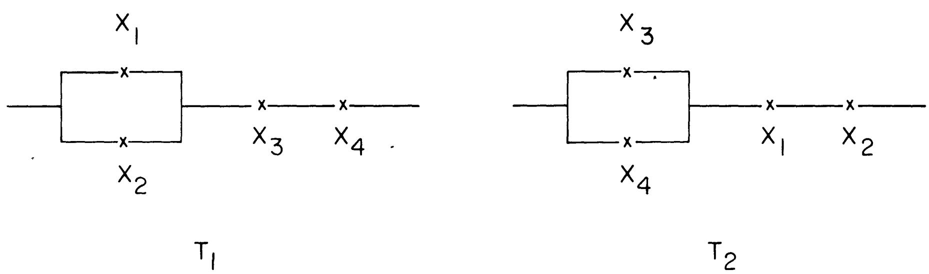

FIG. 2.

Then T_1 and T_2 are dependent but $P(T_1 = T_2) = 0$. For $t_1 \leqslant t_2$,

$$P(T_1 > t_1, T_2 > t_2) = \bar{F}_1(t_2)\bar{F}_2(t_2)\left[\bar{F}_3(t_1)\bar{F}_4(t_2) + \bar{F}_3(t_2)\bar{F}_4(t_1) - \bar{F}_3(t_2)\bar{F}_4(t_2)\right].$$

Here absolute continuity is apparent.

One can verify that the bivariate exponential distribution F given by

$$\bar{F}(x_1, x_2) = \exp\left[-\left(x_1^2 + x_2^2\right)^{\frac{1}{2}}\right], \qquad x_1, x_2 \geqslant 0$$

satisfies Conditions A, B, E, and F, and has a density.

A property of Condition C. The random variables $T_1, \cdots, T_n$ of Condition C are generated from independent random variables $X_1, \cdots, X_k$. However, if $X_1, \cdots, X_k$ are not independent, but only satisfy Condition C, then again, $T_1, \cdots, T_n$ satisfy Condition C. One verifies this from the fact that compositions of coherent life functions are coherent life functions.

Connection between Conditions C and D. If $T_1, \cdots, T_n$ satisfy Condition C, then there exists random variables

$$Z_{11}, \cdots, Z_{1k_1}, Z_{21}, \cdots, Z_{2k_2}, \cdots, Z_{nk_n}$$

having a joint distribution satisfying Condition D such that

$$T_i = \max_{1 \leqslant j \leqslant k_i} Z_{ij}, \qquad i = 1, 2, \cdots, n.$$

This results from the fact that minimal paths of a coherent structure with independent IHRA components have a joint distribution satisfying Condition D.

8. Proofs. In this section, we prove the implications summarized in Figure 1 of Section 4.

D $\Rightarrow$ C. This is trivial.

C $\Rightarrow$ B. This follows from the fact that compositions of coherent life functions are coherent life functions.

B $\Rightarrow$ E. Again, trivial.

A $\Rightarrow$ F. In (ii′), let $z > 0$, $a_i = z/t_i$, and interpret $1/0$ as ∞. Then (ii′) can be written as

$$-\log \bar{F}\left(\frac{\alpha z}{a_1}, \cdots, \frac{\alpha z}{a_n}\right) \leqslant -\alpha \log \bar{F}\left(\frac{z}{a_1}, \cdots, \frac{z}{a_n}\right).$$

Observe that $\bar{F}\left(\frac{z}{a_1}, \cdots, \frac{z}{a_n}\right) = P\{\min a_i T_i > z\} = \bar{G}(z)$, say. Thus we have from (ii′) that

$$-\log \bar{G}(\alpha z) \leqslant -\alpha \log \bar{G}(z);$$

i.e., $\bar{G}$ is IHRA.

F $\Rightarrow$ A. This proof is obtained by reversing the steps of the proof that A $\Rightarrow$ F.

F $\Rightarrow$ E. This is trivial.

C $\Rightarrow$ A. This is the only troublesome implication. The following proof uses ideas of Birnbaum, Esary and Marshall (1966), which become somewhat more involved in the multivariate setting. We require some preliminary notation and lemmas.

