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THE UNIFORM DIMENSION OF THE LEVEL SETS OF A BROWNIAN SHEET

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Let $W_N(\mathbf{t})$ denote the N-parameter Brownian sheet (Wiener process) taking values in R^1. For $0 < T \leqq 1$, set $\Delta(T) = \{\mathbf{t} \in R^N : 0 < t_i \leqq T, i = 1, \cdots, N\}$ and let $E(x, T) = \{\mathbf{t} \in \Delta(T) : W_N(\mathbf{t}) = x\}$, the set of $\mathbf{t}$ where the process is at the level x. Then we show that, with probability one, the Hausdorff dimension of $E(x, T)$ equals $N - \frac{1}{2}$ for all $0 < T \leqq 1$ and every x in the interior of the range of $W_N(\mathbf{t})$, $\mathbf{t} \in \Delta(T)$. This provides an answer to a question raised earlier by Pyke.

1. Introduction and main result. Let W_N, or simply W, denote the N-parameter Brownian sheet (Wiener process) taking values in R^1; i.e., $W_N(\mathbf{t}, \omega)$ is a real valued, Gaussian random field with mean zero and covariance

$$E\{W_N(\mathbf{s}, \omega)W_N(\mathbf{t}, \omega)\} = \prod_{i=1}^N \min(s_i, t_i),$$

where $\mathbf{s} = (s_1, \cdots, s_N)$, $\mathbf{t} = (t_1, \cdots, t_N)$, and $\mathbf{s}, \mathbf{t} \in R_+^N$, $R_+^N = \{\mathbf{t} \in R^N : t_i \geqq 0\}$. For $\mathbf{t} \in R_+^N$, let $\Delta(\mathbf{t})$ denote the N-dimensional rectangle $\{\mathbf{s} \in R_+^N : s_i \leqq t_i\}$, and for $T > 0$, let $\Delta(T)$ denote the cube $\{\mathbf{s} \in R_+^N : s_i \leqq T\}$. In [7], Pyke raised the question of what dimensional properties are possessed almost surely (a.s.) by the zero set $\{\mathbf{t} \in \Delta(1) : W_N(\mathbf{t}) = 0\}$. The arguments generally used to establish such results are, like those in our earlier study [1], of a capacitarian form. However, in this paper, we use arguments based on local time, which will allow us to obtain a result that holds with probability one for all level sets simultaneously. In particular, we shall obtain

THEOREM. *If $E(x, T) = \{\mathbf{t} \in \Delta(T) : W_N(\mathbf{t}) = x\}$ denotes the set of $\mathbf{t}$ where the process is at the level x, then, with probability one,*

$$\dim [E(x, T)] = N - \tfrac{1}{2} \tag{1.1}$$

for all $0 < T \leqq 1$ and every x in the interior of the range of $W_N(\mathbf{t})$ for $\mathbf{t} \in \Delta(T)$.

Note that the one-dimensional process $X(t)$ defined by $X(t) = W_N(1, \cdots, 1, t)$ is a simple Brownian motion, and thus by the law of the iterated logarithm for such processes the point $x = 0$ always lies in the interior of the range of $X(t)$, $t \in [0, T]$ for any $T > 0$. Hence $x = 0$ is also always in the interior of the range of $W_N(\mathbf{t})$, $\mathbf{t} \in \Delta(1)$, and so (1.1) is always true a.s. when $x = 0$. Thus the theorem provides an answer to Pyke's question.

We shall prove the theorem in Section 3. The key to our argument is the observation made by S. M. Berman (for the case $N = 1$) that the "uniform

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irregularity" of the sample paths implies the "uniform regularity" of the corresponding local time. We therefore start by considering the properties of local time for W_N.

2. Some results on the local time of W_N. We commence by defining a local time. Let ε be an arbitrary positive number, and let $W_\varepsilon(\mathbf{t}) = W_N(t_1 + \varepsilon, \cdots, t_N + \varepsilon)$. For any Borel set B of R^1 let $\mu_\varepsilon(\mathbf{t}, B, \omega)$ be the occupation measure of B defined by

$$\mu_\varepsilon(\mathbf{t}, B, \omega) = \lambda_N\{\mathbf{s} \in \Delta(\mathbf{t}) : W_\varepsilon(\mathbf{s}, \omega) \in B\} ,$$

where we use λ_N to denote N-dimensional Lebesgue measure. Then by the lemma of Tran [9] we have that for almost every ω there exists a real valued function $L_\varepsilon(x, \mathbf{t}, \omega)$, jointly continuous in $(x, t_1, \cdots, t_N)$ such that

$$\mu_\varepsilon(\mathbf{t}, B, \omega) = \int_B L_\varepsilon(x, \mathbf{t}, \omega)\, dx .$$

