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SECULAR COEFFICIENTS AND THE HOLOMORPHIC MULTIPLICATIVE CHAOS

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We study the *secular coefficients* of $N \times N$ random unitary matrices U_N drawn from the Circular β-Ensemble which are defined as the coefficients of $\{z^n\}$ in the characteristic polynomial $\det(1 - zU_N^*)$. When $\beta > 4$, we obtain a new class of limiting distributions that arise when both n and N tend to infinity simultaneously. We solve an open problem of Diaconis and Gamburd (*Electron. J. Combin.* **11** (2004/06) 2) by showing that, for $\beta = 2$, the middle coefficient of degree $n = \lfloor \frac{N}{2} \rfloor$ tends to zero as $N \to \infty$. We show how the theory of Gaussian multiplicative chaos (GMC) plays a prominent role in these problems and in the explicit description of the obtained limiting distributions. We extend the remarkable *magic square formula* of (*Electron. J. Combin.* **11** (2004/06) 2) for the moments of secular coefficients to all $\beta > 0$ and analyse the asymptotic behaviour of the moments. We obtain estimates on the order of magnitude of the secular coefficients for all $\beta > 0$, and we prove these estimates are sharp when $\beta \geq 2$ and N is sufficiently large with respect to n. These insights motivated us to introduce a new stochastic object associated with the secular coefficients, which we call *Holomorphic Multiplicative Chaos (HMC)*. Viewing the HMC as a random distribution, we prove a sharp result about its regularity in an appropriate Sobolev space. Our proofs expose and exploit several novel connections with other areas, including random permutations, Tauberian theorems and combinatorics.

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1. Introduction. Let $\beta > 0$ be a fixed parameter, and consider the joint distribution on N points

$$\mathrm{C}\beta\mathrm{E}_N(\vartheta_1, \dots, \vartheta_N) \propto \prod_{1 \le k < j \le N} |e^{i\vartheta_k} - e^{i\vartheta_j}|^\beta, \tag{1.1}$$

where $\vartheta_j \in [0, 2\pi)$ for all $j = 1, \dots, N$. This is known as the *circular β-ensemble*. When $\beta = 2$, this distribution arises as the law of the eigenvalues of a Haar distributed unitary matrix and is better known as the CUE (circular unitary ensemble). When $\beta \neq 2$, it also arises from the eigenvalue distribution of certain random matrix models; see, for example, [38]. Therefore, it makes sense to consider the characteristic polynomial

$$\chi_N(z) = \prod_{j=1}^{N} (1 - ze^{-i\vartheta_j}) \tag{1.2}$$

which would have the representation $\det(1 - zU_N^*)$ if $\{e^{i\vartheta_j} : 1 \le j \le N\}$ were the eigenvalues of a matrix U_N.

The goal of this paper is to formulate and solve a new class of probabilistic and combinatorial questions associated with such characteristic polynomials. Our main quantities of interest will be the so-called *secular coefficients*, defined most simply as the coefficients $c_n^{(N)}$ in the expansion of (1.2) in its Fourier basis,

$$\chi_N(z) = \sum_{n=0}^{N} c_n^{(N)} z^n. \tag{1.3}$$

In a remarkable paper [19], Diaconis and Gamburd studied these coefficients in the $\beta = 2$ setting. They showed that the joint moments of $\{c_n^{(N)}\}_{n \ge 1}$ are related to the enumeration of combinatorial objects known as *magic squares*—integer valued square matrices with prescribed row and column sums. For example, when $N \ge nk$, the moment $\mathbb{E}(|c_n^{(N)}|^{2k})$ is equal to the number of $k \times k$ magic squares whose rows and columns all sum up to n. In general, combinatorial results on magic squares imply that this quantity has order $n^{(k-1)^2}$ as $n \to \infty$, with a multiplicative constant given by the volume of the kth Birkhoff polytope. The determination of these volumes remains a well-studied and challenging problem in the combinatorics community. They have been explicitly computed only for $k \le 10$, and this has required the use of high-performance computers; see [10, 12, 18] for this and other perspectives, and for number theoretical applications, see [17, 37].

Given the rich combinatorial structure associated with the moments of $c_n^{(N)}$, there is a natural probabilistic question that comes to mind: Is there a commensurately richly structured probabilistic object to which $c_n^{(N)}$ converges as $N \to \infty$? What if N and n tend to infinity together, in a suitable way? This problem is mentioned in the same paper of Diaconis and Gamburd (see the discussion below Proposition 4 of [19]); however to our knowledge the answer has remained out of reach conjecturally or otherwise, now for almost 15 years. The purpose of this paper is to begin closing this gap, particularly in the general context of the β-ensembles (1.1).

There is one exception where a limiting distribution for $c_n^{(N)}$ can be obtained with relative ease, namely, when the index n remains finite and we let the matrix size $N \to \infty$, as discussed in [19] for $\beta = 2$. By the Newton formula, which relates the elementary and power sum symmetric functions, each $c_n^{(N)}$ has a finite polynomial dependence on the first n power traces, $p_k := \mathrm{Tr}(U_N^k)$, via

$$c_n^{(N)} = \frac{1}{n!} \det \begin{pmatrix} p_1 & 1 & 0 & \dots & 0 \\ p_2 & p_1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \dots & n-1 \\ p_n & p_{n-1} & p_{n-2} & \dots & p_1 \end{pmatrix}. \tag{1.4}$$

The distributional convergence of these power traces was famously studied by Diaconis and Shahshahani [20] for $\beta = 2$ and by Jiang and Matsumoto [33] for general $\beta > 0$; see also [15]. These authors show that for any fixed n we have the joint convergence in law

$$\{\mathrm{Tr}(U_N^k)\}_{k=1}^n \xrightarrow{d} \sqrt{\frac{2}{\beta}}\{\sqrt{k}\mathcal{N}_k\}_{k=1}^n, \quad N \to \infty, \tag{1.5}$$

where $\{\mathcal{N}_k\}_{k=1}^n$ are i.i.d. standard complex normal random variables, that is, the real and imaginary parts of $\mathcal{N}_k$ are independent normal random variables such that

$$\mathbb{E}(\mathcal{N}_k) = 0, \qquad \mathbb{E}(\mathcal{N}_k^2) = 0, \quad \text{and} \quad \mathbb{E}(|\mathcal{N}_k|^2) = 1. \tag{1.6}$$

Therefore, (1.5) implies that, *for fixed n*, the sequence of random variables $c_n^{(N)}$ converges as $N \to \infty$ to a limit random variable c_n. Furthermore, each c_n is explicitly characterized through a polynomial dependence on a family of i.i.d. Gaussian random variables via formulas (1.5) and (1.4).

In contrast, the situation where the degree $n \to \infty$ turns out to be much more challenging. The two Theorems below address this situation; see the subsequent sections for further results and discussion.

THEOREM 1.1. *Let $N = N_n$ be a sequence such that $N \to \infty$ as $n \to \infty$ and such that $n/N \to 0$. Let $\mathcal{Z}$ denote a standard complex normal random variable and $\mathcal{E}(1)$ denote the standard exponential random variable with parameter* 1, *sampled independently. Then, for any $\beta > 4$, we have the convergence in distribution*

$$\frac{c_n^{(N)}}{\sqrt{\mathbb{E}(|c_n^{(N)}|^2)}} \xrightarrow[n\to\infty]{d} \frac{\mathcal{Z}\mathcal{E}(1)^{-\frac{1}{\beta}}}{\sqrt{\Gamma(1-\frac{2}{\beta})}}, \tag{1.7}$$

where $\Gamma(z)$ is the Gamma function.

A quick computation shows that the right-hand side of (1.7) has finite moments of order $2k$ if and only if $\frac{2k}{\beta} < 1$. In fact, apart from the complex Gaussian $\mathcal{Z}$, the right-hand side of (1.7) can be identified as the square root of the total mass of Gaussian multiplicative chaos (GMC) on the unit circle; see Section 1.3. Our proof makes explicit use of the second moment method ($k = 2$ in this context), and this gives rise to the restriction $\beta > 4$ in the statement of

Theorem 1.1. This is known as the L^2*-phase* in GMC theory. It is natural to expect that (1.7) persists to any $\beta > 2$ and even the critical case $\beta = 2$ after suitably renormalizing on both sides of (1.7).

At the level of tightness, we are able to remove the restriction on β, and our results hold for any $\beta > 0$. We can also relax the condition $n/N \to 0$ stated in Theorem 1.1. As an example we have the following particular case that resolves a problem of Diaconis and Gamburd [19] on the middle secular coefficient, defined by setting $n = \lfloor \frac{N}{2} \rfloor$.

THEOREM 1.2. *Let $\beta = 2$, and set $w_N = \log(1+N)^{-1/4}$. Then we have that $\{c^{(N)}_{\lfloor \frac{N}{2} \rfloor} / w_N\}_{N \geq 1}$ is a tight family of random variables. In particular, $c^{(N)}_{\lfloor \frac{N}{2} \rfloor} \xrightarrow{d} 0$ as $N \to \infty$. More generally, when $n_N \to \infty$ in such a way that $2n_N \leq N$, we have $c^{(N)}_{n_N} \xrightarrow{d} 0$ as $N \to \infty$.*

Note that Theorem 1.2 holds despite the fact that $\mathbb{E}(|c^{(N)}_n|^2) = 1$ identically. A similar phenomenon has been observed in a number theoretical context, namely, in the theory of random multiplicative functions and "better than square root cancellation" [29]. In the regime $0 < \beta < 2$, we will establish a similar class of tightness results which show that the normalization $\sqrt{\mathbb{E}(|c^{(N)}_n|^2)}$ overestimates the correct order of magnitude for the coefficients; see Section 1.5.

1.1. *The holomorphic multiplicative chaos.* Statistical properties of random matrix characteristic polynomials have attracted considerable interest recently, in large part due to an intimate relationship to *logarithmically correlated Gaussian fields* and *Gaussian multiplicative chaos* (GMC); see [22] and [52] for general background on these topics. The connection to characteristic polynomials of random matrices arose quite recently in an influential work of Fyodorov, Hiary and Keating [24, 25]. In particular, the attempts to prove the conjectures in [24, 25], which are still unresolved in full generality, have motivated a number of recent studies on characteristic polynomials of random matrices; for a nonexhaustive list, see, for example, [1, 11, 14, 15, 43, 49, 50, 58], and also on various parallel questions concerning the Riemann zeta function [2, 3, 48, 53]. For some general connections between GMC and random matrices, see [16]. We will show how the GMC theory also plays a prominent role in the analysis of the secular coefficients $c^{(N)}_n$ in (1.3).

To describe formally how a log-correlated Gaussian field can arise from the characteristic polynomial (1.2), we expand the logarithm as

$$\log \chi_N(z) = -\sum_{k=1}^{\infty} \frac{z^k}{k} \operatorname{Tr}(U_N^{-k}). \tag{1.8}$$

By the convergence (1.5), we can identify a candidate limiting Gaussian field by replacing the power traces $\operatorname{Tr}(U_N^{-k})$ with $\sqrt{\frac{2}{\beta}}\sqrt{k}\mathcal{N}_k$, where $\mathcal{N}_k$ are i.i.d. standard complex Gaussian variables, as in (1.6), and for the sake of simplicity, we ignore the minus sign in (1.8), noting the rotational invariance of each $\mathcal{N}_k$. Now, let $G^{\mathbb{C}}$ be the Gaussian analytic function on the unit disc $\mathbb{D}$,

$$G^{\mathbb{C}}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\sqrt{k}} \mathcal{N}_k, \tag{1.9}$$

so that we expect $\chi_N(z)$ to be close to $e^{\sqrt{\frac{2}{\beta}} G^{\mathbb{C}}(z)}$ in a suitable sense. The covariance of the field $G^{\mathbb{C}}$ follows from the simple i.i.d. structure of the variables $\mathcal{N}_k$ as

$$\mathbb{E}[G^{\mathbb{C}}(w) G^{\mathbb{C}}(z)] = 0 \quad \text{and} \quad \mathbb{E}[G^{\mathbb{C}}(w) \overline{G^{\mathbb{C}}(z)}] = -\log(1 - w\overline{z})$$

from which it follows that $G = 2\Re G^{\mathbb{C}}$ and $H = 2\Im G^{\mathbb{C}}$ are identically distributed Gaussian fields with

$$\mathbb{E}\big[G(w)G(z)\big] = -2\log|1 - w\bar{z}| \quad \text{and} \quad \mathbb{E}\big[G(w)H(z)\big] = -2\operatorname{Arg}(1 - w\bar{z}), \tag{1.10}$$

where we take the principal branch of the argument. In particular, when defined on the unit circle $|z| = |w| = 1$, the real-valued field G is a prototypical example of a log-correlated Gaussian field (see, e.g., [32] where it first appeared explicitly).

For trigonometric polynomials ϕ, if we let $z \mapsto \phi(z)$ for $z \in \mathbb{D}$ be the continuous harmonic extension to $\mathbb{D}$, we can define the random distribution

$$(\mathrm{HMC}_\theta, \phi) := \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{\theta}G^{\mathbb{C}}(re^{i\vartheta})}\overline{\phi(r\vartheta)}\, d\vartheta, \tag{1.11}$$

which we will call the *holomorphic multiplicative chaos* or HMC, where from now on we will adopt the notation

$$\theta := \frac{2}{\beta}. \tag{1.12}$$

For the moment we comment that, for trigonometric polynomials, the existence of this limit is trivial and is sufficient to uniquely define the HMC. In particular, if we take $\phi(\vartheta) = e^{in\vartheta}$ for $n \in \mathbb{N}_0$, then, using analyticity and Cauchy's theorem, the limit is just the nth coefficient in the power series expansion of $e^{\sqrt{\theta}G^{\mathbb{C}}(z)}$ at $z = 0$. We define for $n \in \mathbb{N}_0$ the Fourier coefficient of the HMC

$$c_n := \big(\mathrm{HMC}_\theta, \vartheta \mapsto e^{in\vartheta}\big). \tag{1.13}$$

Equivalently, the coefficients c_n can be extracted from a generating function by the formula

$$c_n = \big[z^n\big]e^{\sqrt{\theta}G^{\mathbb{C}}(z)} = \big[z^n\big]\exp\left(\sqrt{\theta}\sum_{k=1}^{\infty}\frac{z^k}{\sqrt{k}}\mathcal{N}_k\right), \tag{1.14}$$

where the notation $[z^n]h(z)$ denotes the coefficient of z^n in the power series expansion of $h(z)$ around the point $z = 0$. We could as well define this for $n \in \mathbb{Z}$, but for negative integers, this would be 0.

The HMC is in some sense the distributional limit of the characteristic polynomial χ_N inside the unit circle. The following result is obtained by combining the equation (7.2), given below, Proposition 3.1 of [15] and the fact that Φ_{N-1}/Φ^*_{N-1} a.s. converges uniformly to zero on compact subsets of the open unit disc which is a consequence of [54], Theorem 1.7.4.

THEOREM 1.3. *For any $\beta > 0$, it is possible to define χ_N and the field $G^{\mathbb{C}}$ on a single probability space in such a way that, for any $r \in (0, 1)$,*

$$\sup_{|z|\le r}\big|\chi_N(z) - e^{\sqrt{\theta}G^{\mathbb{C}}(z)}\big| \xrightarrow[N\to\infty]{\text{a.s.}} 0.$$

The convergence also holds in L^p for any $p \ge 1$.

Then as a corollary of Theorem 1.3 and Cauchy's integral formula, on the probability space therein, each $c_n^{(N)} \xrightarrow[N\to\infty]{\text{a.s.}} c_n$. We shall continue to refer to $\{c_n\}$ as the secular coefficients of the HMC_θ.

While HMC_θ is uniquely determined by its Fourier coefficients and exists for all $\theta > 0$, we will characterize its regularity in the Sobolev sense. We define the Sobolev norms for any $s \in \mathbb{R}$ on the trigonometric polynomials on $\mathbb{T}$ by

$$\|f\|^2_{H^s} = \sum_{n\in\mathbb{Z}}(1 + n^2)^s\big|\widehat{f}(n)\big|^2, \quad \text{where } \widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\vartheta})e^{-in\vartheta}\, d\vartheta. \tag{1.15}$$

The Sobolev spaces H^s for $s \in \mathbb{R}$ are the closures in the space of distributions of the trigonometric polynomials on $\mathbb{T}$ under these norms. For any $s \in \mathbb{R}$, the norms defined in (1.15) are well defined for all functions $f \in L^2$, with the understanding that they may be infinite, and for $s \geq 0$, the space H^s is precisely the subspace of L^2 for which $\|\cdot\|_{H^s} < \infty$. The formulas for the norms (1.15) extend generally to the space of distributions on $\mathbb{T}$ by replacing $\hat{f}(n) := (f, \vartheta \mapsto e^{in\vartheta})$. For any $s \in \mathbb{R}$, the pair H^s and H^{-s} are dual spaces with respect to the natural inner product on $\mathbb{T}$.

THEOREM 1.4. *Define*

$$s_\theta := \begin{cases} -\dfrac{\theta}{2} & \text{if } \theta \leq 1, \\ -\sqrt{\theta} + \dfrac{1}{2} & \text{if } \theta > 1. \end{cases}$$

Then, for any $\theta > 0$, HMC_θ *is in* H^s *almost surely for any* $s < s_\theta$, *and it is almost surely not in* H^s *for any* $s > s_\theta$.

PROOF. See Section 6. □

We remark that, in the variable $\theta = \frac{\gamma^2}{4}$, this Theorem states that HMC_θ is in H^s, provided that

$$-2s + 1 > \begin{cases} 1 + \dfrac{\gamma^2}{4} & \gamma < 2, \\ \gamma & \gamma > 2. \end{cases} \tag{1.16}$$

The threshold on the right-hand side of (1.16) is familiar from the study of the *free energy* in various other log-correlated models (see, e.g., [23]).

1.2. *The combinatorial structure of the moments.* We begin by discussing the moments of the secular coefficients. As we already mentioned, when $\beta = 2$ (or $\theta = 1$), Diaconis and Gamburd [19] characterize these moments in terms of the enumeration of magic squares. We will state a result that generalizes this characterization to arbitrary $\beta > 0$.