Let

$$X_i(t) = 1 \quad \text{if } t < X_i,$$
$$= 0 \quad \text{if } t \geqslant X_i, \qquad i = 1, 2, \cdots, k,$$
$$\mathbf{X}(t) = (X_1(t), \cdots, X_k(t)).$$

Let $\phi_1, \cdots, \phi_n$ be semi-coherent structure functions (of order k) corresponding to the life functions $\tau_1, \cdots, \tau_n$. In case $\phi_i(\mathbf{x}) \equiv 0$, $\tau_i(X_1, \cdots, X_k) \equiv 0$; in case $\phi_i(\mathbf{x}) \equiv 1$, $\tau_i(X_1, \cdots, X_k) \equiv \infty$.

The joint survival function $\bar{H}$ of $\tau_1, \cdots, \tau_n$ is given by

(1) $$\bar{H}(t_1, \cdots, t_n) = E\Pi_{i=1}^n \phi_i(\mathbf{X}(t_i)).$$

Since $X_1, \cdots, X_k$ are statistically independent, so are the rows of the $k \times n$ matrix $\mathbf{X}(\mathbf{t}) = (X_i(t_j))$. Hence, $\bar{H}(t_1, \cdots, t_n)$ depends only on the $k \times n$ matrix

(2) $$\mathbf{P} = E\mathbf{X}(\mathbf{t}) = \left(\bar{F}_i(t_j)\right) \equiv (p_{ij}).$$

Here F_i is the distribution of X_i. Let

(3) $$\bar{H}(t_1, \cdots, t_n) = h(\mathbf{P}).$$

In order to find the partial derivative of $h(\mathbf{P})$ with respect to p_{ij}, suppose for convenience that

$$0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n < t_{n+1} = \infty,$$

let

$$\mathbf{d}_l = (1, \cdots, 1, 0, \cdots, 0)$$

be the vector with first l components equal to unity and last $n - l$ components equal to zero, and let $\mathbf{P}_{ij}$ be obtained from $\mathbf{P}$ by replacing the ith row of $\mathbf{P}$ by $\mathbf{d}_j$.

Since $t_1 \leqslant \cdots \leqslant t_n$, it follows that $X_i(t_j)$ is decreasing in j, and hence

$$\Pi_{j=1}^n \phi_j(\mathbf{X}(t_j)) = \Sigma_{l=0}^n [X_i(t_l) - X_i(t_{l+1})] \Pi_{j=1}^l \phi_j(\mathbf{X}(t_j), 1_i) \Pi_{j=l+1}^n \phi_j(\mathbf{X}(t_j), 0_i),$$

where $(\mathbf{z}, 1_i)$ is the vector $\mathbf{z}$ with ith component replaced by 1, and $(\mathbf{z}, 0_i)$ is the vector $\mathbf{z}$ with ith component replaced by 0. Thus

(4) $$E\Pi_{j=1}^n \phi_j(\mathbf{X}(t_j)) = h(\mathbf{P}) = \Sigma_{l=0}^n (p_{il} - p_{i,l+1}) h(\mathbf{P}_{il}),$$

where $p_{i0} = 1$ and $p_{i,n+1} = 0$. With this representation, it is apparent that

(5) $$\frac{\partial h(\mathbf{P})}{\partial p_{ij}} = h(\mathbf{P}_{ij}) - h(\mathbf{P}_{i,j-1}).$$

In the following, the notation

$$\psi(x) = -x \log x, \qquad 0 \leqslant x \leqslant 1$$

is used.

LEMMA 1. $\Sigma_{i,j} \psi(p_{ij}) \dfrac{\partial h}{\partial p_{ij}} \geqslant \psi(h).$

PROOF. We prove this by induction on k, the order of the semicoherent structures ϕ_i. If $k = 1$, then either $\phi_i(x) \equiv 0$, $\phi_i(x) = x$, or $\phi_i(x) \equiv 1$. Consequently, either $h(\mathbf{P}) = h(p_{11}, p_{12}, \cdots, p_{1n}) = p_{1i}$ for some i, or $h(\mathbf{P}) \equiv 0$, or $h(\mathbf{P}) \equiv 1$. $h(\mathbf{P}) = p_{1i}$ means $\phi_j(x) = 1$ or x, $j = 1, 2, \cdots, i-1$, $\phi_i(x) = x$, and $\phi_j(x) = 1$, $j = i+1, \cdots, n$. If $h(\mathbf{P}) = p_{1i}$, then

$$\Sigma_j \psi(p_{1j}) \frac{\partial h}{\partial p_{1j}} = \psi(p_{1i}) \frac{\partial h}{\partial p_{1i}} = \psi(h).$$

The equality $\Sigma_j \psi(p_{1j})(\partial h/\partial p_{1j}) = \psi(h)$ is trivial if $h \equiv 0$ or $h \equiv 1$.

