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A UNIFORM LAW OF THE ITERATED LOGARITHM FOR CLASSES OF FUNCTIONS

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Let $\{\xi_k, k \geqslant 1\}$ be a sequence of random variables uniformly distributed over [0, 1] and let $\mathfrak{F}$ be a class of functions on [0, 1] with $\int_0^1 f(x)\,dx = 0$. In this paper we give upper and lower bounds for $\sup_{f\in\mathfrak{F}}|\Sigma_{k\leqslant N} f(\xi_k)|$ for the class of functions of variation bounded by 1 and for the class of functions satisfying a Lipschitz condition.

1. Introduction. Let $\{\xi_k, k \geqslant 1\}$ be a sequence of random variables, uniformly distributed over [0, 1], and let $\mathfrak{F}$ be a set of integrable functions on [0, 1] with $\int_0^1 f(x)\,dx = 0$. We are interested in those classes $\mathfrak{F}$ for which we have almost surely

$$\sup_{f\in\mathfrak{F}}|\Sigma_{k\leqslant N} f(\xi_k)| \ll (N \log\log N)^{\frac{1}{2}}. \tag{1.1}$$

If for instance the random variables ξ_k are independent and $\mathfrak{F}$ consists of all the indicators of intervals $[s, t]$ centered of expectation, i.e., $\mathfrak{F} = \{1_{[s,t]}(x) - (t - s);\ 0 \leqslant s < t \leqslant 1\}$, then (1.1) follows from the Chung-Smirnov law of the iterated logarithm for empirical distribution functions. In view of a number-theoretic inequality of Koksma (see (2.7) below) the Chung-Smirnov theorem implies that (1.1) holds for the class $\mathfrak{F}$ of functions whose variation $V(f)$ on [0, 1] does not exceed 1. The results stated so far remain valid for stationary sequences $\{\xi_k, k \geqslant 1\}$ of random variables satisfying a strong mixing condition and having uniform distribution over [0, 1] and for lacunary sequences $\xi_k = \langle n_k x\rangle$. Here x is uniformly distributed over [0, 1], $\langle \varepsilon\rangle$ denotes the fractional part of ε and $\{n_k, k \geqslant 1\}$ is a lacunary sequence of real numbers, i.e., a sequence satisfying

$$n_{k+1}/n_k \geqslant q > 1, \qquad k \geqslant 1. \tag{1.2}$$

The purpose of this paper is to prove (1.1) for the class $\mathfrak{F} = \Lambda_\alpha$ ($\alpha > \frac{1}{2}$) of functions f on [0, 1] with $f(0) = f(1)$, $\int_0^1 f(x)\,dx = 0$ satisfying a Lipschitz condition with exponent α and Lipschitz constant not exceeding 1 and for independent and mixing random variables, as well as for lacunary sequences. Moreover, we shall disprove (1.1) for Λ_α ($\alpha < \frac{1}{2}$) and uniformly distributed independent as well as for lacunary sequences. It is this very last result which is the most difficult one to prove.

We shall divide the material into four sections. In Section 2 we shall collect the results for the class of functions of bounded variation. In Sections 3 and 4 we shall

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treat the classes Λ_α for $\alpha > \frac{1}{2}$ and $\alpha < \frac{1}{2}$ respectively. The case $\alpha = \frac{1}{2}$ remains open. In Section 5 we shall give another proof of (1.1) for Λ_α ($\alpha > \frac{1}{2}$) and independent uniformly distributed random variables by translating the problem into a setting involving Hilbert space valued random variables.

2. Functions of bounded variation. The theorems of this section follow easily from known results. We state and prove them only for the sake of completeness.

Let $\{\xi_k, k \geqslant 1\}$ be a strictly stationary sequence of random variables satisfying a strong mixing condition

$$|P(AB) - P(A)P(B)| \leqslant \rho(n)\downarrow 0 \tag{2.1}$$

for all $A \in \mathcal{F}_1{}^t$ and $B \in \mathcal{F}_{t+n}^{\infty}$ and for all integers $n, t \geqslant 1$. Here $\mathcal{F}_a{}^b$ denotes the σ-field generated by ξ_k ($a \leqslant k \leqslant b$). Let g be a measurable mapping from the space of infinite sequences $(\alpha_1, \alpha_2, \cdots)$ of real numbers into the real line. Define

$$\eta_n = g(\xi_n, \xi_{n+1}, \cdots) \qquad n \geqslant 1 \tag{2.2}$$

and

$$\eta_{mn} = E(\eta_n | \mathcal{F}_n^{n+m}) \qquad m, n \geqslant 1. \tag{2.3}$$

As is usual we assume that η_n can be closely approximated by η_{mn} in the form

$$E|\eta_n - \eta_{mn}| \leqslant \psi(m)\downarrow 0 \tag{2.4}$$

for all $m, n \geqslant 1$.

Let Λ be the class of real valued functions f on $[0, 1]$ whose variation $V(f)$ on $[0, 1]$ does not exceed 1 and for which $\int_0^1 f(x)\, dx = 0$.

THEOREM 2.1. *Let $\{\xi_k, k \geqslant 1\}$ be a strictly stationary sequence of random variables satisfying a strong mixing condition* (2.1) *with*

$$\rho(n) \ll n^{-8}. \tag{2.5}$$

Suppose that the random variables η_n defined by (2.2) *are uniformly distributed over* $[0, 1]$ *and that they satisfy* (2.4) *with*

$$\psi(m) \ll m^{-12}. \tag{2.6}$$

Then

$$\limsup_{N\to\infty} \sup_{f\in\Lambda} \frac{|\sum_{k\leqslant N} f(\eta_k)|}{(N \log\log N)^{\frac{1}{2}}} \leqslant C \quad \text{a.s.}$$

where the constant C only depends on the constants implied by $\ll$ in (2.5) *and* (2.6).

THEOREM 2.2. *Let $\{n_k, k \geqslant 1\}$ be a sequence of real numbers satisfying* (1.2). *Extend the functions of Λ with period* 1. *Then*

$$\limsup_{N\to\infty} \sup_{f\in\Lambda} \frac{|\sum_{k\leqslant N} f(n_k x)|}{(N \log\log N)^{\frac{1}{2}}} \leqslant C$$

for almost all $x \in [0, 1]$. Here constant C depends on q of (1.2) *only.*

The proofs of these two theorems depend on an inequality of Koksma (see Kuipers and Niederreiter (1974), page 143). Let $x_1, \cdots, x_n$ be points in [0, 1]. Denote by $A(N, t)$ the number of indices $k \leqslant N$ with $0 \leqslant x_k \leqslant t$. Here $0 \leqslant t \leqslant 1$. Define the discrepancy

$$D_N = \sup_{0 \leqslant t \leqslant 1}|N^{-1}A(N, t) - t|.$$

(Note that $N^{-1}A(N, t)$ can be considered as the empirical distribution function of the pointmasses in $x_1, \cdots, x_N$.) Let f be a function of variation $V(f)$ on [0, 1]. Then

$$|\Sigma_{k \leqslant N} f(x_k) - N\textstyle\int_0^1 f(x)\,dx| \leqslant V(f) N D_N. \tag{2.7}$$

Now let $F_N(t)$ be the empirical distribution function at stage N of the random variables $\{\xi_k, k \geqslant 1\}$ or of $\{\langle n_k x\rangle, k \geqslant 1\}$. Put

$$\Delta_N = \sup_{0 \leqslant t \leqslant 1}|F_N(t) - t|.$$

Then by Theorems 3.1 and 4.1 of Philipp (1977)

$$\limsup_{N \to \infty} \frac{N\Delta_N}{(N \log \log N)^{\frac{1}{2}}} \leqslant C \tag{2.8}$$

for some constants C with the properties spelled out in Theorem 2.1 and 2.2 above. (If the random variables are independent then (2.8), with $C = 2^{\frac{1}{2}}$, is the Chung-Smirnov theorem.)

We now apply (2.7) with $x_k = \xi_k$ or $x_k = \langle n_k x\rangle$. Then $D_N = \Delta_N$ and thus for each $f \in \Lambda$

$$|\Sigma_{k \leqslant N} f(\xi_k)| \leqslant N\Delta_N.$$

Hence

$$\sup_{f \in \Lambda}|\Sigma_{k \leqslant N} f(\xi_k)| \leqslant N\Delta_N.$$

We divide by $(N \log \log N)^{\frac{1}{2}}$, take the limes superior and apply (2.8). This proves Theorem 2.1. The proof of Theorem 2.2 is the same.

Theorem 2.1 is best possible in the sense that one cannot replace the constant C by $o(1)$, at least not for independent sequences of random variables. A similar remark holds for Theorem 2.2. One can show that $C \geqslant \frac{1}{8}$ in Theorem 2.2. (For the details see Philipp (1975).)

3. The class Λ_α ($\alpha > \frac{1}{2}$). Let Λ_α be the class of real-valued functions f on [0, 1] with $f(0) = f(1)$, $\int_0^1 f(x)\,dx = 0$ and satisfying a Lipschitz condition

$$|f(x) - f(y)| \leqslant |x - y|^\alpha \qquad 0 \leqslant x < y \leqslant 1.$$

THEOREM 3.1. *Theorem* 2.1 *remains valid if we replace the class* Λ *by* Λ_α ($\alpha > \frac{1}{2}$) *and* (2.5) *and* (2.6) *by*

$$\rho(n) \ll n^{-16}$$

and

$$\psi(m) \ll m^{-24}$$

respectively.

