The Annals of Probability
1979, Vol. 7, No. 2, 244–266

EXCURSIONS OF A MARKOV PROCESS[1]

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The excursion straddling t and the first excursion exceeding u in length of a Markov process are compared in a general setting. These results are then specialized to the excursions from a point, and to specific processes. The main tool is the general theory of excursions as developed by Maisonneuve and others.

1. Introduction. During the past few years there has been a considerable body of work dealing with what might be called the general theory of excursions of a Markov process. Perhaps the definitive work in this direction is Maisonneuve [10]. Shortly thereafter Chung [2] made a penetrating analysis of the excursions of Brownian motion from the origin. However, Chung did not make use of the general theory; rather he worked by hand using the special properties of Brownian motion. (See also [4] and [5] for some interesting results on Brownian excursions from the origin.) The purpose of this paper is to apply Maisonneuve's general theory to specific excursions and processes.

We shall now give a rough description of some of our results. Let X be a nice Markov process and let F be a closed, finely perfect subset of its state space; a much more general situation is considered in Sections 2 through 6. If $t > 0$, let $G_t = \sup\{s \leqslant t : X_s \in F\}$ and $D_t = t + T_F \circ \theta_t = \inf\{s > t : X_s \in F\}$. Then the interval $]G_t, D_t[$ is called the excursion interval (from F) straddling t, and $U_s = X(G_t + s)$, $0 < s < L_t$ where $L_t = D_t - G_t$ is called the excursion process. If $u > 0$, let G^u be the left endpoint of the first maximal open interval, I, with length strictly greater than u such that $I \bigcap \{t : X_t \in F\}$ is empty, and let D^u be its right endpoint. Then $]G^u, D^u[$ is the first excursion interval (from F) exceeding u in length. Let $V_s = X(G^u + s)$, $0 < s < L^u = D^u - G^u$ be the corresponding excursion process. After some preliminaries in Sections 2 and 3, we investigate the relationship between these two excursion processes. The main result, Theorem 5.9, states that if $A_t = t - G_t$ is the *age* of the excursion straddling t at time t, then the law of the process (U_s) conditional on $A_t = u$, $0 < u < t$, and $X(G_t)$ is the same as the law of the process (V_s) conditional on $X(G^u)$. Moreover, the finite dimensional distributions of these processes are written down explicitly in Sections 4 and 5. It is also interesting to note that G^u and the process (V_s) are conditionally independent given $X(G^u)$, but that this is not the case for G_t and (U_s). In Section 6, under somewhat stronger hypotheses it is shown that $X(G_t)$ and $X(G^u)$ may be replaced by $X(G_t -)$ and $X(G^u -)$ respectively in the above statements.

Received August 15, 1977.
[1]This research was supported in part by NSF grant MCS76-80623.
AMS 1970 *subject classifications.* Primary 60J40, 60J55; secondary 60J45.
Key words and phrases. Markov processes, excursions, entrance law, exit law, local time.

In Sections 7 through 10 we consider the case where F consists of a single point b. Here, under various hypotheses, one obtains many nice formulas. These results, in turn, may be specialized to specific processes. For example, in Section 10 (see (10.25) and (10.26)) it is shown that if X is a stable process of index $\alpha > 1$ on the real line and $b = 0$, then the distributions of G_t and G^u under P^0 depend only on α and not on the asymmetry parameter β in the characteristic function, (10.16), of X.

I would like to thank M. J. Sharpe for several very helpful discussions, especially in connection with Section 9.

2. Preliminaries. In this section we shall establish our basic notation and recall the fundamental theorem on excursions in the form given by Maisonneuve [10]. Let $(E, \mathcal{E})$ be a Lusinien space and let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be the canonical right continuous realization of a semigroup (P_t) satisfying the "hypothèses droites" of Meyer. We assume the existence of a cemetery $\Delta \in E$ and, as usual, put $\zeta = \inf\{t : X_t = \Delta\}$. Without loss of generality we may assume that $P^x(\zeta = \infty) = 1$ for each $x \neq \Delta$. We shall consistently adopt the familiar notation of Markov process theory. See [1], [6], or [13], for example. In particular, we shall write $X(t)$ for X_t when convenient.

A process $Y = (Y_t)$ is said to be optional (predictable) if for each initial measure μ on E there exists a process Y^μ that is optional (predictable) over $(\Omega, \mathcal{F}_t^\mu, P^\mu)$ such that Y and Y^μ are P^μ indistinguishable. A process Y is *homogeneous* (on $]0, \infty[$) if for each $t \geqslant 0$ and $s > 0$, $Y_s \circ \theta_t = Y_{s+t}$. A subset $\mathfrak{M}$ of $\mathbb{R}^+ \times \Omega$ is homogeneous provided its indicator function is a homogeneous process.

We now fix a *closed, optional, homogeneous random set* $\mathfrak{M} \subset \mathbb{R}^+ \times \Omega$. If $M = M(\omega) = \{t : (t, \omega) \in \mathfrak{M}\} \subset \mathbb{R}^+$ is the ω-section of $\mathfrak{M}$, then the statement that $\mathfrak{M}$ is closed means that $M(\omega)$ is a closed subset of $\mathbb{R}^+$ for each $\omega \in \Omega$. We associate with $\mathfrak{M}$ the following random variables:

$$R = \inf\{s > 0 : s \in M\}$$

$$D_t = t + R \circ \theta_t = \inf\{s > t : s \in M\}. \tag{2.1}$$

As usual, the infimum of the empty set is $+\infty$. It is well known that for each $t \geqslant 0$, D_t is a stopping time and that $t \to D_t$ is right continuous and increasing. Clearly $D_0 = R$, and, since $\mathfrak{M}$ is homogeneous, R is a terminal time. Finally we assume that $R = \infty$ on $\{R \geqslant \zeta\}$. Under these circumstances Meyer [11] has shown that we may suppose that R, and hence each D_t, is $\mathcal{F}^*$ measurable, where $\mathcal{F}^*$ is the σ-algebra of universally measurable sets over $(\Omega, \mathcal{F}^0)$. Therefore in the sequel we *shall assume that R and each D_t are $\mathcal{F}^*$ measurable.*

Since $M(\omega)$ is a closed subset of $\mathbb{R}^+$, its complement in $\mathbb{R}^+$ consists of a countable union of maximal (relatively) open intervals of $\mathbb{R}^+$. We call these maximal open intervals in $\mathbb{R}^+ - M(\omega)$ the *contiguous* intervals of M.

(2.2) DEFINITION. $G = G(\omega)$ is the set of strictly positive left endpoints of the contiguous intervals of $M(\omega)$.

Observe that $G \subset]0, \infty[$ and that

(2.3) $$G = \{s > 0 : D_{s-} = s, D_s > s\}.$$

Also, if $s \in G$, then $D_s = s + R \circ \theta_s$ is the right endpoint of the contiguous interval whose left endpoint is s. Define

(2.4) $$F = \{x : P^x(R = 0) = 1\}.$$

Then F is a finely closed nearly Borel set. We may now state Maisonneuve's theorem in a form convenient for us.

(2.5) THEOREM. (Maisonneuve [10]). *There exist an adapted additive functional B with a bounded one potential and a family of measures $(\tilde{P}^x; x \in E)$ on Ω for which $x \to \tilde{P}^x(\Lambda)$ is $\mathcal{E}^*$ measurable when $\Lambda \in \mathcal{F}^*$, and such that if $Z \geq 0$ is an optional process and $K \geq 0$ is $\mathcal{F}^*$ measurable, then for all $x \in E$*

(2.6) $$E^x \Sigma_{s \in G} Z_s K \circ \theta_s = E^x \int_0^\infty Z_s \tilde{E}^{X(s)}(K)\, dB_s.$$

Moreover, each $\tilde{P}^x$ is σ-finite and satisfies $\tilde{P}^x(R = 0) = 0$, $\tilde{E}^x(1 - e^{-R}) = 1$ for all x. If $x \notin F$, then $\tilde{P}^x = c(x)P^x$ where $c(x) = [E^x(1 - e^{-R})]^{-1}$. The continuous part of B is carried by F and the discontinuous part by $E - F$. For each x the process $(X_t)_{t>0}$ is strong Markov with semigroup (P_t) relative to the measure $\tilde{P}^x$; that is, if T is an $(\mathcal{F}^0_{t+})$ stopping time with $\tilde{P}^x(T = 0) = 0$, then for all positive $\mathcal{F}^0_{T+}$ measurable H and $\mathcal{F}^0$ measurable J one has

(2.7) $$\tilde{E}^x[HJ \circ \theta_T] = \tilde{E}^x[HE^{X(T)}(J)].$$

Note that $1 \leq c(x) < \infty$ for $x \in E - F$.

There is a useful extension of (2.6) that we shall often use. Let $\mathcal{B}$ denote the Borel sets of $\mathbb{R}^+$ and let $K_s(\omega)$ be a positive $\mathcal{B} \times \mathcal{F}^*$ measurable function; then

(2.8) $$E^x \Sigma_{s \in G} Z_s K_s \circ \theta_s = E^x \int_0^\infty Z_s \tilde{E}^{X(s)}(K_s)\, dB_s,$$

for Z a positive optional process.

The reader should consult [10] for the proofs of these results. This paper also contains many other interesting results related to the above facts.

3. Entrance and exit laws. We define the semigroup (Q_t) of the process X killed at R by

(3.1) $$Q_t f(x) = E^x[f(x_t); t < R].$$

If $x \in F$, $Q_t(x, \cdot) = 0$ for all $t \geq 0$. Since F is finely closed it is immediate that $R \leq T_F$ almost surely. Consequently

$$Q_t(x, F) = E^x\{1_F(X_t); t < R\} = 0;$$

that is, the measures $Q_t(x, \cdot)$ do not charge F.

