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ON THE FIELDS OF SOME BROWNIAN MARTINGALES

BY DAVID A. LANE
University of Minnesota

Suppose $\{B_t\}_{t\geqq 0}$ is a standard 1-dimensional Brownian motion, and f is a continuous function with nonaccumulating zero set. For $t \geqq 0$, let $M_t = \int_0^t f(B_s)\, dB_s$. When does M generate the same fields as B? When does M generate the same fields as *some* Brownian motion? The answers to these questions are obtained; they involve the behavior of f around its zeros. Also, *either* M generates the same fields as some Brownian motion, *or* the fields of M support discontinuous martingales.

1. Introduction. Suppose $(\Omega, \mathscr{F}, P)$ is a probability space, and X is a stochastic process on $(\Omega, \mathscr{F}, P)$. Indicate by $\{\mathscr{F}_t^X\}_{t\geqq 0}$ the family of right-continuous, P-complete σ-fields generated by X.

DEFINITION. If X and Y are two processes defined on $(\Omega, \mathscr{F}, P)$, X and Y are *equivalent* if $\mathscr{F}_t^X = \mathscr{F}_t^Y$ for all t.

Now suppose that $\{B_t\}_{t\geqq 0}$ is a standard 1-dimensional Brownian motion on $(\Omega, \mathscr{F}, P)$. If $\{M_t, \mathscr{F}_t^B\}_{t\geqq 0}$ is a local martingale, it is well known that M can be represented as

$$M_t = M_0 + \int_0^t C_s\, dB_s$$

a.s. for each t, where M_0 is a constant and C is an $\mathscr{F}^B$-adapted and measurable process with $\int_0^t C_s^2\, ds < \infty$ for all t (see, for example, Kunita and Watanabe (1967)). In particular, this implies that every $\mathscr{F}^B$-martingale has continuous paths, which in turn implies that every $\mathscr{F}^B$-stopping time is predictable, a strong regularity property of B (see Chung and Walsh (1974) and Lemma 2 below).

The questions considered in this paper—and answered, for a special class of integrands described later—are:

(1) What conditions on $\{C_t\}_{t\geqq 0}$ guarantee that M is equivalent to B?

(2) More generally, when can we find some $\{\mathscr{F}_t^B\}_{t\geqq 0}$-adapted Brownian motion $\{X_t\}_{t\geqq 0}$ such that M is equivalent to X?

(3) If no such X exists, how "bad" are the fields $\{\mathscr{F}_t^M\}_{t\geqq 0}$—in particular, do they support discontinuous martingales?

In the next section, we show that these questions are easily settled for two important classes of integrands, simple functionals and nonrandom functions. We also present some examples indicating the complications which can arise with more complicated integrands. The remainder of the paper is devoted to

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answering the questions for integrands of the form $C_s = f(B_s)$ where f is continuous with nonaccumulating zero set.

2. Some examples.

EXAMPLE 1. Suppose there is a finite time set $\{0 = t_1 < \cdots < t_n < \infty\}$ such that $C_t(\omega) = \sum_{i=1}^n C_{t_i}(\omega) 1_{[t_i, t_{i+1})}(t)$, where C_{t_i} is $\mathscr{F}_{t_i}^B$-adapted and $P(C_{t_i} < \infty) = 1$ for $1 \leqq i \leqq n$. Set $M_t = \int_0^t C_s\, dB_s$.

CLAIM. If $P(C_{t_i} = 0) = 0$, $1 \leqq i \leqq n$, then M is equivalent to B; otherwise, M is equivalent to no Brownian motion.

PROOF. (a) Suppose $P(C_{t_i} = 0) = 0$, $1 \leqq i \leqq n$. Since $\mathscr{F}_0^B$ is trivial, C_0 is a nonzero constant; since $M_t = C_0 B_t$ for $t \leqq t_2$, clearly $\mathscr{F}_t^M = \mathscr{F}_t^B$ for t in $[0, t_2]$. But then C_{t_2} is $\mathscr{F}_{t_2}^M$-adapted, and for t in $[t_2, t_3]$, $B_t = (1/C_0)M_{t_2} + (1/C_{t_2})(M_t - M_{t_2})$, and so B_t is $\mathscr{F}_t^M$-adapted. Thus for t in $[0, t_3]$, $\mathscr{F}_t^B = \mathscr{F}_t^M$. By induction, then, B and M are equivalent.

(b) Suppose $P(C_{t_j} = 0) > 0$ for some $j \leqq n$. Then for any $A \in \mathscr{F}_t^M$, $t \in (t_j, t_{j+1})$, $A \cap \{C_{t_j} = 0\} \in \mathscr{F}_{t_j}^M$. If M were equivalent to a Brownian motion X, we could choose $A = \{X_t - X_{t_j} > 0\}$ and obtain a contradiction, since plainly, $P(A \cap \{C_{t_j} = 0\}) = P(A) \times P(C_{t_j} = 0) > 0$ while $A \cap \{C_{t_j} = 0\} \notin \mathscr{F}_{t_j}^X$!

Now every stochastic integral may be obtained as the limit of integrals with simple integrands (which may be taken to satisfy $P(C_{t_i} = 0) = 0$, $1 \leqq i \leqq n$). Hence *every local martingale on* $\{\mathscr{F}_t^B\}_{t \geqq 0}$ *may be approximated arbitrarily closely* (uniformly on almost all paths in the case of L^2-martingales) *by martingales equivalent to B.*

EXAMPLE 2. Suppose C is nonrandom: that is, $C_t(\omega) = f(t)$ with $\int_0^t f^2(s)\, ds < \infty$ for all t. Set $M_t = \int_0^t f(s)\, dB_s$, M is a Gaussian martingale. Denote Lebesgue measure by "leb."