NOTE. For independent random variables we shall prove (1.1) for a larger class of functions which comes close to $\Lambda_{\frac{1}{2}}$. This will be done in Section 5.

THEOREM 3.2. *Theorem 2.2 remains valid if we replace the class Λ by Λ_α ($\alpha > \frac{1}{2}$).*

REMARK. It is interesting to compare Theorem 3.2 and Theorem 4.2 below with a result of Takahashi (1962) who proved the standard law of the iterated logarithm for all $f \in \Lambda_\alpha$ ($\alpha > 0$) and gap-sequences of integers n_k.

3.1. *A proposition.* The proofs of Theorems 3.1 and 3.2 are based on the following proposition.

PROPOSITION 3.1.1. *Let $\{x_k, k \geq 1\}$ be a sequence of random variables. Suppose that there exist positive constants A, C and $\beta < 1$ such that*

$$P\left\{|\Sigma_{k=H+1}^{H+N} \exp(2\pi i h x_k) \geq AR(N \log\log N)^{\frac{1}{2}}\right\}$$
$$\leq C\left(\exp(-R \log\log N) + R^{-2}N^{-2-\beta}\right) \tag{3.1.1}$$

for all $R \geq 1$ and all integers $H > 0$, $N \geq 1$ and $h \neq 0$ with $|h| \leq N^2$. Then

$$\limsup_{N\to\infty} \sup_{f\in\Lambda_\alpha} \frac{|\Sigma_{k\leq N} f(x_k)|}{(N \log\log N)^{\frac{1}{2}}} \leq A_1 \quad \text{a.s.}$$

where the constant A_1 depends on A, α and β only.

For the proof of the proposition we need two lemmas. Write

$$F(H, N, h) = |\Sigma_{k=H+1}^{H+N} \exp(2\pi i h x_k)|.$$

Define n by

$$2^n \leq N < 2^{n+1}. \tag{3.1.2}$$

LEMMA 3.1.1. *For $N \geq 1$*

$$F(0, N, h) \leq F(0, 2^n, h) + \Sigma_{\frac{1}{2}n \leq l \leq n} F\left(2^n + m_l 2^l, 2^{l-1}, h\right) + N^{\frac{1}{2}} \tag{3.1.3}$$

where m_l are integers with $0 \leq m_l < 2^{n-l}$.

A proof of this lemma can be found in Gaal and Gaal (1964). It is simply based on the dyadic expansion of N. We also put

$$\phi(N) = (N \log\log N)^{\frac{1}{2}} \tag{3.1.4}$$

and define the events (here and throughout $\log^+ x = \log(\max(e, x))$)

$$G(n, h) = \{F(0, 2^n, h) \geq 2A \log^+ |h| \phi(2^n)\}, \tag{3.1.5}$$

$$H(n, m, l, h) = \left\{F(2^n + m2^l, 2^{l-1}, h) \geq 2A \log^+ |h| \phi(2^n) 2^{\frac{1}{2}(l-n)\beta}\right\} \tag{3.1.6}$$

$$G_n = \bigcup_{1 \leq |h| \leq 2^n} G(n, h) \tag{3.1.7}$$

$$H_n = \bigcup_{1 \leq |h| \leq 2^n} \bigcup_{\frac{1}{2}n \leq l \leq n} \bigcup_{m \leq 2^{n-l}} H(n, m, l, h). \tag{3.1.8}$$

LEMMA 3.1.2. *Assume the hypothesis of Proposition* 3.1.1. *Then with probability* 1 *only finitely many* G_n *or* H_n *occur.*

PROOF. We apply (3.1.1) with $R = 2\log^+|h|$, $N = 2^n$ and $H = 0$ and obtain

$$P\{G(n,h)\} \ll \exp(-2\log^+|h|\log n) + 2^{-n(\beta+2)}(\log^+|h|)^{-2}.$$

Thus

$$P(G_n) \ll \Sigma_{|h|\geqslant 1}\exp(-2\log^+|h|\log n) + 2^{-n(\beta+2)}\Sigma_{1\leqslant|h|\leqslant 2^n}(\log^+|h|)^{-2} \ll n^{-2}. \tag{3.1.9}$$

Similarly with $H = 2^n + m2^l$, $N = 2^{l-1}$ and $R = 2\log^+|h|2^{\frac{1}{2}(1-\beta)(n-l)}$

$$P\{H(n,m,l,h)\} \ll \exp\left(-2\log^+|h|\cdot 2^{\frac{1}{2}(1-\beta)(n-l)}\log n\right) + 2^{-l(2+\beta)}(\log^+|h|)^{-2}\cdot 2^{(l-n)(1-\beta)}.$$

Thus

$$P(H_n) \ll \Sigma_{|h|\geqslant 1}\Sigma_{\frac{1}{2}n<l\leqslant n}2^{n-l}\exp\left(-2\log^+|h|2^{\frac{1}{2}(1-\beta)(n-l)}\log n\right) + \Sigma_{1\leqslant|h|\leqslant 2^n}\Sigma_{\frac{1}{2}n<l\leqslant n}2^{n-l}\cdot 2^{-2l(\beta+1)}\cdot 2^{(l-n)(1-\beta)} \ll n^{-2} + 2^{n(1+\beta)}\Sigma_{\frac{1}{2}n<l\leqslant n}2^{-l(2+3\beta)} \ll n^{-2}. \tag{3.1.10}$$

The lemma follows now from (3.1.9), (3.1.10) and the Borel-Cantelli lemma.

We now recall two facts valid for all functions in Λ_α ($\alpha > \frac{1}{2}$). First the coefficients a_n of the Fourier series of f

$$f(x) = \Sigma_{|h|\geqslant 1}a_h\exp(2\pi ihx) \tag{3.1.11}$$

satisfy

$$\Sigma_{|h|\geqslant 1}|a_h|^2|h|(\log^+|h|)^4 \ll 1 \tag{3.1.12}$$

where the constant implied by $\ll$ depends on α only. This result was known in essence to Bernstein. For a proof of (3.1.12) see Zygmund (1935), page 136, formula (3).

For the remainder of the Fourier series we obtain

$$|\Sigma_{|h|\geqslant N}a_h\exp(2\pi ihx)| \ll N^{-\frac{1}{2}} \tag{3.1.13}$$

uniformly in x. Indeed, by a theorem of Lebesgue there is a constant C_1 such that

$$\|\Sigma_{h\geqslant N}a_h\exp(2\pi ihx)\| \leqslant C_1\min_{T_N}\|f - T_N\|\log N$$

where $\|\cdot\|$ denotes the supremum norm and the minimum is extended over all trigonometric polynomials T_N of degree N (see Lorentz (1966), page 54, Theorem 1). But according to a theorem of Jackson there is a constant C_2 such that

$$\min_{T_N}\|f - T_N\| \leqslant C_2\omega\left(f, \frac{1}{n}\right) \leqslant C_3N^{-\alpha}.$$

Here $\omega(f, h)$ is the modulus of continuity of f (see Lorentz (1966), page 56, Theorem 2). (3.1.13) follows now since $\alpha > \frac{1}{2}$.

We finally can prove Proposition 3.1.1. By Lemmas 3.1.1 and 3.1.2, (3.1.2), (3.1.4) and (3.1.5)–(3.1.8),

$$\begin{aligned} F(0, N, h) &\leqslant 2A \log^+|h|\phi(2^n) \\ &\quad + 2A \log^+|h|\phi(2^n)\Sigma_{\frac{1}{2}n\leqslant l<n}2^{\frac{1}{2}(l-n)\beta} + N^{\frac{1}{2}} \\ &\leqslant 2A \log^+|h|\phi(2^n)\left(1 + \left(1 - 2^{-\frac{1}{2}\beta}\right)^{-1}\right) + N^{\frac{1}{2}} \\ &\ll \log^+|h|\phi(N) \quad \text{a.s.} \end{aligned}$$

for all $1 \leqslant |h| \leqslant \frac{1}{2}N$. Hence we have for each $f \in \Lambda_\alpha$ $(\alpha > \frac{1}{2})$

$$\begin{aligned} |\Sigma_{k\leqslant N}f(x_k)| &\leqslant |\Sigma_{1\leqslant|h|\leqslant\frac{1}{2}N}a_h\Sigma_{k\leqslant N}\exp(2\pi ihx_k)| \\ &\quad + \Sigma_{k\leqslant N}|\Sigma_{|h|>\frac{1}{2}N}a_h \exp(2\pi ihx_k)| \\ &\leqslant \left(\Sigma_{1\leqslant|h|\leqslant\frac{1}{2}N}|a_h|^2|h|(\log^+|h|)^4\right)^{\frac{1}{2}} \\ &\quad \times\left(\Sigma_{1\leqslant|h|\leqslant\frac{1}{2}N}|h|^{-1}(\log^+|h|^{-4})F^2(0, h, N)\right)^{\frac{1}{2}} + NN^{-\frac{1}{2}} \\ &\ll \phi(N) + N^{\frac{1}{2}} \ll \phi(N) \quad \text{a.s.} \end{aligned}$$

using Cauchy's inequality and (3.1.11)–(3.1.13). Here the constant implied by $\ll$ only depends on A, α, β. We take the supremum over all $f \in \Lambda_\alpha$ $(\alpha > \frac{1}{2})$, divide both sides by $\phi(N)$ and then take the limes superior as $N \to \infty$. This proves the proposition.