For each $t > 0$ and $x \in E$ define a measure $\tilde{Q}_t(x, \cdot)$ on E by

(3.2) $$\tilde{Q}_t(x, f) = \tilde{E}^x[f(X_t); t < R].$$

Since $\tilde{E}^x(1 - e^{-R}) = 1$, it is immediate that each $\tilde{Q}_t(x, \cdot)$ is a *finite* measure on E and $t \to \tilde{Q}_t(x, 1)$ is a decreasing right continuous function on $]0, \infty[$. We also define

(3.3) $$\Gamma(x, dt) = P^x[R \in dt],$$

and observe that $\Gamma(x,]t, \infty]) = P^x(R > t) = Q_t 1(x)$. By definition $x \in F$ if and only if $\Gamma(x, \cdot) = \varepsilon_0$, while if $x \notin F$, $\Gamma(x, \cdot)$ is a probability on $]0, \infty]$ since $P^x(R = \infty)$ may be strictly positive.

(3.4) PROPOSITION. *For each x, $\{\tilde{Q}_t(x, \cdot)\}$ is an entrance law for (Q_t); that is,*

$$\tilde{Q}_{t+s}(x, \cdot) = \int \tilde{Q}_t(x, dy) Q_s(y, \cdot); \qquad t > 0, s \geqslant 0.$$

In particular $\tilde{Q}_t(x, \cdot)$ does not charge F, and

(3.5) $$\tilde{Q}_{t+s} 1(x) = \int \tilde{Q}_t(x, dy) P^y[R > s].$$

PROOF. By (2.7) if $t > 0$

$$\begin{aligned} \tilde{Q}_{t+s}(x, f) &= \tilde{E}^x[f(X_{t+s}); t + s < R] \\ &= \tilde{E}^x[f(X_s) \circ \theta_t; s < R \circ \theta_t, t < R] \\ &= \tilde{E}^x[E^{X(t)}[f(X_s); s < R]; t < R] = \tilde{Q}_t(x, Q_s f) \end{aligned}$$

proving (3.4).

REMARK. For each x, $Q_t 1(x) = \Gamma(x,]t, \infty])$ is an *exit law* for (Q_t) since $Q_{t+s} 1(x) = Q_t(Q_s 1)(x)$.

4. The excursion straddling t. For each $t > 0$ define

(4.1) $$G_t = \sup\{s \leqslant t : s \in M\},$$

where the supremum of the empty set is taken to be zero. Since $\mathfrak{M}$ is optional, each G_t is $\mathcal{F}$ measurable and clearly $t \to G_t$ is right continuous. If $0 < G_t < t$, then G_t is the left endpoint of a contiguous interval of M. In this case $D_t = t + R \circ \theta_t$ is the right endpoint of the corresponding interval, and following Chung [2], we shall say that $]G_t, D_t[$ is the excursion interval straddling t. Note that if $0 < G_t < t$, then $t \notin M$, and since M is closed $D_t > t$. Thus $t \in]G_t, D_t[$ when $0 < G_t < t$. Let $L_t = D_t - G_t$ be the length of this interval. Of course, D_t and L_t may be infinite, but in any case D_t is a stopping time.

Let H and g be positive and $\mathcal{F}^*$ and $\mathcal{E}$ measurable functions on Ω and E respectively. Observe that $0 < s = G_t < t$ if and only if $s \in G$ and $s < t < s + R \circ \theta_s$. Let Z be a positive optional process and apply (2.8) with

$$K_s = Hg(X_{t-s}) 1_{\{0 < t-s < R\}}$$

to obtain

(4.2)

$$\begin{aligned} E^x\{Z_{G_t} g(X_t) H \circ \theta_{G_t}; 0 < G_t < t\} &= E^x \Sigma_{s \in G} Z_s K_s \circ \theta_s \\ &= E^x \int_{]0, t[} Z_s \tilde{E}^{X(s)}\{Hg(X_{t-s}); t - s < R\}\, dB_s. \end{aligned}$$

This formula contains all of the information about the $U_s^t = U_s = X(G_t + s)$ defined on the random interval $0 < s < L_t$. We shall call the process $(U_s; 0 < s < L_t)$ the *excursion process straddling t*. From (4.2) we shall compute the joint law of G_t, L_t, $X(G_t)$, and this excursion process.

(4.3) LEMMA. *Let $\beta(s, x)$ be a positive $\mathcal{B} \times \mathcal{E}^*$ measurable function on $\mathbb{R}^+ \times E$. Then for each positive optional process Z one has*

$$E^x\left[\beta(G_t, X_{G_t})Z_{G_t}; 0 < G_t < t\right] = E^x \int_{]0,t[} Z_s \beta(s, X_s) \tilde{Q}_{t-s}(X_s, 1)\, dB_s.$$

PROOF. It suffices to prove (4.3) when β and Z are bounded. Suppose first $\beta(s, x) = \varphi(s)f(x)$ with φ and f bounded Borel. Then the process $Z_s\varphi(s)f(X_s)$ is optional and applying (4.2) to this process (take H and g identically one) we obtain

$$E^x\left[Z_{G_t}\varphi(G_t)f(X_{G_t}); 0 < G_t < t\right] = E^x \int_{]0,t[} Z_s \varphi(s) f(X_s) \tilde{Q}_{t-s}(X_s, 1)\, dB_s. \tag{4.4}$$

But for Z and φ fixed both sides of (4.4) are measures in f and so it holds for bounded $f \geq 0$ in $\mathcal{E}^*$. Now for Z fixed both sides of the formula in (4.3) are measures in β and since they agree for $\beta(s, x) = \varphi(s)f(x)$ with φ, f positive and $\mathcal{B}$, $\mathcal{E}^*$ measurable respectively, they agree on $\mathcal{B} \times \mathcal{E}^*$.

Define for a positive H in $\mathcal{F}^*$, $s \geq 0$, and $x \in E$,

$$K(x, s, H) = \tilde{E}^x[H; s < R]. \tag{4.5}$$

For H bounded, $x \to K(x, s, H)$ is $\mathcal{E}^*$ measurable and $s \to K(x, s, H)$ is right continuous on $[0, \infty[$. Consequently $(s, x) \to K(x, s, H)$ is $\mathcal{B} \times \mathcal{E}^*$ measurable for each positive H in $\mathcal{F}^*$. If Z is a positive optional process and $H \geq 0$ is $\mathcal{F}^*$ measurable, then (4.2) implies

$$E^x\left[Z_{G_t} H \circ \theta_{G_t}; 0 < G_t < t\right] = E^x \int_{]0,t[} Z_s K(X_s, t - s, H)\, dB_s. \tag{4.6}$$

But if $0 \leq H \leq 1$, $K(x, s, H) \leq \tilde{Q}_s(x, 1)$ and so if we let $\beta(s, x) = K(x, t - s, H)/\tilde{Q}_{t-s}(x, 1)$ where $0/0 = 1$, then (4.3) applied to the second term in (4.6) yields

$$\begin{aligned} &E^x\left[Z_{G_t} H \circ \theta_{G_t}; 0 < G_t < t\right] \\ &\qquad = E^x\left[Z_{G_t} K(X_{G_t}, t - G_t, H)/\tilde{Q}_{t-G_t}(X_{G_t}, 1); 0 < G_t < t\right]. \end{aligned} \tag{4.7}$$

Let $A_t = t - G_t$ be the *age* of the excursion at time t. Applying (4.7) with $Z_s = \varphi(s)f(X_s)$ for positive Borel functions φ and f we find

$$\begin{aligned} &E^x\left[\varphi(G_t)f(X_{G_t})H \circ \theta_{G_t}; 0 < G_t < t\right] \\ &\qquad = E^x\left[\varphi(G_t)f(X_{G_t})K(X_{G_t}, A_t, H)/\tilde{Q}_{A_t}(X_{G_t}, 1); 0 < G_t < t\right] \\ &\qquad = \int \varphi(s)f(y)K(y, t - s, H)/\tilde{Q}_{t-s}(y, 1)m_t^x(ds, dy), \end{aligned} \tag{4.8}$$

where

$$m_t^x(ds, dy) = P^x\left[G_t \in ds, X_{G_t} \in dy; 0 < G_t < t\right] \tag{4.9}$$

is the joint distribution of (G_t, X_{G_t}) on the set $\{0 < G_t < t\}$ under P^x. Moreover, it is clear from (4.3) that m_t^x does not charge sets of the form $\mathbb{R}^+ \times N$ where N is of B-potential zero and is carried by $]0, t[\times E$.

Next suppose H is of the form

$$H = \Pi_{i=1}^k f_i(X_{s_i})\psi(R)1_{\{s_k < R\}} \tag{4.10}$$

where $0 < s_1 < \cdots < s_k$ and f_i and ψ are bounded positive Borel functions on E and $[0, \infty]$ respectively. If y is a point for which $(X_s)_{s>0}$ is Markov with transition function (P_s) relative to $\tilde{P}^y$, then using the Markov property repeatedly one obtains for $0 < s < t$

$$\begin{aligned} K(y, t-s, H) &= \tilde{E}^y\left[\Pi_{i=1}^k f_i(x_{s_i})\psi(R);\, s_k < R;\, t-s < R\right] \\ &= \int \cdots \int \tilde{Q}_{s_1}(y, dy_1) f_1(y_1) Q_{s_2-s_1}(y_1, dy_2) f_2(y_2) \cdots \\ &\quad \times Q_{s_k-s_{k-1}}(y_{k-1}, dy_k) f_k(y_k) \int_{](t-s)\vee s_k, \infty]} \Gamma(y_k, dr - s_k)\psi(r). \end{aligned} \tag{4.11}$$

Finally combining (4.8) and (4.11) we obtain the following proposition.