CLAIM. If leb $\{s : f(s) = 0\} = 0$, M is equivalent to B; otherwise, M is equivalent to no Brownian motion.

PROOF. (a) If leb $\{s : f(s) = 0\} = 0$, then $B_t = \int_0^t 1/f(s)\, dM_s$, so B and M are equivalent.

(b) Suppose M is equivalent to a Brownian motion X. By Lemma 1 below, X may be represented as a stochastic integral of M: $X_t = \int_0^t D_s\, dM_s$. Then $X_t = \int_0^t D_s f(s)\, dB_s$. Since X is a Brownian motion, $\langle X \rangle_t = t$; since B is a Brownian motion, $\langle X \rangle_t = \int_0^t D^2(s) f^2(s)\, ds$. Taking derivatives, $P(D_s^2 f^2(s) = 1 \text{ a.s. (leb)}) = 1$. In particular, leb $\{s : f(s) = 0\} = 0$.

LEMMA 1. *Suppose f is a nonrandom function with $\int_0^t f^2(s)\, ds < \infty$ for all t and $M_t = \int_0^t f(s)\, dB_s$. Then every local martingale with respect to $\{\mathscr{F}_t^M\}_{t \geqq 0}$ can be represented as a stochastic integral of M—that is, if Y is such a local martingale, then there exists an $\mathscr{F}^M$-adapted measurable process C with $\int_0^t C_s^2\, ds < \infty$ for all t, and constant Y_0 such that $\{Y_0 + \int_0^t C_s\, dM_s\}_{t \geqq 0}$ is a version of Y.*

PROOF. Trivial modification of the special case $f \equiv 1$ as proved in Kallianpur (1977).

In the preceding example, even when $\text{leb}\{s: f(s) = 0\} > 0$ and M was not equivalent to any Brownian motion, Lemma 1 implied that the fields $\{\mathscr{F}_t^M\}_{t\geq 0}$ were still quite regular in that they supported no discontinuous martingales.

When the integrands are random and *not* simple, this no longer is the case as the following example shows.

EXAMPLE 3. Let $\tau = \inf\{s: |B_s| = 1\}$. Set $C_s = 1_{\{\tau < s\}}$. Then $\text{leb}\{s: C_s(\omega) = 0\}$ is just $\tau(\omega)$ and is a.s. positive. As was true in both Examples 1 and 2, this implies that $M = \int C\, dB$ is equivalent to no Brownian motion. But in this case, the fields of M are not so nice: in particular, they support a discontinuous martingale, $Y(t) = E(\tau \mid \mathscr{F}_t^M)$. That Y is discontinuous follows from the obvious fact that τ is an unpredictable $\{\mathscr{F}_t^M\}_{t\geq 0}$-stopping time and Lemma 2.

LEMMA 2. *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\mathscr{F}_t\}_{t\geq 0}$ an increasing family of subfields of $\mathscr{F}$. Suppose τ is an $\{\mathscr{F}_t\}_{t\geq 0}$-stopping time with $E(\tau) < \infty$. If the martingale $\{E(\tau \mid \mathscr{F}_t)\}_{t\geq 0}$ is continuous at τ, τ is predictable.*

PROOF. Immediate from the proof of Proposition 4 of Chung and Walsh (1974).

For the integrands of Examples 1 and 2, either the integrals were equivalent to B or they were equivalent to *no* Brownian motion. For general integrands, this dichotomy fails, as Example 4 shows.

EXAMPLE 4. Consider the integrand B. Then $M_t = \int_0^t B_s\, dB_s = (B_t^2 - t)/2$, so M is not equivalent to B but to $|B|$. As will be shown below in Lemma 4, however, there is a Brownian motion X which *is* equivalent to $|B|$ and hence to M.

Again, in Examples 1 and 2, the condition that the integrand not vanish a.s. (leb) was necessary and sufficient for the integral to be equivalent to a Brownian motion. For general integrands, this is not the case. The theorem of Section 3 shows that there exist integrands $\{D_t\}_{t\geq 0}$ such that $P(\text{leb}\{s: D_s = 0\} = 0) = 1$ and yet $\{\int_0^t D_s\, dB_s\}_{t\geq 0}$ is equivalent to *no* Brownian motion and generates fields which support discontinuous martingales.

3. Some definitions and the statement of the theorem. Let f be a continuous function. Define the *zero set*, Z_f, of f and the *crossing set*, C_f, of f as follows:

$$Z_f = \{x: f(x) = 0\}$$
$$C_f = \{x: f(x) = 0 \text{ and } \lim_{s\downarrow x} \operatorname{sgn} f(s) \neq \lim_{s\uparrow x} \operatorname{sgn} f(s)\},$$

where $\operatorname{sgn} f = 1_{\{f>0\}} - 1_{\{f<0\}}$.

For x in C_f, define

$$\gamma_f(x) = \inf\{s \geq 0: f(x+s) \neq -f(x-s)\}.$$

Let $\gamma C_f = \{s: s = \gamma_f(x) \text{ for some } x \text{ in } C_f\}$.

THEOREM. *Suppose that f is a continuous function, such that* leb $Z_f = 0$ *and* C_f *has no accumulation points. Let* $M_t^f = \int_0^t f(B_s)\, dB_s$. *Then*:

(1) *If* $\gamma C_f = \{0\}$, M^f *is equivalent to* B;

(2) *If* $\gamma C_f = \{0\} \cup \{\infty\}$, M^f *is equivalent to a Brownian motion* X, *which is itself equivalent to* B *reflected in a (perhaps infinite) interval*;

(3) *If* $\gamma C_f \cap (0, \infty) \neq \emptyset$, *then* M^f *is equivalent to no Brownian motion, and* $\{\mathcal{F}_t^{M^f}\}_{t \geqq 0}$ *supports discontinuous martingales.*

In Section 4, three basic lemmas are established; the theorem itself is proved in Sections 5—7.