DEFINITION 1.5. A magic square of size k with row sums $\mu = (\mu_1, \dots, \mu_k)$ and column sums $\nu = (\nu_1, \dots, \nu_k)$ is a $k \times k$ matrix A with entries in $\mathbb{N}_0$ and with the property that

$$\begin{aligned} \sum_{j=1}^{k} A_{ij} &= \mu_i, \quad i = 1, \dots, k, \\ \sum_{i=1}^{k} A_{ij} &= \nu_j, \quad j = 1, \dots, k. \end{aligned} \tag{1.17}$$

Throughout the article we will use the notation $\mathrm{Mag}_{\mu,\nu}$ to denote the collection of all such $k \times k$ magic squares. We recall that nonnegative integer vectors μ are called *compositions* in the combinatorics literature, while the entries $\mu_1, \dots, \mu_k$ are known as *parts*. In what follows, for a general real parameter $\theta > 0$ and $n \in \mathbb{N}$, we define

$$\binom{n + \theta - 1}{n} = \frac{\Gamma(n + \theta)}{\Gamma(\theta)\Gamma(n + 1)}. \tag{1.18}$$

THEOREM 1.6. *Let μ and ν be any compositions with k parts. Then, for any $k \in \mathbb{N}$ and $\theta > 0$, we have*

$$\mathbb{E}\left(\prod_{j=1}^{k} c_{\mu_j}\overline{c_{\nu_j}}\right) = \sum_{A\in \mathrm{Mag}_{\mu,\nu}} \prod_{1\le i,j\le k} \binom{A_{ij}+\theta-1}{A_{ij}}. \tag{1.19}$$

We give the proof in Section 2 and, furthermore, provide a combinatorial connection to Jack functions. When $\beta = 2$ ($\theta = 1$), the right-hand side of (1.19) reduces to the cardinality of $\mathrm{Mag}_{\mu,\nu}$, recovering the result of [19] in the particular case $N = \infty$. Choosing the column and row sums all to equal n, we obtain an expression for the absolute $2k$th moments

$$\mathbb{E}(|c_n|^{2k}) = \sum_{A\in \mathrm{Mag}_{\pi,\pi}} \prod_{1\le i,j\le k} \binom{A_{ij}+\theta-1}{A_{ij}}, \tag{1.20}$$

where π is the composition in which n appears k times, that is, $\mathrm{Mag}_{\pi,\pi}$ is the set of all magic squares of size $k \times k$ with all row and column sums equal to n. We note the special case

$$\mathbb{E}(|c_n|^2) = \binom{n+\theta-1}{\theta-1} \sim \frac{1}{\Gamma(\theta)} n^{\theta-1}. \tag{1.21}$$

It turns out it is possible to use this formula and (1.20) to probe asymptotic behaviour of the moments. We have the following theorem.

THEOREM 1.7. *For any positive integer k and any $\theta > 0$ so that $\theta k < 1$,*

$$\lim_{n\to\infty} \frac{\mathbb{E}(|c_n|^{2k})}{\mathbb{E}(|c_n|^2)^k} = \frac{\Gamma(1-k\theta)}{\Gamma(1-\theta)^k} k! \tag{1.22}$$

PROOF. See Section 2. □

It is a simple computation, recalling that $\theta = \frac{2}{\beta}$, that the moments on the right-hand side of (1.22) are precisely those of the limiting distribution in Theorem 1.1. We mention in passing that besides magic squares, which play an important role in describing HMC_θ, the *Ewens sampling formula*, which defines a classical distribution on random permutations, plays a prominent role in our analysis; see Section 3 for details.

1.3. *From magic squares back to multiplicative chaos.* Theorem 1.7 is strongly reminiscent of the *freezing transition* observed for moments of random energy models [23] (c.f. Remark 2.4 for the behavior of a moment above the critical threshold). Indeed, in [23] it is shown that the *Morris integral*,

$$\frac{\Gamma(1-k\theta)}{\Gamma(1-\theta)^k} = \frac{1}{(2\pi)^k} \int_{[0,2\pi]^k} \prod_{1\le a<b\le k} |e^{i\vartheta_a} - e^{i\vartheta_b}|^{-2\theta}\, d\vartheta, \tag{1.23}$$

describes the appropriately rescaled moments of the partition function of the logarithmically correlated *random energy model* on the unit circle in the high temperature phase; see [23] for details.

The presence of the Morris integral in Theorem 1.7 is, moreover, indicative that the theory of Gaussian multiplicative chaos will be relevant here. There is a substantial literature on this random measure (see [52] for a general overview), but we will be concerned only with the following specific instance: for $\theta \in (0, 1)$,

$$\mathrm{GMC}_\theta(d\vartheta) := \lim_{r\to 1} (1-r^2)^\theta |e^{\sqrt{\theta} G^{\mathbb{C}}(re^{i\vartheta})}|^2\, d\vartheta = \lim_{r\to 1} (1-r^2)^\theta e^{\sqrt{\theta} G(re^{i\vartheta})}\, d\vartheta. \tag{1.24}$$

The existence of this limit as a random measure is shown[1] in [34] where the convergence is shown to hold in L^q for any $0 < q < 1$ (c.f. [15], Proposition 3.1). Moreover, it is shown in [51] that the total mass of this particular random measure has law characterized by the natural analytic continuation of the Morris integral, that is, for any $p > 0$ with $p\theta < 1$,

$$\mathbb{E}(\mathcal{M}_\theta)^p = \frac{\Gamma(1-p\theta)}{\Gamma(1-\theta)^p}, \quad \text{where } \mathcal{M}_\theta := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{GMC}_\theta(d\vartheta), \tag{1.25}$$

with the sense of the limit being an in-probability, weak-* convergence.

We briefly summarize some of the qualitative properties of the GMC. The regime $\theta \in (0, 1)$ is typically referred to as the *subcritical* phase of the GMC_θ. The limiting measure is supported on a set of Hausdorff dimension $1-\theta$. This result is well known in the literature on GMC; see [52], though for the model discussed here, see [15]. The subset $(0, \frac{1}{2})$ is sometimes referred to as the L^2 phase, on account of the mass $\mathcal{M}_\theta$ gaining a finite second moment (and indeed Theorem 1.7 applies as well).

The L^2 phase is technically simpler to manage and has been the setting for some of the first convergence results of $\beta = 2$ characteristic polynomials and powers thereof to the GMC [11, 58]. For a power of the modulus of the CUE ($\beta = 2$) characteristic polynomial, this has been improved to the whole subcritical phase in [49]. For general $\beta > 0$, to the authors' knowledge there are no convergence results of this type, except when the characteristic polynomial is regularized at some small scale; see [42]. Also for an object closely related to the characteristic polynomial, convergence to the GMC was established in [15] for general $\beta \geq 2$.

The value $\theta = 1$ is the *critical* temperature at which it is possible to establish (1.24) under an additional logarithmic renormalization,

$$\mathrm{GMC}_1(d\vartheta) := \lim_{r\to 1} \sqrt{\log \frac{1}{1-r^2}}(1-r^2)e^{G(re^{i\vartheta})}\, d\vartheta, \tag{1.26}$$

with the limit again holding in L^q for any $0 < q < 1$. As for the subcritical case (1.24), this convergence follows from [34], Theorem 1.3, and Appendix B. This measure is known to have Hausdorff dimension 0 and is nonatomic, as proved in [15]. We note, in contrast, that HMC_θ is well defined as a random distribution on the unit circle for all $\theta > 0$ (and indeed on a single probability space), regardless of the phase of the associated GMC.

Returning to consideration of Theorem 1.7, we see that there is a factor of $k!$ beyond the Morris integral term. This strongly suggests a limiting factorization into independent random variables, where the additional factor can be interpreted as the moments of a standard complex normal random variable $k! = \mathbb{E}(|\mathcal{Z}|^{2k})$. We show this is indeed the case in the L^2 phase. The theorem that follows shows this more precisely.

THEOREM 1.8 (L^2-phase). *For any $0 < \theta < \frac{1}{2}$, we have the convergence in distribution*

$$\frac{c_n}{\sqrt{\mathbb{E}(|c_n|^2)}} \xrightarrow[n\to\infty]{d} \sqrt{\mathcal{M}_\theta}\,\mathcal{Z}, \tag{1.27}$$

where $\mathcal{Z}$ and $\mathcal{M}_\theta$ are independent, $\mathcal{Z}$ is standard complex normal and $\mathcal{M}_\theta$ has law (1.25).

PROOF. See Sections 3–5. □

We expect this result to persist to all $\frac{1}{2} \leq \theta < 1$ and also to $\theta = 1$ (the critical temperature) subject to a different normalization. While we do not show this convergence in distribution

[1]The Gaussian field used in [34], equation (6.1), contains a constant term in its Fourier series definition in comparison to our field G. This turns out not to affect the convergence results of [34]; see Appendix B

for $\theta \geq \frac{1}{2}$, we give a sharp estimate for the order of magnitude of the coefficient for larger θ (see Lemma 7.5 and Theorem 1.11).

By the mentioned work [51], the limiting random variable $\mathcal{M}_\theta$, appearing on the right-hand side of (1.27), can be given an explicit characterization. For any $\theta < 1$, it is known that we have the equality in law

$$\mathcal{M}_\theta \stackrel{d}{=} \frac{1}{\Gamma(1-\theta)} \mathcal{E}(1)^{-\theta}, \tag{1.28}$$

where $\mathcal{E}(1)$ is an exponential random variable with parameter 1. Formula (1.28) was initially conjectured by Fyodorov and Bouchaud in [23] and proved quite recently in [51] using techniques of Liouville conformal field theory; see also [15] for an alternative proof. Using the explicit formula (1.28), we see that Theorem 1.8 is the particular case $N = \infty$ of our earlier stated Theorem 1.1.

1.4. *The mass of the chaos.* We note that the presence of the mass of the chaos in the secular coefficients could potentially be anticipated. On the one hand, by the generating function (1.14) and Parseval's identity, we have

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{\theta} G(re^{i\vartheta})} \, d\vartheta. \tag{1.29}$$

On the other hand, by (1.24) and (1.26) for all $0 < \theta \leq 1$,

$$L(r,\theta)(1-r^2)^\theta \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \xrightarrow[r\to 1]{p} \mathcal{M}_\theta, \qquad L(r,\theta) := \begin{cases} \sqrt{\log \dfrac{1}{1-r^2}} & \text{if } \theta = 1, \\ 1 & \text{else.} \end{cases} \tag{1.30}$$

So in a suitably averaged sense, the squared modulus of the secular coefficients gives the total mass of the chaos in the subcritical and critical phases. In comparison, Theorem 1.8, shows that each individual coefficient $\{c_n\}$ already contains much of the information about the mass of the GMC.

We also detour briefly to mention that from (1.30), it is possible to derive other approximations to the total mass. One, which will be important here, is

$$\mathcal{M}_{\theta,n} := (\sqrt{\log n})^{\mathbf{1}\{\theta=1\}} \frac{\Gamma(\theta+1)}{n^\theta} \sum_{q=0}^{n} |c_q|^2. \tag{1.31}$$

If the convergence in (1.30) were almost sure, the Hardy–Littlewood Tauberian theorem would immediately imply that $\mathcal{M}_{\theta,n}$ converges almost surely to $\mathcal{M}_\theta$. We show in Theorem A.2 in the Appendix that this Tauberian theorem generalizes to the setting of convergence in probability, and the following is an immediate consequence of (1.29), (1.30) and Theorem A.2.

LEMMA 1.9. *For any* $0 < \theta \leq 1$, *we have the convergence in probability to the total mass*

$$\mathcal{M}_{\theta,n} \xrightarrow[r\to\infty]{p} \mathcal{M}_\theta, \quad n \to \infty. \tag{1.32}$$

1.5. *The secular coefficients of CβE.* Some of what we have proved for the HMC coefficients $\{c_n\}$ adapt or transfer to the secular coefficients $\{c_n^{(N)}\}$ of $N \times N$ random matrices, as defined in (1.3). In particular, when $n \to \infty$ and $N \to \infty$ in such a way that $n/N \to 0$, Theorem 1.8 transfers directly to $\{c_n^{(N)}\}$.

THEOREM 1.10 (L^2-phase for slowly growing n). *Let $N = N_n$ be chosen in such a way that $n/N \to 0$ as $n \to \infty$. Recalling the notation $\theta := \frac{2}{\beta}$, for any $0 < \theta < \frac{1}{2}$, we have the convergence in distribution*

$$\frac{c_n^{(N)}}{\sqrt{\mathbb{E}(|c_n^{(N)}|^2)}} \xrightarrow[n\to\infty]{d} \sqrt{\mathcal{M}_\theta}\mathcal{Z}, \tag{1.33}$$

where $\mathcal{Z}$ and $\mathcal{M}_\theta$ are independent, $\mathcal{Z}$ is standard complex normal and $\mathcal{M}_\theta$ has law (1.25).

PROOF. See Section 7. □

We remark that by formula (1.28), Theorem 1.10 is simply a restatement of Theorem 1.1, but we have included it here for added clarity in the context of GMC theory. Let us discuss the necessity of the condition $n/N \to 0$ as $n \to \infty$ appearing in Theorem 1.10. The normalization constant $\mathbb{E}(|c_n^{(N)}|^2)$ is known explicitly, due to [27] who obtain for any $\theta > 0$,

$$\mathbb{E}(|c_n^{(N)}|^2) = \binom{N}{n}\frac{\Gamma(n+\theta)\Gamma(N-n+\theta)}{\Gamma(\theta)\Gamma(N+\theta)}. \tag{1.34}$$

We remark in passing that, to our knowledge, the work [27] was likely the first explicit investigation of secular coefficients in the literature. From (1.34) we observe two possible types of asymptotics. If N grows with n at a fast enough rate that $n/N \to 0$ as $n \to \infty$, then

$$\mathbb{E}(|c_n^{(N)}|^2) \sim \frac{1}{\Gamma(\theta)} n^{\theta-1}, \quad n \to \infty \tag{1.35}$$

which matches the asymptotic (1.21). When, however, $n = \kappa N$ with $0 < \kappa < 1$, the asymptotics contain an additional prefactor

$$\mathbb{E}(|c_{\lfloor \kappa N\rfloor}^{(N)}|^2) \sim \frac{(1-\kappa)^{\theta-1}}{\Gamma(\theta)} n^{\theta-1}, \quad N \to \infty. \tag{1.36}$$

This hints that these higher degree coefficients could display a different limiting behavior.

On the other hand, we show that the order of magnitude of these secular coefficients is no larger than that of c_n.

THEOREM 1.11 (Order estimate). *The orders of magnitudes of the secular coefficients can be estimated as follows*:

- (*Subcritical case*) *For any $\theta \in (0,1)$, the families*
  $$\{c_n^{(N)}/n^{(\theta-1)/2} : n, N \in \mathbb{N}, N \ge 2n\} \quad \text{and} \quad \{n^{(\theta-1)/2}/c_n^{(N)} : n, N \in \mathbb{N}, N \ge 2n\}$$
  are tight.
- (*Critical case*) *If $\theta = 1$, the family*
  $$\{c_n^{(N)}/(\log(1+n))^{-1/4} : n, N \in \mathbb{N}, N \ge 2n\}$$
  is tight, and if $N_0(n)$ is a sequence such that
  $$\frac{N_0(n)}{n\sqrt{\log n}(\log\log n)} \underset{n\to\infty}{\longrightarrow} \infty,$$
  then
  $$\{(\log(1+n))^{-1/4}/c_n^{(N)} : n, N \in \mathbb{N}, N \ge N_0(n)\}$$
  is tight.

- (*Supercritical case*) *For any* $\theta \in (1,2)$, *there are constants* $u_\theta, v_\theta > 0$ *so that the family*
$$\{(c_n^{(N)}/(n^{\sqrt{\theta}-1}(\log(1+n))^{-\frac{3}{4}\sqrt{\theta}+v_\theta}) : n, N \in \mathbb{N}, N \geq n^{u_\theta}\}$$
is tight.

PROOF. See Theorems 7.6 and 8.2. □

We make similar statements for c_n for any $\theta > 0$ in Lemma 7.5 and Theorem 8.1—in particular, the bounds we establish are consistent with Theorem 1.8 extending to all $\theta \in (0, 1]$. For $\theta > 1$, we do not have any expectation for the value of u_θ, for which we give an explicit value in Theorem 7.6, to be sharp. In contrast, we may reasonably conjecture the correct value for $v_\theta = 0$; see Remark 7.4 for further discussion.

1.6. *Discussion.* We have analyzed a random distribution, the HMC, which can be considered as a large-N limit of the characteristic polynomials χ_N of the CβE. This limiting process exists for all $\theta > 0$, and we have given the distributional convergence of the Fourier coefficients of this Schwartz distribution.

The HMC$_\theta$, in the case $\theta = 1$ ($\beta = 2$) has appeared explicitly in the work of [53], Appendix C, where it is shown that HMC$_1$ is in H^{-s} for all $s > \frac{1}{2}$ and the characteristic polynomial of the CUE converges to it in law as a random element of H^{-s} for any $s > \frac{1}{2}$. The focus of [53] is rather more related to a local scaling of the characteristic polynomial; see especially [53], Theorem 1.3. Therein, the authors show there exists a sequence $\delta_N \to 0$ and a random Schwartz distribution η on $\mathbb{R}$ so that
$$\chi_N(e^{i\delta_N x})e^{-Y_N} \xrightarrow[N\to\infty]{d} \eta(x),$$
where Y_N is a complex Gaussian variable having a nontrivial dependence on χ_N. The processs η can be formally understood as
$$\eta(x) = \exp\left(\int_0^\infty \frac{e^{-2\pi i x u}}{\sqrt{u}}\, dB_u^{\mathbb{C}}\right)$$
for a complex Brownian motion $B^{\mathbb{C}}$. It is also shown that this chaos appears as the limit of a randomized model of the Riemann ζ-function. The process η is a possible candidate for definition of HMC$_\theta$ on the real line when $\theta = 1$, and could also be a type of local scaling limit of HMC$_\theta$ in a suitable vanishing window of ϑ. It can also be noted that $\log \eta(x)$ appears closely related to the $H = 0$ fractional Brownian motion of [26].

In the vein of possible scaling limits of HMC$_\theta$, we also mention the recent work of [57]. They introduce the *stochastic zeta function* as a limit of characteristic polynomials of the circular β-ensembles. As these characteristic polynomials are explicitly related to the HMC$_\theta$ (see Section 7 and (7.2)), the stochastic zeta function can be viewed as the microscopic scaling limit of a random-matrix regularization of HMC$_\theta$ (note that the stochastic zeta function is a limit of $(\chi_N : N \in \mathbb{N})$ (7.2) and not $(\Phi_N^* : N \in \mathbb{N})$ (7.1)).

