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SOME LOCAL PROPERTIES OF GAUSSIAN VECTOR FIELDS[1]

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Formulas for the Hausdorff dimension of the graph, image, and level sets of Gaussian vector fields are given under general conditions which allow for different local behavior of the components and for dependence among them. Conditions for the field to have a local time, to hit any fixed point, and for the image to have positive Lebesgue measure are given, and relations between these properties are discussed. Applications of the results are given and include a discussion of when differentiable planar fields have critical points at fixed levels.

1. Introduction. In this article, we consider dimension and other local properties of Gaussian vector fields. There has been considerable recent interest in the Hausdorff dimension of the image, graph, and level sets of Gaussian fields, and we refer the reader to Adler (1977) for background and further references. Our results on dimension generalize those of Adler by allowing different local behavior of the components of the vector field and also by allowing dependence among components. The results depend on the determinant of the covariance matrix of two neighboring vectors. This determines the maximum value of the joint density function of the vectors, which is the natural quantity for studying many local properties of general (non-Gaussian) stochastic vector fields. When the dimension of the image set equals the dimension of the range space, we give conditions under which any fixed point in the range is hit with positive probability. The answer appears to be linked to two other local properties, which we discuss briefly. In particular, we give conditions under which the field will have a local time. In the last section, we consider some examples which make use of the above machinery, including a study of the level sets of isotropic mappings from $R^2 \to R$. Conditions are established for the presence or absence of critical points at fixed levels which improve a result of Belyaev (1970) in this setting. In the former case, the dimension of the critical points at a fixed level is considered, but the question is not fully answered.

2. Notation and preliminaries. Let $\mathbf{X}(\mathbf{t}) = (X_1(\mathbf{t}), \cdots, X_d(\mathbf{t}))$ denote a mean zero Gaussian vector field from $R^N \to R^d$, or briefly, an (N, d) field. Although it is not essential for many of the results, to simplify notation we shall always assume that $\mathbf{X}(\mathbf{t})$ has *homogeneous increments*. Denote the incremental variance of X_i by

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$\sigma_i^2(\mathbf{t}) = E(X_i(\mathbf{t}) - X_i(\mathbf{0}))^2$ and say that X_i has index α_i if

$$\alpha_i = \sup\{\alpha : \limsup_{|\mathbf{t}|\to\mathbf{0}}|\mathbf{t}|^{-\alpha}\sigma_i(\mathbf{t}) = 0\}$$
$$= \inf\{\alpha : \liminf_{|\mathbf{t}|\to\mathbf{0}}|\mathbf{t}|^{-\alpha}\sigma_i(\mathbf{t}) = \infty\}$$

where $|\cdot|$ denotes Euclidean distance. Of course we must have $0 \leqslant \alpha_i \leqslant 1$. We shall usually restrict the domain of X to the unit cube $I_N \equiv [0, 1]^N$ and for simplicity shall assume that all $\sigma_i(\mathbf{t})$ are bounded away from zero on $I_N^* \equiv [-1, 1]^N$ for $\mathbf{t}$ bounded away from the origin. It is only essential that this be true for some neighborhood of the origin.

We shall also need to put some restrictions on the type of dependence between coordinates. In particular we shall need the following two conditions: there exists $\varepsilon > 0$, such that for all $\mathbf{t} \in I_N^*$ we have

(1A) $$\frac{\det \operatorname{Cov}(\mathbf{X}(\mathbf{t}) - \mathbf{X}(\mathbf{0}))}{\prod_{i=1}^d \sigma_i^2(\mathbf{t})} \geqslant \varepsilon, \quad \text{or}$$

(1B) $$\frac{\det \operatorname{Cov}(\mathbf{X}(\mathbf{t}), \mathbf{X}(\mathbf{0}))}{\prod_{i=1}^d \sigma_i^2(\mathbf{t})} \geqslant \varepsilon$$

where det Cov(**Y**) denotes the determinant of the covariance matrix of the vector **Y**. Since det Cov(**X**(**t**), **X**(**0**)) = det Cov(**X**(**t**) − **X**(**0**), **X**(**0**)) $\leqslant$ det Cov(**X**(**t**) − **X**(**0**)) det Cov(**X**(**0**)), (1B) is a stronger requirement than (1A), and we shall give a natural example below where (1A) holds and (1B) fails. However, these conditions are often satisfied, for example when coordinates are independent. The examples of Section 5 illustrate a method for verifying them more generally.

In the sequel, we shall use dim to denote Hausdorff dimension, and Im **X** and Gr **X** to denote the image set and graph of $(\mathbf{X}(\mathbf{t}), \mathbf{t} \in I_N)$, respectively.

3. Dimension of the image and graph.

THEOREM 1. *Let* $\mathbf{X}(t)$ *be an* (N, d) *field with coordinates arranged so that their indices satisfy*

(2) $$0 < \alpha_1 \leqslant \alpha_2 \cdots \leqslant \alpha_d.$$

If (1A) *holds then*

(3) $$\dim(\operatorname{Im}\mathbf{X}) = \min\left(d, \frac{N + \sum_{i=1}^d(\alpha_d - \alpha_i)}{\alpha_d}\right)$$

(4) $$\dim(\operatorname{Gr}\mathbf{X}) = \min\left(\frac{N + \sum_{i=1}^d(\alpha_d - \alpha_i)}{\alpha_d}, N + \sum_{i=1}^d(1 - \alpha_i)\right)$$
$$= \dim(\operatorname{Im}\mathbf{X}) \quad \text{if} \quad \dim(\operatorname{Im}\mathbf{X}) < d$$
$$= N + \sum_{i=1}^d(1 - \alpha) \quad \text{if} \quad \dim(\operatorname{Im}\mathbf{X}) = d.$$