4. Three lemmas. Lemma 3 shows that if for each path the integrand vanishes only on a set of zero Lebesgue measure, the fields of the integral are sufficiently rich to support a Brownian motion.

LEMMA 3. *Suppose* C *is a process such that*:

(a) $\{C_t\}_{t \geqq 0}$ *is* $\{\mathcal{F}_t^B\}_{t \geqq 0}$*-adapted and measurable*;

(b) *for all* t, $\int_0^t C_s^2\, ds < \infty$ a.s.; *and*

(c) $P(\text{leb}\{s\colon C_s = 0\} = 0) = 1$.

Then the Brownian motion $X = \int \text{sgn}\, C\, dB$ *generates fields* $\mathcal{F}^X$ *such that, for all* t, $\mathcal{F}_t^X \subseteqq \mathcal{F}_t^{\int C\,dB} \subseteqq \mathcal{F}_t^B$.

PROOF. Set $M = \int C\, dB$. If as usual we denote the quadratic variation of M as $\langle M \rangle \cdot (= \int_0^\cdot C_s^2\, ds)$, then, as is well known, $\langle M \rangle = M^2 - 2 \int M\, dM$ and so is $\mathcal{F}^M$-adapted and measurable. Since $\langle M \rangle$ is differentiable a.e. Lebesgue, we may define

$$D_t(\omega) = \overline{\lim}_{h \downarrow 0} \frac{\langle M \rangle_t - \langle M \rangle_{t-h}}{h}\,;$$

note that D is $\mathcal{F}^M$-predictable.

Set $F_t(\omega) = |D_t|^{-\frac{1}{2}}(\omega)$ (which is well defined, since $P(D_t = C_t^2$ a.e. Lebesgue$) = 1$ implies that $P(\text{leb}\{t\colon D_t(\omega) = 0\} = 0) = 1$. Then we may define the $\mathcal{F}^M$-adapted and measurable process $X_t = \int_0^t F_s\, dM_s$. But $X_t = \int_0^t F_s C_s\, dB_s = \int_0^t (C_s/|D_t|^{\frac{1}{2}})\, dB_s = \int_0^t \text{sgn}\, C_s\, dB_s$ (since the stochastic integral is unaffected by altering the integrand on sets of zero Lebesgue measure). Then $\langle X \rangle_t = \int_0^t [\text{sgn}\, C_s]^2 ds = t$ and so X is an $\mathcal{F}^M$-adapted Brownian motion.

The next lemma is fundamental. For $z \in R$, let $M_t = \int_0^t \text{sgn}\,(B_s - z)\, dB_s$, and let $\tau_z = \inf\{t\colon B_t = z\}$. Clearly, $M_t = B_t$ on $\{\tau_z > t\}$. On $\{\tau_z \leqq t\}$, $M_t = z + |B_t - z| - l_z(t)$, where $l_z(t, \omega) = \lim_{\varepsilon \downarrow 0} 1/2\varepsilon\, \text{leb}\{s \leqq t\colon |B_s(\omega) - z| < \varepsilon\}$, the local time of B at z (this is Tanaka's formula—see McKean (1969)).

Since l_Z is adapted to $\mathcal{F}^{|B-z|}$, so is M. Lemma 4 shows that the *converse* is also true—so that M is equivalent to $|B - z|$. (This lemma was proved independently by M. Yor (1977, Proposition 14).)

LEMMA 4. *Let* $z \in R$, $M_t = \int_0^t \text{sgn}\,(B_s - z)\, dB_s$, *then* M *is equivalent to* $|B - z|$.

PROOF. Clearly, it suffices to prove the lemma for $z = 0$, where it must be shown that $M = \int \operatorname{sgn} B_s \, dB_s = |B| - l_0$ is equivalent to $|B|$. As stated in the preceding paragraph, M is $\mathscr{F}^{|B|}$-adapted; the lemma will be proved once it is shown that $|B|_t$ may be recovered from $\{M_s\}_{s \leqq t}$. Here is a recipe for this recovery for almost all ω:

Fix t. Let $s' = \max\{s \leqq t : B_s(\omega) = 0\}$. Then $r > s'$ implies that $l_0(r, \omega) = l_0(s', \omega)$, since $l_0(\cdot, \omega)$ is an increasing function and the measure $dl_0(\cdot, \omega)$ is concentrated on $Z_{B\cdot}(\omega)$. Since $|B_{s'}(\omega)| = 0$, $M_{s'}(\omega) < M_r(\omega)$. If $r < s'$ then $l_0(r, \omega) \leqq l_0(s', \omega)$, and so $M_r(\omega) \geqq M_{s'}(\omega)$. Hence $s' = \max\{s \leqq t : M_s(\omega) = \min_{0 \leqq r \leqq t} M_r(\omega)\}$. Since $|B_t|(\omega) = M_t(\omega) - M_{s'}(\omega)$, $|B_t|(\omega)$ has been recovered from $\{M_s(\omega)\}_{s \leqq t}$.