For the proof of Theorem 3.1 it remains to show that (3.1.1) holds for functions of mixing random variables as well as for lacunary sequences. This will be done in the next two subsections.

3.2. *Functions of mixing random variables.* Before we begin the proof of (3.1.1) for functions of strongly mixing random variables let us observe that for independent random variables η_k with uniform distribution over [0, 1] relation (3.1.1) follows from the classical exponential bounds. Indeed, $\{\exp(2\pi ih\eta_k), k \geqslant 1\}$ is a sequence of independent identically distributed random variables, centered at expectation with variance $\sigma^2 \leqslant 1$ and bounded by 1. Hence it is enough to check (3.1.1) for $H = 0$ only. Since

$$c = \max_{k\leqslant N}\frac{|\exp(2\pi ih\eta_k)|}{\sigma N^{\frac{1}{2}}} = \sigma^{-1}N^{-\frac{1}{2}}$$

relation (1) on page 255 of Loéve (1963) yields

$$\begin{aligned} P\left\{|\Sigma_{k\leqslant N}\exp(2\pi ih\eta_k)| \geqslant 2R(N\log\log N)^{\frac{1}{2}}\right\} \\ \leqslant \exp\left(-t2R(\log\log N)^{\frac{1}{2}} + \tfrac{1}{2}t^2\left(1 + \tfrac{1}{2}\sigma^{-1}N^{-\frac{1}{2}}t\right)\right) \end{aligned}$$

for all $h \neq 0$ and all $t \leqslant N^{\frac{1}{2}}$. We set $t = (\log\log N)^{\frac{1}{2}}$ and obtain (3.1.1) with $A = 2$ and $C = 1$.

The proof of (3.1.1) for functions of mixing random variables is much more complicated. On the other hand, in a recent paper Philipp (1977) proved a similar inequality for the random variables $1\{s \leqslant \eta_k \leqslant t\} - t + s$ $(0 \leqslant s < t \leqslant 1)$ instead of $\exp(2\pi i h \eta_k)$. Since the proof of (3.1.1) is essentially the same as the one given in Philipp's paper we shall only sketch the proof.

We first observe that for each integer $h \neq 0$ the random variables $\langle h\eta_k \rangle$ are also uniformly distributed over [0, 1] since for $0 \leqslant t \leqslant 1$

$$P\{\langle h\eta_k\rangle \leqslant t\} = \Sigma_{0\leqslant \nu < h} P\left\{\nu h^{-1} \leqslant \eta_k \leqslant (\nu + t)h^{-1}\right\} = t.$$

We put

$$x_n = x_n(h) = \exp(2\pi i h \eta_n) \tag{3.2.1}$$

and

$$x_{mn} = x_{mn}(h) = \exp(2\pi i h \eta_{mn}). \tag{3.2.2}$$

Then

$$E|x_n - x_{mn}| \leqslant 2\pi |h| E|\eta_n - \eta_{mn}| \ll |h|\psi(m). \tag{3.2.3}$$

As in Section 3.3.1 of Philipp (1977) we put

$$\alpha = \frac{1}{120}.$$

(This α has nothing to do with the α in Λ_α.) We define blocks of integers H_j and I_j each consisting of $[j^{100\alpha}]$ consecutive integers. We leave no gaps between the blocks. The order is $H_1, I_1, H_2, I_2, \cdots$. Let $M = M_N$ be the index j of the block H_j or I_j containing N and let h_j be the smallest member of H_j. Then

$$h_M \leqslant N < h_{M+1}$$

and

$$M^{100\alpha+1} \ll N \ll M^{100\alpha+1}.$$

We introduce new random variables y_j and z_j by

$$y_j = \Sigma_{n\in H_j} x_{mn}$$

$$z_j = \Sigma_{n \in I_j} x_{mn}$$

where we set $m = [j^{99\alpha}]$. This notation parallels that of Philipp's (1977) Section 3.3.1. We now state a series of lemmas whose proofs are also entirely parallel.

Lemma 3.2.1. *As* $N \to \infty$

$$P\left\{|\Sigma_{n<h_{M+1}} x_n - \Sigma_{j\leqslant M}(y_j + z_j)| \geqslant RN^{\frac{1}{2}}\right\} \ll R^{-12} N^{-2.8}$$

for all $|h| \leqslant N^2$.

LEMMA 3.2.2. *As* $N \to \infty$

$$\Sigma_{n \in H_M \cup I_M} |x_n| \ll N^{\frac{1}{2}}$$

for all integers h.

LEMMA 3.2.3. *As* $N \to \infty$

$$\Sigma_{j \leqslant M} E y_j^2 \ll N$$

for all $|h| \leqslant N^2$.

Let $\mathfrak{L}_j$ be the σ-field generated by $y_1, \cdots, y_j$.

LEMMA 3.2.4. *The random variables* y_j *can be represented in the form*

$$y_j = Y_j + v_j$$

where $(y_j, \mathfrak{L}_j)$ *is a martingale difference sequence and* $v_j = E(y_j | \mathfrak{L}_{j-1})$ *satisfies*

$$\|v_j\|_8 \ll j^{-100\alpha}.$$

LEMMA 3.2.5. *As* $N \to \infty$

$$P\left\{\Sigma_{j \leqslant M} |v_j| \geqslant R N^{\frac{1}{2}}\right\} \ll R^{-8} N^{-3}.$$

LEMMA 3.2.6. *As* $N \to \infty$

$$P\left\{\Sigma_{j \leqslant M} E\left(Y_j^2 | \mathfrak{L}_{j-1}\right) \geqslant 2RBN\right\} \ll R^{-4} N^{-2.03}.$$

Here $B \geqslant 1$ is the constant implied by $\ll$ in Lemma 3.2.3.

LEMMA 3.2.7. *As* $N \to \infty$

$$P\left\{|\Sigma_{j \leqslant M} Y_j| \geqslant 8RB(N \log \log N)^{\frac{1}{2}}\right\} \ll \exp(-6R \log \log N) + R^{-4} N^{-2.03}$$

for all integers h with $0 < |h| \leqslant N^2$.

The proofs of all these lemmas are practically the same as the corresponding ones in Philipp (1977). Similarly all these lemmas remain valid if the y_j's are replaced by the z_j's. We denote the corresponding martingale difference sequence by $\{Z_j, j \geqslant 1\}$. Then for all integers h with $0 < |h| \leqslant N^2$

$$\begin{aligned} |\Sigma_{n \leqslant N} x_n| &\leqslant |\Sigma_{n < h_{M+1}} x_n - \Sigma_{j \leqslant M}(y_j + z_j)| \\ &\quad + \Sigma_{n=h_M}^{h_{M+1}} |x_n| + \Sigma_{j \leqslant M} |y_j - Y_j| + \Sigma_{j \leqslant M} |z_j - Z_j| \\ &\quad + |\Sigma_{j \leqslant M} Y_j| + |\Sigma_{j \leqslant M} Z_j| \\ &\leqslant 10^3 RB(N \log \log N)^{\frac{1}{2}} \end{aligned}$$

except on a set with probability

$$\ll \exp(-6R \log \log N) + R^{-4} N^{-2.03}.$$

3.3 *Lacunary sequences.* There are several ways to prove (3.1.1) for lacunary sequences. The first one is to modify the argument in Section 4.2 of Philipp (1977). This approach is then similar to the one followed in the previous section. The second way is to modify the proof of Lemma 4 of Takahashi (1962) and combine it with Lemma 7 of Erdös and Gál (1955). However, it is not difficult to prove (3.1.1) directly making use of elementary trigonometric identities and the fact that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 \cos ux \, dx = 0 \qquad u > 2. \tag{3.3.1}$$

The use of this identity in the context of lacunary sequences is due to Hartman (1942).

Before we set out to prove (3.1.1) we shall make the usual simplifications. We shall show (3.1.1) with $A = 4 \log 4/\log q + 1$ and an absolute constant C. Since the sequence $\{hn_{k+H}, k \geqslant 1\}$ is lacunary with the same ratio q it is therefore no loss of generality to assume $H = 0$ and $h = 1$. Next, let

$$r = \left[\frac{\log 4}{\log q}\right] + 1 \tag{3.3.2}$$

so that

$$q^r \geqslant 4.$$

Since each sequence $\{n_{a+kr}, k \geqslant 1\}$ $(1 \leqslant a \leqslant r)$ is lacunary with ratio at least 4 it is enough to establish (3.1.1) with $A = 4$, and an absolute constant C under the additional hypothesis $q \geqslant 4$.