PROOF. Clearly $\dim(\operatorname{Im}\mathbf{X}) \leqslant d$. From Yadrenko (1971) X_i is Lipschitz of order $\beta_i < \alpha_i$ and from (2) we can assume $0 < \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_d$. Split I_N into 2^{Nn}

"cubes" with side 2^{-n}. By the Lipschitz condition, the image of each cube is contained in a rectangular solid with sides proportional to $2^{-n\beta_i}$. Each of these solids can be divided into $2^{n\sum_{i=1}^{d}(\beta_d-\beta_i)}$ cubes of side $2^{-n\beta_d}$. Thus the dimension of the image is less than the infimum of all λ such that

$$2^{Nn}\left[2^{n\sum_{i=1}^{d}(\beta_d-\beta_i)}\right]2^{-n\beta_d\lambda}\to 0.$$

This is true for all

$$\lambda > \frac{N+\sum_{i=1}^{d}(\beta_d-\beta_i)}{\beta_d}.$$

Letting $\beta_i \to \alpha_i$, we have that

$$\dim(\operatorname{Im}\mathbf{X}) \leqslant \frac{N+\sum_{i=1}^{d}(\alpha_d-\alpha_i)}{\alpha_d}.$$

To obtain lower bounds for the dimension, we use the standard capacity arguments as found in Taylor (1955), Kahane (1968), Orey (1970), and Adler (1977). To show that Im $\mathbf{X}$ supports a measure of finite λ-capacity a.s. it is enough to show that

$$\int_{I_N}\int_{I_N}E|\mathbf{X}(\mathbf{t})-\mathbf{X}(\mathbf{s})|^{-\lambda}\,d\mathbf{s}\,d\mathbf{t}<\infty. \tag{5}$$

We will show the integral (5) is finite for all $\lambda < \min\left(d, \left(N+\sum_{i=1}^{d}(\alpha_d-\alpha_i)\right)/\alpha_d\right)$. Since X has homogeneous increments, we need only establish that

$$\int_{I_N^*}E\left[(X_1(\mathbf{t})-X_1(\mathbf{0}))^2+\cdots+(X_d(\mathbf{t})-X_d(\mathbf{0}))^2\right]^{-\lambda/2}d\mathbf{t}<\infty. \tag{6}$$

Now make the change of variables $Y_i(\mathbf{t})=\sigma_i^{-1}(\mathbf{t})(X_i(\mathbf{t})-X_i(\mathbf{0}))$, $i=1,\cdots,d$. By (1A), we have that $\det \operatorname{Cov}(\mathbf{Y}(\mathbf{t}))\geqslant \varepsilon$, so that $\mathbf{Y}(\mathbf{t})$ has a bounded density and it is enough to establish that

$$\int_{I_N^*}\int_{R^d}\left[(y_1\sigma_1(\mathbf{t}))^2+\cdots+(y_d\sigma_d(\mathbf{t}))^2\right]^{-\lambda/2}d\mathbf{y}\,d\mathbf{t} \tag{7}$$

is finite. By assumption the $\sigma_i(\mathbf{t})$ are bounded away from zero for $\mathbf{t}$ away from the origin so that the integral on I_N^* need only be considered over $I_N^{\delta}=[-\delta,\delta]^N$ for any $\delta>0$. Also

$$\sigma_i(\mathbf{t})\geqslant K|\mathbf{t}|^{\beta_i}\quad \text{for}\quad |\mathbf{t}|<\delta\quad \text{for some}\quad \delta>0, K>0$$

with $\beta_i>\alpha_i$ for $\alpha_i<1$ and $\beta_i=1$ when $\alpha_i=1$, and $\beta_1\leqslant\beta_2\cdots\leqslant\beta_d\leqslant 1$. Thus (7) can be replaced by

$$\int_{I_N^{\delta}}|\mathbf{t}|^{-\lambda\beta_1}\int_{R^d}\left[y_1^2+\left(y_2|\mathbf{t}|^{\beta_2-\beta_1}\right)^2+\cdots+\left(y_d|\mathbf{t}|^{\beta_d-\beta_1}\right)^2\right]^{-\lambda/2}d\mathbf{y}\,d\mathbf{t}. \tag{8}$$

The integral dy_1 can now be estimated and is less than a constant times

$$\int_{I_N^{\delta}}|\mathbf{t}|^{-\lambda\beta_1}\int_{R^{d-1}}\left[\left(y_2|\mathbf{t}|^{\beta_2-\beta_1}\right)^2+\cdots+\left(y_d|t|^{\beta_d-\beta_1}\right)^2\right]^{(-\lambda+1)/2}dy_2\cdots dy_d\,d\mathbf{t}.$$

Now iterate this argument for $dy_2,\cdots,dy_{d-1}$ and assume $\lambda<d$ so that the integral dy_d is finite. Then (8) is less than a constant times

$$\int_{I_N^{\delta}}|\mathbf{t}|^{-\lambda\beta_1+(\beta_2-\beta_1)(-\lambda+1)+\cdots+(\beta_d-\beta_{d-1})(-\lambda+d-1)}\,d\mathbf{t}$$

and this is finite if $\lambda < (N + \Sigma_{i=1}^{d}(\beta_d - \beta_1))/\beta_d$. Letting $\beta_i \downarrow \alpha_i$ gives the result.