The next lemma allows the proof of the equivalence between B and M asserted in the theorem to be carried out "locally." It will be stated in a more general setup. Suppose $\{X_t\}_{t \geqq 0}$ is a stochastic process on $(\Omega, \{\mathscr{G}_t\}_{t \geqq 0}, P)$, and $\tau_1 \leqq \tau_2$ are $\{\mathscr{G}_t\}_{t \geqq 0}$-stopping times. For each t, let $Y_t = X_{t \wedge \tau_1}$, so that $\{Y_t\}_{t \geqq 0}$ is also a stochastic process on $(\Omega, \{\mathscr{G}_t\}_{t \geqq 0}, P)$. Set $\mathscr{F}^X_{(\tau_1, \tau_2)} = \bigvee_{t \geqq 0} \mathscr{F}_t^Y$.

LEMMA 5. *Suppose X and M are processes on $(\Omega, \{\mathscr{G}_t\}_{t \geqq 0}, P)$, the fields $\{\mathscr{G}_t\}_{t \geqq 0}$ are right continuous, $\mathscr{G}_0$ is trivial and X has continuous paths. Suppose also that $\{\tau_n\}_{n \geqq 0}$ is a sequence of $\{\mathscr{F}_t^M\}_{t \geqq 0}$-stopping times, $\tau_0 = 0$ and $\tau_n \uparrow \infty$. If for all $n \geqq 0$ and $s \leqq t$, $\mathscr{F}^X_{(\tau_n \wedge s, \tau_{n+1} \wedge s)} \subset \mathscr{F}^M_{(\tau_n \wedge s, \tau_{n+1} \wedge s)}$, then $\mathscr{F}_t^X \subset \mathscr{F}_t^M$.*

PROOF. Since X has continuous paths and $\tau_n \to \infty$, for $s \leqq t$, $X_{\tau_n \wedge s} \to X_s$; hence it suffices to show that $X_{\tau_n \wedge s}$ is $\mathscr{F}_t^M$-measurable. $X_{\tau_n \wedge s} = X_{\tau_0} + (X_{\tau_1 \wedge s} - X_{\tau_0}) + \cdots + (X_{\tau_n \wedge s} - X_{\tau_{n-1} \wedge s})$, and by definition, $(X_{\tau_{k+1} \wedge s} - X_{\tau_k \wedge s})$ is $\mathscr{F}^X_{(\tau_k \wedge s, \tau_{k+1} \wedge s)}$-measurable—and thus, by assumption, is $\mathscr{F}^M_{(\tau_k \wedge s, \tau_{k+1} \wedge s)}$-measurable also. Since each τ_k is an $\{\mathscr{F}_t^M\}_{t \geqq 0}$-stopping time, $\mathscr{F}^M_{(\tau_k \wedge s, \tau_{k+1} \wedge s)} \subset \mathscr{F}_s^M \subset \mathscr{F}_t^M$. Thus $X_{\tau_n \wedge s}$ is $\mathscr{F}_t^M$-measurable, and so $\mathscr{F}_t^X \subset \mathscr{F}_t^M$.

5. Proof of theorem. $\gamma C_f = \{0\}$. For simplicity, we divide the argument into two cases.

CASE 1. $\gamma_f^{-1}\{0\} = 0$. Set $X_t = \int_0^t \operatorname{sgn} f(B_s) \, dB_s$. By Lemma 1, X_t is $\mathscr{F}_t^{M^f}$-measurable. Since $\operatorname{sgn} f(B_s) = (\lim_{x \downarrow 0} \operatorname{sgn} f(x)) \operatorname{sgn} B_s$, X is equivalent to $\int \operatorname{sgn} B_s \, dB_s$, and hence, by Lemma 2, to $|B|$. Thus $|B|_t$ is $\mathscr{F}_t^{M^f}$-measurable for each t.

Since $\langle M^f \rangle_t = \int_0^t f^2(B_s) \, ds$ and f is continuous, $\langle M^f \rangle_t' = f^2(B_t)$ for all t and so $|f(B_t)|$ is $\mathscr{F}_t^{M^f}$-measurable for all t.

Fix t. Let $s'(\omega) = \max\{s \leqq t : |B(s)| = 0\}$. Then s' is $\mathscr{F}_t^{M^f}$-measurable. For s in $(s'(\omega), t)$, either (A): $B_s = +|B_s|$ or (B): $B_s = -|B_s|$.

Since $\gamma(0) = 0$ and $|B_{s'(\omega)}| = 0$, for almost all ω we can find $\bar{s}(\omega)$ in $(s'(\omega), t)$ such that

$$f(|B_{\bar{s}}(\omega)|) \neq -f(-|B_{\bar{s}}(\omega)|) \text{—that is,}$$
$$|f(|B_{\bar{s}}(\omega)|)| \neq |f(-|B_{\bar{s}}(\omega)|)| \,.$$

Comparing $|B_{\bar{s}}(\omega)|$ with $|f(B_{\bar{s}}(\omega))|$ will therefore allow us to determine which of

A or B is true and hence we may determine $B_t(\omega)$. So $\mathscr{F}_t^B \subset \mathscr{F}_t^{M^f}$, and M^f is equivalent to B.

CASE 2. General $\gamma_f^{-1}\{0\}$. Let $\tau_0 \equiv 0$, $\tau_{n+1} = \inf\{t > \tau_n : B_t \in \{C_f - B_{\tau_n}\}\}$ for $n = 0, 1, 2, \cdots$.

CLAIM. *For all n and t*, $\mathscr{F}^B_{(0,\tau_n \wedge t)} \subset \mathscr{F}^{M^f}_{(0,\tau_n \wedge t)}$. Once the claim is established, an easy modification of Lemma 3 shows that, if τ_n is an $\{\mathscr{F}_t^M\}_{t \geq 0}$-stopping time, $\mathscr{F}_t^B = \mathscr{F}_t^{M^f}$.

PROOF OF CLAIM.