Furthermore, we may, and shall, use the version of L_ε given by

$$L_\varepsilon(x, \mathbf{t}, \omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-iux} \int_{\Delta(\mathbf{t})} e^{iuW_\varepsilon(\mathbf{s},\omega)}\, d\mathbf{s}\, du . \tag{2.1}$$

Note that (2.1) can be rewritten as follows (cf. Cairoli and Walsh [4]):

$$\begin{aligned} L_\varepsilon&(x, \mathbf{t}, \omega) \\ &= \int_{s_1=0}^{t_1} \cdots \int_{s_{N-1}=0}^{t_{N-1}} \{(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-iux} \int_{s_N=0}^{t_N} e^{iuW_\varepsilon(\mathbf{s},\omega)}\, ds_N\, du\}\, ds_1 \cdots ds_{N-1} \\ &= \int \cdots \int l_\varepsilon(\mathbf{s} : x, t_N, \omega)\, ds_1 \cdots ds_{N-1} . \end{aligned} \tag{2.2}$$

Here l_ε is simply the local time of the diffusion $Y(t) = W(s_1 + \varepsilon, \cdots, s_{N-1} + \varepsilon, t + \varepsilon)$. By a result of Ray [8] it follows that l_ε is a.s. strictly positive for every x in the interior of the range of $Y(t)$. Now suppose x lies in the interior of the range of $W(\mathbf{t})$, $\mathbf{t} \in \Delta(T)$, for some fixed $T > 0$. Then we can choose some point $\mathbf{s} \in \Delta(T)$ for which $W_\varepsilon(\mathbf{s}) = x$ and for which $W_\varepsilon(\mathbf{s})$ is neither a local maximum nor a local minimum. Consider now the $(N-1)$-dimensional subspace $\Delta(\infty, \cdots, \infty, 0)$ of R_+^N. Let $\mathbf{s}^*$ be the projection of $\mathbf{s}$ into this subspace. Then by the continuity of W_ε there is an open neighbourhood η^* of $\mathbf{s}^*$ in this subspace, of positive λ_{N-1} measure, for which for every $\mathbf{r} \in \eta^*$, x is contained in the interior of the range of the diffusion $W(r_1 + \varepsilon, \cdots, r_{N-1} + \varepsilon, t + \varepsilon)$, $t \in [0, t_N]$. Thus, for every $\mathbf{r} \in \eta^*$, $l_\varepsilon(\mathbf{r} : x, t_N, \omega)$ is a.s. strictly positive, implying, by (2.2), that the same is true of $L_\varepsilon(x, \mathbf{t}, \omega)$. Thus we have

LEMMA 1. *For any* $\mathbf{t} \in \Delta(1)$ *the local time* $L_\varepsilon(x, \mathbf{t}, \omega)$ *is* a.s. *strictly positive on any interval interior to the range of* $W_N(\mathbf{s}, \omega)$, $\mathbf{s} \in \Delta(\mathbf{t})$.

To establish (1.1) we shall need a uniform Hölder condition on the local time. This is obtained in Lemmata 2 and 4, which correspond closely to Theorems 3.1 and 4.1 of Berman [3]. For each $x \in [0, \infty]$, $\mathbf{t} \in R_+^N$, and real valued, continuous, $F: R^{N+1} \to R$ we use $F(\langle(x, \mathbf{t}), (x + h, \mathbf{t} + \mathbf{k})\rangle)$ to denote the usual increment of F over the $(N+1)$-dimensional rectangle $[x, x+h] \times \prod_{i=1}^N [t_i, t_i + k_i]$, while for fixed x, $F(x, \langle\mathbf{t}, \mathbf{t} + \mathbf{k}\rangle)$ denotes the increment of F (considered as a function of N variables only) over $\prod_{i=1}^N [t_i, t_i + k_i]$.