To prove (4), the same Lipschitz argument as for (3) gives $\dim(\mathrm{Gr}\,\mathbf{X}) \leqslant (N + \Sigma_{i=1}^{d}(\alpha_d - \alpha_i))/\alpha_d$. Using cubes of side 2^{-n} to divide up the graph gives $\dim(\mathrm{Gr}\,\mathbf{X}) \leqslant N + \Sigma_{i=1}^{d}(1 - \alpha_i)$. When $\dim(\mathrm{Im}\,\mathbf{X}) < d$, since $\dim(\mathrm{Gr}\,\mathbf{X}) \geqslant \dim(\mathrm{Im}\,\mathbf{X})$, we see that $\dim(\mathrm{Gr}\,\mathbf{X}) = \dim(\mathrm{Im}\,\mathbf{X}) = (N + \Sigma_{i=1}^{d}(\alpha_d - \alpha_i))/\alpha_d \leqslant N + \Sigma_{i=1}^{d}(1 - \alpha_i)$. When $\dim(\mathrm{Im}\,X) = d$, the dimension of the graph can be larger. A use of the capacity argument and similar manipulations as above on the integral

$$\int_{I_N}\int_{I_N} E\left[|\mathbf{X}(\mathbf{t}) - \mathbf{X}(\mathbf{s})|^2 + |\mathbf{t} - \mathbf{s}|^2\right]^{-\lambda/2} d\mathbf{s}\, d\mathbf{t}$$

will verify that $\dim(\mathrm{Gr}\,\mathbf{X}) \geqslant N + \Sigma_{i=1}^{d}(1 - \alpha_i)$. □

4. **Local properties when** $\dim(\mathrm{Im}\,\mathbf{X}) = d$. When the image set has dimension d, it is of interest to consider the following three properties:

(A) Im **X** has positive Lebesgue measure in R^d a.s.,
(B) $\mathbf{X}(\mathbf{t})$ hits any fixed point $\mathbf{u} \in R^d$ with positive probability,
(C) $\mathbf{X}(\mathbf{t})$ has a local time.

These properties are related. Property C implies A for any function. For sample continuous homogeneous Gaussian fields (i.e., $d = 1$), A and B are equivalent. A Fubini argument shows that a weaker form of A is equivalent to a weaker form of B:

LEMMA 1. *$P(\mathbf{X}(\mathbf{t})$ hits $\mathbf{u}$, $\mathbf{t} \in I_N) > 0$ for all $\mathbf{u}$ in some set of positive measure $\Leftrightarrow$ $P(m(\mathrm{Im}\,\mathbf{X}) > 0) > 0$. (Here m denotes Lebesgue measure in R^d.)*

COROLLARY. *$P(\mathbf{X}(\mathbf{t})$ hits $\mathbf{u}) = 0$ a.e. $\mathbf{u} \Leftrightarrow P(m(\mathrm{Im}\,\mathbf{X}) = 0) = 1 \Rightarrow \mathbf{X}$ does not have a local time.*

It seems likely that $P(m(\mathrm{Im}\,\mathbf{X}) = 0) = 0$ or 1 and that A, B and C are equivalent, although the last assertion is open even in the case of stationary Gaussian processes.

We now give some necessary and some sufficient conditions for **X** to hit points. Note that when $\Sigma_{i=1}^{d}\alpha_i > N$, $\dim(\mathrm{Im}\,\mathbf{X}) < d$ so that $m(\mathrm{Im}\,\mathbf{X}) = 0$ a.s. and by the corollary to Lemma 1, $\mathbf{X}(\mathbf{t})$ does not hit a.e. **u** a.s. A direct argument allows us to strengthen this to all **u**.

THEOREM 2. *Let $X(t)$ be a homogeneous (N, d) field with* $\det \mathrm{Cov}(\mathbf{X}(\mathbf{0})) > 0$. *If $\Sigma_{i=1}^{d}\alpha_i > N$, then $P(\mathbf{X}(\mathbf{t})$ hits $\mathbf{u}) = 0$ for all $\mathbf{u}$.*

PROOF.

$$P(\mathbf{X}(\mathbf{t}) = \mathbf{u} \quad \text{some} \quad \mathbf{t} \in I_N)$$

$$\leqslant 2^{Nn} P\left(\mathbf{X}(\mathbf{t}) = \mathbf{u} \quad \text{some} \quad \mathbf{t} \in [0, 2^{-n}]^N\right). \tag{9}$$

Each X_i is Lipschitz of order $\beta_i < \alpha_i$ and we can take $\Sigma_{i=1}^{d}\beta_i > N$. Thus for large n

(9) is less than

$$2^{Nn}P(|X_i(\mathbf{0}) - u_i| < 2^{-n\beta_i},\quad i = 1, \cdots, d)$$
$$\sim 2^{Nn}2^{-n\Sigma_{i=1}^{d}\beta_i}\phi(\mathbf{u}) \to 0 \qquad \text{as} \quad n \to \infty.$$

where $\phi(\mathbf{u})$ is the density function for $(\mathbf{X}(\mathbf{0}))$.

THEOREM 3. *Let* $\mathbf{X}(\mathbf{t})$ *be an* (N, d) *Gaussian vector field (in general nonhomogeneous). If*

$$\int_{I_N}\int_{I_N} \det \operatorname{Cov}(\mathbf{X}(\mathbf{t}) - \mathbf{X}(\mathbf{s}))^{-\frac{1}{2}}\, d\mathbf{s}\, d\mathbf{t} < \infty, \tag{10A}$$

then $\mathbf{X}(\mathbf{t})$ *has a square integrable local time. If* $\mathbf{X}(\mathbf{t})$ *is homogeneous and either* (1A) *and* (10A) *hold or*

$$\int_{I_N^*} \det \operatorname{Cov}(\mathbf{X}(\mathbf{t}), \mathbf{X}(\mathbf{0}))^{-\frac{1}{2}}\, d\mathbf{t} < \infty, \tag{10B}$$

then $\mathbf{X}$ *hits any fixed point* $\mathbf{u}$ *with positive probability for* $\mathbf{t} \in I_N$. *If* $\mathbf{X}$ *is ergodic, then with probability one there exists (a random)* $\mathbf{t} \in R^N$ *such that* $\mathbf{X}(\mathbf{t}) = \mathbf{u}$.