(1) Let $z_1 = \max(z \in C_f \cap (-\infty, 0))$ and $z_2 = \min(z \in C_f \cap (0, \infty))$,

$$\begin{aligned} g(x) &= f(x) \qquad \text{for} \quad x \quad \text{in} \quad [z_1, z_2)\,, \\ &> 0 \qquad \text{and continuous for} \quad x \quad \text{not in} \quad [z_1, z_2]\,, \end{aligned}$$

and $X_t = \int_0^t g(B_s)\, dB_s$. Then $M_{s\wedge\tau_1} = X_{s\wedge\tau_1}$ for all s. If $0 \notin C_f$, we may apply Lemma 1 to conclude that X is equivalent to B. If $0 \in C_f$, Case 1 above implies that X is equivalent to B. In either case, $\mathscr{F}^M_{(0,\tau_1\wedge t)} = \mathscr{F}^X_{(0,\tau_1\wedge t)} = \mathscr{F}^B_{(0,\tau_1\wedge t)}$.

(2) Assume for all t, $\mathscr{F}^M_{(0,\tau_n\wedge t)} = \mathscr{F}^B_{(0,\tau_n\wedge t)}$. Let $Y_s = \int_0^s \operatorname{sgn} f(B_r)\, dB_r$. By Lemma 1, for each s, Y_s is $\mathscr{F}_s^{M^f}$-measurable, and so the process $\{Z_s = Y_{\tau_{n+1}\wedge s} - Y_{\tau_n\wedge s} = \int_{\tau_n\wedge s}^{\tau_{n+1}\wedge s} \operatorname{sgn} f(B_r)\, dB_r\}_{0\leq s\leq t}$ is $\mathscr{F}^{M^f}_{(0,\tau_{n+1}\wedge t)}$-measurable. Note that $f(B_{\tau_n}) = 0$ and for r in (τ_n, τ_{n+1}),

$$\begin{aligned} \operatorname{sgn} f(B_r) &= C(B_{\tau_n}) \qquad \text{if} \quad B_r > B_{\tau_n} \\ &= -C(B_{\tau_n}) \qquad \text{if} \quad B_r < B_{\tau_n} \end{aligned}$$

where C is a function from $C_f \to \{-1, +1\}$.

Let $\tilde{B}_s = B_{\tau_n+s} - B_{\tau_n}$. Then $\{\tilde{B}_s\}_{s\geq 0}$ is Brownian motion, and by Lemma 2, $\int \operatorname{sgn} \tilde{B}\, d\tilde{B}$ is equivalent to $|\tilde{B}| = |B_{\tau_n+\cdot} - B_{\tau_n}|$. Set $\tilde{Z}_s = Z_{\tau_n+s} \cdot 1^{(s)}_{(0,\tau_{n+1}-\tau_n)}$. Then $\tilde{Z}_s = 1^{(s)}_{(0,\tau_{n+1}-\tau_n)} \int_0^s \operatorname{sgn} f(B_{\tau_n+r})\, dB_{\tau_n+r} = 1^{(s)}_{(0,\tau_{n+1}-\tau_n)} \cdot C(B_{\tau_n}) \int_0^s \operatorname{sgn}(\tilde{B}_r)\, d\tilde{B}_r$. Hence $\{|B_{\tau_{n+1}\wedge s} - B_{\tau_n\wedge s}|\}$ is $\mathscr{F}^{M^f}_{(0,\tau_{n+1}\wedge t)}$-measurable.

Since $\gamma_f(B_{\tau_n}) = 0$ and by assumption $B_{\tau_n\wedge t}$ is $\mathscr{F}^{M^f}_{(0,\tau_n\wedge t)}$-measurable, the argument of Case 1 shows that $\{\operatorname{sgn}(B_{\tau_{n+1}\wedge s})\}_{0\leq s\leq t}$ is $\mathscr{F}^{M^f}_{(0,\tau_{n+1}\wedge t)}$-measurable, and we may conclude that $\mathscr{F}^B_{(0\ \tau_{n+1}\wedge t)} \subset \mathscr{F}^{M^f}_{(0,\tau_{n+1}\wedge t)}$ for all t, and hence that τ_n is an $\{\mathscr{F}_t^{M^f}\}_{t\geq 0}$-stopping time for each n. Thus $\mathscr{F}_t^{M^f} = \mathscr{F}_t^B$ for all t.

6. Proof of theorem. $\gamma C_f = \{\infty\} \cup \{0\}$.

CASE 1. $z = \gamma_f^{-1}\{\infty\}$. Define the Brownian motion $X = \int \operatorname{sgn}(B_s - z)\, dB_s$. As Lemma 4 shows, X is equivalent to $|B - z|$.

CLAIM. *M^f is equivalent to X.*

(1) $\mathscr{F}_t^{M^f} \subset \mathscr{F}_t^X$.

(a) First, suppose f is continuously differentiable. Let

$$\begin{aligned} F(x) &= \int_z^x f(y)\, dy \qquad \text{for} \quad x \geq z \\ &= F(2z - x) \qquad \text{for} \quad x < z\,. \end{aligned}$$

Then $F'(x) = f(x)$ for $x > z$, and $F'(z + y) = -F'(z - y)$ for all $y \geqq 0$. Since $\gamma_f(z) = \infty, f(z + y) = -f(z - y)$ for all $y \geqq 0$ also. Thus, $F'(x) = f(x)$ for all x.