We note that for $|x| \leqslant 1$

$$e^x \leqslant 1 + x + x^2. \tag{3.3.3}$$

Let $t > 0$ to be chosen suitably later. We shall first estimate

$$\begin{aligned} &\int_0^1 \exp(t\Sigma_{k \leqslant N} \cos 2\pi n_k x) \, dx \\ &\ll \int_0^1 \left(\frac{\sin x}{x}\right)^2 \Pi_{k \leqslant N}\left(1 + t\cos 2\pi n_k x + t^2 \cos^2 2\pi n_k x\right) dx \\ &\ll \int_0^\infty \left(\frac{\sin x}{x}\right)^2 \Pi_{k \leqslant N}\left(1 + \tfrac{1}{2}t^2 + t\cos 2\pi n_k x + \tfrac{1}{2}t^2 \cos 4\pi n_k x\right) dx \end{aligned} \tag{3.3.4}$$

using (3.3.3) and the fact that $\sin x/x \gg 1$ on [0, 1]. When we multiply out this product we obtain a term $(1 + \frac{1}{2}t^2)^N$ plus a cosine polynomial whose frequencies are bounded from below by

$$\begin{aligned} 2\pi(n_k - 2n_{l_1} - 2n_{l_2} - \cdots) &\geqslant 2\pi n_k(1 - 2q^{l_1 - k} - 2q^{l_2 - k} \ldots) \\ &\geqslant 2\pi n_k\left(1 - 2\left(\frac{1}{4} + \frac{1}{4^2} + \cdots\right)\right) \\ &\geqslant 2\pi n_k/3 \geqslant 2\pi n_1/3 > 2 \end{aligned}$$

if we assume $n_1 \geqslant 1$, as we can do without loss of generality. Here $k > l_1 > l_2 > \cdots$.

Hence by (3.3.1) the integral in (3.3.4) does not exceed

$$\left(1 + \tfrac{1}{2}t^2\right)^N \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx \ll \left(1 + \tfrac{1}{2}t^2\right)^N \ll e^{Nt^2}.$$

Consequently,

$$\begin{aligned} P\left\{\left|\Sigma_{k \leqslant N} \cos 2\pi n_k x\right| \geqslant 2R(N \log\log N)^{\frac{1}{2}}\right\} \\ \leqslant \exp\left(-2Rt(N \log\log N)^{\frac{1}{2}}\right) \\ \cdot \int_0^1 \exp(t\Sigma_{k \leqslant N} \cos 2\pi n_k x)\, dx \\ \ll \exp\left(-2Rt(N \log\log N)^{\frac{1}{2}} + Nt^2\right). \end{aligned} \tag{3.3.5}$$

We set $t = (N^{-1} \log\log N)^{\frac{1}{2}}$ and obtain the bound

$$\ll \exp(-R \log\log N).$$

Of course, the same bound holds, if in (3.3.5) cos is replaced by sin. This proves (3.1.1) in the lacunary case.

4. The class Λ_α ($\alpha < \frac{1}{2}$). In this section we disprove the uniform law of the iterated logarithm for the class Λ_α ($\alpha < \frac{1}{2}$) and for independent uniformly distributed random variables as well as for lacunary sequences. This will be done in Sections 4.1 and 4.2 respectively.

4.1. *Independent random variables.*

THEOREM 4.1. *Let $\{\xi_k, k \geqslant 1\}$ be a sequence of independent random variables uniformly distributed over* [0, 1]. *Then for each $0 < \varepsilon < \frac{1}{4}$ there is a $\delta > 0$ such that*

$$\sup_{f \in \Lambda_\alpha} \Sigma_{k \leqslant N} f(\xi_k) \gg N^{1-\alpha-\varepsilon}$$

with probability at least δ. Here the constant implied $\gg$ depends only on ε.

Notice that the theorem is valid for all $0 < \alpha < 1$ although for $\alpha \geqslant \frac{1}{2}$ it does not give much information.

For the proof of Theorem 4.1 we need the Paley-Zygmund theorem and a martingale inequality due to Burkholder.

LEMMA 4.1.1. *Let X be a nonnegative random variable with finite second moment. Suppose that*

$$\|X\|_2 \leqslant aEX \tag{4.1.1}$$

for some $0 \leqslant a \leqslant 1$. Let $0 \leqslant b \leqslant a$. Then

$$P\{X \geqslant b\|X\|_2\} \geqslant (b - a)^2.$$

The next lemma follows from Burkholder's (1973) Theorem 3.1.

LEMMA 4.1.2. *Let $\{Y_j, \mathfrak{L}_j\}$ be a martingale difference sequence. Then for each $\kappa > 0$ and $M \geqslant 1$*

$$\kappa^{\frac{1}{2}} P\{\Sigma_{j \leqslant M} Y_j^2 > \kappa\} \leqslant 3E|\Sigma_{j \leqslant M} Y_j|.$$

Before we begin the proof of Theorem 4.1 we introduce some notation. Let $H \geqslant 1$ be an integer. For $0 \leqslant h < H$ we define

$$F_h(x) = \sin 2\pi H x \quad \text{if } hH^{-1} \leqslant x \leqslant (h+1)H^{-1}$$
$$= 0 \quad \text{else.} \tag{4.1.2}$$

Let $\alpha > 0$. We observe that for all choices $\delta_h = \pm 1 (0 \leqslant h \leqslant H)$ the functions

$$f(x) = H^{-\alpha} \Sigma_{0 \leqslant h < H} \delta_h F_h(x) \tag{4.1.3}$$

belong to Λ_α. Indeed, f has period 1 and $\int_0^1 f(x)\,dx = 0$. Moreover, since $|f| \leqslant H^{-\alpha}$ we have

$$|f(x) - f(y)| \leqslant 2H^{-\alpha} \leqslant 2|x - y|^\alpha$$

if $|x - y| \geqslant H^{-1}$. If $|x - y| < H^{-1}$ then

$$|F_h(x) - F_h(y)| \leqslant 2\pi H|x - y|.$$

Thus

$$|f(x) - f(y)| \leqslant H^{-\alpha} 2\pi H |x - y|$$
$$\leqslant 2\pi |x - y|^\alpha (|x - y| H)^{1-\alpha}$$
$$\leqslant 2\pi |x - y|^\alpha.$$

Let $N \geqslant 1$ be integer and let $0 < \varepsilon < \frac{1}{4}$. Put

$$H = [N^{1-\varepsilon}] \tag{4.1.4}$$

and for $0 \leqslant h < H$ put

$$S_N = S_N(h) = \Sigma_{k \leqslant N} F_h(\xi_k). \tag{4.1.5}$$

Theorem 4.1 is a consequence of the following lemma.

LEMMA 4.1.3. *As $N \to \infty$*

$$E|S_N| \gg N^{\frac{1}{2}\varepsilon} \tag{4.1.6}$$

and

$$ES_N^2 \ll N^\varepsilon \tag{4.1.7}$$

where the constants implied by $\ll$ are absolute.

Let us postpone the proof of the lemma for a moment and let us first deduce Theorem 4.1 from it. We introduce random variables

$$T_N = N^{-1+\frac{1}{2}\varepsilon} \Sigma_{0 \leqslant h < H} |\Sigma_{k \leqslant N} F_h(\xi_k)|.$$

Then by (4.1.6) and (4.1.4)

$$ET_N \gg N^{-1+\frac{1}{2}\varepsilon} N^{1-\varepsilon} N^{\frac{1}{2}\varepsilon} \gg 1.$$

By (4.1.7), (4.1.4) and Minkowski's inequality

$$\|T_N\|_2 \ll N^{-1+\frac{1}{2}\varepsilon} N^{1-\varepsilon} N^{\frac{1}{2}\varepsilon} \ll 1.$$

Hence by the Paley-Zygmund theorem (Lemma 4.1.1) there is a $\delta > 0$ such that

$$P\{T_N \geqslant \delta\} \geqslant \delta. \tag{4.1.8}$$

Denote by $\sup_{\{\delta_h\}}$ the supremum over all 2^H sets $\{\delta_1, \cdots, \delta_H\}$ with $\delta_h = \pm 1$ for $0 \leqslant h < H$. Then by (4.1.8)

$$\begin{aligned}
&\sup_{\{\delta_h\}} \Sigma_{k \leqslant N} H^{-\alpha} \Sigma_{0 \leqslant h < H} \delta_h F_h(\xi_k) \\
&\quad \geqslant H^{-\alpha} N^{1-\frac{1}{2}\varepsilon} \sup_{\{\delta_h\}} N^{-1+\frac{1}{2}\varepsilon} \Sigma_{0 \leqslant h < H} \delta_h \Sigma_{k \leqslant N} F_h(\xi_k) \\
&\quad = H^{-\alpha} N^{1-\frac{1}{2}\varepsilon} T_N \gg H^{-\alpha} N^{1-\frac{1}{2}\varepsilon} \\
&\quad \gg N^{-\alpha(1-\varepsilon)+1-\frac{1}{2}\varepsilon} \gg N^{1-\alpha-\varepsilon}
\end{aligned}$$

with probability $\geqslant \delta$. The result follows now in view of (4.1.3).

We now turn to the *proof of Lemma* 4.1.3. Since by (4.1.2)

$$E\{F_h^2(\xi_k)\} = \int_{hH^{-1}}^{(h+1)H^{-1}} \sin^2 2\pi H x \, dx = \tfrac{1}{2} H^{-1}, \tag{4.1.9}$$

we have by (4.1.4) and (4.1.5)

$$ES_N^2 = E\{\Sigma_{k \leqslant N} F_h^2(\xi_k)\} = N \cdot \tfrac{1}{2} H^{-1} \sim \tfrac{1}{2} N^{\varepsilon}. \tag{4.1.10}$$

This proves (4.1.7). To prove (4.1.6) we observe that for $0 \leqslant h < H$

$$E\{F_h^4(\xi_k)\} \ll H^{-1}$$

and for $0 \leqslant h < H$ and $1 \leqslant k < l \leqslant N$

$$E\{F_h^2(\xi_k) F_h^2(\xi_l)\} \ll H^{-2}$$

by (4.1.9). Hence by (4.1.4)

$$E\{\Sigma_{k \leqslant N} F_h^2(\xi_k)\}^2 \ll N^2 H^{-2} + N H^{-1} \ll N^{2\varepsilon}. \tag{4.1.11}$$

We apply the Paley-Zygmund theorem (Lemma 4.1.1) with $X = \Sigma_{k \leqslant N} F_h^2(\xi_k)$. Then (4.1.1) is satisfied with some $a > 0$ by (4.1.10) and (4.1.11). Hence we obtain

$$P\{\Sigma_{k \leqslant N} F_h^2(\xi_k) \geqslant \tfrac{1}{2} a N^{\varepsilon}\} \geqslant \tfrac{1}{4} a^2 > 0.$$

Thus by Lemma 4.1.2 with $\kappa = \frac{1}{2} a N^{\varepsilon}$ and (4.1.5)

$$E|S_N| \gg N^{\frac{1}{2}\varepsilon}.$$

This proves (4.1.6) and thus Lemma 4.1.3.