PROOF. The proof of the first assertion is a straightforward generalization of Berman's (1970) approach in the case of processes. It consists of establishing that the Fourier transform of the occupation time measure is square integrable and is not given. To prove the second part, we use an idea of Klein (1976) and exploit the kernel space representation of $L_2(\mathbf{X}(\mathbf{t}), \mathbf{t} \in I_N)$. The identification $\mathbf{X}(\mathbf{t}) \Leftrightarrow R(\cdot, \mathbf{t})$ yields a Hilbert space isomorphism of $L_2(\mathbf{X}(\mathbf{t}), \mathbf{t} \in I_N)$ with inner product generated by $E\langle \mathbf{X}(\mathbf{s}), \mathbf{X}(\mathbf{t})\rangle$ and the kernel space $H(R)$ spanned by $R(\cdot, \mathbf{t})$, $\mathbf{t} \in I_N$ with inner product generated by $\langle R(\cdot, \mathbf{t}), R(\cdot, \mathbf{s})\rangle_{H(R)} = \operatorname{tr} R(\mathbf{t}, \mathbf{s})$. Note that the members of $H(R)$ are $d \times d$ matrix functions. Since $\mathbf{X}(\mathbf{t})$ is homogeneous, it has a spectral representation $\mathbf{X}(\mathbf{t}) = \int_{R^N} e^{i\boldsymbol{\lambda}\mathbf{t}}\, d\xi(\boldsymbol{\lambda})$ and $\mathbf{X}_M(\mathbf{t}) = \int_{|\boldsymbol{\lambda}|<M} e^{i\boldsymbol{\lambda}\mathbf{t}}\, d\xi(\boldsymbol{\lambda})$ will approach $\mathbf{X}(\mathbf{t})$ uniformly in q.m. as $M \to \infty$. Our assumptions imply that $\det \operatorname{Cov}(\mathbf{X}(\mathbf{0})) > 0$ so that $\det \operatorname{Cov}(\mathbf{X}_M(\mathbf{0})) > 0$ for M large enough. Further, $\mathbf{X}_M(\mathbf{0})$ corresponds to a matrix $R_M(\cdot, \mathbf{0}) \in H(R)$ which has analytic components. Now decompose elements in $H(R)$ into their projection onto the subspace spanned by $R_M(\cdot, \mathbf{0})$ and its orthogonal complement. Upon returning to $L_2(\mathbf{X}(\mathbf{t}), \mathbf{t} \in I_N)$, we have

$$\mathbf{X}(\mathbf{t}) = \phi(\mathbf{t})\mathbf{X}_M(\mathbf{0}) + \mathbf{Y}(\mathbf{t})$$

where $\mathbf{Y}(\mathbf{t})$ is independent of $\mathbf{X}_M(\mathbf{0})$, and $\phi(\mathbf{t}) = \operatorname{tr}(R_M(\mathbf{t}, \mathbf{0}))$ is analytic, positive in a neighborhood N of the origin and $\nabla \phi(\mathbf{0}) = \mathbf{0}$. The same is true for $\psi(\mathbf{t}) = (\phi(\mathbf{t}))^{-1}$.

Now let

$$\mathbf{Z}(\mathbf{t}) = \psi(\mathbf{t})(\mathbf{X}(\mathbf{t}) - \mathbf{u}) = \mathbf{X}_M(\mathbf{0}) + \psi(\mathbf{t})(\mathbf{Y}(\mathbf{t}) - \mathbf{u}).$$

If we can show that

$$\int_N\int_N \det \operatorname{Cov}(\mathbf{Z}(\mathbf{t}) - \mathbf{Z}(\mathbf{s}))^{-\frac{1}{2}}\, d\mathbf{s}\, d\mathbf{t} < \infty \tag{11}$$

for some neighborhood N of $\mathbf{0}$, then $\mathbf{Z}(\mathbf{t})$, $\mathbf{t} \in N$, will have a local time, implying that $\text{Im}(\mathbf{Z}(\mathbf{t}))$ and thus $\text{Im}(\psi(\mathbf{t})(\mathbf{Y}(\mathbf{t}) - \mathbf{u}))$ will have positive Lebesgue measure a.s. Since $\mathbf{X}(\mathbf{t})$ hits $\mathbf{u}$ is equivalent to $\mathbf{Z}(\mathbf{t})$ hits $\mathbf{0}$ which in turn is equivalent to $\mathbf{X}_M(\mathbf{0}) \in \text{Im}(\psi(\mathbf{t})(\mathbf{u} - \mathbf{Y}(\mathbf{t})))$, the theorem will be true if (11) holds. When (1A) holds write

$$Y_i(\mathbf{t}, \mathbf{s}) = \frac{X_i(\mathbf{t}) - X_i(\mathbf{s})}{\sigma_i(\mathbf{t} - \mathbf{s})}, \qquad i = 1, \cdots, d$$