From the definition of F, it is clear that there exist continuous functions G and g defined on R^+ such that $F(x) = G|x - z|$ and $F''(x) = g|x - z|$. By Itô's formula,

$$M_t^f = G|B_t - z| - G|z| - \tfrac{1}{2} \textstyle\int_0^t g|B_s - z| \, ds ,$$

so

$$M_t^f \text{ is } \mathscr{F}_t^{|B-z|} (= \mathscr{F}_t^X) \text{ measurable.}$$

(b) If f is *not* continuously differentiable, select a sequence $\{f_n\}$ such that, for each n,

(i) f_n is continuously differentiable;
(ii) $f_n(z + x) = -f_n(z - x)$ for all $x > 0$;
(iii) $\sup_{x \in (-n,n)} |f_n(x) - f(x)| < 1/n$; and
(iv) $\int_0^t f_n^2(B_s) \, ds < \infty$ a.s.

Then $|\int_0^t f_n^2(B_s) \, ds - \int_0^t f^2(B_s) \, ds| < t/n$ on $\{\sup_{s \leq t} B_s \leq k\}$ for $n > k$. Since $P\{\sup_{s \leq t} B_s > k\} \to 0$ as $k \to \infty$, $\int_0^t f_n^2(B_s) \, ds \to \int_0^t f^2(B_s) \, ds$ in probability, and so $\int_0^t f_n(B_s) \, dB_s \to \int_0^t f(B_s) \, dB_s$ in probability also. Thus $\mathscr{F}_t^{Mf} \subset \bigcup_n \mathscr{F}_t^{Mf_n}$ and by the previous paragraph, for each n $\mathscr{F}_t^{Mf_n} \subset \mathscr{F}_t^X$.

(2) $\mathscr{F}_t^X \subset \mathscr{F}_t^{Mf}$.

(a) First, suppose $C_f = \{z\}$ and $\gamma_f(z) = \infty$. Then $\operatorname{sgn} f(B_s) = [\lim_{x \downarrow 0} \operatorname{sgn} f(z + x)] \times \operatorname{sgn}(B_s - z)$, so $\int \operatorname{sgn} f(B_s) \, dB_s$ is equivalent to X. By Lemma 3, $\int \operatorname{sgn} f(B_s) \, dB_s$ is adapted to $\{\mathscr{F}_t^{Mf}\}_{t \geqq 0}$. Hence $\mathscr{F}_t^X \subseteqq \mathscr{F}_t^{Mf}$ for all $t \geqq 0$.

(b) Next, suppose $C_f = \{z\} \cup \gamma_f^{-1}\{0\}$, and $\gamma_f(z) = \infty$. Set $\tau_1 = \inf(t: B_t = z)$. Let $x_1 = \min\{x > 0: z + x \in C_f\}$. For $k = 1, 2, \cdots$, set $\tau_{2k} = \inf\{t > \tau_{2k-1}: |B_t - z| = x\}$ and $\tau_{2k+1} = \inf\{t > \tau_{2k}: B_t = z\}$. The proof is concluded by an appeal to Lemma 5:

Define bounded continuous g with

(i) $g = f$ on $(z - x_1, z + x_1)$;
(ii) $g(z + s) = -g(z - s)$ for all s; and
(iii) $C_g = \{z\}$.

By (a) above, M^g is equivalent to $|B_t - z|$. Since $\mathscr{F}^{Mf}_{(0,\tau_2 \wedge t)} = \mathscr{F}^{Mf}_{(0,\tau_2 \wedge t)}$, $\mathscr{F}^{|B-z|)}_{(0,\tau_2 \wedge t)} = \mathscr{F}^{Mf}_{(0,\tau_2 \wedge t)}$.

Now set $h(x) = f(x)$ for $x \geqq z$ and $= -f(x)$ for $x < z$. Then $C_h = C_f - \{z\}$ and $\gamma C_h = \{0\}$, so M^h is equivalent to B. Since $\mathscr{F}^{Mf}_{(\tau_2 \wedge t, \tau_3 \wedge t)} = \mathscr{F}^{Mh}_{(\tau_2 \wedge t, \tau_3 \wedge t)} = \mathscr{F}^{B}_{(\tau_2 \wedge t, \tau_3 \wedge t)}$, it follows that:

(i) τ_3 is an $\{\mathscr{F}_t^{Mf}\}_{t \geqq 0}$-stopping time; and
(ii) $\mathscr{F}^{|B-z|}_{(\tau_2 \wedge t, \tau_3 \wedge t)} \subset \mathscr{F}^{Mf}_{(\tau_2 \wedge t, \tau_3 \wedge t)}$.

Similar arguments yield: for all n, τ_n is an $\{\mathscr{F}_t^{Mf}\}_{t \geqq 0}$-stopping time, and $\mathscr{F}^{|B-z|}_{(\tau_n \wedge t, \tau_{n+1} \wedge t)} \subset \mathscr{F}^{Mf}_{(\tau_n \wedge t, \tau_{n+1} \wedge t)}$. By Lemma 5, then, $\mathscr{F}_t^{|B-z|} \subset \mathscr{F}_t^{Mf}$ for all $t \geqq 0$. Thus the claim is established for this case: if $\gamma^{-1}\{\infty\}$ consists of just one point z, then M^f is equivalent to $\int \operatorname{sgn}(B_s - z) \, dB_s$.

Case 2. Card $\gamma^{-1}\{\infty\} > 1$. Let $z_1 = \sup\{z : z \in (-\infty, 0] \cap \gamma^{-1}\{\infty\}\}$ and $z_2 = \inf\{z : z \in (0, \infty) \cap \gamma^{-1}\{\infty\}\}$, and set $d = z_2 - z_1$. Then z_1 and z_2 are finite, f is periodic with period $2d$, and $\gamma^{-1}\{\infty\} = \{z_1 + kd, k = 0, \pm 1, \pm 2, \cdots\}$.