4.2. *Lacunary sequences.*

THEOREM 4.2. *Let $\{n_k, k \geqslant 1\}$ be a lacunary sequence of real numbers and let $\alpha < \frac{1}{2}$. Extend the functions $f \in \Lambda_\alpha$ with period* 1. *Then for each $0 < \varepsilon < \frac{1}{4}$ there is a $\delta > 0$ such that*

$$\sup_{f \in \Lambda_\alpha} \Sigma_{k \leqslant N} f(n_k x) \gg N^{1-\alpha-\varepsilon}$$

on a set $E \subset [0, 1]$ of Lebesgue measure $\lambda(E) \geqslant \delta$.

The proof of Theorem 4.2 runs parallel to the proof of Theorem 4.1 but is much more complicated. Again we shall define functions $F_h(x)$, but slightly different from (4.1.2).

We first introduce some notation. Let $H \geqslant 1$ be an integer and let

$$P = [H^{2\varepsilon}]H. \tag{4.2.1}$$

Let $G \in C^\infty(-\infty, \infty)$ with period 1, $G(0) = G(1) = 0$ and all derivatives vanishing at 0 and 1. Suppose that G has a pure sine series and that

$$\textstyle\int_0^1 G^2(x)\,dx = \frac{1}{2}. \tag{4.2.2}$$

For integer $0 \leqslant h < H$ we define

$$\begin{aligned} G_h(x) &= G(Hx) \qquad \text{if} \quad hH^{-1} \leqslant x \leqslant (h+1)H^{-1} \\ &= 0 \qquad\qquad \text{else on } [0, 1]. \end{aligned} \tag{4.2.3}$$

We extend G_h with period 1. We now define the functions

$$F_h(x) = G_h(x) \cdot \cos 2\pi Px. \tag{4.2.4}$$

For all choices $\delta_h = \pm 1$ $(0 \leqslant h < H)$ the functions

$$f(x) = P^{-\alpha} \Sigma_{0 \leqslant h < H} \delta_h F_h(x) \tag{4.2.5}$$

belong to Λ_α and have period 1. We first show that f satisfies a Lipschitz condition with Lipschitz constant less than $\|G'\|_\infty + 2\pi\|G\|_\infty$. To prove this claim we note that $|f| \leqslant P^{-\alpha}\|G\|_\infty$. Thus

$$|f(x) - f(y)| \leqslant 2P^{-\alpha}\|G\|_\infty \leqslant 2\|G\|_\infty |x - y|^\alpha$$

if $|x - y| \geqslant P^{-1}$. If $|x - y| < P^{-1}$ then

$$\begin{aligned} &|G_h(x) \cos 2\pi Px - G_h(y) \cos 2\pi Py| \\ &\qquad \leqslant |G_h(x) - G_h(y)| + |\cos 2\pi Px - \cos 2\pi Py|\ \|G\|_\infty \\ &\qquad \leqslant \|G'\|_\infty |x - y| H + 2\pi P|x - y| \|G\|_\infty \\ &\qquad \leqslant P|x - y|(2\pi\|G\|_\infty + \|G'\|_\infty) = P|x - y|L \quad \text{(say)}. \end{aligned}$$

Consequently,

$$\begin{aligned} |f(x) - f(y)| &\leqslant LP|x - y|P^{-\alpha} \leqslant L|x - y|^\alpha |x - y|^{1-\alpha} P^{1-\alpha} \\ &\leqslant L|x - y|^\alpha. \end{aligned}$$

This proves that the Lipschitz constant is less than

$$\|G'\|_\infty + 2\pi\|G\|_\infty.$$

Finally, $\int_0^1 f(x)\,dx = 0$ because

$$\textstyle\int_0^1 F_h(x)\,dx = H^{-1}\int_0^1 G(x) \cos 2\pi PH^{-1}x\,dx = 0 \tag{4.2.6}$$

since by (4.2.1) PH^{-1} is an integer and since $G(x)$ is assumed to have a pure sine series. Similarly, we obtain

(4.2.7) $$\int_0^1 F_h^2(x)\,dx = H^{-1}\int_0^1 G^2(x)\cos^2 2\pi PH^{-1}x\,dx$$
$$= \tfrac{1}{2}H^{-1}\int_0^1 G^2(x) + \tfrac{1}{2}H^{-1}\int_0^1 G^2(x)\cos 4\pi PH^{-1}x\,dx$$
$$= \tfrac{1}{4}H^{-1}(1 + 0(P^{-1}H)) = \tfrac{1}{4}H^{-1}(1 + 0(H^{-2\varepsilon})),$$

using integration by parts and (4.2.1).

Let $N \geqslant 1$ and put

(4.2.8) $$H = [N^{1-\varepsilon}]$$

so that by (4.2.1)

(4.2.9) $$N^{1+\varepsilon-2\varepsilon^2} \ll P \ll N^{1+\varepsilon-2\varepsilon^2}.$$

Write

(4.2.10) $$S_N = S_N(h) = \Sigma_{k \leqslant N} F_h(n_k x).$$

Theorem 4.2 is a consequence of the following proposition.

PROPOSITION 4.2.1. *As* $N \to \infty$,

(4.2.11) $$E|S_N| \gg N^{\frac{1}{2}\varepsilon}$$

and

(4.2.12) $$ES_N^2 \ll N^{\varepsilon}$$

where the constants implied by $\ll$ *depend only at most on* q, G *and* ε.

Except for a few minor changes the proof that Proposition 4.2.1 implies Theorem 4.2 is identical with the proof that Lemma 4.1.3 implies Theorem 4.1. It will be given in Section 4.2.5. In Section 4.2.1 we collect a few preliminary facts and lemmas. In Section 4.2.2 we introduce blocks of integers and random variables similar to Section 3.1. These block random variables are then approximated by a martingale difference sequence to which Burkholder's inequality (Lemma 4.1.2) is applied. All this is done in Section 4.2.3. The proof of Proposition 4.2.1 is then completed in Section 4.2.4.

4.2.1 *Preliminaries.* We expand G_h into a Fourier series

(4.2.13) $$G_h(x) = \Sigma_{m=1}^{\infty}(a_m \cos 2\pi mx + b_m \sin 2\pi mx)$$

where for each positive integer t

(4.2.14) $$a_m, b_m \ll \min(H^t m^{-t-1}, m^t H^{-t-1}).$$

Here the constant implied by $\ll$ depends only on $G^{(t)}$ as is easily seen, e.g. by integration by parts. We put

(4.2.15) $$G_h^*(x) = \Sigma_{m \leqslant H^{1+3\varepsilon/2}}(a_m \cos 2\pi mx + b_m \sin 2\pi mx)$$

and

(4.2.16) $$F_h^*(x) = G_h^*(x)\cos 2\pi Px.$$

Then by (4.2.14)

(4.2.17) $$F_h(x) - F_h^*(x) \ll \Sigma_{m \geqslant H^{1+3\varepsilon/2}}(|a_m| + |b_m|) \ll H^t \Sigma_{m \geqslant H^{1+3\varepsilon/2}} m^{-t-1} \ll H^{-3}$$

if we choose $t \geqslant 2/\varepsilon$.

Let r_k be the largest integer r with

(4.2.18) $$2^r \leqslant n_k \exp(\log^2 k)$$

and let $\mathfrak{F}_k$ be the σ-field generated by the intervals

$$U_{\nu k} = [\nu 2^{-r_k}, (\nu + 1)2^{-r_k}] \qquad 0 \leqslant \nu < 2^{r_k}.$$

Write

(4.2.19) $$\xi_k = F_h(n_k x).$$

LEMMA 4.2.1. *We have*

(4.2.20) $$E(\xi_{j+k} | \mathfrak{F}_j) \ll H^{-1} \exp(\log^2 j) q^{-k} \quad \text{a.s.}$$

and

(4.2.21) $$E(\xi_{j+k}^2 | \mathfrak{F}_j) = \tfrac{1}{4} H^{-1}(1 + 0(H^{-2\varepsilon} + q^{-k} \exp(\log^2 j))) \quad \text{a.s.}$$

where the constants implied by $\ll$ *and* 0 *are nonrandom and depend on G only.*

PROOF. We prove (4.2.20) first. We estimate

$$\int_{\nu 2^{-r_j}}^{(\nu+1)2^{-r_j}} F_h(n_{j+k}x)\, dx = \int_{[\nu 2^{-r_j}, (\nu+1)2^{-r_j}] \cap [hH^{-1}, (h+1)H^{-1}]} G(Hn_{j+k}x) \cos 2\pi P n_{j+k} x\, dx$$
$$= (Hn_{j+k})^{-1} \int_{[\alpha_\nu, \alpha_{\nu+1}] \cap [hn_{j+k}, (h+1)n_{j+k}]} G(x) \cos 2\pi P H^{-1} x\, dx \ll Hn_{j+k}^{-1}$$

using (4.2.6). Here $\alpha_\nu = \nu 2^{-r_j} n_{j+k} H$. Thus by (4.2.18)