and note that the eigenvalues of $\text{Cov}(\mathbf{Y}(\mathbf{t}, \mathbf{s}))$ are all greater than or equal $\varepsilon > 0$. Now $\text{Cov}(\mathbf{Z}(\mathbf{t}) - \mathbf{Z}(\mathbf{s})) = \text{Cov}(\psi(\mathbf{t})(\mathbf{X}(\mathbf{t}) - \mathbf{X}(\mathbf{s})) + \mathbf{X}(\mathbf{s})(\psi(\mathbf{t}) - \psi(s)))$ and since ψ is analytic and $\nabla \psi(\mathbf{0}) = \mathbf{0}$, we can make

$$\frac{\text{tr Cov}(\mathbf{X}(\mathbf{s})(\psi(\mathbf{t}) - \psi(\mathbf{s})))}{\psi(\mathbf{t})\Pi_{i=1}^{d}\sigma_i^2(\mathbf{t} - \mathbf{s})} \leqslant \frac{\varepsilon}{2}$$

by restricting $\mathbf{s}$ and $\mathbf{t}$ to a sufficiently small neighborhood of the origin. It follows that $\det \text{Cov}(\mathbf{Z}(\mathbf{t}) - \mathbf{Z}(\mathbf{s})) \sim \det \text{Cov}(\mathbf{X}(\mathbf{t}) - \mathbf{X}(\mathbf{s}))$ and (11) holds by assumption (10A). To prove (11) under condition (10B), note that $\det \text{Cov}(\mathbf{Z}(\mathbf{t}) - \mathbf{Z}(\mathbf{s}))$ $\det \text{Cov}(\mathbf{Z}(\mathbf{s})) \geqslant \det \text{Cov}(\mathbf{Z}(\mathbf{t}), \mathbf{Z}(\mathbf{s})) = \psi^2(\mathbf{t})\psi^2(\mathbf{s}) \det \text{Cov}(\mathbf{X}(\mathbf{t}), \mathbf{X}(\mathbf{s}))$. □

When $\mathbf{X}(\mathbf{t})$ is differentiable in q.m. and satisfies (1A), it has a local time and hits points when $N > d$. When $N < d$, Im X does not have positive Lebesgue measure in R^d and cannot have a local time. The case $N = d$ requires separate attention and is handled by the following theorem.

THEOREM 4. *Let* $\mathbf{X}(\mathbf{t})$ *be a homogeneous* (N, N) *field whose first order partial derivatives exist in* q.m. *If the support of the determinant of the spectral measure* $F(\lambda)$ *of* $\mathbf{X}$ *is not contained in a subspace, then* $X(t)$ *has a local time and will hit any fixed point* $\mathbf{u}$ *with positive probability. The condition on* $F(\boldsymbol{\lambda})$ *holds, for example, if* F *has an absolutely continuous component which is positive definite on a set of positive measure.*

PROOF. Let $J\mathbf{X}(\mathbf{t})$ denote the Jacobian of $\mathbf{X}$ at $\mathbf{t}$. We have that $E|\det J\mathbf{X}(\mathbf{t})| < \infty$ so that $\int_{I_N}|\det J\mathbf{X}(\mathbf{t})|\, d\mathbf{t}$ exists a.s. and from Federer (1969, page 241) we have

$$\int_{I_N}|\det J\mathbf{X}(\mathbf{t})|\, d\mathbf{t} = \int_{R^N}\mathcal{H}^0(\mathbf{X}^{-1}(\mathbf{u}) \cap I_N)\, d\mathbf{u}$$

where $\mathcal{H}^0$ is counting measure (Hausdorff zero dimensional measure). If the set $C = \{\mathbf{t} : \det J\mathbf{X}(\mathbf{t}) = 0\}$ has Lebesgue measure zero a.s., then the argument of Geman and Horowitz (1973, Theorem 1) can be used to see that for a.e. $\mathbf{u}$, $\mathbf{X}^{-1}(\mathbf{u}) = \{\mathbf{t}_1, \cdots, \mathbf{t}_n\}$ is finite and

$$\text{(12)} \qquad \alpha(\mathbf{u}) = I_{\mathbf{X}(C)^c}(u)\int_{I_N \cap \mathbf{X}^{-1}(\mathbf{u})}|\det J\mathbf{X}(\mathbf{t})|^{-1}\, d\mathcal{H}^0(\mathbf{t})$$
$$= \Sigma_{i=1}^{n}|\det J\mathbf{X}(\mathbf{t}_n)|^{-1} \qquad \text{for} \quad \mathbf{u} \notin \mathbf{X}(C)$$

is well defined and is a local time. (See also Federer (1969, page 244)). It remains to establish $m(C) = 0$. If not, then

$$P(m\{\mathbf{t} : \det J\mathbf{X}(\mathbf{t}) = 0\} > 0) > 0$$

so that

$$\int\int I_{\{\det J\mathbf{X}(\mathbf{t},\,\omega)\,=\,0\}}\, d\mathbf{t}\, dP(\omega) > 0$$

and a Fubini argument gives

$$\int P(\det J\mathbf{X}(\mathbf{t}) = 0)\, d\mathbf{t} > 0$$

whence $P(\det J\mathbf{X}(\mathbf{t}) = 0)$ is a positive constant by homogeneity. For $\det J\mathbf{X}(\mathbf{t})$ to be zero, the gradient vectors $(\nabla X_1(\mathbf{t}), \cdots, \nabla X_N(\mathbf{t}))$ must all lie in a subspace of dimension $\leqslant N - 1$. Since $\mathbf{X}(\mathbf{t})$ is Gaussian, this has probability one or zero according to whether or not

$$\det \operatorname{Cov}(\nabla X_1(\mathbf{t}), \cdots, \nabla X_N(\mathbf{t})) = 0. \tag{13}$$