(4.12) PROPOSITION. *Let* $0 < s_1 < \cdots < s_k < r$, $0 < \tau < t$, $t - \tau < r$, *and* $U_s = X(G_t + s)$ *be the excursion process straddling* t. *Then*

$$\begin{aligned} &P^x\left[G_t \in d\tau, X_{G_t} \in dy;\, U_{s_1} \in dy_1, \cdots, U_{s_k} \in dy_k, L_t \in dr\right] \\ &= m_t^x(d\tau, dy)\left[\tilde{Q}_{t-\tau}(y, 1)\right]^{-1}\tilde{Q}_{s_1}(y, dy_1) Q_{s_2-s_1}(y_1, dy_2) \\ &\quad \times \cdots Q_{s_k-s_{k-1}}(y_{k-1}, dy_k)\Gamma(y_k, dr - s_k). \end{aligned}$$

(4.13) COROLLARY. *If* $0 < s_1 < \cdots < s_k < r$ *and* $0 < u < (t \wedge r)$, *then*

$$\begin{aligned} &P^x\left[U_{s_1} \in dy_1, \cdots, U_{s_k} \in dy_k, L_t \in dr \mid A_t = u, X_{G_t} = y\right] \\ &= \left[\tilde{Q}_u(y, 1)\right]^{-1}\tilde{Q}_{s_1}(y, dy_1) \cdots Q_{s_k-s_{k-1}}(y_{k-1}, dy_k)\Gamma(y_k, dr - s_k), \end{aligned}$$

and

$$P^x\left[L_t > r \mid A_t = u, X_{G_t} = y\right] = \tilde{Q}_r(y, 1)/\tilde{Q}_u(y, 1).$$

In particular the process $(U_s)_{s>0}$ *under* P^x *and conditional on* $A_t = u$ *and* $X(G_t) = y$ *is an inhomogeneous Markov process with state space* $E - F$ *and whose entrance law* $\rho_s(dy)$ *and transition function* $P[s, y; t, dz]$, $0 \leqslant s < \tau$ *are given by* ($a^+ = \sup(a, 0)$ *for* $a \in \mathbb{R}$)

$$\rho_s(dy) = \frac{Q_s(x, dy)Q_{(u-s)^+}(y, 1)}{\tilde{Q}_u(x, 1)} \tag{4.14}$$

$$P[s; y; \tau, dz] = \frac{Q_{\tau-s}(y; dz)Q_{(u-\tau)^+}(z, 1)}{Q_{(u-s)^+}(y, 1)}.$$

PROOF. This follows directly from (4.12).

5. The first excursion exceeding u in length. Let $u > 0$ be fixed. Let $]G^u, D^u[$ be the first contiguous interval of M whose length strictly exceeds u. If no such interval exists we set $G^u = D^u = \infty$. If $G^u < \infty$, then $L^u = D^u - G^u = R \circ \theta_{G^u} > u$. It is known [3] that $T^u = u + G^u$ and D^u are stopping times. Clearly $G^u \in G$ if $0 < G^u < \infty$. Observe that if $s > 0$, then $s = G^u < \infty$ if and only if $s \in G$, $s < D^u$, and $R \circ \theta_s > u$. (For later reference, note that if almost surely M has no isolated points, then one may replace $s < D^u$ by $s \leqslant D^u$ in the preceding sentence.) Let H and g be positive and respectively $\mathcal{F}^*$ and $\mathcal{E}$ measurable. If Z is a positive optional process, then $Z 1_{[0, D^u[}$ is optional since D^u is a stopping time. Apply (2.6) to this optional process and with $K = Hg(X_u)1_{\{R>u\}}$ to obtain

(5.1)

$$E^x[Z_{G^u}g(X_{T^u})H \circ \theta_{G^u}; 0 < G^u < \infty] = E^x\Sigma_{s\in G}Z_sK \circ \theta_s 1_{[0, D^u[}(s)$$
$$= E^x\int_{[0, D^u[}Z_s\tilde{E}^{X(s)}[Hg(X_u); u < R]\, dB_s.$$

If we set

$$B_s^u = \int_{[0, s]}1_{[0, D^u[}(t)\, dB_t, \tag{5.2}$$

then (5.1) may be written

(5.3)

$$E^x[Z_{G^u}g(X_{T^u})H \circ \theta_{G^u}; 0 < G^u < \infty] = E^x\int_0^\infty Z_s\tilde{E}^{X(s)}[Hg(X_u); u < R]\, dB_s^u.$$

Arguing exactly as in the proof of (4.3) and (4.7) one obtains:

$$E^x[\beta(G^u, X_{G^u})Z_{G^u}; 0 < G^u < \infty] = E^x\int_0^\infty Z_s\beta(s, X_s)\tilde{Q}_u(X_s, 1)\, dB_s^u \tag{5.4}$$

for β a positive $\mathcal{B} \times \mathcal{E}^*$ measurable function and Z a positive optional process; and

(5.5)

$$E^x[Z_{G^u}H \circ \theta_{G^u}; 0 < G^u < \infty] = E^x\int_0^\infty Z_sK(X_s, u, H)\, dB_s^u$$
$$= E^x[Z_{G^u}K(X_{G^u}, u, H)/\tilde{Q}_u(X_{G^u}, 1); 0 < G^u < \infty]$$

for Z a positive optional process and H a positive $\mathcal{F}^*$ measurable random variable where $K(x, s, H)$ is defined in (4.5).

Let

$$M_u^x(ds, dy) = P^x[G^u \in ds, X(G^u) \in dy; 0 < G^u < \infty] \tag{5.6}$$

be the distribution of $(G^u, X(G^u))$ under P^x on the set $\{0 < G^u < \infty\}$. Note that (5.4) gives an alternate expression for M_u^x.

Define $V_s^u = V_s = X(G^u + s)$ on the random interval $0 < s < L^u$. Then we shall call $(V_s; 0 < s < L^u)$ the *excursion process* on the interval $]G^u, D^u[$. Arguing exactly as in the proof of (4.12) and (4.13) one obtains the following result.

(5.7) PROPOSITION.

(i) *Let* $0 < s_1 < \cdots < s_k < r$, $0 < s < \infty$, *and* $0 < u < r$.

Then

$$P^x[G^u \in ds, X_{G^u} \in dy, V_{s_1} \in dy_1, \cdots, V_{s_k} \in dy_k, L^u \in dr]$$
$$= M_u^x(ds, dy)[\tilde{Q}_u(y, 1)]^{-1}\tilde{Q}_{s_1}(y, dy_1)Q_{s_2-s_1}(y_1, dy_2)$$
$$\times \cdots Q_{s_k-s_{k-1}}(y_{k-1}, dy_k)\Gamma(y_k, dr - s_k).$$

(ii) *If* $0 < s_1 < \cdots < s_k < r$, $0 < s < \infty$, *and* $0 < u < r$, *then*

$$P^x[V_{s_1} \in dy_1, \cdots, V_{s_k} \in dy_k, L^u \in dr | G^u = s, X_{G^u} = y]$$
$$= [\tilde{Q}_u(y, 1)]^{-1}\tilde{Q}_{s_1}(y, dy_1) \cdots Q_{s_k-s_{k-1}}(y_{k-1}, dy_k)\Gamma(y_k, dr - s_k).$$

But the right-hand side of (5.7 ii) does not depend on s. Consequently if $0 < s_1 < \cdots < s_k < r$ and $u < t$ one has

$$(5.8) \quad P^x[V_{s_1} \in dy_1, \cdots, V_{s_k} \in dy_k, L^u \in dr, G^u < \infty | X_{G^u} = y]$$
$$= [\tilde{Q}_u(y, 1)]^{-1}\tilde{Q}_{s_1}(y, dy_1) \cdots Q_{s_k-s_{k-1}}(y_{k-1,} {}_{dy_k})\Gamma(y_k, dr - s_k),$$

and G^u and the excursion process $(V_s, 0 < s < L^u, L^u)$ are conditionally independent given $X(G^u)$ with respect to each P^x.

Comparing (4.13) and (5.8) we obtain the following result.

(5.9) THEOREM. *Fix* $t > 0$ *and* $0 < u < t$. *Then the excursion process straddling* t, $(U_s, 0 < s < L_t, L_t)$ *conditional on* $A_t = u$ *and* $X(G_t)$ *has the same distribution under each* P^x *as the excursion process on the interval* $]G^u, D^u[$, $(V_s, 0 < s < L^u, L^u)$ *conditional on* $X(G^u)$.

6. The predictable version. Under suitable conditions one can obtain a "predictable" version of Theorem 2.5. Conditions under which this is possible are given in [9] – see especially the discussion on page 379 and pages 393–394; also example (4.13) on page 398 is illuminating in this matter. In the present paper we shall not strive for the utmost generality, but shall be content with a result that covers the applications we have in mind.

In the remainder of this paper we assume that X *is a Hunt process and that there exists a* σ*-finite reference measure* ξ. The assumption that almost surely $\zeta = \infty$ is still in force. Recall that a nearly Borel subset N of E is *admissible* provided $T_N = \inf\{t > 0 : X_{t-} \in N\}$ almost surely. It is known that every finely perfect set is admissible, and that if X satisfies Hunt's hypothesis (B)—in particular under duality assumptions—every nearly Borel set is admissible.

We now suppose that

$$(6.1) \qquad M = \{t : X_t \in F\}^-$$

where "$-$" denotes closure and F is a finely closed admissible set. Under these conditions $R = T_F$ almost surely and $\{x : P^x(R = 0) = 1\} = F^r$—the set of points regular for F. (Note that F^r is the set which was denoted by F in Section 2.)

Under these circumstances one may take the dual predictable projection in (2.6) and obtain the following result.