(4.2.22)
$$E(\xi_{j+k} | \mathfrak{F}_j) = \Sigma_{\nu=0}^{2^{r_j}-1} 1(U_{\nu j}) 2^{r_j} \int_{\nu 2^{-r_j}}^{(\nu+1)2^{-r_j}} G_h(n_{j+k}x) \cos 2\pi P n_{j+k} x\, dx$$
$$\ll (Hn_{j+k})^{-1} 2^{r_j} \ll H^{-1} \exp(\log^2 j) q^{-k}.$$

This proves (4.2.20). To prove (4.2.21) put $\beta_\nu = \nu 2^{-r_j} n_{j+k}$. Then

$$\int_{\nu 2^{-r_j}}^{(\nu+1)2^{-r_j}} F_h^2(n_{j+k}x)\, dx = n_{j+k}^{-1} \int_{\beta_\nu}^{\beta_{\nu+1}} F_h^2(x)\, dx$$
$$= n_{j+k}^{-1}(2^{-r_j} n_{j+k} + \theta) \int_0^1 F_h^2(x)\, dx$$
$$= 2^{-r_j} \tfrac{1}{4} H^{-1}(1 + 0(H^{-2\varepsilon} + q^{-k} \exp(\log^2 j)))$$

by (4.2.7) and (4.2.18). Here θ is a constant bounded by 1. (4.2.21) follows now as in (4.2.22).

LEMMA 4.2.2. *For $l > k \geqslant 0$*

$$E(\xi_{j+k}\xi_{j+l}|\mathfrak{F}_j) \ll P^{-1}q^{-l}\exp(\log^2 j) + H^{-3} \quad \text{a.s.}$$

where the constant implied by $\ll$ is nonrandom and only depends on G, ε and q.

PROOF. We first estimate

$$\int_{\nu 2^{-r_j}}^{(\nu+1)2^{-r_j}} \cos 2\pi\big((P \pm s)n_{j+l} \pm (P \pm r)n_{j+k}\big)x\,dx \tag{4.2.23}$$

where we assume

$$r, s \leqslant H^{1+3\varepsilon/2}. \tag{4.2.24}$$

Now by (4.2.24)

$$\begin{aligned}(P \pm s)n_{j+l} \pm (P \pm r)n_{j+k} &\geqslant Pn_{j+l}\big(1 - sP^{-1} - (1 + rP^{-1})q^{k-l}\big)\\ &\geqslant \tfrac{1}{2}(1 - q^{-1})Pn_{j+l}\end{aligned} \tag{4.2.25}$$

if H is sufficiently large. Hence (4.2.23) equals

$$\begin{aligned}(Pn_{j+l})^{-1}\int_{\rho_\nu}^{\rho_{\nu+1}} &\cos 2\pi\big((1 \pm sP^{-1}) \pm (1 \pm rP^{-1})n_{j+k}n_{j+l}^{-1}\big)x\,dx\\ &\ll (Pn_{j+l})^{-1}\end{aligned}$$

where $\rho_\nu = \nu 2^{-r_j}Pn_{j+l}$. The same estimate holds with sine instead of cosine. Consequently (we shall drop the index h from F_h^* and F_h),

$$\begin{aligned}\int_{\nu 2^{-r_j}}^{(\nu+1)2^{-r_j}} &F^*(n_{j+k}x)F^*(n_{j+l}x)\,dx\\ &\ll \Sigma_{r,s\leqslant H^{1+3\varepsilon/2}}(|a_r| + |b_r|)(|a_s| + |b_s|)(Pn_{j+l})^{-1}\\ &\ll (Pn_{j+l})^{-1}\end{aligned} \tag{4.2.26}$$

since by (4.2.14)

$$\Sigma_{m\geqslant 1}|a_m| \ll 1 \quad \text{and} \quad \Sigma_{m\geqslant 1}|b_m| \ll 1$$

with absolute constants implied by $\ll$. Next by (4.2.17) and (4.2.26)

$$\int_{\nu 2^{-r_j}}^{(\nu+1)2^{-r_j}} F(n_{j+k}x)F(n_{j+l}x)\,dx \ll H^{-3}2^{-r_j} + (Pn_{j+l})^{-1}.$$

Hence as in (4.2.22)

$$\begin{aligned}E(\xi_{j+k}\xi_{j+l}|\mathfrak{F}_j) &\ll H^{-3} + (Pn_{j+l})^{-1}2^{r_j}\\ &\ll H^{-3} + P^{-1}q^{-l}\exp(\log^2 j) \quad \text{a.s.}\end{aligned}$$

4.2.2. *Introduction of the blocks.* We now define inductively blocks I_j of consecutive integers each containing $[j^{\frac{1}{4}\varepsilon}]$ integers leaving no gaps between the blocks. We put $h_j = \min I_j$ so that $I_j = [h_j, h_{j+1})$ and

$$j^{1+\frac{1}{4}\varepsilon} \ll h_j \ll j^{1+\frac{1}{4}\varepsilon}. \tag{4.2.27}$$

Write

$$x_\nu = E(\xi_\nu | \mathfrak{F}_{h_{j+1}}) \qquad \text{if } \nu \in I_j, \tag{4.2.28}$$

$$w_j = \Sigma_{\nu \in I_j} \xi_\nu, \qquad y_j = \Sigma_{\nu \in I_j} x_\nu \tag{4.2.29}$$

so that

$$y_j = E(w_j | \mathfrak{F}_{h_{j+1}}). \tag{4.2.30}$$

LEMMA 4.2.3. *As $j \to \infty$*

$$y_j - w_j \ll P.\exp(-\log^2 j) \quad \text{a.s.}$$

where the constant implied by $\ll$ is nonrandom and depends on G and on q only.

PROOF. By elementary calculus and (4.2.18)

$$\xi_\nu - x_\nu \ll Pn_\nu 2^{-r_{h_j+1}} \ll Pq^{\nu - h_{j+1}} \exp(-\log^2 j) \quad \text{a.s.} \tag{4.2.31}$$

Thus

$$w_j - y_j \ll P\exp(-\log^2 j)\Sigma_{\nu < h_{j+1}} q^{\nu - h_{j+1}} \ll P.\exp(-\log^2 j) \quad \text{a.s.}$$

LEMMA 4.2.4. *For $i < j$ and $j \to \infty$*

$$Ew_j^4 \ll H^{-3} j^\varepsilon + H^{-1} j^{3\varepsilon/4} \tag{4.2.32}$$

and

$$E(w_j^2 | \mathfrak{F}_{h_i}) \ll H^{-1} j^{\frac{1}{4}\varepsilon} \tag{4.2.33}$$

where the constants implied by $\ll$ only depend on q, ε, and G.

PROOF. We prove (4.2.32) first. We show that there is no loss of generality in proving (4.2.32) under the additional hypothesis $q \geqslant 4$. Indeed, if we define r by (3.2.2) then by Minkowski's inequality

$$\|w_j\|_4 = \|\Sigma_{a=0}^{r-1} \Sigma_{k\,:\,h_j \leqslant kr+a < h_{j+1}} \xi_{kr+a}\|_4$$
$$\ll \Sigma_{a<r} \|\Sigma_{k\,:\,h_j \leqslant kr+a < h_{j+1}} \xi_{kr+a}\|_4.$$

This proves our claim since each sequence $\{n_{kr+a}, k \geqslant 1\}$ has ratio at least 4.

From now on the proof of Lemma 4.2.4 follows the pattern of the proof of Lemma 4.2.2, but is more complicated. We first estimate for $\nu_1 \leqslant \nu_2 \leqslant \nu_3 < \nu_4$ the integrals

$$\int_0^1 \cos 2\pi\{n_{\nu_4}(P \pm r_4) \pm n_{\nu_3}(P \pm r_3) \pm n_{\nu_2}(P \pm r_2) \pm n_{\nu_1}(P \pm r_1)\} x\, dx \tag{4.2.34}$$

with $r_i \leqslant H^{1+3\varepsilon/2}$ $(1 \leqslant i \leqslant 4)$. We observe that as in (4.2.25) the argument in the cosine is at least $\frac{1}{5} Pn_{\nu_4}$, since we assumed $q \geqslant 4$. Hence

$$(4.2.34) \ll (Pn_{\nu_4})^{-1}.$$

The same bound holds with sine instead of cosine. Write

$$\xi_\nu^* = F_h^*(n_\nu x).$$

Then we obtain as in (4.2.26)

$$E(\xi^*_{\nu_1}\xi^*_{\nu_2}\xi^*_{\nu_3}\xi^*_{\nu_4}) \ll (Pn_{\nu_4})^{-1}.$$

Hence by a repeated application of (4.2.17)

$$E(\xi_{\nu_1}\xi_{\nu_2}\xi_{\nu_3}\xi_{\nu_4}) \ll H^{-3} + (Pn_4)^{-1} \qquad \nu_1 \leqslant \nu_2 \leqslant \nu_3 < \nu_4. \tag{4.2.35}$$