Now suppose the lemma holds for semicoherent structure functions $\phi_1, \cdots, \phi_n$ of order $k-1$. We compute, using (4) and then applying the induction hypothesis as follows, remembering $p_{1l} \geqslant p_{1,l+1}$:

$$\begin{aligned}
&\Sigma_{i,j} \psi(p_{ij}) \frac{\partial h}{\partial p_{ij}} \\
&= \Sigma_{i,j} \psi(p_{ij}) \frac{\partial}{\partial p_{ij}} \Sigma_{l=0}^n (p_{1l} - p_{1,l+1}) h(\mathbf{P}_{1l}) \\
&= \Sigma_{i=2}^k \Sigma_{j=1}^n \psi(p_{ij}) \Sigma_{l=0}^n (p_{1l} - p_{1,l+1}) \frac{\partial}{\partial p_{ij}} h(\mathbf{P}_{1l}) \\
&\quad + \Sigma_{j=1}^n \psi(p_{1j}) \left[\frac{\partial}{\partial p_{1j}} (p_{1,j-1} - p_{1j}) h(\mathbf{P}_{1,j-1}) + \frac{\partial}{\partial p_{1j}} (p_{1j} - p_{1,j+1}) h(\mathbf{P}_{1j}) \right] \\
&\geqslant \Sigma_{l=0}^n (p_{1l} - p_{1,l+1}) \psi[h(\mathbf{P}_{1l})] + \Sigma_{j=1}^n \psi(p_{1j}) [h(\mathbf{P}_{1j}) - h(\mathbf{P}_{1,j-1})].
\end{aligned}$$

To simplify the notation, write

$$h_j = h(\mathbf{P}_{1j}) \quad \text{and} \quad p_{1j} = p_j, \qquad j = 1, 2, \cdots, n.$$

Then the inequality becomes, after recombining terms,

$$\Sigma_{i,j} \psi(p_{ij}) \frac{\partial h}{\partial p_{ij}} \geqslant \psi(h_0) + \Sigma_{j=1}^n [\psi(p_j h_j) - \psi(p_j h_{j-1})]. \tag{6}$$

From the concavity of ψ, it follows that

$$\psi(p_j h_j) - \psi(p_j h_{j-1}) \geqslant \psi[h_0 + \Sigma_{i=1}^j (h_i - h_{i-1}) p_i] - \psi[h_0 + \Sigma_{i=1}^{j-1} (h_i - h_{i-1}) p_i].$$

Summing both sides gives

$$\Sigma_{j=1}^n [\psi(p_j h_j) - \psi(p_j h_{j-1})] \geqslant \psi[h_0 + \Sigma_{i=1}^n (h_i - h_{i-1}) p_i] - \psi(0),$$

which, together with (6), is

$$\Sigma_{i,j} \psi(p_{ij}) \frac{\partial h}{\partial p_{ij}} \geqslant \psi[h_0 + \Sigma_{i=1}^n (h_i - h_{i-1}) p_i].$$

From (4), we see that this is the inequality we set out to prove. □

Let η be the real-valued function of $k \times n$ matrices $U = (u_{ij})$ defined by

$$\eta(\mathbf{U}) = -\log h(e^{-u_{ij}}).$$

LEMMA 2. $\eta(\alpha\mathbf{U}) \leqslant \alpha\eta(\mathbf{U})$ *whenever* $\alpha \in [0, 1]$.