LEMMA 2. *Let* $Y(x, \mathbf{t})$, $x \in [0, 1)$, $\mathbf{t} \in \Delta(1)$, *be an* $(N + 1)$*-dimensional random field. Suppose there are positive constants* r, b, c, d *such that*:

$$(2.3) \qquad E|Y(x + h, \mathbf{t}) - Y(x, \mathbf{t})|^r \leqq b|h|^{1+c} \quad \text{for} \quad x, \quad x + h \in [0, 1], \quad \mathbf{t} \in \Delta(1),$$

$$(2.4) \qquad E|Y(x, \langle \mathbf{t}, \mathbf{t} + \mathbf{k} \rangle)|^r \leqq b \prod_{i=1}^N |k_i|^{1+d} \quad \text{for} \quad x \in [0, 1], \quad \mathbf{t}, \quad \mathbf{t} + \mathbf{k} \in \Delta(1),$$

$$(2.5) \qquad E|Y(\langle (x, \mathbf{t}), (x + h, \mathbf{t} + \mathbf{k}) \rangle)|^r \leqq b|h|^{1+c} \prod_{i=1}^N |k_i|^{1+d} \quad \text{for} \quad x, \quad x+h \in [0, 1], \quad \mathbf{t}, \quad \mathbf{t}+\mathbf{k} \in \Delta(1).$$

Then for every $\gamma < d/r$ *there exists a version of* Y, *and random variables* η *and* ξ *which are* a.s. *positive and finite, such that for all* $x \in [0, 1]$, $\mathbf{t}$, $\mathbf{t} + \mathbf{k} \in \Delta(1)$, *and* $\max |k_i| < \eta$,

$$(2.6) \qquad |Y(x, \langle \mathbf{t}, \mathbf{t} + \mathbf{k} \rangle)| \leqq \xi \prod_{i=1}^N |k_i|^\gamma .$$

PROOF. Without losing any generality, we shall establish (2.6) for the notationally simple case $k_i = k$ for all i. Let $\mathbf{i}$ denote an integer lattice point in R_+^{N+1}, i.e., each of its components is a nonnegative integer. For each such $\mathbf{i}$, let $\mathbf{i}_n$ denote the point $(i_1 2^{-n}, \cdots, i_{N+1} 2^{-n})$. Now choose $\alpha \in (0, 1)$ and put

$$A_{n\mathbf{i}} = \{\omega : Y(\langle (\mathbf{i} - \mathbf{1})_n, \mathbf{i}_n \rangle)| \geqq 2^{-(\alpha n/r)(c+Nd)}\} .$$

Then by (2.5) and a suitable form of Tchebychev's inequality,

$$P\{A_{n\mathbf{i}}\} \leqq b 2^{-n(N+1)} 2^{-nc(1-\alpha)} 2^{-nNd(1-\alpha)} .$$

Since $\sum_{\mathbf{i}} P\{A_{n\mathbf{i}}\} < \infty$ for all n, it follows from the Borel–Cantelli lemma that there exists an a.s. finite, positive random variable ν for which

$$\max_{1 \leqq i_1 \cdots i_{N+1} \leqq 2^n} |Y(\langle (\mathbf{i} - \mathbf{1})_n, \mathbf{i}_n \rangle)| \leqq 2^{-(\alpha n/r)(c+Nd)}$$

for all $n \geqq \nu$. From this it follows in the standard fashion (cf. [5], [10]) that there exists a constant D_1 and an a.s. finite random variable η' such that for all $\gamma_1 < \alpha c/r$, $\gamma_2 < \alpha d/r$, and h and $\mathbf{k}$ for which $|h| < \eta'$, $|k| < \eta'$

$$(2.7) \qquad |Y(\langle (x, \mathbf{t}), (x + h, \mathbf{t} + \mathbf{k}) \rangle)| \leqq D_1 |h|^{\gamma_1} |k|^{N\gamma_2} .$$

Letting α approach 1 we see that (2.7) in fact holds for all $\gamma_1 < c/r$, $\gamma_2 < d/r$.

By Corollary 2 of Theorem 2 of [10], (2.4) implies that for each $x \in [0, 1]$ and $\gamma_2 < d/r$ there exists a finite constant D_2 and a.s. finite, positive, random η such that

$$(2.8) \qquad |Y(x, \langle \mathbf{t}, \mathbf{t} + \mathbf{k} \rangle)| \leqq D_2 |k|^{N\gamma_2}$$

whenever $|k| < \eta$. It follows from the fact that (2.4) is uniform in x that D_2 is independent of x, so that to establish (2.6) we need only construct an η also independent of x. However, given (2.7) and (2.8) we can follow the form of the construction given on pages 72–73 of [3] to produce such an η, as well as an appropriate ξ, and thus complete the proof of the lemma.