To show that (13) does not hold, we show the $N^2 \times N^2$ matrix $A = (a_{i_1 i_2 j_1 j_2})$ with

$$\begin{aligned} a_{i_1 i_2 j_1 j_2} &= E\left(\frac{\partial}{\partial t_{i_1}} X_{i_2}(t) \frac{\partial}{\partial t_{j_1}} X_{j_2}(t)\right) \\ &= \int \lambda_{i_1} \lambda_{j_1}\, dF_{i_2 j_2}(\boldsymbol{\lambda}) \end{aligned}$$

is positive definite where F_{ij} are the components of the spectral measure F. For any complex numbers $x_{i_1 i_2}$, $i_1 = 1, \cdots, N$, $i_2 = 1, \cdots, N$,

$$\begin{aligned} (14)\quad &\Sigma_{i_1, i_2} \Sigma_{j_1, j_2} x_{i_1 i_2} \overline{x_{j_1 j_2}}\, a_{i_1 i_2 j_1 j_2} \\ &\qquad = \Sigma_{i_2} \Sigma_{j_2} \int_{R^N} (\Sigma_{i_1} \lambda_{i_1} x_{i_1 i_2}) \left(\overline{\Sigma_{j_1} \lambda_{j_1} x_{j_1 j_2}}\right) dF_{i_2 j_2}(\boldsymbol{\lambda}). \end{aligned}$$

Since $\Sigma_{i_1} \lambda_{i_1} x_{i_1 i_2}$ is zero only on a subspace and F is nonnegative definite a.s. (Rozanov (1967)), our assumptions guarantee that (14) is positive. This completes the proof of the fact that $\mathbf{X}$ has local time.

To show that $\mathbf{X}(\mathbf{t})$ hits points we use the Hilbert space methods of Theorem 3 to write $\mathbf{X}(t) = \phi(\mathbf{t})\mathbf{X}(\mathbf{0}) + \mathbf{Y}(\mathbf{t})$ where $\mathbf{X}(\mathbf{0})$ and $\mathbf{Y}(\mathbf{t})$ are independent and $\phi(\mathbf{t}) = \operatorname{tr} R(\mathbf{t}, \mathbf{0})$ is continuously differentiable and $\nabla \phi(\mathbf{0}) = \mathbf{0}$. Thus

$$\left|\det J\left(\frac{\mathbf{X}(\mathbf{t}) - \mathbf{u}}{\phi(\mathbf{t})}\right)\right|_{\mathbf{t}=0} = \left|\frac{\det J\mathbf{X}(\mathbf{0})}{\phi(\mathbf{0})}\right| > 0 \quad \text{a.s.}$$

so that $\operatorname{Im}\left(\dfrac{\mathbf{X}(\mathbf{t}) - \mathbf{u}}{\phi(\mathbf{t})}\right)$ has positive Lebesgue measure and the proof is completed as with Theorem 3. □

To get results on the dimension of the level sets, we are forced to strengthen our dependence condition (1A).

THEOREM 5. *If* $\mathbf{X}(\mathbf{t})$ *is a homogeneous* (N, d) *field and* (1B) *holds, then for almost all* $\mathbf{u} \in R^d$

$$\begin{aligned} \dim(\mathbf{X}^{-1}(\mathbf{u})) &= \max(0, \dim(\operatorname{Gr} \mathbf{X}) - d) \\ &= 0 \qquad\qquad\qquad \text{if} \quad \dim \operatorname{Gr} \mathbf{X} \leqslant d \\ &= N - \Sigma_{i=1}^{d} \alpha_i \qquad \text{if} \quad \dim \operatorname{Gr} \mathbf{X} > d \end{aligned}$$

with positive probability.

PROOF. The proof follows Adler (1977) closely and is not given.

REMARK. In Theorems 4 and 5, if we replace the domain by R^N and assume $\mathbf{X}$ is ergodic, the results will hold with probability one.

5. Some examples.

(i) If $X(t)$ and $Y(t)$ are independent stationary Gaussian processes, each with index α, then the dimension of the range of the vector process is $\min(\alpha^{-1}, 2)$. When $\alpha < \frac{1}{2}$, the vector process is a space filling curve and will hit any fixed point with positive probability. Also $\dim X^{-1}(\mathbf{u}) = 1 - 2\alpha$ for a.e. $\mathbf{u} \in R^2$ with positive probability.

(ii) Let $X(t)$ be an ergodic stationary Gaussian process with index α and assume that it is locally nondeterministic (cf. Berman (1973) and Cuzick (1977)). Assume that t_i, $i = 1, \cdots, d$, are distinct and let $\mathbf{X}(t) = (X(t + t_1), \cdots, X(t + t_d))$. Then by applying Theorems 2 and 3 to $\mathbf{X}(t)$ we obtain that for any $\mathbf{u} = (u_1, \cdots, u_d)$

$$\begin{aligned} P(X(t + t_i) = u_i, i = 1, \cdots, d, \quad \text{some} \quad t \in R) &= 0 \quad \text{if} \quad \alpha > d^{-1} \\ &= 1 \quad \text{if} \quad \alpha < d^{-1}. \end{aligned}$$

In particular the probability of two zeros at a specified separation is unity if $\alpha < \frac{1}{2}$.

(iii) In this example we consider the level sets of homogeneous (2, 1) fields. Belyaev (1970) has shown that when $X(\mathbf{t})$ is a twice continuously differentiable homogeneous Gaussian field, then the level sets $\mathbf{X}^{-1}(u) \cap I_N$ are a.s. a finite union of nonintersecting rectifiable curves. When $\nabla X(\mathbf{t})$ does not exist, under a few additional assumptions Adler (1977) has proven that $\dim \mathbf{X}^{-1}(u) > 1$ so that the "contours" are very badly behaved. Adler's results also establish that when $\nabla X(\mathbf{t})$ exists in q.m. then the dimension of the level sets is one. Thus it is of interest to study the structure of the level sets when X possesses one derivative but not two.