(6.2) THEOREM. *Let M be of the form* (6.1). *Then there exist a predictable additive functional $l = (l_t)$ with a finite* 1*-potential and a family of measures $(\bar{P}^x;\ x \in E)$ on Ω with $x \to \bar{P}^x(\Lambda)$ $\mathcal{E}^*$ measurable whenever $\Lambda \in \mathcal{F}^*$—if, in addition, E is Polish $x \to \bar{P}^x(\Lambda)$ is Borel whenever $\Lambda \in \mathcal{F}^0$—and such that if Z is a positive predictable process and $K_s(\omega)$ is $\mathcal{B} \times \mathcal{F}^*$ measurable, then*

$$E^x \Sigma_{s \in G} Z_s K_s(\theta_s) = E^x \int_0^\infty Z_s \bar{E}^{X(s)}[K_s]\, dl_s, \tag{6.3}$$

for all x. Moreover, $\bar{E}^x(1 - e^{-R}) \leqslant 1$ and $\bar{P}^x(R = 0) = 0$ for all x.

One may prove the strong Markov property of the measures $\bar{P}^x$ (i.e., (2.7)) exactly as the strong Markov property of the measures $\tilde{P}^x$ is proved in [10]. Let f be the indicator of the complement of $\bar{F}$. Because F is admissible it follows that almost surely if t is an isolated point of M, then both X_t and X_{t-} are in F. As a result almost surely $Z_s = f(X_{s-}) = 0$ for $s \in G$. Apply (6.3) with this Z and $K_s = (1 - e^{-R})$ independent of s to obtain

$$0 = E^x \int_0^\infty f(X_{s-}) \bar{E}^{X(s)}(1 - e^{-R})\, dl_s = E^x \int_0^\infty f(X_s) \bar{E}^{X(s)}(1 - e^{-R})\, dl_s.$$

But $0 < \bar{E}^y(1 - e^{-R}) \leqslant 1$ for all y, and hence l is carried by $\bar{F}$. Finally it is easy to see and well known that if F is projective, in particular if F is finely perfect and closed, then l is continuous.

Define for $t > 0$

$$\bar{Q}_t(x, f) = \bar{E}^x[f(X_t);\ t < R]. \tag{6.4}$$

Then it follows just as in Section 3 that for each x, $\{\bar{Q}_t(x, \cdot)\}_{t>0}$ is an entrance law for (Q_t), and that $\bar{Q}_t(x, \cdot)$ does not charge $\underline{F^r}$. In fact all of the results of Proposition 3.4 are valid with $\tilde{Q}$ replaced by $\bar{Q}$; of course, F^r is now playing the role of F in (3.4).

The results of Sections 4 and 5 may be recast in the present situation. However, in order to carry over the argument in the first paragraph of Section 5 one seems to need an additional hypothesis. For example, if almost surely M contains no isolated points, then $0 < s = G^u < \infty$ if and only if $s \in G$, $R \circ \theta_s > u$, and $s \leqslant D^u$. Now $1_{[0, D^u]}$ is predictable and so one obtains the following analog of (5.1):

$$E^x[Z_{G^u} g(X_{T^u}) H \circ \theta_{G^u};\ 0 < G^u < \infty] = E^x \int_{[0, D^u]} Z_s \bar{E}^{X(s)}[H g(X_u);\ u < R]\, dl_s \tag{6.5}$$

for positive predictable Z, and g and H as in (5.1). Once this result is established the remainder of Section 5 goes just as before. In particular, if F is finely perfect, then almost surely M has no isolated points.

In rephrasing the results of Sections 4 and 5 one replaces $\tilde{Q}$, B, $X(G_t)$, and $X(G^u)$ by $\bar{Q}$, l, $X(G_t -)$, and $X(G^u -)$ respectively. For example, (5.9) becomes the following.

(6.6) THEOREM. *Assume M is of the form* (6.1) *and has no isolated points almost surely. Fix $t > 0$ and $0 < u < t$. Then the excursion process straddling t, $(U_s, 0 < s < L_t, L_t)$ conditional on $A_t = u$ and $X(G_t -)$ has the same distribution as the excursion process on the interval $]G^u, D^u[$, $(V_s, 0 < s < L^u, L^u)$ conditional on $X(G^u -)$. In particular, if $0 < s_1 < \cdots < s_k < r$ and $u < r$, then*

$$P^x\left[V_{s_1} \in dy_1, \cdots, V_{s_k} \in dy_k, L^u \in dr, G^u < \infty | X(G^u -) = y\right]$$
$$= \left[\overline{Q}_u(y, 1)\right]^{-1}\overline{Q}_{s_1}(y, dy_1)Q_{s_2-s_1}(y_1, dy_2) \cdots \times Q_{s_{k-1}-s_k}(y_{k-1}, dy_k)\Gamma(y_k, dr - s_k).$$

Moreover G^u and $(V_s, 0 < s < L^u, L^u)$ are conditionally independent given $X(G^u -)$. In addition $(V_s)_{s>0}$ conditional on $X(G^u -) = y$ under P^x is an inhomogeneous Markov process whose entrance law and transition function are given by (4.14) *with $\tilde{Q}$ replaced by $\overline{Q}$.*

In the sequel we shall use the predictable versions of the results of Sections 3, 4 and 5 when M is of the form (6.1) with F finely perfect without special mention. If, in addition to being finely perfect, F is closed, then, as remarked above (6.4), l is a *continuous* additive functional with a finite 1-potential that is carried by F.

7. Excursions from a point. As in Section 6 we assume that X is a Hunt process with a σ-finite reference measure ξ. We fix a point $b \in E$ and we assume that b is regular (for itself), that is, $P^b(T_b = 0) = 1$ where $T_b = \inf\{t > 0 : X_t = b\}$ is the hitting time of b. Then $F = \{b\}$ is a finely perfect closed set, and so all of results of the previous sections apply to

(7.1) $$M = \{t : X_t = b\}^- .$$

Under these assumptions l is a continuous additive functional carried by $\{b\}$, and so l is a *local time* for b. If we define $P^* = \overline{P}^b$, then, because l is carried by $\{b\}$, (6.3) takes the form

(7.2) $$E^x\Sigma_{s\in G}Z_sK_s(\theta_s) = E^x\textstyle\int_0^\infty Z_sE^*\left[K_s\right] dl_s$$

for Z a positive predictable process and $K_s(\omega)$ a positive $\mathcal{B} \times \mathcal{F}^*$ measurable function. Similarly we shall write

(7.3) $$Q_t^*f = \overline{Q}_t(b, f) = E^*\left[f(X_t); t < R\right]$$

for the associated entrance law. In the present case $R = T_b$ almost surely and we shall usually write T_b in place of R in our formulas.

The continuous additive functional l is determined by its λ-potential, u^λ, which is finite for $\lambda > 0$ and given by

(7.4) $$u^\lambda(x) = E^x\textstyle\int_0^\infty e^{-\lambda t}\, dl_t = E^x\int_{T_b}^\infty e^{-\lambda t}\, dl_t$$
$$= E^x(e^{-\lambda T_b})E^b\textstyle\int_0^\infty e^{-\lambda t}\, dl_t = u^\lambda(b)E^x(e^{-\lambda T_b}).$$

Let $\tau = (\tau_t)$ be the right continuous inverse of l. Then it is well known that τ is a strictly increasing subordinator under P^b, and so

(7.5) $$E^b\{e^{-\lambda\tau_t}\} = e^{-tg(\lambda)},$$

(7.6) $$g(\lambda) = \gamma\lambda + \int_{]0,\infty]} e^{-\lambda s}\nu(ds)$$

where $\gamma \geqslant 0$ and ν is a measure on $]0, \infty]$ satisfying $\int(1 \wedge s)\nu(ds) < \infty$. The possible mass of ν at ∞ corresponds to a jump to ∞ by τ. This, in turn, corresponds to $\sup\{t : X_t = b\} < \infty$.

In the sequel the function

(7.7) $$h(u) = \nu(]u, \infty])$$

will play an important role. Clearly h is a right continuous decreasing function on $]0, \infty]$ with $\lim_{u\to\infty} h(u) = \nu(\{\infty\})$ and satisfying $\int_0^t h(u)\, du < \infty$ for each $t < \infty$. In particular, $\lim_{u\to 0} uh(u) = 0$, although $h(0+)$ may be infinite. In fact, since $\tau_t > 0$, if $t > 0$, $h(0+) = \infty$ if $\gamma = 0$. By a simple change of variables

(7.8) $$u^\lambda(b) = E^b\int_0^\infty e^{-\lambda t}\, dl_t = E^b\int_0^\infty e^{-\lambda\tau_t}\, dt = 1/g(\lambda).$$

Also of importance is the measure $m(dt) = dE^b(l_t)$. If $\varphi \geqslant 0$, then

$$\int\varphi(t)m(dt) = E^b\int\varphi(t)\, dl_t = E^b\int\varphi(\tau_t)\, dt;$$

so that m is just the *potential measure* of τ. It follows from (7.8) that

(7.9) $$\int_0^\infty e^{-\lambda t}m(dt) = 1/g(\lambda).$$

If we assume a bit more about the process X, we can say more about m. Suppose first that:

(7.10) *Each point $x \in E$ is regular and $\psi(x, y) = E^x(e^{-T_y})$ is jointly Borel measurable.*

Under (7.10) there exists a local time l^x for each $x \in E$ satisfying almost surely ($\xi(dx) = dx$)

(7.11) $$\int_0^t f(X_s)\, ds = \int f(x)l_t^x\, dx$$

for all bounded Borel f and $t \geqslant 0$ simultaneously. Moreover, if we define

(7.12) $$u^\lambda(x, y) = E^x\int_0^\infty e^{-\lambda t}\, dl_t^y, \qquad \lambda > 0,$$

then it follows from (7.11) that

(7.13) $$E^x\int_0^\infty e^{-\lambda t}f(X_t)\, dt = \int u^\lambda(x, y)f(y)\, dy$$

for $\lambda > 0$ and positive Borel f. See [7] for a proof of these facts. Comparing (7.4) and (7.12) we see that we may suppose that $l = l^b$ and then $u^\lambda(x) = u^\lambda(x, b)$. (This may involve replacing P^* by cP^* where c is a constant.) Next suppose that

X has a transition density $p(t, x, y)$ with respect to ξ such that

(7.14) $$u^\lambda(x, y) = \int_0^\infty e^{-\lambda t}p(t, x, y)\, dt$$

for all x, y in E and $\lambda > 0$.