Another class of related objects which have been considered are the *complex multiplicative chaoses*. For example, consider the random distribution CGMC$_\theta$, given (hypothetically) by
$$(\mathrm{CGMC}_\theta, \phi) = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{\theta/4}(G_1(re^{i\vartheta})+iG_2(re^{i\vartheta}))}\phi(\vartheta)\, d\vartheta, \tag{1.37}$$
where G_1 and G_2 are i.i.d. copies of G from (1.10). This roughly fits within the frameworks of [41] and [40], although the technical assumptions on the manner of regularization of $G(e^{i\vartheta})$ are not precisely the same. We comment that in the language of [41] that CGMC$_\theta$ would

be in *Phase I* for $\theta \in (0,1)$, in *Phase II* for $\theta > 1$ and at the triple point for $\theta = 1$. We mention briefly that there is other related work on complex multiplicative cascades [8, 9] and imaginary chaoses in [6, 36].

When $\theta < 1$, an adaptation of [41], Theorem 3.1, would show the limit in (1.37) exists, and so the CGMC_θ is well defined.[2] We note that, for $\theta \geq 1$, [41], Conjectures 5.2,5.3, suggest that an additional logarithmic normalization is required for convergence, analogously to logarithmic factors needed for convergence of the critical GMC. Indeed, in work of [30, 46] (see also [31]), an analogous statement is proven for a complex random energy model built from branching processes. Note this in constrast to HMC_θ *which requires no further normalization* to converge for any $\theta > 0$.

Because the correlations are relatively weak between the real and imaginary parts of the field which define HMC_θ, we expect that some results carry to HMC_θ from the theory of the complex multiplicative chaoses, for example the multifractality for $\theta < 1$, $q < 1$,

$$\mathbb{E}\big[\big|(\mathrm{CGMC}_\theta, \mathbf{1}\{|\vartheta| \leq r\})\big|^q\big] \underset{r\to 0}{\sim} C_q r^{q-\theta q^2},$$

(see [41], Theorem 3.6, [7], and see also the related work on the regularity of the complex Gaussian multiplicative chaoses in [35]). We expect the same to hold for HMC_θ.

QUESTION 1.12. For $\theta < 1$, $q < 1$, does it hold that

$$\mathbb{E}\big[\big|(\mathrm{HMC}_\theta, \mathbf{1}\{|\vartheta| \leq r\})\big|^q\big] \underset{r\to 0}{\sim} C_q r^{q-\theta q^2}?$$

We conclude by mentioning some unsolved questions on the properties of HMC_θ. We have considered the L^2-phase of θ, that is, $\theta < \frac{1}{2}$. We assume that Theorem 1.10 generalizes without alteration to $\theta \in (0,1)$, the L^1-phase and in addition to the critical value $\theta = 1$ after introducing an appropriate logarithmic factor.

QUESTION 1.13. Show that Theorem 1.8 generalizes to all $\theta \in (0,1)$ and that it can be adapted to hold at the critical point $\theta = 1$.

For supercritical $\theta > 1$, it is reasonable to assume that there is still distributional convergence of c_n, but the exact form of the limit is unclear.

We have also shown distributional convergence of the secular coefficients $c_n^{(N)}$ when $n/N \to 0$ in the L^2-phase. This relies on making a comparison between c_n and $c_n^{(N)}$ which is weaker when $n/N \to c \in (0,1)$. So we pose the following question.

QUESTION 1.14. What is the distributional limit of $c_n^{(N)}/\sqrt{\mathbb{E}(|c_n^{(N)}|^2)}$ in the subcritical (or even L^2) phase when $n/N \to c \in (0,1)$ as $N \to \infty$?

One feature of the secular coefficients $c_n^{(N)}$ is that they may be expressed as combinations of certain conditional expectations of c_n and c_{N-n+1} (see (7.12)). Beyond this, it would be interesting to know the joint behavior of the secular coefficients $\{c_n\}$.

QUESTION 1.15. For $\theta \in (0,1)$, to what does $(n^{(1-\theta)/2} c_{n+m} : m \in \mathbb{Z})$ converge as $n \to \infty$ in the sense of finite-dimensional marginals?

[2] From the theory in [9], using the series truncation regularization in place of the harmonic regularization, the existence of the limit follows.

We conclude with one final question of a metamathematical nature. For $r \in (0,1)$, the function $\vartheta \mapsto e^{\sqrt{\theta} G^{\mathbb{C}}(re^{i\vartheta})}$ from $[0, 2\pi)$ to $\mathbb{C}$ can be seen, after suitable normalization, as the (random) wave function of the position of a particle which lives on $[0, 2\pi)$. Then, if one makes a quantum measurement of the position of the particle, the outcome of the measurement is distributed according to the (random) probability measure with density

$$\vartheta \mapsto \frac{e^{\sqrt{\theta} G(re^{i\vartheta})}}{\int_0^{2\pi} e^{\sqrt{\theta} G(re^{i\vartheta})}\, d\vartheta}$$

with respect to the Lebesgue measure on $[0, 2\pi)$. Now, if we are in the subcritical or the critical phases ($\theta \le 1$), and if we take the limit $r \to 1$, we see that

$$\vartheta \mapsto e^{\sqrt{\theta} G^{\mathbb{C}}(e^{i\vartheta})} = \sum_{n \ge 0} c_n e^{ni\vartheta}$$

can be formally seen as the wave function of a particle such that a quantum measurement gives an outcome distributed according to the (random) probability measure $\mathrm{GMC}_\theta / \mathcal{M}_\theta$. Formally, this wave function is an eigenfunction for the operator

$$\begin{aligned} \Psi(\vartheta) \mapsto & \frac{1}{i} \frac{d\Psi(\vartheta)}{d\vartheta} - \frac{\sqrt{\theta}}{i} \frac{dG^{\mathbb{C}}(e^{i\vartheta})}{d\vartheta} \Psi(\vartheta) \\ &= \frac{1}{i} \frac{d\Psi(\vartheta)}{d\vartheta} - \left(\sum_{k=1}^{\infty} \sqrt{k\theta} e^{i\vartheta k} \mathcal{N}_k \right) \Psi(\vartheta). \end{aligned}$$

Multiplying by the adjoint, we formally get a self-adjoint operator for which $\mathrm{HMC}_\theta / \sqrt{\mathcal{M}_\theta}$ is an eigenstate. However, the physical meaning of this operator is unclear to us. A discussion on the problem of a massless two-dimensional Dirac fermion in a static random magnetic field is provided by Carpentier and Le Doussal in [13] (see Section VI.A.), where the exponential of a Gaussian logarithmically correlated field is involved in the wave functions which are obtained.

1.7. *Organization*. The structure of this paper is as follows. In Section 2 we compute the joint moments of the secular coefficients $\{c_n\}$, which we express in terms of magic squares. We then relate these to Jack functions and compute their asymptotics, proving Theorems 1.6 and 1.7. In Section 3 we make a connection between the moments of the secular coefficients and the Ewens sampling formula. We then review known estimates of the Ewens sampling formula.

In Sections 4, 4.2, 5 and 5.2, we prove Theorem 1.8. We do this by ultimately using the martingale central limit theorem. However, c_n itself is not suitable for a direct application. So in Section 4, we find a related random variable $\tilde{c}_n^{(\delta)}$ which is a close approximation to c_n and whose Doob martingale with respect to a natural filtration has easily understood increments. In Section 4.2, we give the proof of this normal approximation for $\tilde{c}_n^{(\delta)}$ and then a proof of Theorem 1.8, contingent on showing the bracket process of the Doob martingale for $\tilde{c}_n^{(\delta)}$ stabilizes in the $n \to \infty$ followed by $\delta \to 0$ limit. In Section 5 we compute moments of *secular coefficients with restricted cycle count*, whose meaning will become apparent, and finally in Section 5.2, we prove the convergence needed to complete the proof of Theorem 1.8.

In Section 6 we prove Theorem 1.4 on the regularity of HMC_θ. In Section 7 we give the precise connection between the characteristic polynomial χ_N and HMC_θ. We also show Theorems 1.10 and 1.11. Finally in Section 8, we prove the sharpness of the estimates in Section 7 in the regime $\theta \in (0, 1]$.

2. Moments of the secular coefficients. The purpose of this section will be to prove Theorems 1.6 and 1.7 on the moments of c_n for general $\beta > 0$. We also discuss combinatorial properties of the moments and their relation to Jack functions.

PROOF OF THEOREM 1.6. Recall from (1.14) that c_n can be extracted from a generating function according to the formula

$$c_n = [z^n]\exp\left(\sqrt{\theta}\sum_{j=1}^{\infty}\frac{\mathcal{N}_j}{\sqrt{j}}z^j\right). \tag{2.1}$$

Denoting the left-hand side of (1.19) as $\mathcal{R}^{(k)}_{(\mu,\nu)}$, we have

$$\mathcal{R}^{(k)}_{(\mu,\nu)} = [z_1^{\mu_1}\dots z_k^{\mu_k} w_1^{\nu_1}\dots w_k^{\nu_k}]\mathbb{E}(F^{(k)}(\vec{z},\vec{w})), \tag{2.2}$$

where

$$F^{(k)}(\vec{z},\vec{w}) = \exp\left(\sqrt{\theta}\sum_{r=1}^{k}\sum_{j=1}^{\infty}\left(\frac{\mathcal{N}_j}{\sqrt{j}}z_r^j + \frac{\overline{\mathcal{N}_j}}{\sqrt{j}}w_r^j\right)\right). \tag{2.3}$$

A simple Gaussian computation using independence of the family $\{\mathcal{N}_k\}_{k=1}^{\infty}$ shows that

$$\mathbb{E}(F^{(k)}(\vec{z},\vec{w})) = \prod_{r_1,r_2=1}^{k}\frac{1}{(1-z_{r_1}w_{r_2})^{\theta}}. \tag{2.4}$$

Expanding (2.4) with the Newton binomial formula, we obtain

$$\begin{aligned}\prod_{r_1,r_2=1}^{k}\frac{1}{(1-z_{r_1}w_{r_2})^{\theta}} &= \sum_{A}\prod_{i,j=1}^{k}\binom{A_{ij}+\theta-1}{\theta-1}\\ &\quad\times\prod_{i=1}^{k} z_i^{\sum_{j=1}^{k}A_{ij}}\prod_{j=1}^{k} w_j^{\sum_{i=1}^{k}A_{ij}},\end{aligned} \tag{2.5}$$

where the sum runs over the set of all $k \times k$ matrices A whose entries are nonnegative integers. Equating coefficients according to (2.2) fixes the row and column sums appearing in (2.5) and completes the proof of the Theorem. $\square$

We remark that Diaconis and Gamburd [19] prove this result specifically for the coefficients $c_n^{(N)}$ with $\theta = 1$, related to random unitary matrices. In contrast to the above computation, they exploited the known orthogonality of the Schur functions and explicit results associated with the RSK correspondence. When $N = \infty$, their result recovers ours for the coefficients c_n with $\theta = 1$, but if $\theta \neq 1$, their result is distinct from ours. Despite this, in the following we discuss an interpretation of our result for general $\theta > 0$ in terms of Jack functions (which extend the Schur functions to any $\theta > 0$).

2.1. *Connection to Jack functions.* We briefly recall some symmetric function notation. We follow [55] and [45] for all notational conventions. We refer the reader to [55] for a concise reminder of the definitions.

Let Λ be the algebra of all symmetric formal power series in a countably infinite family of indeterminates. For any partition λ we let $\mathfrak{p}_\lambda$ be the power sum symmetric function, $\mathfrak{e}_\lambda$ be the elementary symmetric function, and $\mathfrak{m}_\lambda$ be the monomial symmetric function.

For any partition $\lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \dots)$, we let

$$z_\lambda = 1^{m_1}\cdot 2^{m_2}\cdot 3^{m_3}\cdots m_1!m_2!m_3!\cdots.$$

We also define $\ell(\lambda)$ to be the length of a partition. We define an inner product on Λ by

$$\langle \mathfrak{p}_\lambda, \mathfrak{p}_\mu \rangle = \mathbf{1}_{\lambda=\mu} z_\lambda \theta^{\ell(\lambda)}.$$

The Jack functions $\{P_\lambda^\theta\}$ form another basis Λ, which can be uniquely defined by (c.f. [55], Theorem 1.1):

1. $\langle P_\lambda^\theta, P_\mu^\theta \rangle = 0$ if $\lambda \neq \mu$.
2. Expanding the Jack function into monomial basis,

$$P_\lambda^\theta = \sum u_{\lambda\mu}(\alpha)\mathfrak{m}_\mu,$$

all nonzero coefficients $u_{\lambda\mu}(\alpha)$ satisfy $\mu \leq \lambda$ where $\leq$ is the dominance ordering (also known as the "natural ordering" in [45], p.6).

3. The leading coefficient $u_{\lambda\lambda} = 1$.

When $\theta = 1$, the Jack polynomials coincide with the Schur polynomials, which we denote by $\mathfrak{s}_\lambda$.

For any symmetric functions p, g, we define another inner product

$$\langle p, g \rangle_n = \frac{1}{Z_{n,\beta}} \int_{\mathbb{T}} p(x)\overline{g(x)} \prod_{i \neq j} |x_i - x_j|^{1/\theta}\, dx, \tag{2.6}$$

that is to say integration against the circular–β ensemble. Here we have specialized the functions p and g by sending all $x_j = 0$ for $j > n$. The $Z_{n,\beta}$ is the usual normalization so that $\langle 1, 1 \rangle = 1$. Then, for any symmetric functions, $\langle p, g \rangle_n \to \langle p, g \rangle$ as $n \to \infty$ (see the discussion below (10.38) in [45]). Furthermore, one has that the polynomials $\{P_\lambda^\theta : \ell(\lambda) \leq n\}$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_n$ [45], (10.36).

The Kostka numbers $K_{\lambda\mu}$ can be defined as

$$\mathfrak{s}_\lambda = \sum_\mu K_{\lambda\mu} \mathfrak{m}_\mu.$$

As a corollary (apply the ω involution to [56], Corollary 7.12.4)

$$\mathfrak{e}_\mu = \sum_\lambda K_{\lambda'\mu} \mathfrak{s}_\lambda.$$

Then it is possible to generalize these coefficients to the Jack setting by defining them as the unique coefficients so that, for all partitions μ,

$$\mathfrak{e}_\mu = \sum_\lambda K_{\lambda'\mu}^\theta P_\lambda^\theta. \tag{2.7}$$

The proof of [19] exploited an identity for the Kostka numbers which follows from the RSK bijection (see [56], Section 7.11). This is given by

$$\sum_\lambda K_{\lambda\mu} K_{\lambda\nu} = |\mathrm{Mag}_{\mu,\nu}|;$$

see [56], Corollary 7.12.3.

As a corollary of Theorem 1.6, we get a new proof of this fact as well as a generalization to all θ. We mention that these connection coefficients are useful for the exact evaluation of some moments of β-ensembles [47].

THEOREM 2.1.

$$\sum_\lambda K_{\lambda'\mu}^\theta K_{\lambda'\nu}^\theta \langle P_\lambda^\theta, P_\lambda^\theta \rangle = \sum_{A \in \mathrm{Mag}_{\mu,\nu}} \prod_{i,j} \binom{\theta + A_{ij} - 1}{A_{ij}}.$$

PROOF. Recall χ_N is the characteristic polynomial of a circular-β random matrix, and so

$$\chi_N(t) = \sum_{k=0}^{N} \mathfrak{e}_k(\lambda) t^k,$$

with λ distributed as CβE. Then from Theorem 1.3,

$$\mathbb{E}\big(\mathfrak{e}_\mu(\lambda)\overline{\mathfrak{e}_\nu(\lambda)}\big) \to \mathbb{E} \prod_{j=1}^{\infty} c_j^{\mu_j} \overline{c_j}^{\nu_j}.$$

On the other hand,

$$\mathbb{E}\big(\mathfrak{e}_\mu(\lambda)\overline{\mathfrak{e}_\nu(\lambda)}\big) = \langle \mathfrak{e}_\mu, \mathfrak{e}_\nu \rangle_N \to \langle \mathfrak{e}_\mu, \mathfrak{e}_\nu \rangle,$$

and hence by (2.7) and Theorem 1.6

$$\sum_\lambda K^\theta_{\lambda'\mu} K^\theta_{\lambda'\nu} \langle P^\theta_\lambda, P^\theta_\lambda \rangle = \sum_{A \in \mathrm{Mag}_{\mu,\nu}} \prod_{i,j} \binom{\theta + A_{ij} - 1}{A_{ij}}.$$

The normalization constant $\langle P^\theta_\lambda, P^\theta_\lambda \rangle$ is given in [45], (10.16). $\square$

2.2. *Asymptotics of the moments.* In this subsection we will give the proof of Theorem 1.7. It is instructive to begin with a particular case, so we first discuss the case of the fourth moment, or $k = 2$ in Theorem 1.7. Then we show how to generalize the approach to all positive integers k.

EXAMPLE 2.2. Using Theorem 1.6, we can, for instance, take $k = 2$, $\mu_1 = \mu_2 = n$ and $\nu_1 = \nu_2 = n$ with all other exponents equal to zero. In this case one is summing over all 2×2 magic squares whose row and column sums are equal to n. Such magic squares are parameterized by a single variable (denoted here j), and Theorem 1.6 yields

$$\mathbb{E}(|c_n|^4) = \sum_{j=0}^{n} \binom{j + \theta - 1}{\theta - 1}^2 \binom{n - j + \theta - 1}{\theta - 1}^2. \tag{2.8}$$

When $\theta = 1$, the right-hand side of (2.8) is given by $n + 1$ (the number of 2×2 magic squares), as obtained in [19]. For general $\theta > 0$, we can compute the asymptotics as $n \to \infty$ from the sum representation (2.8) as follows.