Belyaev's method consists of showing that there are no critical points at fixed levels. The existence of a critical point at level u is equivalent to the vector field $(X(\mathbf{t}), \nabla X(\mathbf{t}))$ hitting the point $(u, \mathbf{0})$. We shall make use of Theorems 2 and 3 to determine when this event occurs. For simplicity we assume that

(15) $X(\mathbf{t})$ is an *isotropic* (2, 1) field, and with $\nabla X = (X_1, X_2)$ existing, and $\sigma_1^2(\mathbf{t}) = \mathrm{Var}(X_1(\mathbf{t}) - X_1(\mathbf{0}))$ is such that $\sigma_1^2(t_1, 0)$ is regularly varying at the origin with exponent 2α, $0 < \alpha < 1$.

Our basic estimate is provided by the following lemma.

LEMMA 2. *Under* (15), *we have that*

$$\det \mathrm{Cov}(X(\mathbf{t}) - X(\mathbf{0}), X_1(\mathbf{t}) - X_1(\mathbf{0}), X_2(\mathbf{t}) - X_2(\mathbf{0})) \sim |\mathbf{t}|^2\sigma^4(\mathbf{t}) \tag{16}$$

$$\det \mathrm{Cov}(X(\mathbf{0}), X(\mathbf{t}), X_1(\mathbf{0}), X_1(\mathbf{t}), X_2(\mathbf{0}), X_2(\mathbf{t})) \sim |\mathbf{t}|^2\sigma^6(\mathbf{t}) \tag{17}$$

where $f(\mathbf{t}) \sim g(\mathbf{t})$ *means that*

$$0 < \liminf_{\mathbf{t}\to\mathbf{0}} \frac{f(\mathbf{t})}{g(\mathbf{t})} \leqslant \limsup_{\mathbf{t}\to\mathbf{0}} \frac{f(\mathbf{t})}{g(\mathbf{t})} < \infty.$$

Before proving Lemma 2, we require two further lemmas. We shall use the spectral representation of isotropic correlation functions, which for planar fields yields the representation (Cramér and Leadbetter (1967, page 168))

$$\rho(\mathbf{t}) = EX(\mathbf{0})X(\mathbf{t}) = \int_0^\infty J_0(|\mathbf{t}|\lambda)\, dF(\lambda)$$

where J_0 is a Bessel function.

LEMMA 3. *Under* (15), *as* $\mathbf{t} \to \mathbf{0}$

$$\frac{dF\left(\frac{\lambda}{|\mathbf{t}|}\right)}{|\mathbf{t}|^2\sigma_1^2(\mathbf{t})} \to_\omega K_\alpha^{-1}(\phi)\lambda^{-(3+2\alpha)}\, d\lambda$$

where $\phi = \arctan(t_2/t_1)$ *and*

$$K_\alpha(\phi) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\cos^2(\theta + \phi)(1 - \cos(\lambda\cos\theta))\lambda^{-(1+2\alpha)}\, d\theta\, d\lambda.$$

For $0 < \alpha < 1$, $K_\alpha(\phi)$ *is bounded away from zero and infinity for all* ϕ.

PROOF. Let $\mathbf{t}^* = (t_1, 0)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. Since σ_1^2 is regularly varying, we have

$$\frac{\sigma_1^2(s\mathbf{t}^*)}{\sigma_1^2(\mathbf{t}^*)} \to s^{2\alpha} \qquad \mathbf{t}^* \to \mathbf{0}.$$

Also

$$\frac{\sigma_1^2(s\mathbf{t}^*)}{\sigma_1^2(\mathbf{t}^*)} = \int\int_{-\infty}^\infty \lambda_1^2|e^{i\lambda_1 st_1} - 1|^2\frac{dF^*(\lambda_1, \lambda_2)}{\sigma_1^2(\mathbf{t}^*)}$$

where F^* is the measure F in rectangular coordinates. If we make the polar change of variables $\lambda_1 = |\boldsymbol{\lambda}|\cos\theta$, and the substitution $|\boldsymbol{\lambda}| \to \lambda/t_1$, this equals

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty\lambda^2\cos^2\theta(1 - \cos(s\lambda\cos\theta))\frac{dF\left(\frac{\lambda}{t_1}\right)}{t_1^2\sigma_1^2(\mathbf{t}^*)}\, d\theta.$$

However,

$$K_\alpha(0)s^{2\alpha} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty\lambda^2\cos^2\theta(1 - \cos(s\lambda\cos\theta))\lambda^{-(3+2\alpha)}\, d\lambda\, d\theta$$

so that by the continuity theorem

$$\frac{dF\left(\frac{\lambda}{t_1}\right)}{t_1^2\sigma_1^2(t^*)} \to_\omega K_\alpha^{-1}(0)\lambda^{-(3+2\alpha)}\, d\lambda.$$