By (4.2.7)

$$E(\xi_{\nu_1}\xi_{\nu_2}\xi^2_{\nu_3}) \ll E\xi^2_{\nu_3} \ll H^{-1}. \tag{4.2.36}$$

Now

$$Ew_j^4 \ll \Sigma_{\nu_1\leqslant\nu_2\leqslant\nu_3\leqslant\nu_4\in I_j}|E(\xi_{\nu_1}\xi_{\nu_2}\xi_{\nu_3}\xi_{\nu_4})|.$$

We split the sum into two parts Σ_1 and Σ_2 where Σ_1 sums over all $\nu_i \in I_j$ $(1 \leqslant i \leqslant 4)$ with $\nu_1 \leqslant \nu_2 \leqslant \nu_3 < \nu_4$ and Σ_2 sums over all $\nu_i \in I_j$ $(1 \leqslant i \leqslant 4)$ with $\nu_1 \leqslant \nu_2 \leqslant \nu_3 = \nu_4$. By (4.2.35) and (4.2.1)

$$\begin{aligned} \Sigma_1 &\ll H^{-3}j^{\varepsilon} + P^{-1}\Sigma_{\nu_1\leqslant\nu_2\leqslant\nu_3<\nu_4\in I_j}q^{-\nu_4} \\ &\ll H^{-3}j^{\varepsilon} + P^{-1}\Sigma_{\nu\geqslant j^{1+\frac{1}{4}\varepsilon}}\nu^3 q^{-\nu} \\ &\ll H^{-3}j^{\varepsilon} + H^{-1-2\varepsilon}q^{-j}. \end{aligned} \tag{4.2.37}$$

Using (4.2.36) we obtain

$$\Sigma_2 \ll H^{-1}j^{3\varepsilon/4}. \tag{4.2.38}$$

(4.2.32) follows now from (4.2.37) and (4.2.38).

To prove (4.2.33) we use Lemmas 4.2.1 and 4.2.2. We obtain

$$\begin{aligned} E(w_j^2|\mathcal{F}_{h_j}) &\ll \Sigma_{\nu\in I_j}E(\xi_\nu^2|\mathcal{F}_{h_j}) + \Sigma_{\mu<\nu\in I_j}|E(\xi_\mu\xi_\nu|\mathcal{F}_{h_j})| \\ &\ll j^{\frac{1}{4}\varepsilon}H^{-1} + \Sigma_{h_j\leqslant\mu<\nu\leqslant h_j+\log^3 h_j}\|\xi_\mu\|_2\|\xi_\nu\|_2 \\ &\quad + P^{-1}j^{\frac{1}{4}\varepsilon}\Sigma_{h_j+\log^3 h_j<\nu\leqslant h_{j+1}}q^{h_j-\nu}\exp(\log^2 h_j) \\ &\ll j^{\frac{1}{4}\varepsilon}H^{-1} \quad \text{a.s.} \end{aligned}$$

The last estimate follows from (4.2.21) and (4.2.1). This concludes the proof of Lemma 4.2.4.

4.2.3. *The martingale representation.* We shall use the following lemma.

LEMMA 4.2.5. *Let* $\{y_j, j \geqslant 1\}$ *be a sequence of random variables and let* $\{\mathfrak{L}_j, j \geqslant 1\}$ *be a nondecreasing sequence of σ-fields such that* y_j *is* $\mathfrak{L}_j$*-measurable. (Here* $\mathfrak{L}_0$ *denotes the trivial σ-field.) Suppose that*

$$\Sigma_{k=0}^{\infty}E|E(y_{j+k}|\mathfrak{L}_j)| < \infty$$

for each $j \geqslant 1$. *Then for each* $j \geqslant 1$

$$y_j = Y_j + u_j - u_{j+1}$$

where $\{Y_j, \mathfrak{L}_j, j \geqslant 1\}$ *is a martingale difference sequence and*

$$u_j = \Sigma_{k=0}^{\infty} E(y_{j+k} | \mathfrak{L}_{j-1}).$$

This is Lemma 2.1 on page 7 of Philipp and Stout (1975). We apply this lemma to the random variables y_j defined in (4.2.29) and to the σ-fields $\mathfrak{L}_j$ generated by $y_1, \cdots, y_j$.

LEMMA 4.2.6. *Let the* y_j *be the random variables defined in* (4.2.29). *Then we can represent the* y_j *by*

$$y_j = Y_j + u_j - u_{j+1}$$

where $\{Y_j, \mathfrak{L}_j, j \geqslant 1\}$ *is a martingale difference sequence and*

$$u_j \ll H^{-1} \log^2 j \quad \text{a.s.}$$

The constant implied by $\ll$ *is nonrandom and depends on* ε, q *and* G *only.*

PROOF. Let us first estimate u_j given by Lemma 4.2.5. Using (4.2.28), Lemma 4.2.1 and (4.2.27) we obtain

$$\begin{aligned} u_j &= \Sigma_{k=0}^{\infty} E(y_{j+k} | \mathfrak{L}_{j-1}) \\ &= \Sigma_{k=0}^{\infty} \Sigma_{\nu \in I_{j+k}} E\big(E(\xi_\nu | \mathfrak{F}_{h_{j+1}}) | \mathfrak{L}_{j-1}\big) \\ &= \Sigma_{k=0}^{\infty} \Sigma_{\nu \in I_{j+k}} E(\xi_\nu | \mathfrak{L}_{j-1}) \\ &= \Sigma_{k=0}^{\infty} \Sigma_{\nu \in I_{j+k}} E\big(E(\xi_\nu | \mathfrak{F}_{h_j}) | \mathfrak{L}_{j-1}\big) \\ &\ll H^{-1} J + H^{-1} \Sigma_{\nu \geqslant h_j + J} \exp(\log^2 h_j) q^{h_j - \nu} \\ &\ll H^{-1} \log^2 h_j \ll H^{-1} \log^2 j \quad \text{a.s.} \end{aligned}$$

Here J is the smallest integer r with $\exp(\log^2 h_j) q^{-r} \leqslant 1$. The above calculations show that the series defining u_j is a.s. absolutely convergent and thus Lemma 4.2.5 applies.

4.2.4. *Proof of Proposition 4.2.1.* Let $N \geqslant 1$. Define $M = M_N$ by the requirement $N \in I_M$. Then by (4.2.27)

$$M^{1+\frac{1}{4}\varepsilon} \ll N \ll M^{1+\frac{1}{4}\varepsilon}. \tag{4.2.39}$$

Recall that in (4.2.8) we defined H in terms of N. We need three more estimates on Y_j^2 and Y_j which we state as lemmas.

LEMMA 4.2.7. *As* $N \to \infty$

$$\Sigma_{j \leqslant M} E Y_j^2 - \tfrac{1}{4} N^{\varepsilon} \ll N^{5\varepsilon/6}$$

where the constant implied by $\ll$ *depends on* ε, q *and* G *only.*

PROOF. Since $\{Y_j, \mathfrak{L}_j\}$ is a martingale difference sequence,

$$\begin{aligned} \Sigma_{j \leqslant M} E Y_j^2 &= E(\Sigma_{j \leqslant M} Y_j)^2 \\ &= E\{\Sigma_{\nu \leqslant h_M} \xi_\nu + u_1 - u_{M+1} + \Sigma_{j \leqslant M}(y_j - w_j)\}^2 \end{aligned}$$

by (4.2.29) and Lemma 4.2.6. Hence by Minkowski's inequality, Lemmas 4.2.6, 4.2.3, (4.2.39) and (4.2.9)

$$\begin{aligned}(\Sigma_{j\leqslant M}EY_j^2)^{\frac{1}{2}} - \|\Sigma_{\nu\leqslant h_M}\xi_\nu\|_2 & \\ \leqslant H^{-1}\log^2 N + \Big(\Sigma_{j\leqslant N^{\frac{1}{4}\varepsilon}} + \Sigma_{j>N^{\frac{1}{4}\varepsilon}}\Big)|y_j - w_j| & \\ \ll N^{-1+2\varepsilon} + N^{\frac{1}{4}\varepsilon(1+\frac{1}{4}\varepsilon)} + P\exp\big(-N^{\frac{1}{4}\varepsilon}\big) & \\ \ll N^{\varepsilon/3}. & \end{aligned} \tag{4.2.40}$$

We now estimate using Lemmas 4.2.1 and 4.2.2

$$\begin{aligned}E(\Sigma_{\nu\leqslant h_M}\xi_\nu)^2 - \tfrac{1}{4}NH^{-1} &\ll \Sigma_{\nu\leqslant h_M}E\xi_\nu^2 - \tfrac{1}{4}NH^{-1} + \Sigma_{\mu<\nu<h_M}|E(\xi_\mu\xi_\nu)| \\ &\ll M^{\frac{1}{4}\varepsilon}H^{-1} + H^{-1-2\varepsilon}N + H^{-1}\Sigma_{k\geqslant 1}q^{-\frac{1}{2}k} + H^{-3}N^2 \\ &\quad + P^{-1}\Sigma_{\mu<\nu\leqslant h_M}q^{-\frac{1}{2}\nu} \ll 1.\end{aligned} \tag{4.2.41}$$

Hence by (4.2.40), (4.2.8) and (4.2.9)

$$\Sigma_{j\leqslant M}EY_j^2 - \tfrac{1}{4}N^\varepsilon \ll N^{5\varepsilon/6}.$$

LEMMA 4.2.8. *As $N\to\infty$*

$$E\big(\Sigma_{j\leqslant M}Y_j^2\big)^2 \ll N^{2\varepsilon}$$

where the constant implied by $\ll$ depends on ε, q and G only.