If $z = z_1 + 2kd \pm r$, k an integer, $r \in [0, d]$, say $z \bmod f = r$. Let $\tau_1 = \inf(t: B_t = z_1$ or $B_t = z_2)$ and define, for $t \geqq 0$, $X_t = B_t 1_{(\tau_1 > t)} + (B \bmod f) \cdot 1_{(\tau_1 \leqq t)}$.

(1) $\mathscr{F}_t^{M^f} \subset \mathscr{F}_t^X$ for all $t \geqq 0$. Suppose h is continuously differentiable and $\gamma_h^{-1}\{\infty\} = \gamma_f^{-1}\{\infty\}$. Let $H(x) = \int_{z_1}^{x} h(y)\,dy$ for $x \in [z_1, z_1 + 2d]$, and extend H by periodicity. There exist functions G and g defined on $[0, d]$ such that for all $H(z) = G(z \bmod h)$ and $H''(z) = g(z \bmod h)$. Applying Itô's formula, $M_t^h = G(B_t \bmod h) - G(0 \bmod h) - \frac{1}{2}\int_0^t g(B_s \bmod h)\,ds$. Thus $\mathscr{F}_t^{M^h} \subset \mathscr{F}_t^X$. As in Case 1, we may approximate continuous f with continuously differentiable h satisfying $\gamma_f^{-1}\{\infty\} = \gamma_h^{-1}\{\infty\}$, and so $\mathscr{F}_t^{M^f} \subset \mathscr{F}_t^X$ for all $t \geqq 0$.

(2) $\mathscr{F}_t^X \subset \mathscr{F}_t^{M^f}$ for all t. Let $\tau_{n+1} = \inf\{t > \tau_n : B_t \in \{\gamma^{-1}\{\infty\} - B_{\tau_n}\}\}$, $n = 1, 2, \cdots$. Then $\mathscr{F}^{M^f}_{(0,\tau_1 \wedge t)} = \mathscr{F}^{B}_{(0,\tau_1 \wedge t)}$ and for all n, $\mathscr{F}^{M^f}_{(\tau_n \wedge t, \tau_{n+1} \wedge t)} = \mathscr{F}^{|B - B_{\tau_n}|}_{(\tau_n \wedge t, \tau_{n+1} \wedge t)} = \mathscr{F}^{X}_{(\tau_n \wedge t, \tau_{n+1} \wedge t)}$ by Case 1 above. Since the only discontinuity of X occurs at τ_1, we may apply Lemma 3 to conclude that $\mathscr{F}_t^X \subset \mathscr{F}_t^{M^f}$ for all t.

(3) $\mathscr{F}_t^X = \mathscr{F}_t^Y$ for all $t \geqq 0$, where $Y_t = \int_0^t g(B_s)\,dB_s$,

$$\begin{aligned} g(x) &= 1 \qquad \text{on} \quad (z_1, z_2) \\ &= -1 \qquad \text{on} \quad (z_2, z_2 + d)\,, \end{aligned}$$

and g is periodic with period $2d$. Since $\langle Y \rangle_t = t$, Y is a Brownian motion. To show that Y is equivalent to X, apply Lemma 3 with the stopping times τ_n as in (2) above.

Thus M^f is equivalent to the Brownian motion Y, which is itself equivalent to B reflected in the period interval (z_1, z_2).

7. Proof of theorem. $\gamma_f^{-1}(0, \infty) \neq \varnothing$. The structure of C_f clutters up the argument in this case, but the basic idea is quite simple. If $\gamma_f^{-1}(0, \infty)$ is not empty, it is possible to construct a square-integrable $\{\mathscr{F}_t^{M^f}\}_{t \geqq 0}$-stopping time which is not predictable. Hence $\{\mathscr{F}_t^{M^f}\}_{t \geqq 0}$ cannot be generated by any Brownian motion, and—as Lemma 2 shows—these fields support discontinuous martingales.

The argument is given in detail in the simplest possible case (Case 1), while the extension to more complicated crossing and zero sets (Cases 2 and 3) is merely sketched.

Case 1. $C_f = \{0\}$ and $0 < \gamma_f(0) < \infty$. Set $\tau_1 = \inf(t: |B_t| = \gamma_f(0))$, and $\tau_2 = \tau_1 1_{\{B_{\tau_1} = \gamma_f(0)\}} + \infty \cdot 1_{\{B_{\tau_1} = -\gamma_f(0)\}}$. Set

$$\begin{aligned} g(x) &= f(x) \qquad & x \ \text{ in } \ [-\gamma_f(0), \gamma_f(0)] \\ &= f(\gamma_f(0)) \qquad & x \ \text{ in } \ [\gamma_f(0), \infty] \\ &= -f(\gamma_f(0)) \qquad & x \ \text{ in } \ (-\infty, -\gamma_f(0))\,. \end{aligned}$$

Then M^g is equivalent to $|B|$, and $\mathscr{F}^{M^f}_{(0,\tau_1)} = \mathscr{F}^{M^g}_{(0,\tau_1)} = \mathscr{F}^{|B|}_{(0,\tau_1)}$. Thus τ_1 is an $\{\mathscr{F}_t^{M^f}\}_{t \geqq 0}$-stopping time. By Lemma 3, for all $t \geqq 0$, $|f(B_t)|$ is $\mathscr{F}_t^{M^f}$-measurable, and by Lemmas 3 and 4, $|B_t|$ is $\mathscr{F}_t^{M^f}$-measurable also. Hence by the argument in Section 5, for all t, $A = \{\omega : \tau_1(\omega) < t \text{ and } B(\tau_1) = \gamma(0)\} = \{\tau_2 < t\}$ is in $\mathscr{F}_t^{Mf}$. By the right continuity of $\mathscr{F}_t^{Mf}$, $\{\tau_2 \leqq t\} \in \mathscr{F}_t^{Mf}$, and so τ_2 is an $\{\mathscr{F}_t^{Mf}\}_{t \geqq 0}$-stopping time.