LEMMA 2.3. *For any* $0 < \theta < 1/2$, *we have the following*:

$$\frac{\mathbb{E}(|c_n|^4)}{\mathbb{E}(|c_n|^2)^2} \sim 2\frac{\Gamma(1 - 2\theta)}{\Gamma(1 - \theta)^2}, \quad n \to \infty, \tag{2.9}$$

where we recall the normalization

$$\mathbb{E}(|c_n|^2) = \binom{n + \theta - 1}{\theta - 1} \sim \frac{1}{\Gamma(\theta)} n^{\theta - 1}. \tag{2.10}$$

PROOF. Fix $0 < \delta < 1$, and split the sum in (2.8) according to whether $j \le \lfloor \delta n \rfloor$, $\lfloor \delta n \rfloor + 1 \le j \le n - \lfloor \delta n \rfloor - 1$ or $n - \lfloor \delta n \rfloor \le j \le n$, denoting each sum $\mathcal{S}_1$, $\mathcal{S}_2$ or $\mathcal{S}_3$, respectively. Then in the sum $\mathcal{S}_1$, the term $n - j$ is always large so that the second binomial coefficient is uniformly bounded by $c_{\delta,\theta}n^{\theta-1}$ for some constant $c_{\delta,\theta} > 0$. Applying dominated convergence then gives

$$\lim_{n\to\infty}\frac{\mathcal{S}_1}{(\mathbb{E}(|c_n|^2))^2} = \sum_{j=0}^{\infty}\binom{j+\theta-1}{\theta-1}^2 \lim_{n\to\infty}\frac{\binom{n-j+\theta-1}{\theta-1}^2}{\binom{n+\theta-1}{\theta-1}^2} \tag{2.11}$$

$$= \sum_{j=0}^{\infty}\binom{j+\theta-1}{\theta-1}^2 \tag{2.12}$$

$$= \frac{\Gamma(1-2\theta)}{\Gamma(1-\theta)^2}, \tag{2.13}$$

where in (2.13) we used Lemma C.1. An identical argument holds for the sum $\mathcal{S}_3$ (by symmetry of $j \to n-j$), and this gives the factor 2 in (2.9). The sum $\mathcal{S}_2$ is negligible: the second binomial coefficient is still bounded by $c_{\theta,\delta}n^{\theta-1}$, which gives the same order of magnitude, but now both the upper and lower limits of the sum are growing. Since (2.12) converges, we have that $n^{-2(\theta-1)}\mathcal{S}_2 \to 0$. This completes the proof of the Lemma. □

REMARK 2.4. When $\theta \geq 1/2$, this argument breaks down because the sum in (2.13) is divergent which leads to slightly different asymptotic behaviour. When $\theta > 1/2$, the sum $\mathcal{S}_2$ can be approximated by a Riemann integral which now gives the main contribution: As $n \to \infty$, we have

$$\begin{aligned}\mathbb{E}(|c_n|^4)|_{\theta>1/2} &\sim n^{4(\theta-1)+1}\frac{1}{\Gamma(\theta)^4}\int_0^1 x^{2(\theta-1)}(1-x)^{2(\theta-1)}\,dx \\ &= n^{4(\theta-1)+1}\frac{\Gamma(2\theta-1)^2}{\Gamma(4\theta-2)\Gamma(\theta)^4}\end{aligned} \tag{2.14}$$

and conversely note that this integral becomes divergent when $\theta \leq 1/2$, with the leading power of n matching at the transition $\theta = 1/2$. In the case $\theta = 1/2$, one can show that

$$\mathbb{E}(|c_n|^4)|_{\theta=1/2} \sim \frac{2}{\pi^2}\frac{\log n}{n}, \quad n \to \infty. \tag{2.15}$$

The fact that the argument leading to (2.9) can be generalized to all higher moments is the subject of the next result.

THEOREM 2.5. *Let k be a positive integer such that $k\theta < 1$. Then*

$$\lim_{n\to\infty}\frac{\mathbb{E}(|c_n|^{2k})}{(\mathbb{E}(|c_n|^2))^k} = k!\frac{\Gamma(1-k\theta)}{\Gamma(1-\theta)^k}.$$

PROOF. Recall from Theorem 1.6

$$\mathbb{E}(|c_n|^{2k}) = \sum_{A\in \mathrm{Mag}_{\pi,\pi}}\prod_{i,j}\binom{\theta+A_{ij}-1}{A_{ij}}, \tag{2.16}$$

where π is the partition which has k parts of length n. In particular, the magic squares $\mathrm{Mag}_{\pi,\pi}$ are $k\times k$ and have all row and column sums equal to n. Fix a $\delta > 0$. Let $E_{\delta,n} \subset \mathrm{Mag}_{\pi,\pi}$ be those magic squares in which there is a row i for which there are two j so that $A_{ij} > \delta n$. We claim that

$$\sum_{A\in E_{\delta,n}}\prod_{i,j}\binom{\theta+A_{ij}-1}{A_{ij}} = o(n^{k(\theta-1)}). \tag{2.17}$$

We shall return to this point, but for the moment we give the proof contingent on (2.17).

Note that, for δ sufficiently small, each $A \in \mathrm{Mag}_{\pi,\pi} \setminus E_{\delta,n}$ has in each row exactly one entry with size larger than $n(1-k\delta)$. Again for δ sufficiently small, this implies that each column additionally has exactly one such entry. Thus, for any such A there is a $k \times k$ permutation matrix P, the support of which coincides with the entries of A larger than $n(1-k\delta)$.

So for each permutation σ, we can define $S_\sigma \subset \mathrm{Mag}_{\pi,\pi} \setminus E_{\delta,n}$ as those magic squares A for which $A_{ij} \ge n(1-k\delta)$ for some i, j if and only if $j = \sigma(i)$. For δ sufficiently small, we, moreover, have that $\{S_\sigma : \sigma \text{ is a permutation of } [k]\}$ is a partition of $\mathrm{Mag}_{\pi,\pi} \setminus E_{\delta,n}$.

For any $A \in S_\sigma$, we may make a uniform estimate (over S_σ)

$$\prod_{i,j} \binom{\theta + A_{ij} - 1}{A_{ij}} = \prod_{j=\sigma(i)} \left\{ \binom{\theta + n - 1}{n} \bigl(1 + O(\delta)\bigr) \right\} \cdot \prod_{j \ne \sigma(i)} \left\{ \binom{\theta + A_{ij} - 1}{A_{ij}} \right\}.$$

Now, by permuting the rows of the magic squares, we can bijectively map S_σ to S_{Id}. Moreover, permuting the rows of a magic square does not alter the weight of the filling in the display above, and we can, therefore, conclude, using (2.17),

$$\begin{aligned} \mathbb{E}\bigl(|c_n|^{2k}\bigr) = k! \sum_{A \in S_{\mathrm{Id}}} & \left\{ \binom{\theta + n - 1}{n} \bigl(1 + O(\delta)\bigr) \right\}^k \\ & \cdot \prod_{i \ne j} \left\{ \binom{\theta + A_{ij} - 1}{A_{ij}} \right\} + o\bigl(n^{k(\theta-1)}\bigr). \end{aligned} \tag{2.18}$$

Let H_k be the $k \times k$ nonnegative integer matrices A in which for each $1 \le i \le k$

$$\sum_j A_{ij} = \sum_j A_{ji}.$$

We claim that

$$\sum_{A \in H_k} \prod_{i \ne j} \binom{\theta + A_{ij} - 1}{A_{ij}} = \frac{\Gamma(1 - k\theta)}{\Gamma(1-\theta)^k}. \tag{2.19}$$

From the Newton binomial formula, we have that, for $\{z_i\}$ and $\{y_j\}$ in the unit disk,

$$\prod_{i \ne j} \left(\frac{1}{1 - z_i \overline{y_j}} \right)^\theta = \sum_{(\mu_j),(\nu_j) \in \mathbb{N}^k} \prod_j z_j^{\mu_j} \overline{y_j}^{\nu_j} \cdot \left\{ \sum_{A \in \mathrm{Mag}_{0,\mu,\nu}} \prod_{i \ne j} \binom{\theta + A_{ij} - 1}{A_{ij}} \right\},$$

where $\mathrm{Mag}_{0,\mu,\nu} \subset \mathrm{Mag}_{\mu,\nu}$ have 0 on the diagonal. Hence, on setting $z_j = re^{i\omega_j}$ and $y_j = re^{i\omega_j}$ for $r \in (0,1)$ and averaging over all ω_j, we have

$$\begin{aligned} & \frac{1}{(2\pi)^k} \int \prod_{i \ne j} \left(\frac{1}{1 - r^2 e^{\sqrt{-1}(\omega_i - \omega_j)}} \right)^\theta d\omega \\ & \quad = \sum_{(\mu_j) \in \mathbb{N}^k} r^{2\sum \mu_j} \cdot \left\{ \sum_{A \in \mathrm{Mag}_0(\mu,\mu)} \prod_{i \ne j} \binom{\theta + A_{ij} - 1}{A_{ij}} \right\}. \end{aligned}$$

This integral on the left-hand side is convergent on sending $r \to 1$, and, moreover, gives exactly the Morris integral (1.23). The right-hand side, meanwhile, converges to the left-hand side of (2.19) which completes the proof of (2.19).

As (2.19) is convergent, it follows from dominated convergence that

$$\lim_{n \to \infty} n^{-k(\theta-1)} \sum_{A \in S_{\mathrm{Id}}} \prod_{i=j} \left\{ \binom{\theta + n - 1}{n} \right\} \cdot \prod_{i \ne j} \left\{ \binom{\theta + A_{ij} - 1}{A_{ij}} \right\} = \frac{\Gamma(1 - k\theta)}{\Gamma(1-\theta)^k \Gamma(\theta)^k}.$$

Hence, on combining this with (2.18) and sending $\delta \to 0$, we conclude that

$$\lim_{n\to\infty} n^{-k(\theta-1)}\mathbb{E}(|c_n|^{2k}) = \frac{k!\Gamma(1-k\theta)}{\Gamma(1-\theta)^k\Gamma(\theta)^k}.$$

Finally we turn to the proof of (2.17). By the Birkhoff–von Neumann theorem, the doubly stochastic matrices are the convex hull of the permutation matrices. It follows that, for every $A \in \mathrm{Mag}_{\pi,\pi}$, there is a permutation σ such that each entry of $A_{i\sigma(i)}$ has size at least $n/k!$ Hence, by symmetry (c.f. the argument above (2.18)) we can restrict the sum in (2.17) to those $A \in E_{\delta,n}$ whose every diagonal entry is at least $n/k!$. Denote this subset of $E_{\delta,n}$ by $E'_{\delta,n}$. In short, we have the bound

$$\sum_{A\in E_{\delta,n}} \prod_{i,j} \binom{\theta + A_{ij} - 1}{A_{ij}} \leq k! \sum_{A\in E'_{\delta,n}} \prod_{i} \left\{ C_\theta \left(\frac{n}{k!}\right)^{\theta-1} \right\} \prod_{i\neq j} \left\{ \binom{\theta + A_{ij} - 1}{A_{ij}} \right\}$$

for some absolute constant C_θ. Hence, once more using the absolute convergence of (2.19) and dominated convergence, (2.17) follows. □

3. The Ewens sampling formula. In this section we describe a connection between secular coefficients and random permutations that we believe is interesting in its own right. While detailing this, we shall revise some of the key results about random permutations, as these will be useful in the subsequent sections.

The secular coefficients defined in (1.13) can be given the following explicit formula:

$$c_n = \sum_{\vec{m}\in S_n} \prod_{k=1}^{n} \frac{\mathcal{N}_k^{m_k}}{m_k!} \left(\frac{\theta}{k}\right)^{m_k/2}. \tag{3.1}$$

The summation is over the set S_n of all compositions of n, that is, $\vec{m} = (m_1, m_2, \dots, m_n)$ is such that each m_k is a nonnegative integer satisfying

$$\sum_{k=1}^{n} k m_k = n. \tag{3.2}$$

From (3.1) we have that c_n is measurable with respect to $\mathscr{G}_n := \sigma\{\mathcal{N}_1, \dots, \mathcal{N}_n\}$. So, going forward, we will make use of the filtration $\mathscr{G} = (\mathscr{G}_n : n \in \mathbb{N})$.

Taking the L^2-norm of (3.1) gives

$$\mathbb{E}(|c_n|^2) = \sum_{\vec{l}\in S_n, \vec{m}\in S_n} \prod_{k=1}^{n} \frac{\mathbb{E}(\mathcal{N}_k^{m_k}\overline{\mathcal{N}_k}^{l_k})}{m_k! l_k!} \left(\frac{\theta}{k}\right)^{(m_k+l_k)/2}, \tag{3.3}$$

where we used independence of the family $\{\mathcal{N}_k\}_{k=1}^{\infty}$ to take the expectation inside the product. Next, applying the Gaussian formula

$$\mathbb{E}(\mathcal{N}_k^{m_k}\overline{\mathcal{N}_k}^{l_k}) = \mathbb{1}_{m_k=l_k}(m_k)! \tag{3.4}$$

implies that the compositions in the sum (3.3) must coincide in order to give a nonzero term. This gives

$$\mathbb{E}(|c_n|^2) = \sum_{\vec{m}\in S_n} \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{\theta}{k}\right)^{m_k}. \tag{3.5}$$

Given a permutation σ on n symbols, we can characterize it using its cycle structure $(m_1, \dots, m_n)$, where m_j denotes the number of cycles in σ having length j and $\sum_{j=1}^{n} j m_j = n$. The summation in (3.5) is well known in the theory of random permutations where it appears as the normalizing factor for the Ewens sampling formula.

DEFINITION 3.1. The *Ewens sampling formula* is a probability distribution on cycle counts $\vec{M}^{(n)} = (M_1, \dots, M_n)$ given by

$$\mathbb{P}(\vec{M}^{(n)} = \vec{m}) = \frac{\mathbb{1}_{\sum_{k=1}^n km_k = n}}{\binom{n+\theta-1}{\theta-1}} \prod_{k=1}^n \left(\frac{\theta}{k}\right)^{m_k} \frac{1}{m_k!}. \tag{3.6}$$

The fact that (3.6) is normalized gives the explicit form of the sum in (3.5),

$$\mathbb{E}(|c_n|^2) = \binom{n+\theta-1}{\theta-1}. \tag{3.7}$$

In general, if we restrict the summation in (3.1) to a subset $P \subset S_n$ and denote this $c_{n,P}$, an identical computation holds. This yields the fundamental correspondence

$$\mathbb{E}(|c_{n,P}|^2) = \binom{n+\theta-1}{\theta-1} \mathbb{P}(\vec{M}^{(n)} \in P), \tag{3.8}$$

where on the left-hand side of (3.8) the expectation is taken over the Gaussian random variables in (3.1), while on the right-hand side $\vec{M}$ follows the Ewens sampling formula in (3.6) with parameter θ.

We can also use the Ewens sampling formula to describe conditional expectations of $|c_n|^2$. In analogy with (3.3), for $q \le n$,

$$\mathbb{E}(|c_n|^2 \mid \mathscr{G}_q) = \sum_{\vec{l} \in S_n, \vec{m} \in S_n} \prod_{k=1}^n \frac{\mathbb{E}(\mathcal{N}_k^{m_k} \overline{\mathcal{N}_k}^{l_k} \mid \mathscr{G}_q)}{m_k! l_k!} \left(\frac{\theta}{k}\right)^{(m_k+l_k)/2}. \tag{3.9}$$

The only nonzero pairs $(\vec{l}, \vec{m})$ in this sum have $m_k = l_k$ for $k > q$, and this allows us to greatly simplify this expression. Let us define

$$c_{n,q} = \sum_{\substack{(m_k):1 \le k \le q \\ \sum_{k=1}^q km_k = n}} \prod_{k=1}^q \frac{\theta^{m_k/2}}{m_k! k^{m_k/2}} \mathcal{N}_k^{m_k}. \tag{3.10}$$

REMARK 3.2. This (3.10) is a special case of $c_{n,P}$ (c.f. (3.8)) in which P are those partitions with no parts greater than q. This is also equivalent to setting the Gaussians $\mathcal{N}_k = 0$ for all $k > q$ in the sum (3.1). In particular, $c_{n,q}$ is $\mathscr{G}_q$–measruable.

In terms of (3.10), we can, therefore, give the sum formula by partitioning (3.9) according to $r = \sum_{k=1}^q km_k$. Note that when $r \ge n-q$, we have no way to complete the partition, except by choosing all larger $m_k = 0$, and so

$$\mathbb{E}(|c_n|^2 \mid \mathscr{G}_q) = |c_{n,q}|^2 + \sum_{r=0}^{n-q-1} |c_{r,q}|^2 \sum_{\substack{(m_k):q<k\le n \\ \sum_{k=q+1}^{n-q-1} km_k = n-r}} \prod_{k=q+1}^n \frac{\theta^{m_k}}{m_k! k^{m_k}}.$$

Note that we may have no $m_k > 0$ for $k > n-r$ in the inner sum, and so using (3.6),

$$\begin{aligned} \mathbb{E}(|c_n|^2 \mid \mathscr{G}_q) = |c_{n,q}|^2 + \sum_{r=0}^n |c_{r,q}|^2 \binom{n-r+\theta-1}{\theta-1} \mathbb{P}\big[M_j^{(n-r)} \\ = 0, \quad \text{for all } 1 \le j \le q\big]. \end{aligned} \tag{3.11}$$

We note that sum need only run to $n-q$, as the probability therein is 0 for larger r. We will do an asymptotic analysis of this conditional expectation in Section 7.1.

3.1. *Properties of the Ewens sampling formula.* We remark that the case $\theta = 1$ in (3.6) corresponds to the uniform measure on the set of all permutations, while the general case $\theta > 0$ corresponds to a tilting of the uniform measure. For any $\theta > 0$, a wealth of results are known concerning statistical properties of the cycle counts; see the text [5] from which we will borrow from repeatedly in what follows. The main point exploited in [5] is that, apart from the indicator function, (3.6) is a conditional joint law of n independent random variables $(Z_1, \dots, Z_n)$, where each Z_k is Poisson distributed with parameter θ/k. Therefore, statistics of the cycle counts can be reduced to studying the independent random variables $(Z_1, \dots, Z_n)$ paired with the condition that $T_{0n} = n$ where $T_{0n} = \sum_{j=1}^n jZ_j$. We will refer to this as the *conditioning relation*.

In fact, the random variable T_{0n} and its limiting distribution play an important role in what follows, and we record some of the key results about it from [5].

LEMMA 3.3. *Suppose that $r = r_n \in \mathbb{N}$ satisfies $r/n \to y \in (0, \infty)$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} n\mathbb{P}(T_{0n} = r) = p_\theta(y), \tag{3.12}$$

where $p_\theta(y)$ is an (explicit) probability density function satisfying the following properties:

1. *An explicit formula at $y = 1$,*

$$p_\theta(1) = \frac{e^{-\gamma_{\mathrm{E}}\theta}}{\Gamma(\theta)}, \tag{3.13}$$

where γ_{E} is the Euler–Mascheroni constant.

2. *Rapid decay at $y = +\infty$,*

$$\sup_{y\ge n} p_\theta(y) \le \frac{\theta^n}{n!} \tag{3.14}$$

3. *The derivative identity, for $x \notin \{0, 1\}$*

$$\frac{d}{dx}\left[x^{1-\theta} p_\theta(x)\right] = -\theta x^{-\theta} p_\theta(x-1). \tag{3.15}$$

PROOF. These are proved in [5], Section 4, using size biasing techniques. $\square$

We will also make use of the following finite n uniform bound.