Then from (7.8) and (7.12), $u^\lambda(b, b) = [g(\lambda)]^{-1}$, and hence (7.9) implies

(7.15) $$m(dt) = p(t, b, b)\, dt.$$

We explicitly shall state when we are assuming that (7.10) and (7.14) hold, and we shall refer to this situation as the *nice case.*

In general, it is known, see [8] for example, that if 1_b denotes the indicator of $\{b\}$, then $\int_0^t 1_b(X_s)\, ds = \gamma l_t$, and hence

(7.16) $$U^\lambda(b, \{b\}) = E^b \int_0^\infty e^{-\lambda t} 1_b(X_t)\, dt = \gamma E^b \int_0^\infty e^{-\lambda t}\, dl_t = \gamma / g(\lambda).$$

In particular $\gamma = 0$ if and only if $\{b\}$ is of potential zero, and if $\gamma > 0$, then

(7.17) $$m(dt) = \gamma^{-1} P_t(b, \{b\})\, dt.$$

We now specialize the results of the previous sections to the present situation. Then it is clear that (Q_t) is the semigroup of the process $(X, T_b) - X$ killed when it first hits b. We first note that if $t > 0$, then almost surely

(7.18) $$\{G_t = t\} = \{G_t = t = D_t\} = \{X_t = b\}.$$

To see this suppose firstly that $G_t = t > 0$. Then either $X_t = b$ or $X_{t-} = b$, and since the quasi-left-continuity of X implies that $X_t = X_{t-}$ almost surely, we see that almost surely $\{G_t = t\} \subset \{X_t = b\}$. On the other hand if $\{X_t = b\}$, then $G_t = t$, and $D_t = t$ almost surely on $\{X_t = b\}$ since b is regular. Combining these observations yields (7.18).

Clearly $\{G_t = 0\} = \{T_b > t\}$. *Here and henceforth all equalities of this type are understood to hold almost surely unless explicitly stated otherwise.* We now apply the predictable version of (4.2) with Z and H identically one to obtain in light of (7.18) and the previous remark (we decompose Ω as the disjoint union of $\{G_t = 0\}$, $\{G_t = t\}$, and $\{0 < G_t < t\}$)

(7.19) $$P_t f(x) = Q_t f(x) + E^x\{f(X_t); X_t = b\} + \int_0^t Q^*_{t-s} f\, m^x(ds),$$

where $m^x(ds) = dE^x(l_s)$. Note that $m^b = m$. Of course, (7.19) is the celebrated "last exit" decomposition. Let $x = b$ in (7.19). Then, since $Q_t(b, \cdot) = 0$, taking Laplace transforms we obtain in view of (7.9) and (7.16)

(7.20) $$\mu^\lambda(f) = g(\lambda) U^\lambda f(b) - \gamma f(b)$$

where $\mu^\lambda(f)$ is the Laplace transform of $Q^*_t f$. Clearly (7.20) determines the entrance law (Q^*_t). In particular if $f = 1$, then $U^\lambda 1 = \lambda^{-1}$ and so (7.20) becomes

(7.21) $$\mu^\lambda(1) = \frac{g(\lambda) - \gamma\lambda}{\lambda}.$$

On the other hand

$$\begin{aligned} g(\lambda) - \gamma\lambda &= \int_{(0,\infty]} (1 - e^{-\lambda s}) \nu(ds) \\ &= h(\infty) - \int_0^\infty (1 - e^{-\lambda s})\, d[h(s) - h(\infty)] \\ &= h(\infty) + \lambda \int_0^\infty e^{-\lambda s} [h(s) - h(\infty)]\, ds \\ &= \lambda \int_0^\infty e^{-\lambda s} h(s)\, ds. \end{aligned}$$

Since both $Q^*_t 1$ and $h(t)$ are right continuous we have made the following identification.

(7.22) PROPOSITION. $Q_t^* 1 = h(t)$.

In the present case $\Gamma(x, dt) = P^x[T_b \in dt]$, and it follows from (7.4) and (7.8) that

$$\int_0^\infty e^{-\lambda t}\Gamma(x, dt) = u^\lambda(x)/u^\lambda(b) = u^\lambda(x)g(\lambda). \tag{7.23}$$

Clearly (7.23) determines the exit law $\Gamma(x, \cdot)$. In the nice case $u^\lambda(x) = u^\lambda(x, b) = \int_0^\infty e^{-\lambda t}p(t, x, b)\, dt$.

Next we consider the excursion straddling t where $t > 0$ is fixed. Since b is regular, $P^b(G_t = 0) = P^b(T_b > t) = 0$. Applying (7.2) with $Z_s = \varphi(s)$ and $K_s = f(X_{t-s})\psi(s + T_b)1_{\{0<t-s<T_b\}}$, and using (7.18) results in

$$(7.24)\quad E^b[\varphi(G_t)f(X_t)\psi(D_t)] = \varphi(t)f(b)\psi(t)P^b[X_t = b] + \int_0^t \varphi(s)E^*[f(X_{t-s})\psi(s + T_b);\, t - s < T_b]m(ds).$$

Because of the Markov property of P^* the last term in (7.24) may be written

$$\int_0^t \varphi(s)m(ds)\int Q_{t-s}^*(dy)f(y)\int \psi(t + r)\Gamma(y, dr).$$

Consequently if $0 < s \leqslant t \leqslant r$ one has

$$(7.25)\quad P^b[G_t \in ds, X_t \in dy, D_t \in dr] = \varepsilon_t(ds)\varepsilon_t(dr)\varepsilon_b(dy)P_t(b, \{b\}) + m(ds)Q_{t-s}^*(dy)\Gamma(y, dr - t)1_{\{0<s<t<r\}}.$$

Integrating (7.25) over the appropriate variables and using (7.22) and (3.5) for Q_t^* one obtains the following formulas. Recall from (7.7) that $\nu(dr) = -\,dh(r)$ is the Lévy measure of τ.

$$P^b[G_t \in ds, D_t \in dr] = \varepsilon_t(ds)\varepsilon_t(dr)P_t(b, \{b\}) + m(ds)\nu(dr - s)1_{\{0<s<t<r\}}, \tag{7.26}$$

$$P^b(G_t \in ds) = \varepsilon_t(ds)P_t(b, \{b\}) + h(t - s)m(ds)1_{\{0<s<t\}}, \tag{7.27}$$

$$P^b(G_t \in ds, X_t \in dy) = \varepsilon_t(ds)\varepsilon_b(dy)P_t(b, \{b\}) + m(ds)Q_{t-s}^*(dy)1_{\{0<s<t\}}, \tag{7.28}$$

$$P^b[X_t \in dy | G_t = s] = \frac{Q_{t-s}^*(dy)}{h(t - s)}, \qquad 0 < s < t. \tag{7.29}$$

For (7.29) to be valid one must require $h(u) > 0$ for all $u < t$. These formulas take an especially pleasing form in the nice case.

We turn next to the first excursion exceeding $u > 0$ in length. It is easy to see, using the fact that l_{G^u} is the time of the first jump of τ which exceeds u in length, that $P^b[G^u = \infty] = \lim_{t\to\infty} e^{-th(u)}$. Hence $P^b(G^u = \infty) = 0$ if $h(u) > 0$ and $P^b(G^u = \infty) = 1$ if $h(u) = 0$. Thus, in what follows, we shall suppose that $u > 0$ and $h(u) > 0$, or equivalently that $u > 0$ and $P^b(G^u < \infty) = 1$. Recall that $T^u = u +$

G^u. Applying (7.2) with $Z_s = \varphi(s)1_{[0, D^u]}(s)$ and $K_s = f(X_u)\psi(s + T_b)1_{\{u<T_b\}}$ we obtain

(7.30)

$$E^b\{\varphi(G^u)f[X(T^u)]\psi(D^u)\} = E^b\int_0^{D^u}\varphi(s)E^*[f(X_u)\psi(s + T_b); u < T_b]\, dl_s.$$

But l_s is constant on the interval $[G^u, D^u]$, and so introducing the measure

$$m^u(ds) = dE^b(l_{s\wedge D^u}) = dE^b(l_{s\wedge G^u}), \tag{7.31}$$

and using the Markov property of P^*, (7.30) yields

$$P^b[G^u \in ds, X(T^u) \in dy, D^u \in dr] = m^u(ds)Q_u^*(dy)\Gamma(y, dr - u - s); \quad 0 < u + s < r. \tag{7.32}$$

Once again this formula may be specialized in many ways. We record here only the following:

$$P^b[G^u \in ds] = h(u)m^u(ds) \tag{7.33}$$

$$P^b[X(T^u) \in dy, L^u \in dr] = [h(u)]^{-1}Q_u^*(dy)\Gamma(y, dr - u), \quad 0 < u < r \tag{7.34}$$

where, as before, $L^u = D^u - G^u$ is the length of the first excursion exceeding u in length. It is now immediate that G^u and $(X(T^u), L^u)$ are independent under P^b.

Since the definition of m^u involves D^u (or G^u), formula (7.32) is less satisfactory than (7.25) in which m is obtained directly from l, and in the nice case, directly from the transition function of X by (7.15). In Section 9 we shall express m^u, or at least its Laplace transform, directly in terms of ν. See (9.3). Note, however, that from (7.33) the total mass of m^u is $[h(u)]^{-1}$.