PROOF. By Lemmas 4.2.6 and 4.2.4 and Jensen's inequality

$$\begin{aligned}EY_j^4 &\ll Ey_j^4 + H^{-4}\log^8 j \ll Ew_j^4 + H^{-4}\log^8 j \\ &\ll H^{-3}j^\varepsilon + H^{-1}j^{3\varepsilon/4}.\end{aligned}$$

Hence by (4.2.39) and (4.2.8)

$$\Sigma_{j\leqslant M}EY_j^4 \ll H^{-3}M^{1+\varepsilon} + H^{-1}M^{1+3\varepsilon/4} \ll N^{3\varepsilon/2}. \tag{4.2.42}$$

Next we estimate for $i<j$

$$\begin{aligned}E\big(Y_i^2Y_j^2\big) &= E\big(E\big(Y_i^2Y_j^2|\mathcal{F}_{h_i}\big)\big) = E\big(Y_i^2E\big(Y_j^2|\mathcal{F}_{h_i}\big)\big) \\ &\ll EY_i^2\|E\big(Y_j^2|\mathcal{F}_{h_i}\big)\|_\infty.\end{aligned} \tag{4.2.43}$$

By Lemma 4.2.6

$$Y_j^2 \ll y_j^2 + H^{-2}\log^4 j. \tag{4.2.44}$$

Hence by Lemma 4.2.4

$$\begin{aligned}E\big(Y_j^2|\mathcal{F}_{h_i}\big) &\ll E\big(y_j^2|\mathcal{F}_{h_i}\big) + H^{-2}\log^4 j \\ &\ll E\big(w_j^2|\mathcal{F}_{h_i}\big) + H^{-2}\log^4 j \\ &\ll H^{-1}j^{\frac{1}{4}\varepsilon}.\end{aligned} \tag{4.2.45}$$

Also by (4.2.44) and Lemma 4.2.4

$$EY_i^2 \ll Ey_i^2 + H^{-2}\log^4 i$$
$$\ll Ew_i^2 + H^{-2}\log^4 i$$
$$\ll H^{-1} i^{\frac{1}{4}\varepsilon}.$$

Thus by (4.2.43) and (4.2.45) for $i < j$

$$E\left(Y_i^2 Y_j^2\right) \ll H^{-2} j^{\frac{1}{2}\varepsilon}.$$

Hence by (4.2.42), (4.2.39) and (4.2.8)

$$E\left(\Sigma_{j \leqslant M} Y_j^2\right)^2 \ll H^{-2} M^{2+\frac{1}{2}\varepsilon} \ll N^{2\varepsilon}.$$

LEMMA 4.2.9. *As* $N \to \infty$

$$E\left|\Sigma_{j \leqslant M} Y_j\right| \gg N^{\frac{1}{2}\varepsilon}.$$

PROOF. We apply the Paley-Zygmund theorem (Lemma 4.1.1) with $b = \frac{1}{2}a$ and $X = \Sigma_{j \leqslant M} Y_j^2$. Then by Lemmas 4.2.7 and 4.2.8 relation (4.1.1) is satisfied for some $a > 0$ depending only on q, ε and G. Hence with $b = \frac{1}{2}a$ we obtain that

$$P\left\{\Sigma_{j \leqslant M} Y_j^2 \geqslant \tfrac{1}{2}aN^{\varepsilon}\right\} \geqslant \tfrac{1}{4}a^2 > 0.$$

Thus by Lemma 4.1.2 with $\kappa = \frac{1}{2}aN^{\varepsilon}$

$$E\left|\Sigma_{j \leqslant M} Y_j\right| \gg N^{\frac{1}{2}\varepsilon}.$$

Finally, we can prove Proposition 4.2.1. By Lemmas 4.2.6, 4.2.3, (4.2.10), (4.2.19) and (4.2.29)

$$S_N = \Sigma_{j \leqslant M} y_j + \Sigma_{j \leqslant \log N}(w_j - y_j) + \Sigma_{\log N < j \leqslant M}(w_j - y_j) - \Sigma_{\nu = N+1}^{h_{M+1}} \xi_\nu$$
$$= \Sigma_{j \leqslant M} Y_j + u_1 - u_{M+1} + 0(\log^2 N) + P\Sigma_{\log N < j \leqslant M} \exp(-\log^2 j) + 0\left(M^{\frac{1}{4}\varepsilon}\right)$$
$$= \Sigma_{j \leqslant M} Y_j + 0\left(N^{\frac{1}{4}\varepsilon}\right).$$

Hence (4.2.11) follows from Lemma 4.2.9 and (4.2.12) follows from Lemma 4.2.7 and the orthogonality of the Y_j.

4.2.5. *Proof of Theorem 4.2.* From now on the proof of Theorem 4.2 is almost identical with the proof of Theorem 4.1. As in Section 4.1 we introduce the random variables

$$T_N = N^{-1+\frac{1}{2}\varepsilon}\Sigma_{0 \leqslant h < H}\left|\Sigma_{k \leqslant N} F_h(n_k x)\right|.$$

Using Proposition 4.2.1 we prove

$$P\{T_N \geqslant \delta\} \geqslant \delta$$

in the same way as in (4.1.8). Hence, as in Section 4.1, but using (4.2.9) instead we obtain

$$\sup_{\{\delta_h\}} \Sigma_{k \leqslant N} P^{-\alpha} \Sigma_{0 \leqslant h < H} \delta_h F_h(n_k x)$$
$$\gg P^{-\alpha} N^{1-\frac{1}{2}\varepsilon} \gg N^{-\alpha(1+\varepsilon-2\varepsilon^2)+1-\frac{1}{2}\varepsilon} \gg N^{1-\alpha-\varepsilon\left(\alpha+\frac{1}{2}\right)} \gg N^{1-\alpha-\varepsilon}$$

with probability $\geqslant \delta$. Theorem 4.2 follows now in view of (4.2.5).

5. The uniform law of the iterated logarithm and Hilbert space valued random variables. In this section we prove a theorem for independent random variables which, in view of (3.1.12), is slightly stronger than Theorem 3.1.

THEOREM 5.1. *Let* $\{\xi_k, k \geqslant 1\}$ *be a sequence of independent random variables uniformly distributed over* $[0, 1]$. *Let* Λ *be the class of continuous functions* f *on* $[0, 1]$ *with* $\int_0^1 f(x)\,dx = 0$ *and whose Fourier coefficients* a_h *satisfy*

$$\Sigma_{|h|\geqslant 1}|a_h^2|\,|h|(\log^+|h|)^2 \leqslant 1. \tag{5.1}$$

Then the conclusion of Theorem 2.1 *remains valid.*

PROOF. Let

$$\rho_h = |h|^{-\frac{1}{2}} \log^{-1}(e + |h|) \qquad h \neq 0$$
$$\rho_0 = 0.$$

Then

$$\Sigma_h \rho_h^2 = b < \infty.$$

We define a sequence $\{y_n, n \geqslant 1\}$ of l^2-valued random variables

$$y_n = \{\rho_h e^{2\pi i h \xi_n}\}_{h=-\infty}^{\infty}.$$

Clearly, $\{y_n, n \geqslant 1\}$ is a sequence of independent random variables centered at expectations and uniformly bounded by $b^{\frac{1}{2}}$. Hence by Theorem 3.1 of Kuelbs and Kurtz (1974) the sequence $\{y_n, n \geqslant 1\}$ satisfies the compact law of the iterated logarithm and consequently the bounded law of the iterated logarithm, i.e.,

$$\|\Sigma_{n\leqslant N} y_n\| \ll (N \log\log N)^{\frac{1}{2}} \quad \text{a.s.}$$

Thus by Cauchy's inequality and (5.1) we obtain for all $f \in \Lambda$

$$|\Sigma_{n\leqslant N} f(\xi_n)| = |\Sigma_{h=-\infty}^{\infty} a_h \rho_h^{-1} \Sigma_{n\leqslant N} \rho_h e^{2\pi i h \xi_n}|$$
$$\leqslant (\Sigma_{h=-\infty}^{\infty} a_h^2 \rho_h^{-2})^{\frac{1}{2}} \|\Sigma_{n\leqslant N} y_n\| \ll (N \log\log N)^{\frac{1}{2}} \quad \text{a.s.}$$

This proves Theorem 5.1.

REMARKS. The application of Theorem 3.1 of Kuelbs and Kurtz was kindly suggested to us by Kuelbs. In an earlier version of this paper we had proved Theorem 5.1 via the following exponential bound for sums of independent bounded Hilbert space valued random variables which might be of independent interest.

A refinement of Theorem 5.1 was recently obtained by Kuelbs and Philipp (1977).

PROPOSITION 5.1. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables with values in a separable Hilbert space* H *and* $EX_n = 0$, $n \geqslant 1$. *Denote the norm in* H *by* $\|\cdot\|$. *Suppose that*

$$\|X_j\| \leqslant b_j \tag{5.2}$$

for some real b_j. Then for each real t

$$E(\exp(t\|\Sigma_{n \leqslant N} X_n\|)) \leqslant 3 + 2\exp(2t^2 \Sigma_{n \leqslant N} b_n^2). \tag{5.3}$$

When H has dimension 1 this is only slightly weaker than one of the classical inequalities.

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