To establish the lemma for general $\mathbf{t}$ write

$$\frac{dF\left(\frac{\lambda}{|\mathbf{t}|}\right)}{|\mathbf{t}|^2\sigma_1^2(\mathbf{t})} = \frac{dF\left(\frac{\lambda}{|\mathbf{t}|}\right)}{|\mathbf{t}|^2\sigma_1^2(|\mathbf{t}|, 0)}\frac{\sigma_1^2(|\mathbf{t}|, 0)}{\sigma_1^2(\mathbf{t})}$$

and check that

$$\frac{\sigma_1^2(\mathbf{t})}{\sigma_1^2(|\mathbf{t}|, 0)} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\lambda^2 \cos^2\theta(1 - \cos(\lambda\cos(\theta - \phi)))\frac{dF\left(\frac{\lambda}{|\mathbf{t}|}\right)}{|\mathbf{t}|^2\sigma_1^2(|\mathbf{t}|, 0)}\, d\theta$$

$$\to K_\alpha(\phi)K_\alpha^{-1}(0) \qquad \text{as} \quad \mathbf{t} \to \mathbf{0}. \qquad \square$$

LEMMA 4. *Assume f_n are complex valued analytic functions of N real arguments and that $f_n \to f$ uniformly on compact sets. Also assume G_n are measures on R^N and that there exists a nontrivial absolutely continuous measure G such that* lim $\inf_{n\to\infty} G_n(B) \geqslant G(B)$ *for some open set $B \in R^N$. If*

$$\int |f_n|^2\, dG_n \to 0 \quad \text{as} \quad n \to \infty$$

then

$$f \equiv 0.$$

PROOF. This is a multivariate version of Cuzick (1978, Lemma 1).

PROOF OF LEMMA 2. The upper bound for (16) is obvious since $\mathrm{Var}(X_1(\mathbf{t}) - X_1(\mathbf{0})) \sim \mathrm{Var}(X_2(\mathbf{t}) - X_2(\mathbf{0}))$ and the determinant is less than the product of the diagonal entries. The lower bound uses the method of Berman (1973, Section 6) and will follow if

$$\text{(18)} \quad \liminf_{\mathbf{t}\to\mathbf{0}} \mathrm{Var}\left(a\frac{X(\mathbf{t}) - X(\mathbf{0})}{|\mathbf{t}|} + b\frac{X_1(\mathbf{t}) - X_1(\mathbf{0})}{\sigma_1(\mathbf{t})} + c\frac{X_2(\mathbf{t}) - X_2(\mathbf{0})}{\sigma_1(\mathbf{t})}\right) = 0$$

implies that $a = b = c = 0$. The variance in (18) can be written in terms of the spectral distribution function as

$$\int_0^{\infty}\left|a\frac{(e^{i\lambda\mathbf{t}} - 1)}{|\mathbf{t}|} + b\lambda_1\frac{(e^{i\lambda\mathbf{t}} - 1)}{\sigma_1(\mathbf{t})} + c\lambda_2\frac{(e^{i\lambda\mathbf{t}} - 1)}{\sigma_1(\mathbf{t})}\right|^2 dF(|\lambda|)$$

which, after the substitution $\lambda \to \lambda/|\mathbf{t}|$, equals

$$\int_0^{\infty}|a(e^{i\lambda\tau} - 1)\sigma_1(t) + b\lambda_1(e^{i\lambda\tau} - 1) + c\lambda_2(e^{i\lambda\tau} - 1)|^2\frac{dF\left(\frac{|\lambda|}{|\mathbf{t}|}\right)}{|\mathbf{t}|^2\sigma_1^2(\mathbf{t})}$$

where $\tau = \mathbf{t}/|\mathbf{t}|$. By taking a subsequence, we can assume $\tau \to \tau^*$ and an application of Lemmas 3 and 4 gives $b\lambda_1(e^{i\lambda\tau^*} - 1) + c\lambda_2(e^{i\lambda\tau^*} - 1) \equiv 0$ so that $b = c = 0$ and then $a = 0$. To prove (17) write it as

$$\text{(19)} \quad |\mathbf{t}|^2\sigma_1^6(\mathbf{t})\det \mathrm{Cov}\left[X(\mathbf{0}), X_1(\mathbf{0}), \frac{X[\mathbf{t}, \mathbf{0}] - X_1(\mathbf{0})}{\sigma_1(\mathbf{t})},\right.$$

$$\left.\frac{X_1(\mathbf{t}) - X[\mathbf{t}, \mathbf{0}]}{\sigma_1(\mathbf{t})}, X_2(\mathbf{0}), \frac{X_2(\mathbf{t}) - X_2(\mathbf{0})}{\sigma_1(\mathbf{t})}\right]$$

where $X[\mathbf{t}, \mathbf{s}] = (X(\mathbf{t}) - X(\mathbf{s}))/|\mathbf{t} - \mathbf{s}|$. Again all terms in the covariance matrix of (19) are bounded which gives the upper bound. The lower bound is verified in the same manner as above. (See Cuzick (1978) for a similar derivation.) □

We can now return to the level sets of $X(\mathbf{t})$. When $\alpha > \frac{1}{2}$, the image of $(X(\mathbf{t}), X_1(\mathbf{t}), X_2(\mathbf{t}))$ has dimension $4 - 2\alpha < 3$ so that this vector does not hit points and the level sets $\mathbf{X}^{-1}(u)$ consist of a finite number of nonintersecting rectifiable arcs. When $\alpha < \frac{1}{2}$ the point $(u, 0, 0)$ is hit with positive probability and thus critical points do occur at the level u and $\mathbf{X}^{-1}(u)$ is a topologically more complicated object. Since (1B) is not satisfied, we cannot apply Theorem 5 to give the dimension of critical points at a.e. level. However the same arguments can be used to give the bounds

$$\max(0, 1 - 3\alpha) \leqslant \dim(\mathbf{X}^{-1}(u) \cap \nabla \mathbf{X}^{-1}(\mathbf{0})) \leqslant \max(0, 1 - 2\alpha)$$

for a.e. u. The upper bound would appear to be the correct dimension, but Theorem 5 is limited because it requires (1B) instead of (1A).

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