LEMMA 3.4 (Lemma 4.12(i) in [5]). *If $0 \le \theta \le 1$, then*

$$\max_{k\ge 0} \mathbb{P}(T_{0n} = k) \le e^{-\theta h(n+1)}, \tag{3.16}$$

where $h(n+1)$ is the harmonic number

$$h(n+1) = \sum_{j=1}^n \frac{1}{j}. \tag{3.17}$$

Now, let $L^{(n)}$ denote the length of the longest cycle. This quantity is closely related to the distribution of T_{0n}, because by the conditioning relation

$$\begin{aligned} \mathbb{P}(L^{(n)} \le r) &= \mathbb{P}(\{M_{r+1} = 0\} \wedge \dots \wedge \{M_n = 0\}) \\ &= \mathbb{P}(\{Z_{r+1} = 0\} \wedge \dots \wedge \{Z_n = 0\} | T_{0n} = n) \\ &= \mathbb{P}(\{Z_{r+1} = 0\} \wedge \dots \wedge \{Z_n = 0\}) \frac{\mathbb{P}(T_{0r} = n)}{\mathbb{P}(T_{0n} = n)}. \end{aligned} \tag{3.18}$$

Now, Lemma 3.3 gives the following, as quoted in [5], Lemma 4.23, and attributed to Kingman (1977).

LEMMA 3.5. *As $n \to \infty$, we have the convergence in distribution $n^{-1}L^{(n)} \xrightarrow{d} L^{(\infty)}$, where $L^{(\infty)}$ is a random variable with distribution function F_θ given by*

$$F_\theta(x) = e^{\gamma_{\mathrm{E}}\theta} x^{\theta-1}\Gamma(\theta) p_\theta(1/x) \quad \textit{for all } x \in (0,1]. \tag{3.19}$$

We also need a tail bound that controls the probability that the largest cycle is unusually small.

LEMMA 3.6. *For any $\theta > 0$, there is a constant $c_\theta > 0$ so that*

$$\mathbb{P}(L^{(n)} \le n/\log n) \le n^{-c_\theta \log\log n}.$$

PROOF. Using (3.18) and the convergence of $\mathbb{P}(T_{0n} = n)$, there is some constant C_θ so that, for any $r \in \mathbb{N}$,

$$\mathbb{P}(L^{(n)} \le r) \le \frac{\mathbb{P}(T_{0r} = n)}{\mathbb{P}(T_{0n} = n)} \le C_\theta \mathbb{P}(T_{0r} = n).$$

When r is much smaller than n, this probability becomes very small. Using standard concentration results for functionals of Poisson fields (see [59], Proposition 3.1, or [44]), there is a constant C_θ sufficiently large that, for all $t > 0$ and all $r \in \mathbb{N}$,

$$\mathbb{P}(T_{0r} > rt) \le \exp\left(-\frac{t}{4}\log\left(1 + \frac{t}{C_\theta}\right)\right).$$

The proof follows by taking $r = \lfloor n/\log n \rfloor$ and $t = \lfloor \log n \rfloor$. □

To analyze (3.11), we will also need information on the shortest cycle $S^{(n)}$ in a Ewens distributed permutation. From the asymptotic independence of the cycle counts in a Ewens permutation, one expects $\mathbb{P}(S^{(n)} > q) \to e^{-\theta h(q+1)}$ as $n \to \infty$. We will make use of a nonasymptotic bound that has this behavior.

LEMMA 3.7. *For all $\theta > 0$, there is a constant $C_\theta > 0$ so that, for all $n, q \in \mathbb{N}$,*

$$\mathbb{P}(S^{(n)} > q) \le \frac{1}{\binom{q-1+\theta}{\theta}} \le C_\theta e^{-\theta h(q+1)}.$$

PROOF. We use the *Feller* description of the Ewens sampling formula, which we now describe. Let $\xi_1, \xi_2, \xi_3, \ldots$ be independent Bernoulli variables having parameter

$$\mathbb{P}(\xi_j = 1) = \frac{\theta}{\theta + j - 1}, \quad \text{for all } j \in \mathbb{N}.$$

For any n, define a *spacing of length ℓ* in the vector

$$(1, \xi_n, \xi_{n-1}, \ldots, \xi_3, \xi_2, 1)$$

as a pair $\{k, k+\ell\}$, where $\xi_k = \xi_{k+\ell} = 1$ and $\xi_j = 0$ for all j with $k < j < k+\ell$. In the case $k+\ell = n+1$, we instead use 1 in place of $\xi_{k+\ell}$. The Feller description of the Ewens sampling formula (see [4], Section 3) states that the random vector $(\#\{\text{spacings of length } \ell\} | 1 \le \ell \le n)$ has the same distribution as $(\vec{M}_\ell^{(n)} | 1 \le \ell \le n)$.

In particular, we have the inequality

$$\begin{aligned}\mathbb{P}\big[S^{(n)} > q\big] &\le \mathbb{P}[\xi_k = 0, \text{ for all } 2 \le k \le q] \\ &= \prod_{k=2}^{q}\Big(1 - \frac{\theta}{\theta + k - 1}\Big) \\ &= \frac{\Gamma(q)\Gamma(\theta+1)}{\Gamma(\theta+q)} \\ &= \frac{1}{\binom{q-1+\theta}{\theta}}.\end{aligned}$$

To derive the final inequality, using second line of the display above and bounding

$$\prod_{k=2}^{q}\Big(1 - \frac{\theta}{\theta + k - 1}\Big) \le \exp\Big(-\sum_{k=2}^{q}\frac{\theta}{\theta + k - 1}\Big) \le \exp\big(-\theta h(q)\big).$$

On replacing $h(q)$ by $h(q+1)$, we only decrease the bound by a factor at most e^{θ}, and hence the claimed bound following with $C_\theta = e^{\theta}$. $\square$

4. Martingale approximation and convergence. We begin this section by finding an approximate martingale structure in the sum (3.1). This opens the possibility of applying a central limit theorem for martingales, and we explain why this is relevant for our proof of Theorem 1.8. The results on permutations in the previous section will be used to arrive at the martingale approximation, as we now show.

4.1. *Martingale approximation.* Given a composition m, we define

$$\nu(m) := \max\{k = 1, \dots, n \mid m_k \ge 1\}. \tag{4.1}$$

If the m_k are interpreted as cycle counts, the quantity $\nu(m)$ is the length of the largest cycle in the corresponding permutation. Given $\delta > 0$, consider permutations whose largest cycle is smaller than $\lfloor \delta n \rfloor$,

$$P_\delta := \big\{m \in S_n \mid \nu(m) < \lfloor \delta n \rfloor\big\}. \tag{4.2}$$

We will show that the contribution of P_δ to the sum in (3.1) can be neglected for large n and small δ. In the sum over the remaining terms P_δ^{c}, we define another negligible set, the set of all compositions where there are multiple longest cycles of the same length,

$$R := \{m \in S_n \mid m_{\nu(m)} \ge 2\}. \tag{4.3}$$

We define $\tilde{c}_n^{(\delta)}$ to be the sum over compositions whose largest cycle is *greater* than $\lfloor \delta n \rfloor$, and where there is only one such largest cycle, in other words $\tilde{c}_n^{(\delta)} := c_{n, P_\delta^{\mathrm{c}} \cap R^{\mathrm{c}}}$. We have

$$c_n = \tilde{c}_n^{(\delta)} + c_{n,P_\delta} + c_{n,P_\delta^{\mathrm{c}} \cap R}, \tag{4.4}$$

and summing over the possible lengths of the longest cycle, we can decompose the sum as

$$\tilde{c}_n^{(\delta)} = \sum_{q=\lfloor \delta n \rfloor}^{n} \mathcal{N}_q \sqrt{\frac{\theta}{q}} c_{n-q,q-1}, \tag{4.5}$$

where $c_{n-q,q-1}$ are as in (3.10). Note that $c_{n-q,q-1}$ only depends on the first $q-1$ Gaussians. Consequently, $\tilde{c}_n^{(\delta)}$ is a martingale with respect to the filtration $\mathscr{G} = (\sigma\{\mathcal{N}_1, \dots, \mathcal{N}_n\} : n \in \mathbb{N})$.

LEMMA 4.1 (Martingale approximation). *Let $0 < \delta < 1$, and assume $\theta \in [0, 1]$. Then c_n is well approximated by the martingale $\tilde{c}_n^{(\delta)}$, given by (4.5), in the following sense. After proper normalization the error terms c_{n,P_δ} and $c_{n,P_\delta^c \cap R}$ in (4.4) have L^2-norm satisfying*

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{\mathbb{E}(|c_{n,P_\delta}|^2)}{\mathbb{E}(|c_n|^2)} = 0, \tag{4.6}$$

$$\frac{\mathbb{E}(|c_{n,P_\delta^c \cap R}|^2)}{\mathbb{E}(|c_n|^2)} = O_\delta(n^{-\theta}), \quad n \to \infty. \tag{4.7}$$

Taken together, using (4.4), we have

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{\mathbb{E}(|c_n - \tilde{c}_n^{(\delta)}|^2)}{\mathbb{E}(|c_n|^2)} = 0. \tag{4.8}$$

PROOF. By the correspondence (3.8), we have

$$\frac{\mathbb{E}(|c_{n,P_\delta}|^2)}{\mathbb{E}(|c_n|^2)} = \mathbb{P}(L_1^{(n)} < \lfloor \delta n \rfloor), \tag{4.9}$$

where the right-hand side is the probability that the longest cycle is bounded by $\lfloor \delta n \rfloor$. By Lemma 3.5 we have

$$\lim_{n \to \infty} \mathbb{P}(L_1^{(n)} < \lfloor \delta n \rfloor) = e^{\gamma_E \theta} \delta^{\theta - 1} \Gamma(\theta) p_\theta(1/\delta), \tag{4.10}$$

and the fast decay (3.14) gives

$$\lim_{\delta \to 0} \lim_{n \to \infty} \mathbb{P}(L_1^{(n)} < \lfloor \delta n \rfloor) = 0 \tag{4.11}$$

which proves statement (4.6). For (4.7) we have

$$\begin{aligned} \mathbb{P}(P^\delta \cap R) &= \sum_{q=\lfloor \delta n \rfloor}^{n} \mathbb{P}(\{m_q \geq 2\} \wedge \{\nu(m) = q\}) \\ &= \sum_{q=\lfloor \delta n \rfloor}^{n} \mathbb{P}(\{m_q \geq 2\} \wedge \{m_{q+1} = 0\} \wedge \ldots \wedge \{m_n = 0\}) \\ &= \sum_{q=\lfloor \delta n \rfloor}^{n} \sum_{\ell=2}^{\infty} \mathbb{P}(\{Z_q = \ell\} \wedge \{Z_{q+1} = 0\} \wedge \ldots \wedge \{Z_n = 0\}) \\ &\quad \times \frac{\mathbb{P}(T_{0(q-1)} = n - q\ell)}{\mathbb{P}(T_{0n} = n)}, \end{aligned} \tag{4.12}$$

where in the last step we employed the conditioning relation and we interpret contributions to the sum above as zero, whenever $q\ell > n$. In order to bound the quantities above, first take $r = n$ in (3.12) to see that $\mathbb{P}(T_{0n} = n) \sim n^{-1} p_\theta(1)$. By Lemma 3.4 we have

$$\mathbb{P}(T_{0(q-1)} = n - q\ell) \leq e^{-\theta h(q)}, \tag{4.13}$$

while

$$\mathbb{P}(\{Z_{q+1} = 0\} \wedge \ldots \wedge \{Z_n = 0\}) = e^{-\theta h(n+1) + \theta h(q+1)}. \tag{4.14}$$

A simple bound on the harmonic number gives $e^{-\theta h(n+1)} \leq c_\theta n^{-\theta}$ for some positive constant c_θ. Combining these facts gives, for some other constant c_θ, the bound

$$\mathbb{P}(P^\delta \cap R) \leq n^{1-\theta} c_\theta \sum_{q=\lfloor \delta n \rfloor}^{n} \mathbb{P}(Z_q \geq 2) = O_\delta(n^{-\theta}), \quad n \to \infty, \tag{4.15}$$

where we used that $\mathbb{P}(Z_q \geq 2) \sim \frac{\theta^2}{2q^2}$ as $q \to \infty$. This completes the proof of the martingale approximation. $\square$

4.2. *Martingale central limit theorem.* Having demonstrated the martingale approximation, in this section we will discuss the type of central limit theorems we can apply. In the following we will give a proof of Theorem 1.8, contingent on a certain L^2 estimate that will be dealt with separately in a later section.

To recap, Lemma 4.1 states that if we define increments

$$Z_{n,q} := \frac{1}{\sqrt{\mathbb{E}(|c_n|^2)}} \mathcal{N}_q \sqrt{\frac{\theta}{q}} c_{n-q,q-1}, \tag{4.16}$$

then the quantity

$$\frac{\tilde{c}_n^{(\delta)}}{\sqrt{\mathbb{E}(|c_n|^2)}} = \sum_{q=\lfloor \delta n \rfloor}^{n} Z_{n,q} \tag{4.17}$$

is a good approximation of the normalized coefficient $c_n/\sqrt{\mathbb{E}(|c_n|^2)}$. Now, we see by construction (recalling Remark 3.2) that $c_{n-q,q-1}$ only depends on the first $q-1$ Gaussians and this implies that the random variables $\mathcal{N}_q$ and $c_{n-q,q-1}$ are independent. We have

$$\mathbb{E}(Z_{n,q} | \mathscr{G}_{q-1}) = 0, \tag{4.18}$$

in other words $Z_{n,q}$ are the increments of a martingale with respect to the filtration generated by the first $q-1$ Gaussians.

In order to get convergence in distribution of $\tilde{c}_n^{(\delta)}$ (and, therefore, of c_n), we will apply a central limit theorem for martingales. The majority of these limit theorems rely to a large extent on the analysis of a quantity known as the *bracket process* (sometimes also referred to as the *conditional variance* [28]). In our setting it is given by

$$\mathcal{M}_{\theta,\delta,n} := \sum_{q=\lfloor \delta n \rfloor}^{n} \mathbb{E}\big(|Z_{n,q}|^2 | \mathscr{G}_{q-1}\big) = \frac{1}{\mathbb{E}(|c_n|^2)} \sum_{q=\lfloor \delta n \rfloor}^{n} \frac{\theta}{q} |c_{n-q,q-1}|^2. \tag{4.19}$$

A key hypothesis usually involves showing that this type of quantity converges in probability to a constant as $n \to \infty$. An interesting feature here is that $\mathcal{M}_{\theta,\delta,n}$ will not have a deterministic limit, and so we need a sufficiently general form of the martingale CLT that allows for fluctuations in the limit $n \to \infty$ of the bracket process. These limiting fluctuations will be described in terms of the total mass of Gaussian multiplicative chaos and are ultimately responsible for the structure of the distribution given in Theorem 1.8. The appropriate CLT is the following.

THEOREM 4.2 (Martingale Central Limit Theorem—Section 3.2 in [28]). *Let $\{X_{n,q}\}_{q=1}^{n}$ for $n \geq 1$ be an array of real-valued martingale increments with respect to a filtration $\mathcal{F}_{n,q}$ indexed by q. Define the random variables*

$$\nu_n := \sum_{q=1}^{n} \mathbb{E}\big(X_{n,q}^2 \mid \mathcal{F}_{n,q-1}\big), \tag{4.20}$$

$$\xi_n := \sum_{q=1}^{n} \mathbb{E}\big(X_{n,q}^2 \mathbb{1}_{|X_{n,q}|>\epsilon} \mid \mathcal{F}_{n,q-1}\big). \tag{4.21}$$

Suppose we have the convergence in probability $\nu_n \xrightarrow{p} \nu$, *where* ν *is an a.s. finite random variable, and* $\xi_n \xrightarrow{p} 0$. *Then we have the convergence in distribution*

$$\sum_{q=1}^{n} X_{n,q} \xrightarrow{d} \sqrt{\nu} N_{\mathbb{R}}, \quad n \to \infty, \tag{4.22}$$

where $N_{\mathbb{R}}$ *is a standard (real) Gaussian, independent of* ν.

Note that [28] only deals with real-valued random variables, while (4.16) are complex. Although it is probably straightforward to generalise their result to the complex case, for our particular problem the simple i.i.d. structure of the real and imaginary parts of $\mathcal{N}_q$ allows us to apply Theorem 4.2 directly.

COROLLARY 4.3. *Suppose we have the convergence in probability of the quantity defined in* (4.19),

$$\mathcal{M}_{\theta,\delta,n} \xrightarrow{p} \nu, \quad n \to \infty, \tag{4.23}$$

where ν *is a.s. finite and*

$$\frac{1}{\mathbb{E}(|c_n|^2)^2} \sum_{q=\lfloor \delta n \rfloor}^{n} \frac{\theta^2}{q^2} \mathbb{E}|c_{n-q,q-1}|^4 \to 0, \quad n \to \infty. \tag{4.24}$$

Then it follows that we have the convergence in distribution,

$$\frac{\tilde{c}_n^{(\delta)}}{\sqrt{\mathbb{E}(|c_n|^2)}} \xrightarrow{d} \sqrt{\nu}\mathcal{N}_1, \quad n \to \infty, \tag{4.25}$$

where $\mathcal{N}_1$ *is a standard complex Gaussian, independent of* ν.