We now translate Theorem 6.6 to the present case. As before $U_s = X(G_t + s)$, $0 < s < L_t$ and $V_s = X(G^u + s), 0 < s < L^u$ are the two excursion processes. Under the current assumptions $X(G_t -) = b = X(G^u -)$ almost surely P^b. As before $A_t = t - G_t$ is the age of the excursion straddling t at time t. Observe that $0 < A_t < t$ almost surely P^b.

(7.35) THEOREM. *Let $0 < u < t$. Then the process $(U_s, 0 < s < L_t; L_t)$ conditional on $A_t = u$ has the same law as $(V_s, 0 < s < L^u; L^u)$ which is given by*

$$P^b[V_{s_1} \in dy_1, \cdots, V_{s_k} \in dy_k, L^u \in dr]$$
$$= [h(u)]^{-1}Q_{s_1}^*(dy_1)Q_{s_2-s_1}(y_1, dy_2) \cdots Q_{s_k-s_{k-1}}(y_{k-1}, dy_k)\Gamma(y_k, dr - s_k)$$

where $0 < s_1 < \cdots < s_k < r$ and $u < r$. Moreover G^u and the process $(V_s, 0 < s < L^u; L^u)$ are independent.

(7.36) REMARK. Of course, (6.3) also contains the joint distribution of $X(G_t)$ or $X(G^u)$ with any of the previously discussed variables. Define a measure $Q_t^{\#}$ on

$E \times E$ for $t > 0$ by

$$(7.37) \qquad Q_t^{\#}(a) = E^*[a(X_0, X_t); t < R]$$

for $a(x, y)$ a positive $\mathcal{E} \times \mathcal{E}$ measurable function. Note that $Q_t^* f = Q_t^{\#}(1 \otimes f)$ where $(1 \otimes f)(x, y) = f(y)$. Then, for example,

$$(7.38) \qquad P^b[G^u \in ds, X(G^u) \in dx, X(T^u) \in dy, D^u \in dr] = m^u(ds) Q_u^{\#}(dx, dy) \Gamma(y, dr - u - s)$$

for $0 < u + s < r$. Similarly one may express the law of $(V_s, 0 \leqslant s < L^u; L^u)$, and also the analogous quantities involving G_t. We leave it to the interested reader to write down the appropriate formulas.

8. An invariant measure. The assumptions in this section are the same as in Section 7. The first part of the following theorem is due to Silverstein [12]. However, our proof differs in some details from his. We would like to thank Professor Silverstein for pointing out an error in our original version of this theorem.

(8.1) THEOREM. *Let* $\mu(f) = \int_0^\infty Q_t^*(f)\, dt$. *Then* $\eta = \mu + \gamma\varepsilon_b$ *is an invariant measure for* (P_t) *if and only if* $h(\infty) = 0$. *Let* $A = \{x : P^x(T_b < \infty) > 0\}$. *Then* μ *is* σ*-finite on* A *and if* $h(\infty) = 0$, μ *is carried by* A.

(8.2) REMARKS. If $h(\infty) = 0$, then (8.1) states that η is a σ-finite invariant measure for (P_t); that is, $\eta P_t = \eta$ for all $t \geqslant 0$. It is immediate from the discussion in the first part of Section 7 that each of the following is equivalent to $h(\infty) = 0$:

(i) $P^*(T_b = \infty) = \lim_{t \to \infty} Q_t^* 1 = 0$.

(ii) $g(0+) = 0$.

(iii) $E^b(l_\infty) = \infty$.

In the nice case each of these is equivalent to $\int_0^\infty p(t, b, b)\, dt = \infty$.

(8.3) REMARK. If $P^x(T_b = \infty) = 0$ for all x, then $h(\infty) = 0$. To see this let $t > 0$ and observe

$$P^*[T_b = \infty] = P^*[T_b \circ \theta_t = \infty, t < T_b] = E^*[P^{X(t)}(T_b = \infty); t < T_b] = 0.$$

Note that this also shows that if $\varphi(x) = P^x[T_b = \infty]$, then $\varphi = 0$ a.e. Q_t^* for each $t > 0$ whenever $h(\infty) = 0$.

We turn now to the proof of (8.1). Since $Q_t^* 1 = h(t)$, it is immediate that

$$(8.4) \qquad \mu^\beta(f) = \int_0^\infty e^{-\beta t} Q_t^* f\, dt = E^* \int_0^{T_b} e^{-\beta t} f(X_t)\, dt$$

is a finite measure if $\beta > 0$. Clearly $\mu^\beta(f) \uparrow \mu(f)$ as $\beta \downarrow 0$ when $f \geqslant 0$. Let f be positive, bounded, and continuous. Then for $\beta > 0$ and $\lambda > 0$ using the Markov property for P^* we find

$$\mu^\beta(U^\lambda f) = E^* \int_0^{T_b} e^{-\beta t} E^{X(t)} \int_0^\infty e^{-\lambda s} f(X_s)\, ds\, dt$$
$$= E^* \int_0^{T_b} e^{-(\beta-\lambda)t} \int_t^\infty e^{-\lambda s} f(X_s)\, ds\, dt = J_1 + J_2,$$

where

$$J_1 = E^* \int_0^{T_b} e^{-(\beta-\lambda)t} \int_t^{T_b} e^{-\lambda s} f(X_s)\, ds\, dt$$
$$= (\lambda - \beta)^{-1} [\, \mu^\beta(f) - \mu^\lambda(f) \,],$$

and, since $P^*(T_b = 0) = 0$,

$$J_2 = E^* \int_0^{T_b} e^{-(\beta-\lambda)t} \int_{T_b}^\infty e^{-\lambda s} f(X_s)\, ds\, dt$$
$$= U^\lambda f(b) (\beta - \lambda)^{-1} [\, \beta \mu^\beta(1) - \lambda \mu^\lambda(1) \,].$$

Combining these formulas with (7.20) and (7.21) yields

$$(8.5)\quad \mu^\beta(U^\lambda f) = (\lambda - \beta)^{-1} [\, \mu^\beta(f) - \beta \mu^\beta(1) U^\lambda f(b) + \gamma f(b) - \gamma \lambda U^\lambda f(b) \,].$$

Since f is continuous, $t \to P_t f(x)$ is right continuous, and hence we may invert (8.5) with respect to λ to obtain

$$(8.6)\qquad \mu^\beta(P_t f) = e^{\beta t} [\, \mu^\beta(f) + \gamma f(b) \,] - \gamma P_t f(b)$$
$$- \beta [\, \mu^\beta(1) + \gamma \,] \int_0^t e^{\beta(t-s)} P_s f(b)\, ds.$$

Let $\beta \downarrow 0$. Then $\mu^\beta \uparrow \mu$ and $\beta \mu^\beta(1) = E^*(1 - e^{-\beta T_b}) \to P^*(T_b = \infty) = h(\infty)$. Therefore (8.6) becomes

$$(8.7)\qquad \mu(P_t f) + \gamma P_t f(b) = \mu(f) + \gamma f(b) - h(\infty) \int_0^t P_s f(b)\, ds,$$

and this implies that $\eta = \mu + \gamma \varepsilon_b$ is invariant for (P_t) if and only if $h(\infty) = 0$.

Let $\varphi(x) = P^x(T_b = \infty)$. Then $1_{A^c} \leqslant \varphi$. If $h(\infty) = 0$, then

$$Q_t^*(A^c) \leqslant Q_t^*(\varphi) = E^*[\, P^{X(t)}(T_b = \infty);\, t < T_b \,]$$
$$= P^*[\, T_b \circ \theta_t = \infty;\, t < T_b \,] \leqslant P^*(T_b = \infty) = 0,$$

and so $\mu(A^c) = 0$.

To complete the proof of (8.1) it remains to show that μ is σ-finite on A. Let $\psi(x) = E^x(e^{-T_b})$ and let $A_n = \{\psi > 1/n\}$. It then follows from (III-5.16) of [1] applied to the multiplicative functional $t \to 1_{[0,\, T_b[}(t)$ that $V 1_{A_n}$ is bounded where $Vf(x) = E^x \int_0^{T_b} f(X_t)\, dt$ is the potential operator for (X, T_b).

Let $\alpha_n = \sup(1, \sup V 1_{A_n})$. Then $1 \leqslant \alpha_n < \infty$. Let $f = \sum_{n=2}^\infty (\alpha_n 2^n)^{-1} 1_{A_n}$. Then f is bounded by 1, f is *strictly* positive on $A = \cup A_n$, and $Vf \leqslant 1$. Therefore

$$\int_0^1 Q_t^* f\, dt \leqslant \int_0^1 Q_t^* 1\, dt = \int_0^1 h(t)\, dt < \infty,$$

and

$$\int_1^\infty Q_t^* f\,dt = \int_0^\infty Q_{t+1}^* f\,dt = \int_0^\infty Q_1^* Q_t f\,dt$$
$$= Q_1^* Vf \leqslant Q_1^* 1 < \infty.$$

Hence $\mu(f) = \int_0^\infty Q_t^* f\,dt < \infty$ and since $f > 0$ on A, μ is σ-finite on A.

9. The measure m^u. In this section we shall compute the Laplace transform of the measure m^u defined in (7.31). In view of (7.33) this is the same as computing the Laplace transform of the distribution of G^u. It is interesting that even in the case of Brownian motion it does not seem to be possible to explicitly invert these transforms. The results of this section are valid for any subordinator τ whose right continuous inverse l is continuous, but we shall state them within the framework of Section 7.

We fix $u > 0$ with $h(u) > 0$ and define

(9.1) $$g(u, \lambda) = \gamma\lambda + \int_{]0, u]}(1 - e^{-\lambda r})\nu(dr).$$

Thus $g(u, \cdot)$ is the subordinator exponent that corresponds to truncating the Lévy measure ν at u.