Suppose σ is an $\{\mathscr{F}_t^{Mf}\}$-stopping time and $\sigma < \tau_2$ a.s. Let $S_1 = \{\omega : \sigma(\omega) > \tau_1(\omega)\}$ and $S_2 = \{\omega : \text{for all } t, B_t(\omega) = -B_t(\omega^*) \text{ for some } \omega^* \text{ in } S_1\}$. Then $P(S_2) = P(S_1)$. Suppose $\omega \in S_2$, $\omega^* \in S_1$ as in the definition of S_2. Since τ_1 is an $\{\mathscr{F}_t^{|B|}\}_{t \geqq 0}$-stopping time, $\tau_1(\omega) = \tau_1(\omega^*)$. Since $\omega^* \in S_1$, $B_{\tau_1}(\omega^*) = -\gamma_f(0)$ and so $B_{\tau_1}(\omega) = \gamma_f(0)$ and therefore $\tau_1(\omega) = \tau_2(\omega)$. So $\underline{(1) \sigma(\omega) < \tau_1(\omega)}$. But σ is an $\{\mathscr{F}_t^{Mf}\}_{t \geqq 0}$-stopping time, and $\mathscr{F}^{Mf}_{(0,\tau_1 \wedge t)} = \mathscr{F}^{|B|}_{(0,\tau_1 \wedge t)}$; thus (1) implies that $\sigma(\omega) = \sigma(\omega^*)$. But then $\sigma(\omega^*) < \tau_1(\omega) = \tau_1(\omega^*)$, a contradiction since $\omega^* \in S_1$. So we must have $P(S_2) = 0$; that is, $\sigma < \tau_1$ a.s. Since we can find $C > 0$ such that $P(\tau_2 - \tau_1 > C) > 0$, for any $\{\mathscr{F}_t^{Mf}\}_{t \geqq 0}$-stopping time $\sigma < \tau_2$ a.s., $P(\tau_2 - \sigma > C) > 0$. Hence τ_2 is not $\{\mathscr{F}_t^{Mf}\}_{t \geqq 0}$-predictable. Neither is $\tau = \tau_2 \wedge (\tau_1 + 1)$, and τ is square integrable.

CASE 2. General C_f, with $C_f = z_f$. Define z_1 and z_2 as follows:

(i) If $(-\infty, 0] \cap \gamma_f^{-1}(0, \infty) = \varnothing$, $z_1 = -\infty$. Otherwise $z_1 \in (-\infty, 0] \cap \gamma_f^{-1}(0, \infty)$ and if $z \in (-\infty, 0] \cap \gamma_f^{-1}(0, \infty)$, then $z_1 - \gamma_f(z_1) > z - \gamma_f(z)$.

(ii) If $(0, \infty) \cap \gamma_f^{-1}(0, \infty) = \varnothing$, $z_2 = \infty$. Otherwise $z_2 \in (0, \infty) \cap \gamma_f^{-1}(0, \infty)$, and if $z \in (0, \infty) \cap \gamma_f^{-1}(0, \infty)$, $z_2 + \gamma_f(z_2) < z + \gamma_f(z)$.

For finite z_i, let $I_i = (z_i - \gamma_f(z_i), z_i + \gamma_f(z_i))$ $i = 1, 2$.

Now we define the appropriate stopping times: let $\sigma = \inf\{t : B_t = z_1 \text{ or } B_t = z_2\}$ and $\tau_1 = \inf\{t > \sigma : |B_t - B_\sigma| > \gamma_f(B_\sigma)\}$ and $\tau_2 = \inf\{t > \sigma : B_t = B_\sigma + \gamma_f(B_\sigma)\}$. Set

$$\begin{aligned} \tau &= \tau_2 \wedge (\tau_1 + 1) && \text{if } z_1 \text{ and } z_2 \text{ are finite;} \\ &= 100 \wedge \tau_2 \wedge (\tau_1 + 1) && \text{if either } z_1 \text{ or } z_2 \text{ are infinite.} \end{aligned}$$

It can easily (if tediously) be checked using all the results and techniques above, that τ_1 and τ_2 are $\{\mathscr{F}_t^{Mf}\}_{t \geqq 0}$-stopping times and that τ_2 is not predictable; again, $\{E(\tau \mid \mathscr{F}_t^{Mf})\}_{t \geqq 0}$ is a discontinuous martingale.

CASE 3. General C_f, leb $(z_f) = 0$. Basically, the only new problem is the following: suppose there exist $\varepsilon < x_0 < \infty$, such that $F(x) = f(-x)$ for $|x| \leqq x_0$, and for all $0 \leqq r \leqq \varepsilon$, $f(-x_0 - r) = -f(-x_0 + r)$ while $f(x_0 - r) = f(x_0 + r)$. In this situation $\gamma_f(0) = x_0$, but $|f|$ is locally symmetric around both $-x_0$ and x_0. However, it is still the case that B_τ is $\mathscr{F}_\tau^{Mf}$-measurable, where $\tau = \inf\{t : |B|_t = x_0\}$, and this is exactly the condition needed to construct a discontinuous martingale on $\{\mathscr{F}_t^{Mf}\}_{t \geqq 0}$. The proof is similar in spirit to the proof given above, but rather tedious, and so will be omitted.

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SCHOOL OF STATISTICS
270 VINCENT HALL
UNIVERSITY OF MINNESOTA
260 CHURCH STREET S.E.
MINNEAPOLIS, MINNESOTA 55455