LEMMA 3. *There exists a finite $B > 0$ such that the multiple integral*

$$\int_{\Delta(\mathbf{t},\mathbf{t}+\mathbf{k})} \cdots \int_{\Delta(\mathbf{t},\mathbf{t}+\mathbf{k})} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{n} |u^j|^{\delta} \times E\{\exp[i \textstyle\sum_{j=1}^{n} u^j W_\varepsilon(\mathbf{t}^j)]\} \prod_{j=1}^{n} du^j \prod_{j=1}^{n} d\mathbf{t}^j \tag{2.9}$$

is at most equal to

$$B|k_1 \cdots k_N|^{n(N-\frac{1}{2}-\delta)} \tag{2.10}$$

for all δ such that $0 \leqq \delta < N - \frac{1}{2}$ and all $\mathbf{t}$ and $\mathbf{k}$ such that $\mathbf{t}, \mathbf{t} + \mathbf{k} \in \Delta(1)$.

PROOF. Tran [9] has shown that (2.9) is bounded by

$$B'\{\int_0^{k_N} \cdots \int_0^{k_1} [t_1 + \cdots + t_N]^{-(\delta+\frac{1}{2})} dt_1 \cdots dt_{N-1}\}^n$$

for $\mathbf{t}$, $\mathbf{k}$ and δ as described in the theorem, and some finite positive B'. Performing the integration then establishes (2.10).

LEMMA 4. *For every $\varepsilon > 0$ there is a version of the local time $L_\varepsilon(x, \mathbf{t}, \omega)$ jointly continuous in all $N + 1$ variables, such that for every interval $[a, b] \in R_+^1$ and rectangle $\langle \mathbf{c}, \mathbf{d} \rangle = \prod_{i=1}^{N} [c_i, d_i] \in R_+^N$, and every $\gamma < \frac{1}{2}$, there exist random variables η and ξ which are* a.s. *positive and finite for which*

$$|L_\varepsilon(x, \langle \mathbf{t}, \mathbf{t} + \mathbf{k} \rangle)| \leqq \xi \prod_{i=1}^{N} |k_i|^{\gamma} \tag{2.11}$$

for all $x \in [a, b]$, $\mathbf{t}, \mathbf{t} + \mathbf{k} \in \langle \mathbf{c}, \mathbf{d} \rangle$, and all $\mathbf{k}$ for which $\max |k_i| < \eta$.

PROOF. For simplicity set $a = c_1 = \cdots = c_N = 0$, $b = d_1 = \cdots = d_N = 1$. We need only show that the conditions of Lemma 2 are satisfied by L_ε. Choose $\delta \in (0, N - \frac{1}{2}]$. Then by Tran [9], (page 29) and Lemma 3 we have that for large enough integral n

$$E|L_\varepsilon(\langle (x, \mathbf{t}), (x + h, \mathbf{t} + \mathbf{k}) \rangle)|^n \leqq B_n |h|^{n\delta} \prod_{i=1}^{N} |k_i|^{n(N-\frac{1}{2}-\delta)} \tag{2.12}$$

for some finite positive B_n. Furthermore, since $L_\varepsilon(x, t_1, \cdots, t_N) = 0$ if any $t_i = 0$, it follows from (2.12) by putting $\mathbf{t} = \mathbf{0}$ and $\mathbf{k} = \mathbf{t}$ in that expression that

$$E|L_\varepsilon(x + h, \mathbf{t}) - L_\varepsilon(x, \mathbf{t})|^n \leqq B_n |h|^{n\delta}. \tag{2.13}$$

Finally, noting that from the definition of L_ε that the nth moment of $L_\varepsilon(x, \langle \mathbf{t}, \mathbf{t} + \mathbf{k} \rangle)$ is a maximum at $x = 0$ for fixed $\mathbf{t}$ and $\mathbf{k}$ we obtain, using (2.12) and letting δ tend to zero, that for large enough n

$$E|L_\varepsilon(x, \langle \mathbf{t}, \mathbf{t} + \mathbf{k} \rangle)|^n \leqq B_n \prod_{i=1}^{N} |k_i|^{n(N-\frac{1}{2})}. \tag{2.14}$$

We now choose specific values of n and δ. For a given $\gamma < \frac{1}{2}$, let δ be a positive number satisfying $\delta < N - \frac{1}{2} - \gamma$. Then choose n (even) so large that the following three inequalities hold:

$$n(N - \tfrac{1}{2} - \delta) > 1$$
$$n\delta > 1$$
$$\gamma < (N - \tfrac{1}{2} - \delta) - n^{-1}.$$

Put $b = B_n$, $r = n$, $c = n\delta - 1$, $d = n(N - \frac{1}{2} - \delta) - 1$. These are all positive.