(9.2) PROPOSITION. $E^b(e^{-\lambda G^u}) = \dfrac{h(u)}{h(u) + g(u, \lambda)}.$

PROOF. We write $\tau_t = \tau_t^1 + \tau_t^2$ where τ^1 and τ^2 are independent subordinators with τ^1 having exponent $g(u, \lambda)$ and τ^2 having exponent $\int_{]u, \infty]}(1 - e^{-\lambda r})\nu(dr)$. Thus τ^1 contains all the jumps of τ which are less than or equal to u while τ^2 is a compound Poisson process containing all the jumps of τ which exceed u. Let R be the time of the first jump of τ^2. Then R is the time of the first jump of τ that exceeds u and hence

$$P^b[R \in dt] = h(u)e^{-th(u)}\,dt.$$

Clearly $R = l_{G^u}$ and so $G^u = \tau_{R-} = \tau_{R-}^1$. But R and τ^1 are independent and so

$$P^b[G^u \leqslant v] = P^b[\tau_{R-}^1 \leqslant v] = \int_0^\infty P^b[\tau_{t-}^1 \leqslant v]h(u)e^{-th(u)}\,dt$$
$$= h(u)\int_0^\infty P^b[\tau_t^1 \leqslant v]e^{-th(u)}\,dt,$$

since $t \to \tau_t^1$ has at most a countable number of discontinuities. Taking Laplace transforms we obtain

$$E^b[e^{-\lambda G^u}] = h(u)\int_0^\infty E^b[e^{-\lambda\tau_t^1}]e^{-th(u)}\,dt$$
$$= h(u)\int_0^\infty e^{-tg(u,\lambda)}e^{-th(u)}\,dt = \frac{h(u)}{h(u) + g(u, \lambda)},$$

proving (9.2).

(9.3) COROLLARY.

(i) $\int_0^\infty e^{-\lambda t} m^u(dt) = [h(u) + g(u, \lambda)]^{-1}$.

(ii) $E^b(e^{-\lambda D^u}) = \dfrac{1}{h(u) + g(u, \lambda)}\int_{]u, \infty]} e^{-\lambda r}\nu(dr).$

PROOF. The first assertion is an immediate consequence of (9.2) and (7.33). For the second note that $D^u = G^u + L^u$ and, as remarked below (7.34), G^u and L^u are independent. But from (7.34), $P^b[L^u \in dr] = [h(u)]^{-1}\nu(dr)$ for $r > u$, and so (9.3 ii) follows from (9.2).

10. Additional hypotheses and examples. Under additional hypotheses that are often satisfied in specific examples the preceding results take an especially pleasing form. In this section we shall present a sampling of such results. However, we shall not strive to obtain results under minimal hypotheses.

Throughout this section we assume that we are in the nice case; that is, (7.10) and (7.14) hold. Moreover, we assume that ξ does not charge points. In particular this implies that $P_t(x, \{b\}) = 0$ for each $x \in E$, $b \in E$, and $t > 0$. In addition, we impose the following regularity assumptions on the transition density $p(t, x, y)$ in (7.14):

(10.1) REGULARITY ASSUMPTIONS.

(i) $p(t, x, y)$ is continuous on $]0, \infty[\times E \times E$.

(ii) Given disjoint compact subsets K and L of E, $p(t, x, y) \to 0$ as $t \downarrow 0$ uniformly for $x \in K, y \in L$.

(iii) Given $\varepsilon > 0$, $M_\varepsilon = \sup\{p(t, x, y) : t \geqslant \varepsilon, x \in E, y \in E\} < \infty$.

If we put $p(0, x, y) = 0$ for $x \neq y$, then it is evident under (10.1) that $p(t, x, y)$ is *uniformly continuous* on $[0, s] \times K \times L$ for $s < \infty$, K and L disjoint compact subsets of E.

If $t > 0$, then

$$P^x(T_b = t) \leqslant P_t(x, \{b\}) = 0.$$

Therefore, using the strong Markov property, we have

$$P_tf(x) = Q_tf(x) + E^x\{P_{t-T_b}f(b);\ T_b < t\} \tag{10.2}$$

for $t > 0$ and $f \geqslant 0$. Define

$$r(t, x, y) = E^x\{p(t - T_b, b, y);\ T_b < t\}. \tag{10.3}$$

Since $(s, y) \to p(s, b, y)$ is uniformly continuous on $[0, t_0] \times K$ for $t_0 < \infty$ and K a compact subset of E which does not contain b, it is easy to see that the family $\{r(\cdot, x, \cdot) : x \in E\}$ is *equicontinuous* on $[0, \infty[\times E_b$ where $r(0, x, y) = 0$ and $E_b = E - \{b\}$. Now (10.2) implies that $r(t, x, y) \leqslant p(t, x, y)$ almost everywhere in y and hence this inequality holds everywhere on E_b. (Actually it is easy to see that this inequality holds everywhere on E if $t > 0$.) Define for $t > 0$

$$q(t, x, y) = p(t, x, y) - r(t, x, y). \tag{10.4}$$

Then $(t, y) \to q(t, x, y)$ is continuous on $]0, \infty[\times E_b$ and on $[0, \infty[\times(E - \{b, x\})$. Clearly $q(t, x, \cdot)$ is a density for the measure $Q_t(x, \cdot)$. Recall that if $x \neq b$, $Q_t(x, \cdot)$ does not charge $\{b\}$ and that if $x = b$, $Q_t(x, \cdot) = 0$. But $r(t, b, y) = p(t, b, y)$, and so $q(t, b, y) = 0$ for $t \geqslant 0$ and y in E_b.

Next observe that

$$\int r(t-\varepsilon, x, z)p(\varepsilon, z, y)\,dz = E^x\{p(t-T_b, b, y);\ T_b < t-\varepsilon\}\uparrow r(t, x, y)$$

as $\varepsilon\downarrow 0$, and hence

$$\int q(t-\varepsilon, x, z)p(\varepsilon, z, y)\,dz \downarrow q(t, x, y). \tag{10.5}$$

It follows from the semigroup property of (Q_t) that for t and $s > 0$

$$q(t+s, x, y) = \int q(t, x, z)q(s, z, y)\,dz \tag{10.6}$$

almost everywhere in y. Replace s by $s-\varepsilon$ in (10.6), multiply both sides by $p(\varepsilon, y, w)$, and integrate over y. Then let $\varepsilon\downarrow 0$ and use (10.5) and the monotone convergence theorem to see that (10.6) holds identically in y. Hence $q(t, x, y)$ for $t > 0$, $x \neq b$, and $y \neq b$ is a *good* transition density for $Q_t(x, \cdot)$.

If $t > 0$, define for $0 < s < t$

$$q^*(t, b, y) = \int Q_s^*(dx)q(t-s, x, y). \tag{10.7}$$

Because of (10.6) and the fact that (Q_t^*) is an entrance law for (Q_t), it is clear that $q^*(t, b, y)$ does not depend on the choice of s, $0 < s < t$. Moreover, it is evident that $q^*(t, b, \cdot)$ is a density for $Q_t^*(\cdot)$ with respect to $\xi(dy) = dy$. Finally using (10.1-i), (10.1-iii), and the equicontinuity of $\{r(\cdot, x, \cdot);\ x \in E\}$ it is easily seen that $(t, y) \to q^*(t, b, y)$ is continuous on $]0, \infty[\times E_b$. Also $\sup\{q^*(t, b, y) : t \geqslant \varepsilon, y \in E\} < \infty$ for each $\varepsilon > 0$ since $q(t, x, y) \leqslant p(t, x, y)$ and Q_s^* is a finite measure. We have included the explicit dependence of q^* on b in our notation since in the following paragraphs we shall consider q^* as a function defined on $]0, \infty[\times \tilde{E}$ where $\tilde{E} = \{(x, y) \in E \times E : x \neq y\}$. This is possible since each point $b \in E$ satisfies our hypotheses.

We shall now assume the existence of a *dual process* $\hat{X}$ relative to ξ satisfying the hypotheses of Section VI-1 of [1]. We shall use the standard notation of [1] without special mention. It follows easily from (10.1) that $\hat{X}$ has a transition density $\hat{p}(t, x, y)$ given by $\hat{p}(t, x, y) = p(t, y, x)$ and we assume $\int p(t, x, y)\,dx = 1$ for $t > 0$ and $y \in E$. Also (VI-1.25) of [1] implies that b is regular if and only if it is coregular; that is, regular for $\hat{X}$. Since (10.1) is symmetric in x and y, $\hat{X}$ satisfies all of the hypotheses imposed on X in the first part of this section. Let $\hat{l}^b$ be a local time for $\hat{X}$ at b normalized so that

$$\hat{E}^x \int_0^\infty e^{-\lambda t}\,d\hat{l}_t^b = \hat{u}^\lambda(x, b) = u^\lambda(b, x). \tag{10.8}$$

The subordinator exponent of $\hat{\tau}^b$ under $\hat{P}^b$ is then given by

$$\hat{g}(b, \lambda) = \hat{u}^\lambda(b, b)^{-1} = u^\lambda(b, b)^{-1} = g(b, \lambda). \tag{10.9}$$

Thus for the Lévy measure $\hat{\nu}^b$ we have $\hat{\nu}^b = \nu^b$, and hence $\hat{h}(b, t) = h(b, t)$. Note that we have now introduced the dependence of g, ν, and h on b; so that $g(b, \lambda)$ is what we previously wrote as $g(\lambda)$ and so on. Since $\hat{X}$ satisfies the same hypotheses as X we can construct the density $\hat{q}(t, x, y) = \hat{q}^b(t, x, y)$ for the process $\hat{X}$ killed when it first hits b—denoted by $(\hat{X}, \hat{T}_b)$—just as before. Using the well-known fact that the potential kernels $v^\lambda(x, y)$ and $\hat{v}^\lambda(x, y)$ for (X, T_b) and $(\hat{X}, \hat{T}_b)$ satisfy

$v^\lambda(x,y) = \hat{v}^\lambda(y,x)$ on $E_b \times E_b$, it follows that $\hat{q}(t,x,y) = q(t,y,x)$ on $E_b \times E_b$. Finally we let $\hat{q}^*(t,b,y)$ be the density for the entrance law $\hat{Q}_t^* = \hat{Q}_t^*(b,\cdot)$ defined by the formula dual to (10.7). Again we regard $\hat{q}^*$ as defined on $]0,\infty[\times\tilde{E}$ where $\tilde{E} = \{(x,y) \in E \times E : x \neq y\}$.

