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AN INEQUALITY IN p-FUNCTIONS

BY DAVID MOUNTFORD

Brighton College

An extension is derived of a p-function inequality due to Blackwell and Freedman.

A p-function (see Kingman (1972)) is a function satisfying an infinite family of inequalities of which the first two are

$$0 \leqslant p(t) \leqslant 1 \qquad t \geqslant 0 \tag{1}$$

and

$$p(s)p(t) \leqslant p(s+t) \leqslant p(s)p(t) + 1 - p(t) \qquad s, t \geqslant 0. \tag{2}$$

If

$$\lim_{t \to 0+} p(t) = 1$$

the p-function is said to be standard.

Blackwell and Freedman (1968) have shown that if p is a standard diagonal Markov transition function,

$$\{1 + p(1)\}/2 \geqslant \int_0^1 p(t)\, dt. \tag{3}$$

They also show that this result may be generalized to

$$\{1 + p(s)\}/2 \geqslant \int_0^1 p(t)\, dt \qquad 0 \leqslant s \leqslant 1 \tag{4}$$

provided

$$\int_0^1 p(t)\, dt \geqslant \tfrac{3}{4}. \tag{5}$$

It is well known that these results also hold if p is any standard p-function. The aim of this note is to prove that, for an arbitrary standard p-function, (4) is still true even when the condition (5) does not hold. We shall need the following result of Kingman (1972, page 100):

$$\int_0^u p(t)\, dt \geqslant \int_s^{u+s} p(t)\, dt \qquad u, s \geqslant 0. \tag{6}$$

From (1) and (2) we have

$$1 + p(s) \geqslant 1 + p(s)p(t) \geqslant p(t) + p(s+t)$$

and integrating with respect to t between 0 and $1 - s$ gives

$$(1-s)\{1 + p(s)\} \geqslant \int_0^{1-s} p(t)\, dt + \int_s^1 p(t)\, dt.$$

Applying (6) with $u = 1 - s$ now yields

$$(1-s)\{1 + p(s)\} \geqslant 2\int_s^1 p(t)\, dt. \tag{7}$$

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The time scale of a p-function is quite arbitrary and scaling (4) by a factor of s gives

$$s\{1 + p(s)\} \geqslant 2\textstyle\int_0^s p(t)\,dt. \tag{8}$$

Adding (7) and (8) and dividing through by 2 gives

$$\{1 + p(s)\}/2 \geqslant \textstyle\int_0^1 p(t)\,dt$$

as required.

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Brighton College
Eastern Road
Brighton
BN2 2AL
United Kingdom