PROOF. This inequality is equivalent to the statement that

$$\frac{\eta(\alpha\mathbf{U})}{\alpha} \text{ is increasing in } \alpha > 0.$$

That this ratio has a nonnegative derivative is equivalent to the inequality of Lemma 1. ▯

Suppose that Condition C holds. Let R be the joint hazard function of $\tau_1, \cdots, \tau_n$ (i.e., $R(\mathbf{t}) = -\log \bar{H}(\mathbf{t})$) and let R_i be the hazard function of X_i, $i = 1, 2, \cdots, k$. By first using the fact that each X_i is IHRA and η is increasing in each argument and then using the inequality of Lemma 2, we obtain

$$R(\alpha t_1, \cdots, \alpha t_n) = \eta\big[\big(R_i(\alpha t_j)\big)\big] \leqslant \eta\big[\alpha\big(R_i(t_j)\big)\big] \leqslant \alpha\eta\big[\big(R_i(t_j)\big)\big] = \alpha R(t_1, \cdots, t_n).$$

Consequently, Condition A is satisfied. This completes the proof that $\mathrm{C} \Rightarrow \mathrm{A}$.

9. Counterexamples. Our aim here is to show that no implications can be added to Figure 1 of Section 4. We do this by exhibiting counterexamples; e.g., to show $\mathrm{C} \not\Rightarrow \mathrm{D}$, we exhibit a distribution satisfying Condition C but not Condition D.

$\mathrm{C} \not\Rightarrow \mathrm{D}$. First, observe that if (T_1, T_2) satisfies Condition D, then

$$T_1 = \min(X, Z), \qquad T_2 = \min(Y, Z)$$

where $X = \min_{j \in S_1 - S_2} X_j$, $Y = \min_{j \in S_2 - S_1} X_j$, and $Z = \min_{j \in S_1 \cap S_2} X_j$. (Take the minimum over an empty set to be ∞.) Consequently, the joint survival function of T_1 and T_2 has the form

$$P(T_1 > t_1, T_2 > t_2) = \bar{F}_X(t_1)\bar{F}_Y(t_2)\bar{F}_Z(\max[t_1, t_2]),$$

so that

$$\text{(7)} \quad P(T_1 \leqslant t_1, T_2 \leqslant t_2)$$
$$= 1 - \bar{F}_X(t_1)\bar{F}_Z(t_1) - \bar{F}_Y(t_2)\bar{F}_Z(t_2) + \bar{F}_X(t_1)\bar{F}_Y(t_2)\bar{F}_Z(\max[t_1, t_2]).$$

Now consider random variables T_1, T_2 of the form

$$T_1 = \max(U, W), \qquad T_2 = \max(V, W),$$

where U, V and W are independent and uniformly distributed on $[0, 1]$. Then (T_1, T_2) satisfies Condition C, and

$$P(T_1 \leqslant t_1, T_2 \leqslant t_2) = t_1 t_2 \min(t_1, t_2), \qquad 0 \leqslant t_1, t_2 \leqslant 1.$$

The assumption that this joint distribution has the form (7) leads to the contradication that $\bar{F}_x(t_1)$ depends upon t_2.

A, $\mathrm{F} \not\Rightarrow \mathrm{B}$. (Consequently, A, $\mathrm{F} \not\Rightarrow \mathrm{C}$ and $\mathrm{E} \not\Rightarrow \mathrm{B}$.) Suppose that (T_1, T_2) has density

$$\begin{aligned} f(t_1, t_2) &= a \quad \text{if } t_1 \geqslant 0, t_2 \geqslant 0 \quad \text{and} \quad t_1 + t_2 \leqslant \tfrac{1}{4} \\ &= b \quad \text{if } 0 \leqslant t_1 \leqslant 1, 0 \leqslant t_2 \leqslant 1 \quad \text{and} \quad t_1 + t_2 \geqslant \tfrac{3}{4} \\ &= 0 \quad \text{elsewhere,} \end{aligned}$$

where $a = 32/47$ and $b = 2a = 64/47$. Here, $\max(T_1, T_2)$ has support that is not an interval, so $\max(T_1, T_2)$ is not IHRA and Condition B is violated. On the other hand $\min(\alpha_1 T_1, \alpha_2 T_2)$ is IHRA (even IHR) whenever $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$. To verify this, we find that $\bar{F}_m(x) = P(T_1 > x, T_2 > mx)$ has, for $0 \leqslant m \leqslant 1$, the density

$$\begin{aligned} f_m(x) &= a[3(m+1) + 4x(1-m)^2]/4, && 0 \leqslant x \leqslant 1/[4(m+1)], \\ &= b[1 + m + 4(1+m^2)x]/4, && 1/[4(m+1)] \leqslant x \leqslant 3/[4(m+1)] \\ &= b(1 + m - 2mx), && 3/[4(m+1)] \leqslant x \leqslant 1 \\ &= 0, && \text{elsewhere.} \end{aligned}$$

It is easy to check that f_m is IHR. The case $m > 1$ follows by symmetry.