The next proposition gives explicit representations for q^* and $\hat{q}^*$, and also for the exit laws $\Gamma^b(x,\cdot)$ and $\hat{\Gamma}^b(x,\cdot)$.

(10.10) PROPOSITION. *If $t > 0$ and $x \neq y$, then*
(i) $q^*(t,x,y) = (d/dt)\int_0^t h(x,t-s)p(s,x,y)\,ds$
(ii) $\hat{q}^*(t,x,y) = (d/dt)\int_0^t h(x,t-s)p(s,y,x)\,ds$.
If $x \neq y$, then on $]0,\infty[$
(iii) $\Gamma^y(x,dt) = P^x[T_y \in dt] = \hat{q}^*(t,y,x)\,dt$
(iv) $\hat{\Gamma}^y(x,dt) = \hat{P}^x[\hat{T}_y \in dt] = q^*(t,y,x)\,dt$.

REMARK. If $u^\lambda(x,x)$ does not depend on x, which is the case, for example, if X has stationary independent increments, then ν^x and $h(x,\cdot)$ do not depend on x; and consequently (i) and (ii) of (10.10) imply that $q^*(t,x,y) = \hat{q}^*(t,y,x)$. Proposition (10.10) takes an especially nice form in this situation.

PROOF. Fix $b \in E$. Then (7.20) implies that for each $\lambda > 0$

$$\int_0^\infty e^{-\lambda t} q^*(t,b,y)\,dt = g(b,\lambda)u^\lambda(b,y) \tag{10.11}$$

almost everywhere in y. But the monotone convergence theorem implies that the integral in (10.11) is right continuous in λ while the right side is continuous. Therefore one can find a fixed set $E_0 \subset E$ with $\xi(E_0) = 0$ such that if $y \notin E_0$, then (10.11) holds for all $\lambda > 0$.

We shall next show that (10.11) holds for each $y \in E_b$ and each $\lambda > 0$. From (10.7)

$$q^*(t,b,y) = \int q^*(s,b,z)q(t-s,z,y)\,dz$$

for $0 < s < t$ and $y \in E$. Integrating in s over $]0,t[$, and then taking Laplace transforms, we obtain in light of (10.11)

$$\int_0^\infty e^{-\lambda t} t q^*(t,b,y)\,dt = g(b,\lambda)\int u^\lambda(b,z)v^\lambda(z,y)\,dz \tag{10.12}$$

identically in $\lambda > 0$ and y. But the right side, and hence the left side, of (10.12) is cofinely continuous in y on E_b since it is λ-excessive for the semigroup $(\hat{Q}_t^b)$. Fix $y \notin E_0$. Then (10.11) holds identically in $\lambda > 0$ for this y, and, using the monotone convergence theorem, it is easy to see that for $\lambda > 0$

$$\begin{aligned}\int_0^\infty e^{-\lambda t} t q^*(t,b,y)\,dt &= -\big(g(b,\lambda)u^\lambda(b,y)\big)' \\ &= -g(b,\lambda)'u^\lambda(b,y) + g(b,\lambda)\int_0^\infty e^{-\lambda t} t p(t,b,y)\,dt,\end{aligned} \tag{10.13}$$

where the prime denotes differentiation with respect to λ. But the right side of (10.13) is continuous in y while the left side is cofinely continuous on E_b. Since

(10.13) holds for all $y \notin E_0$ and $\xi(E_0) = 0$, it follows that (10.13) holds identically for $\lambda > 0$ and $y \in E_b$. Then integrate (10.13) over $]\lambda, \infty[$ to obtain (10.11) identically for $\lambda > 0$ and $y \in E_b$, because

$$g(b, \lambda)u^\lambda(b, y) = \hat{g}(b, \lambda)\hat{u}^\lambda(y, b) = \hat{E}^y(e^{-\lambda \hat{T}_b}) \to 0$$

as $\lambda \to \infty$ if $y \neq b$.

Next divide (10.11) by λ and take inverse transforms to obtain

$$(10.14) \qquad \int_0^t q^*(s, b, y)\, ds = \int_0^t h(b, t - s)p(s, b, y)\, ds$$

for $y \neq b$ and $t > 0$. (Under our assumptions, it is easily checked that both sides of (10.14) are continuous in t.) But the left side of (10.14) is differentiable in t because $s \to q^*(s, b, y)$ is continuous on $]0, \infty[$, and consequently differentiating (10.14) we obtain the first (and by duality, the second) assertion of (10.10).

In the present situation (7.4) may be written

$$E^x(e^{-\lambda T_b}) = g(b, \lambda)u^\lambda(x, b) = \hat{g}(b, \lambda)\hat{u}^\lambda(b, x),$$

and comparing this with the dual of (10.11) we obtain (10.10-iii). This completes the proof of (10.10).

REMARK. If $y \neq x$, then the measure $\Gamma^y(x, \cdot)$ does not charge $\{0\}$, but $\Gamma^y(x, \{\infty\}) = 1 - \int_0^\infty \hat{q}^*(t, y, x)\, dt$ may be positive.

Under the present assumptions the results of Section 7 take an especially pleasing and symmetric form. We leave it to the interested reader to write out such formulas using Proposition 10.10.

We close this section by discussing the stable processes of index $\alpha > 1$ on the real line. Let X be such a process. Then X is a real valued process with stationary independent increments whose distribution is determined by

$$(10.15) \qquad E^0(e^{i\eta X(t)}) = E^x\{e^{i\eta(X(t) - X(0))}\} = e^{-t\psi(\eta)}$$

$$(10.16) \qquad \psi(\eta) = \tfrac{1}{2}|\eta|^\alpha\left[1 + i\beta \operatorname{sgn} \eta \tan \frac{\pi\alpha}{2}\right],$$

where $1 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$. Because of the factor $\frac{1}{2}$ in (10.16), X is the standard one dimensional Brownian motion when $\alpha = 2$. Let $p(t, x) = p^{\alpha, \beta}(t, x)$ be the density of X_t with respect to Lebesgue measure under P^0. Then $p(t, x, y) = p(t, y - x)$ is the transition density of X_t. It is immediate from (10.15) and (10.16) that

$$(10.17) \qquad p(t, x) = t^{-1/\alpha}p(1, t^{-1/\alpha}x),$$

$$(10.18) \qquad p^{\alpha, \beta}(1, 0) \leq p^{\alpha, 0}(1, 0) = 2^{1/\alpha}\Gamma\left(\frac{1}{\alpha}\right)/\alpha\pi.$$

It is now evident that X satisfies all of the hypotheses of Proposition 10.10. From (10.17)

$$(10.19) \qquad u^\lambda(b, b) = u^\lambda(0, 0) = p(1, 0)\Gamma(1 - 1/\alpha)\lambda^{1/\alpha - 1},$$

and so by (7.8)

$$g(b, \lambda) = g(\lambda) = [p(1, 0)\Gamma(1 - 1/\alpha)]^{-1}\lambda^{1-1/\alpha}. \tag{10.20}$$

Consequently (7.21) gives

$$h(b, t) = h(t) = \frac{\sin \pi/\alpha}{\pi p(1, 0)} t^{1/\alpha - 1}. \tag{10.21}$$

By (10.10), $q^*(t, x, y)$ is a function of $y - x$ in the present case, say $q^*(t, \cdot)$, and $\hat{q}^*(t, x, y) = q^*(t, y, x) = q^*(t, x - y)$. The function $q^*(t, z)$ is given by

$$q^*(t, z) = \frac{\sin \pi/\alpha}{\pi p(1, 0)} \frac{d}{dt} \int_0^t (t - s)^{1/\alpha - 1} p(s, z)\, ds \tag{10.22}$$

for $t > 0$ and $z \neq 0$, or by its Laplace transform

$$\int e^{-\lambda t} q^*(t, z)\, dt = g(\lambda) u^{\lambda}(0, z). \tag{10.23}$$

In this case of Brownian motion ($\alpha = 2$) one has the explicit evaluation

$$q^*(t, z) = \frac{|z|}{(2\pi t^3)^{\frac{1}{2}}} e^{-(z^2/2t)}, \; z \neq 0, \; t > 0. \tag{10.24}$$

From (9.2) one obtains in the present case

$$E^0(e^{-\lambda G^u}) = \left[e^{-\lambda u} + (\lambda u)^{1-1/\alpha} \gamma\left(\frac{1}{\alpha}, \lambda u\right) \right]^{-1} \tag{10.25}$$

where $\gamma(\sigma, x) = \int_0^x t^{\sigma - 1} e^{-t}\, dt$ is the incomplete gamma function, $\sigma > 0$. It is of some interest to note that (10.25) states that the distribution of G^u depends only on α and is independent of β. On the other hand we obtain the familiar explicit formula

$$P^0(G_t \in ds) = \pi^{-1} \sin \frac{\pi}{\alpha} \; (t - s)^{1/\alpha - 1} s^{-1/\alpha}\, ds, \qquad 0 < s < t, \tag{10.26}$$

from (7.27) for the distribution of G_t. Again it depends only on α and is independent of β.

It follows readily from (10.25) and (10.26) that

$$E^0(G^u) = (\alpha - 1)u; \; E^0(G_t) = \left(\frac{\alpha - 1}{\alpha}\right)t.$$

The interested reader may find it amusing to write down the corresponding quantities for the asymmetric Cauchy processes.

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