Then substituting these into (2.12)—(2.14) ensure that conditions (2.3)—(2.5) are satisfied by L_ε, and we are done.

The following result is an N-dimensional version of Lemma 6.1 of [3]. For any compact set $A \subset R^N$ we call a function $G: A \to R^1$ a distribution function if it is bounded on A and has nonnegative increments on rectangles in R^N.

LEMMA 5. *Let $G(\mathbf{t})$ be a distribution function on a compact set $A \subset R^N$, satisfying a Hölder condition of order $\gamma < 1$ at each point of A, If $B \subset A$, and* $\dim[B] = \beta$, *where $\beta \leqq N - 1 + \gamma$, then*

$$\int_B dG(\mathbf{t}) = 0 .$$

PROOF. If $\beta < N - 1$ the result is trivial. Suppose $\beta \geqq N - 1$. From the definition of Hausdorff dimension for every $\beta' > \beta$ and every n there is a covering of B by open balls I_{nk}, of diameter d_{nk}, $k = 1, 2, \cdots$ such that $d_{nk} \leqq n^{-1}$ for each k and

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} (d_{nk})^{\beta'} < \infty .$$

If $\gamma > \beta' - N + 1$, then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} (d_{nk})^{N-1+\gamma} = 0 .$$

For any $\beta \in [N - 1, N - 1 + \gamma]$ choose β' so that $\beta < \beta' < N - 1 + \gamma$. Then by the above limit result and the uniform Hölder condition on G we have

$$\begin{aligned}\int_B dG(\mathbf{t}) &\leqq \limsup_{n\to\infty} \sum_{k=1}^{\infty} \int_{I_{nk}} dG(\mathbf{t}) \\ &\leqq \text{const.} \lim_{n\to\infty} \sum_{k=1}^{\infty} (d_{nk})^{N-1+\gamma} \\ &= 0\end{aligned}$$

LEMMA 6. *With probability one, the set*

$$\{x: \dim[\mathbf{s} \in \Delta(\mathbf{t}): W_N(\mathbf{s}) = x] < \beta\}$$

is included in the set of zeroes of $L_\varepsilon(x, \mathbf{t})$ for any $\mathbf{t} \in \Delta(1)$ and any $\beta \leqq N - \frac{1}{2}$.

PROOF. The local time L_ε is, for every $\varepsilon > 0$, a.s. a distribution function in $\mathbf{t}$ over $\Delta(1)$ for every x. Furthermore, it follows from Lemma 1.5 of [3] and (2.2) that, for each x, $L_\varepsilon(x, \mathbf{t})$ has its support contained in $\{\mathbf{s} \in \Delta(\mathbf{t}): W_\varepsilon(\mathbf{s}) = x\}$. If this set has dimension less than β, then since L_ε satisfies an a.s. Hölder condition of order γ for every $\gamma < \frac{1}{2}$ (Lemma 4) it follows from Lemma 5 that $L_\varepsilon(x, \mathbf{t}) = 0$ a.s., and the lemma is established.

The following lemma will be used to provide an upper bound on the dimension of the level sets of W. It corresponds to the type of bound provided by results such as Lemma 7.2 of [3] in the case $N = 1$, although it is actually considerably stronger than that result, and may in fact be of some interest in its own right. Defining the local time of a nonrandom function exactly as for a random function, we can state