B $\not\Rightarrow$ A, F. (Consequently, B $\not\Rightarrow$ C and E $\not\Rightarrow$ A, F.) To generate a distribution which satisfies Condition B but not Condition A, let T_1 be uniformly distributed on [0, 1] and let $T_2 = T_1 + \frac{1}{2}$ if $0 \leqslant T_1 \leqslant \frac{1}{2}$, $T_2 = T_1 - \frac{1}{2}$ if $\frac{1}{2} < T_1 \leqslant 1$. The joint distribution F of T_1 and T_2 has support on the diagonal lines of Figure 3 and is given by

$$\begin{aligned} F(t_1, t_2) &= 0 && \text{if (a) } 0 \leqslant t_1, t_2 \leqslant \tfrac{1}{2}, \\ &= t_1 + t_2 - 1 && \text{if (b) } \tfrac{1}{2} \leqslant t_1, t_2 \leqslant 1, \\ &= t_1 - \tfrac{1}{2} && \text{if (c) } t_2 \leqslant \tfrac{1}{2} \leqslant t_1 \text{ and } t_1 - t_2 - \tfrac{1}{2} \leqslant 0, \\ &= t_2 && \text{if (d) } t_2 \leqslant \tfrac{1}{2} \leqslant t_1 \text{ and } t_1 - t_2 - \tfrac{1}{2} \geqslant 0, \\ &= t_2 - \tfrac{1}{2} && \text{if (e) } t_1 \leqslant \tfrac{1}{2} \leqslant t_2 \text{ and } t_2 - t_1 - \tfrac{1}{2} \leqslant 0, \\ &= t_1 && \text{if (f) } t_1 \leqslant \tfrac{1}{2} \leqslant t_2 \text{ and } t_2 - t_1 - \tfrac{1}{2} \geqslant 0. \end{aligned}$$

The regions (a)–(f) are indicated in Figure 3.

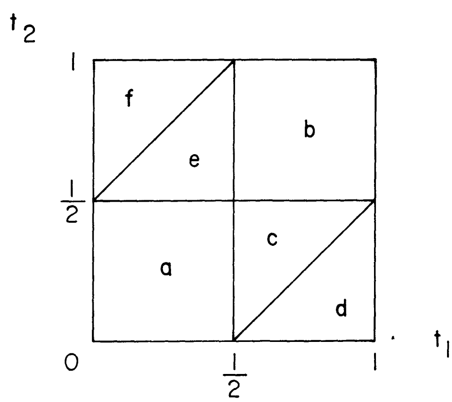

FIG. 3.

It is easily verified that F satisfies Condition B. In fact, T_1, T_2, $\min(T_1, T_2)$ and $\max(T_1, T_2)$ are all uniformly distributed, hence IHRA.

To check Condition F, one computes that if $a > 2$, then

$$
\begin{aligned}
P\{\min(T_1, aT_2) \leqslant z\} &= \frac{a+1}{a} z && \text{if } z \leqslant \tfrac{1}{2}, \\
&= \frac{2z+a}{2a} && \text{if } \tfrac{1}{2} \leqslant z \leqslant a/2(a-1), \\
&= z && \text{if } a/2(a-1) \leqslant z \leqslant 1.
\end{aligned}
$$

The derivative of $-z^{-1} \log P\{\min(T_1, aT_2) \leqslant z\}$, for $\frac{1}{2} < z < a/2(a-1)$ is nonnegative if and only if

$$\frac{2z}{2z+a} \leqslant \log \frac{2z+a}{2a}$$

which is violated for a sufficiently large. Hence, Condition F (and Condition A) do not hold.

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