PROOF. For any pair $(a, b) \in \mathbb{R}^2$, consider the real-valued martingale

$$\frac{\tilde{c}_{n,a,b}}{\sqrt{\mathbb{E}(|c_n|^2)}} := \frac{1}{\sqrt{\mathbb{E}(|c_n|^2)}} \big(a \operatorname{Re}(\tilde{c}_n^{(\delta)}) + b \operatorname{Im}(\tilde{c}_n^{(\delta)})\big) = \sum_{q=\lfloor \delta n \rfloor}^{n} X_{n,q} \tag{4.26}$$

with filtration generated by the real and imaginary parts of $\mathcal{N}_1, \dots, \mathcal{N}_{q-1}$ and, by (4.16), increments given by

$$X_{n,q} = \frac{1}{\sqrt{\mathbb{E}(|c_n|^2)}} \sqrt{\frac{\theta}{q}} \big(a \operatorname{Re}(\mathcal{N}_q c_{n-q,q-1}) + b \operatorname{Im}(\mathcal{N}_q c_{n-q,q-1})\big). \tag{4.27}$$

We will now check the conditions of Theorem 4.2. A straightforward computation using $\mathbb{E}(\mathcal{N}_q^2) = \mathbb{E}(\overline{\mathcal{N}_q}^2) = 0$ and $\mathbb{E}(|\mathcal{N}_q|^2) = 1$ shows that

$$\nu_n := \sum_{q=\lfloor \delta n \rfloor}^{n} \mathbb{E}\big(X_{n,q}^2 | \mathcal{G}_{q-1}\big) = \frac{a^2 + b^2}{2} \mathcal{M}_{\theta,\delta,n}. \tag{4.28}$$

So (4.23) implies that

$$\nu_n \xrightarrow{p} \frac{a^2 + b^2}{2} \nu. \tag{4.29}$$

For the Lindeberg-type condition, we have the bound

$$X_{n,q}^2 \mathbb{1}_{|X_{n,q}| > \epsilon} \le \frac{1}{\epsilon^2} |X_{n,q}|^4 \le \frac{(a^2 + b^2)^2}{\epsilon^2} |Z_{n,q}|^4, \tag{4.30}$$

where the last inequality follows from Cauchy–Schwarz. By (4.24) we have

$$\mathbb{E}|\xi_n| \le 2\theta^2 \frac{(a^2+b^2)^2}{\mathbb{E}(|c_n|^2)\epsilon^2} \sum_{q=\lfloor \delta n \rfloor}^{n} \frac{\mathbb{E}(|c_{n-q,q-1}|^4)}{q^2} \to 0 \tag{4.31}$$

which implies $\xi_n \xrightarrow{p} 0$. Now, Theorem 4.2 applies and shows that we have the convergence in distribution

$$\begin{aligned}\frac{\tilde{c}_{n,a,b}}{\sqrt{\mathbb{E}(|c_n|^2)}} &\xrightarrow{d} \sqrt{\nu}\sqrt{\frac{a^2+b^2}{2}} N_{\mathbb{R}} \\ &\stackrel{d}{=} \sqrt{\nu}\biggl(a\frac{N_{\mathbb{R}}^{(1)}}{\sqrt{2}} + b\frac{N_{\mathbb{R}}^{(2)}}{\sqrt{2}}\biggr),\end{aligned} \tag{4.32}$$

where $N_{\mathbb{R}}$, $N_{\mathbb{R}}^{(1)}$ and $N_{\mathbb{R}}^{(2)}$ are independent and identically distributed standard (real) Gaussians. This establishes the joint convergence of the real and imaginary parts of $\tilde{c}_n^{(\delta)}$ to the appropriate limit and concludes the proof of (4.25). □

We start by checking the Lindeberg condition (4.24) which is relatively straightforward.

LEMMA 4.4 (Lindeberg condition). *The condition* (4.24) *is satisfied for any* $0 < \theta < 1$.

PROOF. Recalling the explicit normalization (1.21) for $\mathbb{E}(|c_n|^2)$, we have

$$\begin{aligned}&\frac{\theta^2}{\binom{n+\theta-1}{\theta-1}^2} \sum_{q=\lfloor \delta n \rfloor}^{n} \frac{\mathbb{E}(|c_{n-q,q-1}|^4)}{q^2} \\ &\quad\le \frac{\theta^2}{\binom{n+\theta-1}{\theta-1}^2 \lfloor \delta n \rfloor^2} \sum_{q=0}^{n} \mathbb{E}(|c_q|^4) \\ &\quad= \frac{\theta^2}{\binom{n+\theta-1}{\theta-1}^2 \lfloor \delta n \rfloor^2} \sum_{k=0}^{n} \binom{k+\theta-1}{\theta-1}^2 \sum_{q=0}^{n-k} \binom{q+\theta-1}{\theta-1}^2 \\ &\quad\le \frac{\theta^2}{\binom{n+\theta-1}{\theta-1}^2 \lfloor \delta n \rfloor^2} \biggl(\sum_{k=0}^{n} \binom{k+\theta-1}{\theta-1}^2\biggr)^2.\end{aligned} \tag{4.33}$$

To obtain the first inequality above, we used that $\mathbb{E}(|c_{n-q,q-1}|^4) \le \mathbb{E}(|c_n|^4)$. This follows from an explicit computation of $\mathbb{E}(|c_{n-q,q-1}|^4)$ using (3.4): since one obtains a sum involving only positive terms, we get the bound in terms of (2.8) by removing the cycle constraint. Alternatively, one can obtain this bound by comparing the explicit moment formulas (2.8) and (5.1). The middle equality uses (2.8) and reorders the sum. Now, consider the final sum in (4.33). If $\theta < 1/2$, it converges as $n \to \infty$ by Lemma C.1. Then by Stirling's formula, the n-dependent part of the expression behaves as

$$\frac{1}{\binom{n+\theta-1}{\theta-1}^2 n^2} \sim n^{-2\theta} \frac{1}{\Gamma(\theta)}, \quad n \to \infty \tag{4.34}$$

which tends to zero as $n \to \infty$. If $\theta \geq 1/2$, the sum is divergent as $n \to \infty$. To estimate its order we replace the summand with its asymptotic behaviour which is proportional to $k^{2(\theta-1)}$ as $k \to \infty$. Substituting this into the sum, we obtain a bound of order $\log^2(n)/n$ if $\theta = 1/2$ and of order $n^{2(\theta-1)}$ if $\theta \in (1/2, 1)$. In each case (4.24) follows. $\square$

Verifying (4.23) and identifying the limit ν turns out to be more difficult. It will turn out that $\mathcal{M}_{\theta,\delta,n}$ has a similar behaviour to the quantity $\mathcal{M}_{\theta,n}$, defined in (1.31), premultiplied by a certain explicit constant C_δ. To see how this constant arises, we begin with the following warm up exercise.

LEMMA 4.5. *Consider the quantity $\mathcal{M}_{\theta,\delta,n}$ defined in* (4.19), *and set*

$$C_\delta := \theta \int_\delta^1 (1-x)^{\theta-1} \mathbb{P}\left(L^{(\infty)} \leq \frac{x}{1-x}\right) \frac{dx}{x}, \tag{4.35}$$

where $L^{(\infty)}$ is the limiting random variable from Lemma 3.5. *Then we have*

$$\lim_{n\to\infty} \mathbb{E}(\mathcal{M}_{\theta,\delta,n}) = C_\delta. \tag{4.36}$$

Furthermore, the constant C_δ can be explicitly computed as

$$C_\delta = 1 - \Gamma(\theta) e^{\gamma_{\mathrm{E}}\theta} \delta^{\theta-1} p_\theta(1/\delta) \tag{4.37}$$

and satisfies the bound

$$C_\delta = 1 + O(\delta), \quad \delta \to 0. \tag{4.38}$$

PROOF. By the correspondence (3.8), we have the identity

$$\begin{aligned} \mathbb{E}(|c_{n-q,q-1}|^2) &= \sum_{\substack{(m_k):1\leq k\leq q-1 \\ \sum_{k=1}^{q-1} k m_k = n-q}} \prod_{k=1}^{q-1} \frac{\theta^{m_k}}{m_k! k^{m_k}} \\ &= \binom{n-q+\theta-1}{\theta-1} \mathbb{P}(L^{(n-q)} \leq q-1), \end{aligned} \tag{4.39}$$

where $L^{(n-q)}$ is the longest cycle in a Ewens distributed random permutation of length $n-q$. Furthermore, in the regime of interest, $q = \lfloor \delta n \rfloor, \dots, n$ so that q and $n-q$ are proportional to n. In this regime Lemma 3.5 applies: if $q/n \to x \in (\delta, 1)$, then

$$\lim_{n\to\infty} \mathbb{P}(L^{(n-q)} \leq q-1) = \mathbb{P}\left(L^{(\infty)} \leq \frac{x}{1-x}\right), \tag{4.40}$$

and consequently, the expectation of $\mathcal{M}_{\theta,\delta,n}$ converges to a Riemann integral,

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}(\mathcal{M}_{\theta,\delta,n}) &= \lim_{n\to\infty} \frac{\theta}{n} \sum_{q=\lfloor \delta n \rfloor}^{n} \frac{\mathbb{E}(|c_{n-q,q-1}|^2)}{\binom{n+\theta-1}{\theta-1} \frac{q}{n}} \\ &= \lim_{n\to\infty} \frac{\theta}{n} \sum_{q=\lfloor \delta n \rfloor}^{n} \frac{\binom{n-q+\theta-1}{\theta-1}}{\binom{n+\theta-1}{\theta-1}} \frac{\mathbb{P}(L^{(n-q)} \leq q-1)}{\frac{q}{n}} \\ &= \theta \int_\delta^1 (1-x)^{\theta-1} \mathbb{P}\left(L^{(\infty)} \leq \frac{x}{1-x}\right) \frac{dx}{x} \\ &= C_\delta. \end{aligned} \tag{4.41}$$

To compute C_δ, we express the probability in the integrand of (4.35) in terms of the function $p_\theta(y)$ and use the properties in Lemma 3.3. We have

$$\begin{aligned} C_\delta &= \theta\Gamma(\theta)e^{\gamma_{\mathrm{E}}\theta}\int_\delta^1 x^{\theta-2}p_\theta(1/x-1)\,dx \\ &= \Gamma(\theta)e^{\gamma_{\mathrm{E}}\theta}\int_1^{1/\delta}\theta x^{-\theta}p_\theta(x-1)\,dx \\ &= -\Gamma(\theta)e^{\gamma_{\mathrm{E}}\theta}\int_1^{1/\delta}\frac{d}{dx}\big(x^{1-\theta}p_\theta(x)\big)\,dx \\ &= \Gamma(\theta)e^{\gamma_{\mathrm{E}}\theta}p_\theta(1)-\Gamma(\theta)e^{\gamma_{\mathrm{E}}\theta}\delta^{\theta-1}p_\theta(1/\delta). \end{aligned} \tag{4.42}$$

The identity (4.37) now follows from the explicit formula $p_\theta(1)=e^{-\gamma_{\mathrm{E}}\theta}/\Gamma(\theta)$, and the limiting behaviour $C_\delta = 1+O(\delta)$ follows from the rapid decay in (3.14). □

We will now describe how these ideas lead to the proof of Theorem 1.8. In the following sections, we will generalise the argument of Lemma 4.5 to show that, for any $0<\theta<\frac{1}{2}$, we have

$$\lim_{n\to\infty}\mathbb{E}\big(|\mathcal{M}_{\theta,\delta,n}-C_\delta\mathcal{M}_{\theta,n}|^2\big)=0, \tag{4.43}$$

allowing us to replace the convergence in probability of $\mathcal{M}_{\theta,\delta,n}$ with that of $C_\delta\mathcal{M}_{\theta,n}$ in (1.31). For the latter the appropriate limit is given in Lemma 1.9, and combined with (4.43), this implies that

$$\mathcal{M}_{\theta,\delta,n}\xrightarrow{p}C_\delta\mathcal{M}_\theta,\quad n\to\infty \tag{4.44}$$

which is the identification of ν. Now, the martingale central limit theorem of Corollary 4.3 gives a limiting distribution for $\tilde{c}_n^{(\delta)}$,

$$\frac{\tilde{c}_n^{(\delta)}}{\sqrt{\mathbb{E}(|c_n|^2)}}\xrightarrow{d}\sqrt{C_\delta\mathcal{M}_\theta}\mathcal{N}_1\quad n\to\infty. \tag{4.45}$$

The proof of Theorem 1.8, contingent on (4.43), now follows from a standard approximation argument.

PROOF OF THEOREM 1.8 ASSUMING (4.43). Let us define the appropriately normalized version of c_n as

$$\hat{c}_n := \frac{c_n}{\sqrt{\mathbb{E}(|c_n|^2)}}. \tag{4.46}$$

The joint characteristic functions of the real and imaginary parts of $\hat{c}_n$ and the proposed limiting random variable are

$$\begin{aligned} \Phi_n(s,t) &:= \mathbb{E}\big(e^{is\,\mathrm{Re}(\hat{c}_n)+it\,\mathrm{Im}(\hat{c}_n)}\big), \\ \Phi(s,t) &:= \mathbb{E}\big(e^{is\,\mathrm{Re}(\sqrt{\mathcal{M}_\theta}\mathcal{N}_1)+it\,\mathrm{Im}(\sqrt{\mathcal{M}_\theta}\mathcal{N}_1)}\big). \end{aligned} \tag{4.47}$$

For our martingale approximation $\tilde{c}_n^{(\delta)}$, similarly, we define

$$\hat{\tilde{c}}_n^{(\delta)} := \frac{\tilde{c}_n^{(\delta)}}{\sqrt{\mathbb{E}(|c_n|^2)}} \tag{4.48}$$

and the corresponding characteristic functions

$$
\begin{aligned}
\tilde{\Phi}_{n,\delta}(s,t) &:= \mathbb{E}\big(e^{is\operatorname{Re}(\hat{\hat{c}}_n^{(\delta)})+it\operatorname{Im}(\hat{\hat{c}}_n^{(\delta)})}\big), \\
\tilde{\Phi}_{\delta}(s,t) &:= \mathbb{E}\big(e^{is\operatorname{Re}(\sqrt{C_\delta\mathcal{M}_\theta}\mathcal{N}_1)+it\operatorname{Im}(\sqrt{C_\delta\mathcal{M}_\theta}\mathcal{N}_1)}\big).
\end{aligned} \tag{4.49}
$$

Then we have

$$
\begin{aligned}
\big|\Phi_n(s,t)-\Phi(s,t)\big| \le \big|\Phi_n(s,t)-\tilde{\Phi}_{n,\delta}(s,t)\big| + \big|\tilde{\Phi}_{n,\delta}(s,t)-\tilde{\Phi}_{\delta}(s,t)\big| \\
+ \big|\tilde{\Phi}_{\delta}(s,t)-\Phi(s,t)\big|.
\end{aligned} \tag{4.50}
$$

By the convergence in distribution (4.45), the term $|\tilde{\Phi}_{n,\delta}(s,t)-\tilde{\Phi}_{\delta}(s,t)| \to 0$ as $n\to\infty$, while by the asymptotics (4.38) we have $|\tilde{\Phi}_{\delta}(s,t)-\Phi(s,t)|\to 0$ as $\delta\to 0$. A standard bound on the exponential function (see Lemma A.3) gives the inequality

$$
\big|\Phi_n(s,t)-\tilde{\Phi}_{n,\delta}(s,t)\big| \le \big(|t|+|s|\big)\sqrt{\mathbb{E}\big(|\hat{c}_n-\hat{\hat{c}}_n^{(\delta)}|^2\big)}. \tag{4.51}
$$

By Lemma 4.1 the expression on the right-hand side of (4.51) tends to zero in the limit $n\to\infty$ followed by $\delta\to 0$. This implies that $|\Phi_n(s,t)-\Phi(s,t)|\to 0$ as $n\to\infty$, and the statement of Theorem 1.8 follows. □

5. Convergence of the bracket process and the proof of L^2-convergence. The goal of this section will be to verify (4.43). To do this, we will need to calculate some higher moments of secular coefficients with a cycle constraint, namely, of the $c_{n,q}$ defined in (3.10). Then we use these results to analyze the second moment in (4.43).

5.1. *Moments of secular coefficients with a cycle constraint.* The following Lemma generalises the second moment formula (4.39) and the magic square formula (1.19).

LEMMA 5.1. *Let m be a positive integer, and assume without loss of generality that the positive integers $q_1\le\cdots\le q_n$ are ordered. Recall that $L_1^{(n)}$ is the longest cycle of a random permutation of length n under the Ewens measure (3.6). Then we have*

$$
\mathbb{E}\left(\prod_{i=1}^m |c_{n_i,q_i}|^2\right) = \sum_{A\in\mathrm{Mag}_{\vec{n},\vec{n}}}\prod_{l_1,l_2=1}^m \binom{A_{l_1,l_2}+\theta-1}{\theta-1}\mathbb{P}\big(L_1^{(A_{l_1,l_2})}\le q_{l_1\wedge l_2}\big), \tag{5.1}
$$

where we used the notation $l_1\wedge l_2 := \min(l_1,l_2)$.

PROOF. The strategy of the proof is very similar to that of Theorem 1.6. By definition (3.10) the coefficients $c_{n,q}$ can be extracted from a generating function

$$
c_{n,q} = [z^n]\exp\left(\sqrt{\theta}\sum_{k=1}^q \frac{\mathcal{N}_k}{\sqrt{k}}z^k\right) \tag{5.2}
$$

so that the left-hand side of (5.1) is $[z_1^{n_1}\dots z_m^{n_m}\overline{w_1}^{n_1}\dots\overline{w_m}^{n_m}]\mathbb{E}(F_{\vec{q}}^{(m)}(\vec{z},\vec{\overline{w}}))$, where

$$
F_{\vec{q}}^{(m)}(\vec{z},\vec{\overline{w}}) := \exp\left(\sqrt{\theta}\sum_{i=1}^m\left(\sum_{k=1}^{q_i}\frac{\mathcal{N}_k}{\sqrt{k}}z_i^k + \frac{\overline{\mathcal{N}_k}}{\sqrt{k}}\overline{w_i}^k\right)\right). \tag{5.3}
$$

Using that $q_1\le\cdots\le q_m$, we can rearrange the summation (interchange the i and k indices) using the identity

$$
\sum_{i=1}^m\sum_{k=1}^{q_i} a_{k,i} = \sum_{i=1}^m\sum_{k=q_{i-1}+1}^{q_i}\sum_{r=i}^m a_{k,r} \tag{5.4}
$$

valid for arbitrary $a_{k,i}$ where we set $q_0 := 0$. Then, using independence, we get

$$\begin{aligned}\mathbb{E}\big(F_{\vec{q}}^{(m)}(\vec{z},\vec{w})\big) &= \prod_{i=1}^{m}\mathbb{E}\Bigg(\exp\Bigg(\sqrt{\theta}\sum_{k=q_{i-1}+1}^{q_i}\sum_{r=i}^{m}\Big(\frac{\mathcal{N}_k}{\sqrt{k}}z_r^k+\frac{\overline{\mathcal{N}_k}}{\sqrt{k}}\overline{w_r}^k\Big)\Bigg)\Bigg)\\
&= \exp\Bigg(\theta\sum_{i=1}^{m}\sum_{k=q_{i-1}+1}^{q_i}\sum_{l_1,l_2=i}^{m}\frac{(z_{l_1}\overline{w_{l_2}})^k}{k}\Bigg) \qquad (5.5)\\
&= \prod_{l_1,l_2=1}^{m}\exp\Bigg(\theta\sum_{k=1}^{q_{l_1\wedge l_2}}\frac{(z_{l_1}\overline{w_{l_2}})^k}{k}\Bigg),\end{aligned}$$

where to obtain the last line, we again rearranged the sums using the identity

$$\sum_{i=1}^{m}\sum_{k=q_{i-1}+1}^{q_i}\sum_{r_1,r_2=i}^{m}a_{k,r_1,r_2}=\sum_{r_1,r_2=1}^{m}\sum_{k=1}^{q_{r_1\wedge r_2}}a_{k,r_1,r_2}, \tag{5.6}$$

valid for arbitrary a_{k,r_1,r_2}. Now, to pick out coefficients, we expand each exponential in (5.5) using

$$\begin{aligned}&\exp\Bigg(\theta\sum_{k=1}^{q_{l_1\wedge l_2}}\frac{(z_{l_1}\overline{w_{l_2}})^k}{k}\Bigg)\\
&\quad=\sum_{A_{l_1,l_2}=0}^{\infty}(z_{l_1}\overline{w_{l_2}})^{A_{l_1,l_2}}\sum_{\substack{(m_k):1\le k\le q_{l_1\wedge l_2}\\ \sum_{k=1}^{q_{l_1\wedge l_2}}km_k=A_{l_1,l_2}}}\prod_{k=1}^{q_{l_1\wedge l_2}}\Big(\frac{\theta}{k}\Big)^{m_k}\frac{1}{m_k!} \qquad (5.7)\\
&\quad=\sum_{A_{l_1,l_2}=0}^{\infty}(z_{l_1}\overline{w_{l_2}})^{A_{l_1,l_2}}\binom{A_{l_1,l_2}+\theta-1}{\theta-1}\mathbb{P}\big(L_1^{(A_{l_1,l_2})}\le q_{l_1\wedge l_2}\big).\end{aligned}$$