LEMMA 7. *Let $F: R^N \to R^1$ satisfy a uniform Hölder condition of every order less than γ, $0 < \gamma < 1$, on $\Delta(1)$, and for every $\varepsilon > 0$ possess a jointly continuous local* *time L_ε, Then for every real x*

$$\dim [\mathbf{t} \in \Delta(1): F(\mathbf{t}) = x] \leqq N - \gamma .$$

PROOF. For each $n \leqq 1$ and each integer lattice point $\mathbf{i} = (i_1, \cdots, i_N)$ for which $1 \leqq i_j \leqq 2^n$ for each j, set

$$J_{n\mathbf{i}} = \{\mathbf{t} \in \Delta(1): |t_j - (i_j - \tfrac{1}{2})2^{-n}| \leqq 2^{-(n+1)}, j = 1, \cdots, N\} ,$$
$$J^*_{n\mathbf{i}} = \{\mathbf{t} \in \Delta(1): |t_j - (i_j - \tfrac{1}{2})2^{-n}| < (2 - n^{-1})^{-(n+1)}, j = 1, \cdots, N\} .$$

Then $J_{n\mathbf{i}} \subset J^*_{n\mathbf{i}}$, and the $J^*_{n\mathbf{i}}$ are open. Furthermore, the sets $J^*_{n\mathbf{i}}$ for which at least one point of the set $F^{-1}(x) = \{\mathbf{t} \in \Delta(1): F(\mathbf{t}) = x\}$ lies in $J_{n\mathbf{i}}$ form an open covering of $F^{-1}(x)$. For any β, $0 < \beta < N$, the sum of the βth powers of the diameters of these $J^*_{n\mathbf{i}}$ is proportional to

$$(2.15) \qquad (2 - n^{-1})^{-n\beta} \times \#\{\mathbf{i}: F(\mathbf{t}) = x \quad \text{for some} \quad \mathbf{t} \in J_{n\mathbf{i}}\} .$$

By hypothesis, for every $\delta < \gamma$ there exists a $D > 0$ such that

$$|F(\mathbf{t}) - F(\mathbf{s})| \leqq D|\mathbf{t} - \mathbf{s}|^\delta , \qquad \text{for} \quad \mathbf{s}, \mathbf{t} \in J_{n\mathbf{i}} ,$$

for large enough n and any $\mathbf{i}$. Thus, for large enough n (2.15) is not greater than

$$(2.16) \qquad (2 - n^{-1})^{-n\beta} \times \#\{\mathbf{i}: |F(\mathbf{t}) - x| \leqq D2^{-n\delta} , \quad \text{for all} \quad \mathbf{t} \in J_{n\mathbf{i}}] .$$

We shall now establish that (2.16) tends to zero as $n \to \infty$, if $\beta > N - \delta$. Since δ is an arbitrary number smaller than γ, this would complete the proof. Assume that (2.16) does not tend to zero, so that for some $\eta > 0$ (2.16) exceeds η for infinitely many n. Then from the definition of local time it is clear that for small enough ε the following inequality also holds for infinitely many n.

$$(2 - n^{-1})^{-n\beta}2^{nN} \int_{x-D2^{-n\delta}}^{x+D2^{-n\delta}} L_\varepsilon(y, \mathbf{1})\, dy > \eta .$$

But this contradicts the continuity of L_ε, which implies that the left-hand side of the above inequality is not greater than

$$2D \max_y L_\varepsilon(y, \mathbf{1})2^{-n(\delta-N)}(2 - n^{-1})^{-n\beta}$$
$$= 2D \max_y L_\varepsilon(y, \mathbf{1})2^{-n(\delta+\beta-N)}(1 - 1/(2n))^{-n\beta}$$

which tends to zero as $n \to \infty$. This completes the proof of the lemma.

3. Proof of the theorem. Without losing any generality, we shall establish (1.1) for the case $T = 1$. Let $W^{-1}(x)$ denote the set $\{\mathbf{t} \in \Delta(1): W_N(\mathbf{t}) = x\}$ and RW the interior of the range of $W_N(\mathbf{t})$, $\mathbf{t} \in \Delta(1)$. For each $\varepsilon > 0$ we have by Lemma 1 that $L_\varepsilon(x, \mathbf{1}) > 0$ for all $x \in RW$ a.s. so that by Lemma 6

$$(3.1) \qquad \dim [W^{-1}(x)] \geqq N - \tfrac{1}{2} \qquad \text{for all} \quad x \in RW \quad \text{a.s.}$$

However, it follows from the results of [6] that W_N satisfies an a.s. uniform Hölder condition over $\Delta(1)$ of order γ for every $\gamma < \frac{1}{2}$. Thus, by Lemmas 4 and 7 we have

$$\dim W^{-1}(x) \leqq N - \tfrac{1}{2} \qquad \text{for all} \quad x , \quad \text{a.s.} ,$$

which, together with (3.1), completes the proof.

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