Inserting this into (5.5) leads to the representation (5.1). □

For the proof of (4.43), we will need the case $m=2$, corresponding to fourth moments of the secular coefficients and involving sums over 2×2 magic squares (in analogy with Example 2.2). Let us assume that $q_1\le q_2$. Then (5.1) implies that

$$\begin{aligned}&\mathbb{E}\big(|c_{n_1,q_1}|^2|c_{n_2,q_2}|^2\big)\\
&\quad=\sum_{k=0}^{n_1\wedge n_2}\binom{k+\theta-1}{\theta-1}^2\mathbb{P}\big(L_1^{(k)}\le q_1\big)^2 \qquad (5.8)\\
&\qquad\times\binom{n_1-k+\theta-1}{\theta-1}\mathbb{P}\big(L_1^{(n_1-k)}\le q_1\big)\binom{n_2-k+\theta-1}{\theta-1}\mathbb{P}\big(L_1^{(n_2-k)}\le q_2\big).\end{aligned}$$

5.2. *The L^2-phase and proof of* (4.43). We recall the definition of the two quantities,

$$\mathcal{M}_{\theta,n}:=\frac{\Gamma(\theta+1)}{n^\theta}\sum_{q=0}^{n}|c_q|^2, \tag{5.9}$$

$$\mathcal{M}_{\theta,\delta,n}:=\frac{1}{\binom{n+\theta-1}{\theta-1}}\sum_{q=\lfloor\delta n\rfloor}^{n}|c_{n-q,q-1}|^2\frac{\theta}{q} \tag{5.10}$$

and the deterministic constant

$$C_\delta := \theta \int_\delta^1 \frac{(1-x)^{\theta-1}\mathbb{P}(L_1^{(\infty)} \le x/(1-x))}{x}\,dx. \tag{5.11}$$

Our goal is to prove (4.43) for any $0 < \theta < 1/2$. The condition $\theta < 1/2$ is a technical requirement in the proof to ensure the convergence of the infinite sum

$$\sum_{k=0}^{\infty} \binom{k+\theta-1}{\theta-1}^2 = \mathbb{E}(\mathcal{M}_\theta^2) = \frac{\Gamma(1-2\theta)}{\Gamma(1-\theta)^2} \tag{5.12}$$

or, equivalently, existence of the second moment of the total mass. For a proof of this identity, see Lemma C.1.

THEOREM 5.2 (L^2-convergence). *Fix $0 < \theta < \frac{1}{2}$ and $\delta > 0$. Then the following limit holds*:

$$\lim_{n\to\infty} \mathbb{E}(|\mathcal{M}_{\theta,\delta,n} - C_\delta \mathcal{M}_{\theta,n}|^2) = 0. \tag{5.13}$$

PROOF. We start by expanding the square in (5.13) and endeavour to compute the three terms $\mathcal{S}_{1,n} := \mathbb{E}(\mathcal{M}_{\theta,\delta,n}^2)$, $\mathcal{S}_{2,n} := \mathbb{E}(\mathcal{M}_{\theta,n}^2)$ and $\mathcal{S}_{3,n} := \mathbb{E}(\mathcal{M}_{\theta,\delta,n}\mathcal{M}_{\theta,n})$. It is clearly sufficient to show that the limit of each of these quantities exists and is given by the second moment of the total mass, up to the appropriate factor of C_δ. It will be convenient to abbreviate

$$P_{n,k} := \mathbb{P}\big(L_1^{(n)} \le k-1\big) \tag{5.14}$$

as the probability that the longest cycle in a Ewens distributed random permutation of size n is less than or equal to $k-1$.

Starting with the second moment of $\mathcal{M}_{\theta,\delta,n}$ in (5.10), we have

$$\mathcal{S}_{1,n} = \frac{\theta^2}{\binom{n+\theta-1}{\theta-1}^2} \sum_{q_1=\lfloor\delta n\rfloor}^{n} \sum_{q_2=\lfloor\delta n\rfloor}^{n} \frac{\mathbb{E}(|c_{n-q_1,q_1-1}|^2|c_{n-q_2,q_2-1}|^2)}{q_1q_2}. \tag{5.15}$$

Because of the obvious symmetry in q_1 and q_2, it will be convenient to consider the contribution of such sums on the region $q_1 \le q_2$, which we denote $\mathcal{S}_{1,n}(q_1 \le q_2)$ and similarly for the other sums. Applying (5.8) and interchanging the order of summation, we obtain

$$\begin{aligned}\mathcal{S}_{1,n}(q_1 \le q_2) = \sum_{q_1=\lfloor\delta n\rfloor}^{n} \sum_{q_2=q_1}^{n} \sum_{k=0}^{n-q_2} &\binom{k+\theta-1}{\theta-1}^2 P_{k,q_1}^2 \frac{\theta^2}{q_1q_2} \frac{1}{\binom{n+\theta-1}{\theta-1}^2} \\ &\times \binom{n-q_1-k+\theta-1}{\theta-1} P_{n-q_1-k,q_1} \\ &\times \binom{n-q_2-k+\theta-1}{\theta-1} P_{n-q_2-k,q_2}.\end{aligned} \tag{5.16}$$

To estimate the sum (5.16) as $n \to \infty$, the idea is to treat k as fixed while q_1 and q_2 are large, applying dominated convergence to bring the limit $n \to \infty$ inside the sum over k.

As in (4.41), the sums over q_1 and q_2 will always be treated as Riemann sum approximations. For $\mathcal{S}_{1,n}(q_1 \le q_2)$, the double sum over q_1 and q_2 is symmetric (we can assume that $P_{k,q_1}^2 = 1$ for fixed k and n large enough, since $q_1 \ge \lfloor\delta n\rfloor$, which is larger than k), and we can write it as $\frac{1}{2}$ times the square of the same sum appearing in (4.41) (neglecting for now the diagonal $q_1 = q_2$, see below). This immediately yields the limit as $\frac{1}{2}C_\delta^2$. By symmetry the same reasoning and limiting value applies to the sum with $q_1 \geq q_2$. Assuming we can apply dominated convergence to the sum over k, we get

$$\lim_{n\to\infty} \mathcal{S}_{1,n} = C_\delta^2 \sum_{k=0}^{\infty} \binom{k+\theta-1}{\theta-1}^2 = C_\delta^2 \mathbb{E}(\mathcal{M}_\theta^2), \tag{5.17}$$

as required.

Now, we will justify the dominated convergence by finding a uniform and summable bound in (5.16). Notice that the sums over q_1 and q_2 are uniformly bounded by

$$\begin{aligned}&\frac{\theta^2}{\lfloor \delta n\rfloor^2 \binom{n+\theta-1}{\theta-1}^2} \sum_{q_1=\lfloor \delta n\rfloor}^{n-k} \sum_{q_2=q_1}^{n-k} \binom{n-q_1-k+\theta-1}{\theta-1}\binom{n-q_2-k+\theta-1}{\theta-1} \\ &= \frac{\theta^2}{\lfloor \delta n\rfloor^2 \binom{n+\theta-1}{\theta-1}^2} \sum_{q_1=0}^{n-\lfloor \delta n\rfloor-k} \binom{q_1+\theta-1}{\theta-1}\binom{q_1+\theta}{\theta},\end{aligned}$$

where we used the binomial sum identity (C.4). The final expression can be bounded uniformly by extending the summation to $q_1 = n$ and noting that the summand is of order $q_1^{2\theta-1}$ as $q_1 \to \infty$. If we took just the terms where $q_2 = q_1$, we would have the bound

$$\begin{aligned}&\frac{\theta^2}{\binom{n+\theta-1}{\theta-1}^2}\frac{1}{\lfloor \delta n\rfloor^2} \sum_{q_1=\lfloor \delta n\rfloor}^{n-k} \binom{n-q_1-k+\theta-1}{\theta-1}^2 \\ &\leq \frac{\theta^2}{\binom{n+\theta-1}{\theta-1}^2}\frac{1}{\lfloor \delta n\rfloor^2} \sum_{q_1=0}^{n-\lfloor \delta n\rfloor} \binom{q_1+\theta-1}{\theta-1}^2 \to 0, \quad n\to\infty.\end{aligned} \tag{5.18}$$

This justifies overcounting terms, where $q_2 = q_1$, and completes the proof of (5.17).

Now, we consider the second moment $\mathbb{E}(\mathcal{M}_{\theta,n}^2)$. Again using (5.8) in the unconstrained case (or (1.19)), we obtain

$$\begin{aligned}\mathcal{S}_{2,n} = {}& \frac{\Gamma(\theta+1)^2}{n^{2\theta}} \sum_{q_1=0}^{n}\sum_{q_2=0}^{n}\sum_{k=0}^{q_1\wedge q_2} \binom{k+\theta-1}{\theta-1}^2 \\ &\times \binom{q_1-k+\theta-1}{\theta-1}\binom{q_2-k+\theta-1}{\theta-1}.\end{aligned} \tag{5.19}$$

An application of Lemma C.2 shows that the contribution to (5.19) from the region $q_1 \leq q_2$ is equal to

$$\mathcal{S}_{2,n}(q_1 \leq q_2) = \frac{\Gamma(\theta+1)^2}{n^{2\theta}} \sum_{q_1=0}^{n}\sum_{k=0}^{q_1} \binom{k+\theta-1}{\theta-1}^2 \tag{5.20}$$

$$\times \binom{q_1-k+\theta-1}{\theta-1}\left(\binom{n-k+\theta}{\theta} - \binom{q_1-k+\theta-1}{\theta}\right). \tag{5.21}$$

Interchanging the summation and summing over q_1, again using Lemma C.2, we obtain

$$\mathcal{S}_{2,n}(q_1 \leq q_2) = \frac{\Gamma(\theta+1)^2}{n^{2\theta}} \sum_{k=0}^{n} \binom{k+\theta-1}{\theta-1}^2 \binom{n-k+\theta}{\theta}^2 \tag{5.22}$$

$$-\frac{\Gamma(\theta+1)^2}{n^{2\theta}}\sum_{k=0}^{n}\binom{k+\theta-1}{\theta-1}^2\sum_{q_1=0}^{n-k}\binom{q_1+\theta-1}{\theta-1}\binom{q_1+\theta-1}{\theta}. \tag{5.23}$$

By dominated convergence, the first term (5.22) converges to $\mathbb{E}(\mathcal{M}_\theta^2)$, where a suitable bound is obtained by noticing that the second binomial coefficient is decreasing in k. For the second term (5.23), a uniform upper bound is obtained by summing up to $q_1 = n$ and noting that the summand is of order $q_1^{2\theta-1}$ with a corresponding sum of order $n^{2\theta}$ as $n \to \infty$. Then dominated convergence implies that (5.23) has a limit given by

$$\begin{aligned}
&\sum_{k=0}^{\infty}\binom{k+\theta-1}{\theta-1}^2\lim_{n\to\infty}\frac{\Gamma(\theta+1)^2}{n^{2\theta}}\sum_{q_1=0}^{n-k}\binom{q_1+\theta-1}{\theta-1}\binom{q_1+\theta-1}{\theta}\\
&\quad=\sum_{k=0}^{\infty}\binom{k+\theta-1}{\theta-1}^2\lim_{n\to\infty}\frac{\theta}{n^{2\theta}}\sum_{q_1=1}^{n}q_1^{2\theta-1}\\
&\quad=\frac{1}{2}\sum_{k=0}^{\infty}\binom{k+\theta-1}{\theta-1}^2=\frac{1}{2}\mathbb{E}(\mathcal{M}_\theta^2).
\end{aligned} \tag{5.24}$$

An analogous computation shows that the contribution to (5.19) from the diagonal $q_1 = q_2$ is order $O(n^{-1})$. By symmetry in q_1 and q_2, we obtain

$$\lim_{n\to\infty}\mathcal{S}_{2,n}=\sum_{k=0}^{\infty}\binom{k+\theta-1}{\theta-1}^2=\mathbb{E}(\mathcal{M}_\theta^2). \tag{5.25}$$

It remains to consider the mixed term $\mathbb{E}(\mathcal{M}_{\theta,\delta,n}\mathcal{M}_{\theta,n})$ denoted $\mathcal{S}_{3,n}$. By definition we have

$$\begin{aligned}
\mathcal{S}_{3,n}&=\mathbb{E}(\mathcal{M}_{\theta,\delta,n}\mathcal{M}_{\theta,n})\\
&=\frac{\Gamma(\theta+1)}{n^\theta}\frac{1}{\binom{n+\theta-1}{\theta-1}}\sum_{q_1=0}^{n}\sum_{q_2=\lfloor\delta n\rfloor}^{n}\mathbb{E}\big(|c_{q_1}|^2|c_{n-q_2,q_2-1}|^2\big)\frac{\theta}{q_2}.
\end{aligned} \tag{5.26}$$

We split this up as two sums, one for values of $q_1 = 0,\dots,n-q_2$ and one for values of $q_1 = n-q_2+1,\dots,n$. Interchanging the order of summation, the first sum gives

$$\begin{aligned}
&\mathcal{S}_{3,n}(q_1\le n-q_2)\\
&\quad=\frac{\Gamma(\theta+1)}{n^\theta\binom{n+\theta-1}{\theta-1}}\sum_{k=0}^{n-\lfloor\delta n\rfloor}\sum_{q_2=\lfloor\delta n\rfloor}^{n-k}\sum_{q_1=k}^{n-q_2}\binom{k+\theta-1}{\theta-1}^2\\
&\qquad\times P_{k,q_2}^2\binom{q_1-k+\theta-1}{\theta-1}\binom{n-q_2-k+\theta-1}{\theta-1}P_{n-q_2-k,q_2}\frac{\theta}{q_2}.
\end{aligned} \tag{5.27}$$

The sum over q_1 in (5.27) is handled with Lemma C.2, and we have

$$\begin{aligned}
&\mathcal{S}_{3,n}(q_1\le n-q_2)\\
&\quad=\frac{\Gamma(\theta+1)}{n^\theta\binom{n+\theta-1}{\theta-1}}\sum_{k=0}^{n-\lfloor\delta n\rfloor}\sum_{q_2=\lfloor\delta n\rfloor}^{n-k}\binom{k+\theta-1}{\theta-1}^2\\
&\qquad\times P_{k,q_2}^2\binom{n-q_2-k+\theta}{\theta}\binom{n-q_2-k+\theta-1}{\theta-1}P_{n-q_2-k,q_2}\frac{\theta}{q_2}.
\end{aligned} \tag{5.28}$$

Now, we look at the contribution to $\mathcal{S}_{3,n}$ indexed by $q_1 = n - q_2 + 1, \dots, n$. This gives

$$\begin{aligned}
&\mathcal{S}_{3,n}(q_1 > n - q_2) \\
&\quad = \frac{\Gamma(\theta+1)}{n^\theta \binom{n+\theta-1}{\theta-1}} \sum_{q_2=\lfloor \delta n \rfloor}^{n} \sum_{q_1=n-q_2+1}^{n} \sum_{k=0}^{n-q_2} \binom{k+\theta-1}{\theta-1}^2 \\
&\qquad \times P_{k,q_2}^2 \binom{q_1-k+\theta-1}{\theta-1} \binom{n-q_2-k+\theta-1}{\theta-1} P_{n-q_2-k,q_2} \frac{\theta}{q_2}.
\end{aligned} \tag{5.29}$$

Summing over q_1 with Lemma C.2 and interchanging the order of summation gives the identity

$$\begin{aligned}
\mathcal{S}_{3,n}(q_1 > n - q_2) = {}& \frac{\Gamma(\theta+1)}{n^\theta \binom{n+\theta-1}{\theta-1}} \sum_{k=0}^{n-\lfloor \delta n \rfloor} \sum_{q_2=\lfloor \delta n \rfloor}^{n-k} \binom{k+\theta-1}{\theta-1}^2 P_{k,q_2}^2 \\
& \times \left(\binom{n-k+\theta}{\theta} - \binom{n-q_2-k+\theta}{\theta} \right) \\
& \times \binom{n-q_2-k+\theta-1}{\theta-1} \frac{\theta}{q_2} P_{n-q_2-k,q_2}.
\end{aligned} \tag{5.30}$$

Combining with (5.28) yields a cancellation, and we are left with the identity

$$\begin{aligned}
\mathcal{S}_{3,n} = {}& \sum_{k=0}^{n-\lfloor \delta n \rfloor} \sum_{q_2=\lfloor \delta n \rfloor}^{n-k} \binom{k+\theta-1}{\theta-1}^2 P_{k,q_2}^2 \\
& \times \Gamma(\theta+1) \frac{\binom{n-k+\theta}{\theta}}{n^\theta} \frac{\binom{n-q_2-k+\theta-1}{\theta-1}}{\binom{n+\theta-1}{\theta-1}} P_{n-q_2-k,q_2} \frac{\theta}{q_2}.
\end{aligned} \tag{5.31}$$

As in the treatment of (5.16) (see also (4.41)), for $n \to \infty$ and fixed k, the kth term of (5.31) is approximated by a Riemann integral and $P_{k,q_2} = 1$. Applying dominated convergence then gives the limit as $n \to \infty$ of (5.31) as $C_\delta \mathbb{E}(\mathcal{M}_\theta^2)$, as required. For a suitable dominating function, note that apart from the factor $\binom{k+\theta-1}{\theta-1}^2$, the $k$þterm in (5.31) is uniformly bounded by

$$\frac{\theta}{\lfloor \delta n \rfloor} \Gamma(\theta+1) \frac{\binom{n-k+\theta}{\theta}}{n^\theta} \sum_{q_2=\lfloor \delta n \rfloor}^{n-k} \frac{\binom{n-q_2+\theta-1}{\theta-1}}{\binom{n+\theta-1}{\theta-1}} \tag{5.32}$$

$$\le \frac{\theta}{\lfloor \delta n \rfloor} \Gamma(\theta+1) \frac{\binom{n+\theta}{\theta}}{n^\theta} \frac{\binom{n-\lfloor \delta n \rfloor+\theta}{\theta}}{\binom{n+\theta-1}{\theta-1}}, \tag{5.33}$$

where we used Lemma C.2 and monotonicity of the binomial coefficients as a function of k. The final bound is independent of k and bounded in n. This shows that

$$\lim_{n\to\infty} \mathcal{S}_{3,n} = C_\delta \sum_{k=0}^{\infty} \binom{k+\theta-1}{\theta-1}^2 = C_\delta \mathbb{E}(\mathcal{M}_\theta^2), \tag{5.34}$$

as required. Putting all three limits for the moments together completes the proof of the theorem. □

6. Regularity of the holomorphic multiplicative chaos. We have to determine, for $s \in \mathbb{R}$, if the series

$$A_{s,\theta} := \sum_{n=0}^{\infty} (1+n^2)^s |c_n|^2 \tag{6.1}$$

is convergent. We will study the cases $\theta \le 1$ and $\theta > 1$ separately.

6.1. *Subcritical and critical cases*, $\theta \in (0, 1]$. By (1.21) we have that $\mathbb{E}[|c_n|^2] \le C_\theta (1+n)^{\theta-1}$ for some constant $C_\theta > 0$, and then

$$\mathbb{E}[A_{s,\theta}] \le C_\theta \sum_{n=0}^{\infty} (1+n)^{2s+\theta-1}$$

which is finite as soon as $s < -\theta/2$. Hence, HMC_θ is a.s. in H^s for all $s < -\theta/2$. Notice that this reasoning remains true in the supercritical phase, but the bound $-\theta/2$ is not optimal for this phase. Let us now show that HMC_θ is a.s. not in H^{-s} for $s > -\theta/2$.

By Parseval's identity, for $1/2 < r < 1$,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{\theta} G(re^{i\vartheta})}\, d\vartheta. \tag{6.2}$$

For any $s < 0$, the map $x \mapsto x^{2s} r^{-2x}$ from $(0, \infty)$ to $\mathbb{R}$ has a logarithmic derivative $2s/x - 2\log r$, and then it reaches its minimum at $x = s/\log r$. Hence, for all $n \ge 1$,

$$(1+n^2)^s r^{-2n} \ge 2^s n^{2s} r^{-2n} \ge 2^s (s/\log r)^{2s} r^{-2s/\log r} \ge C_s |\log r|^{-2s},$$

where $C_s > 0$ depends only on s. Hence, for all $r \in (1/2, 1)$, shrinking C_s, as needed to account for $n = 0$ term and the 2π,

$$A_{s,\theta} \ge \sum_{n=0}^{\infty} (1+n^2)^s r^{-2n} r^{2n} |c_n|^2 \ge \frac{C_s}{|\log r|^{2s}} \int_0^{2\pi} e^{\sqrt{\theta} G(re^{i\vartheta})}\, d\vartheta. \tag{6.3}$$

Now, it is proven in [34] that, for $\theta \in (0, 1]$, the quantity

$$(1-r^2)^\theta |\log(1-r^2)|^{(1/2)\mathbf{1}_{\theta=1}} \int_0^{2\pi} e^{\sqrt{\theta} G(re^{i\vartheta})}\, d\vartheta$$

converges in probability to the total mass of the GMC_θ, when r goes to 1, and then converges a.s. to a nonzero limit along a subsequence. Since for $s \in (-\theta/2, 0)$,

$$|\log r|^{-2s} (1-r^2)^{-\theta} |\log(1-r^2)|^{-(1/2)\mathbf{1}_{\theta=1}}$$

tends to infinity, when $r \to 1$, we deduce that $A_{s,\theta}$ is a.s. infinite.

6.2. *Super-critical case*, $\theta > 1$. Let us prove that HMC_θ is a.s. in H^s for all $s < -\sqrt{\theta} + 1/2$. One checks that, up to a positive multiplicative constant C_s depending only on s, if we set $u_n^* = \lfloor \log(2+n)/\log 2 \rfloor$

$$A_{s,\theta}/C_s \le \sum_{n=0}^{\infty} 2^{2su_n^*} (1-2^{-u_n^*})^{2n} |c_n|^2 \le \sum_{u=1}^{\infty} 2^{2su} \sum_{n=0}^{\infty} (1-2^{-u})^{2n} |c_n|^2.$$

Hence, $A_{s,\theta}$ is bounded, up to a constant depending on s, by

$$\sum_{u=1}^{\infty} 2^{2su} \int_0^{2\pi} e^{\sqrt{\theta} G((1-2^{-u})e^{i\vartheta})}\, d\vartheta.$$

We have that $G((1-2^{-u})e^{i\vartheta})$ is a centered Gaussian variable with variance

$$-2\log\bigl(1-(1-2^{-u})^2\bigr) = -2\log\bigl(2^{1-u}-2^{-2u}\bigr) = 2u\log 2 + \mathcal{O}(1).$$

Hence, for u large enough, the probability that $G((1-2^{-u})e^{i\vartheta}) \geq 2u\log 2 + 10\log u$ is dominated by

$$\begin{aligned}
\mathbb{P}\biggl[\mathcal{N}(0,1) \geq \frac{2u\log 2 + 10\log u}{\sqrt{2u\log 2 + \mathcal{O}(1)}}\biggr] &\leq e^{-\frac{(2u\log 2+10\log u)^2}{4u\log 2+\mathcal{O}(1)}} \\
&\leq e^{-\frac{4u^2(\log 2)^2+40u(\log 2)(\log u)}{4u\log 2+\mathcal{O}(1)}} \\
&\leq e^{-(u\log 2+10\log u)(1+\mathcal{O}(1/u))} = \mathcal{O}\bigl(2^{-u}u^{-10}\bigr).
\end{aligned}$$

By Borel–Cantelli lemma, we have, almost surely,

$$\sup_{\vartheta\in 2\pi\mathbb{Z}/2^u} G\bigl((1-2^{-u})e^{i\vartheta}\bigr) \leq 2u\log 2 + 10\log u$$

for u large enough. We let $\mathcal{G}_u$ be the event the previous display holds. In order to prove that $A_{s,\theta}$ is a.s. finite, it is then sufficient to show that

$$\sum_{u=1}^{\infty} 2^{2su} \int_0^{2\pi} e^{\sqrt{\theta} G((1-2^{-u})e^{i\vartheta})} \mathbf{1}\{\mathcal{G}_u\}\, d\vartheta < \infty$$

almost surely. It is then enough to have

$$\sum_{u=1}^{\infty} 2^{2su} \int_0^{2\pi} \mathbb{E}\bigl[e^{\sqrt{\theta} G((1-2^{-u})e^{i\vartheta})} \mathbf{1}_{\mathcal{G}_u}\bigr]\, d\vartheta < \infty.$$

We let $\chi(u,\vartheta)$ be a multiple of $(2\pi)/2^u$ minimizing its distance to ϑ. Using a change of measure formula, we can bound the expectation inside the integral by

$$\begin{aligned}
&\mathbb{E}\bigl[e^{\sqrt{\theta} G((1-2^{-u})e^{i\vartheta})} \mathbf{1}_{\mathcal{G}_u}\bigr] \\
&\quad\leq \mathbb{E}\bigl[e^{\sqrt{\theta} G((1-2^{-u})e^{i\vartheta})} \mathbf{1}_{G((1-2^{-u})e^{i\chi(u,\vartheta)})\leq 2u\log 2+10\log u}\bigr] \\
&\quad= \mathbb{E}\bigl[e^{\sqrt{\theta} G((1-2^{-u})e^{i\vartheta})}\bigr]\mathbb{P}\bigl[\widetilde{G}\bigl((1-2^{-u})e^{i\chi(u,\vartheta)}\bigr) \leq 2u\log 2 + 10\log u\bigr].
\end{aligned}$$

Here $\widetilde{G}((1-2^{-u})e^{i\chi(u,\vartheta)})$ is the Gaussian variable obtained from $G((1-2^{-u})e^{i\chi(u,\vartheta)})$ after changing the underlying probability measure by a density proportional to $e^{\sqrt{\theta} G(1-2^{-u})e^{i\vartheta})}$. By Girsanov's theorem this change of measure introduces a drift

$$\operatorname{Cov}\bigl(G\bigl((1-2^{-u})e^{i\chi(u,\vartheta)}\bigr), \sqrt{\theta} G\bigl((1-2^{-u})e^{i\vartheta}\bigr)\bigr) = -2\sqrt{\theta}\log\bigl|1-(1-2^{-u})^2 e^{iv}\bigr|,$$

where

$$|v| = \bigl|\chi(u,\vartheta)-\vartheta\bigr| \leq 2\pi/2^u = \mathcal{O}\bigl(2^{-u}\bigr).$$

Hence,

$$\bigl|1-(1-2^{-u})^2 e^{iv}\bigr| = \mathcal{O}\bigl(2^{-u}\bigr)$$

which implies

$$\mathrm{Cov}\big(G\big((1-2^{-u})e^{i\chi(u,\vartheta)}\big), \sqrt{\theta}G\big((1-2^{-u})e^{i\vartheta}\big)\big) \geq \sqrt{\theta}\big(2u\log 2 + \mathcal{O}(1)\big).$$

We then get, for u large enough depending on θ,

$$\begin{aligned}
&\mathbb{E}\big[e^{\sqrt{\theta}G((1-2^{-u})e^{i\vartheta})}\mathbf{1}_{G((1-2^{-u})e^{i\chi(u,\vartheta)})\leq 2u\log 2+10\log u}\big] \\
&\quad\leq \mathbb{E}\big[e^{\sqrt{\theta}G((1-2^{-u})e^{i\vartheta})}\big] \\
&\qquad\times \mathbb{P}\big[G\big((1-2^{-u})e^{i\chi(u,\vartheta)}\big) + \sqrt{\theta}\big(2u\log 2 + \mathcal{O}(1)\big) \leq 2u\log 2 + 10\log u\big]. \\
&\quad\leq e^{\theta(u\log 2+\mathcal{O}(1))} \cdot e^{-(2u(\sqrt{\theta}-1)\log 2+\mathcal{O}(\log u))^2/2(2u\log 2+\mathcal{O}(1))} \\
&\quad\leq e^{\theta(u\log 2+\mathcal{O}(1))} \cdot e^{-(u(\sqrt{\theta}-1)^2\log 2+\mathcal{O}(\sqrt{\theta}(\log u)))(1+\mathcal{O}(u^{-1}))} \\
&\quad\leq C_\theta 2^{u(\theta-(\sqrt{\theta}-1)^2)}u^{m_\theta} = C_\theta 2^{u(2\sqrt{\theta}-1)}u^{m_\theta}
\end{aligned}$$

for $C_\theta, m_\theta > 0$ depending only on θ. Hence,

$$\sum_{u=1}^{\infty} 2^{2su}\int_0^{2\pi} \mathbb{E}\big[e^{\sqrt{\theta}G((1-2^{-u})e^{i\vartheta})}\mathbf{1}_{G((1-2^{-u})e^{i\chi(u,\vartheta)})\leq 2u\log 2+10\log u}\big]\,d\vartheta < \infty$$

as soon as $2s + 2\sqrt{\theta} - 1 < 0$, which implies that, for $\theta > 1$, HMC_θ is a.s. in H^s for all $s < -\sqrt{\theta} + 1/2$.

On the other hand, we have seen from (6.3)

$$A_{s,\theta} \geq \frac{C_s}{|\log r|^{2s}}\int_0^{2\pi} e^{\sqrt{\theta}G(re^{i\vartheta})}\,d\vartheta.$$

for all $r \in (1/2, 1)$. Since G is the real part of a holomorphic function, it is harmonic on the unit disc, and then by Jensen's inequality, for all $\vartheta_0 \in \mathbb{R}$,

$$e^{\sqrt{\theta}G(r^2e^{i\vartheta_0})} \leq \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{\theta}G(re^{i(\vartheta_0+\vartheta)})}P_r(\vartheta)\,d\vartheta,$$

where P_r is the Poisson kernel

$$P_r(\vartheta) = \sum_{p\in\mathbb{Z}} r^{|p|}e^{ip\vartheta}.$$

The Poisson kernel is bounded by $(1+r)/(1-r)$, and then

$$\begin{aligned}
2e^{\sqrt{\theta}\sup_{\vartheta_0\in\mathbb{R}} G(r^2e^{i\vartheta_0})} &\leq \sup_{\vartheta_0\in\mathbb{R}} \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{\theta}G(re^{i(\vartheta_0+\vartheta)})}P_r(\vartheta)\,d\vartheta \\
&\leq \frac{1+r}{2\pi(1-r)}\int_0^{2\pi} e^{\sqrt{\theta}G(re^{i\vartheta})}\,d\vartheta.
\end{aligned}$$

Hence, using (6.3), there is a constant $C_s > 0$ so that, for any $r \in (\frac{1}{2}, 1)$,

$$A_{s,\theta} \geq C_s(1-r)^{1-2s}e^{\sqrt{\theta}\sup_{\vartheta\in\mathbb{R}} G(r^2e^{i\vartheta})}.$$

With probability going to 1, when $r \to 1$, the maximum of the logarithmically correlated field $G(r^2e^{i\vartheta})$ is larger than $2\log(1/(1-r)) - 10\log(\log(1/(1-r)))$ (using the results of [21] on a net of $\lfloor 1/(1-r)\rfloor$ equally spaced points of the circle of radius r^2: notice that we do not need to control G between the points of the net because we are considering lower bounds of the maximum), and then $A_{s,\theta}$ is bounded from below by $(1-r)^{1-2s-2\sqrt{\theta}}\log(1/(1-r))^{m_\theta}$ for m_θ depending only on θ. For $s > -\sqrt{\theta} + 1/2$, the bound tends to infinity, when $r \to 1$, which shows that HMC_θ is a.s. not in H^s.

7. The circular β-ensemble. In this section we develop some of the properties of the secular coefficients of CβE $\{c_n^{(N)}\}$. We begin by recalling the *Verblunsky* coefficients, which will allow us to formulate the exact relationship, due to [38]. We let $\{\alpha_n : n \in \mathbb{N}_0\}$ be independent complex random variables on the unit disk, with each α_n rotationally invariant in law and $|\alpha_n|^2$ distributed like Beta$(1, \frac{\beta(n+1)}{2})$. The *Szegő* recurrence is, for all $N \geq 0$,

$$\begin{pmatrix} \Phi_{N+1}(z) \\ \Phi_{N+1}^*(z) \end{pmatrix} := \begin{pmatrix} z & -\overline{\alpha_N} \\ -\alpha_N z & 1 \end{pmatrix} \begin{pmatrix} \Phi_N(z) \\ \Phi_N^*(z) \end{pmatrix}, \qquad \begin{cases} \Phi_0(z) \equiv 1, \\ \Phi_N^*(z) = z^N \overline{\Phi_N(1/\overline{z})}, \end{cases} \tag{7.1}$$

where Φ_N^* and Φ_N are polynomials of degree at most, N. Note that Φ_N^* and Φ_N are related by being reversals of one another, in that their vector of coefficients is reversed and conjugated. Now, we give the connection to the $C\beta E$ characteristic polynomial, as defined in (1.2), due to [38]. We let η be uniformly distributed random variable on the unit circle independent of $\{\alpha_n : n \in \mathbb{N}_0\}$. The characteristic polynomial χ_N has the distribution of

$$\chi_N(z) = \Phi_{N-1}^*(z) - \eta z \Phi_{N-1}(z). \tag{7.2}$$

From [15], Proposition 3.1, we have almost sure convergence as $N \to \infty$ of $\Phi_N^*(z)$ to a process $\Phi_\infty^*(z)$ on the unit disk uniformly on compact sets. Moreover, this limit is none other than the log-Gaussian process $e^{\sqrt{\theta} G^{\mathbb{C}}(z)}$. In [15], Proposition 3.1, the following uniform estimate is also proven:

$$\mathbb{E}|\Phi_N^*(z)|^p \leq (1 - |z|^2)^{-\theta p^2/4} \quad \text{for all } z \in \mathbb{D}, p > 0, N \in \mathbb{N}. \tag{7.3}$$

We let $\mathscr{F}$ be the filtration $\mathscr{F} = (\mathscr{F}_N := \sigma(\alpha_0, \dots, \alpha_{N-1}) : N \geq 0)$. Then the process $\{\Phi_N^*(z)\}$ is adapted to $\mathscr{F}$ and, moreover, is a complex martingale. We now define $\mathscr{M}_{n,N}$ as the coefficient of degree n of the polynomial Φ_N^*. By Cauchy's theorem and the bound (7.3), we have that

$$\mathscr{M}_{n,N} = \frac{1}{2\pi i} \oint \Phi_N^*(z) z^{-n-1}\, dz, \tag{7.4}$$

for a fixed simple closed contour enclosing 0 in the unit disk, and that $(\mathscr{M}_{n,N})_{N \geq 0}$ forms a uniformly integrable martingale adapted to $\mathscr{F}_N$. Hence, we have the representation

$$\mathscr{M}_{n,N} = \mathbb{E}[c_n \mid \mathscr{F}_N] \xrightarrow[N \to \infty]{\text{a.s.}} c_n.$$

Using the Szegő recurrence (7.1) and (7.4), we have the identity

$$\begin{aligned} \mathscr{M}_{n,N+1} &= \mathscr{M}_{n,N} - \alpha_N \frac{1}{2\pi i} \oint \Phi_N(z) z^{-n}\, dz \\ &= \mathscr{M}_{n,N} - \alpha_N \overline{\mathscr{M}_{N-n+1,N}}. \end{aligned} \tag{7.5}$$

We will let $\mathfrak{B}_{n,N}$ be the bracket process of $\mathscr{M}_{n,N}$, that is, for $n \geq 1$, and for any $N \geq n-1$,

$$\begin{aligned} \mathfrak{B}_{n,N} &:= \sum_{j=n-1}^{N-1} \mathbb{E}[|\mathscr{M}_{n,j+1} - \mathscr{M}_{n,j}|^2 \mid \mathscr{F}_j] \\ &= \sum_{j=n-1}^{N-1} |\mathscr{M}_{j-n+1,j}|^2 \mathbb{E}(|\alpha_j|^2) = \sum_{j=n-1}^{N-1} \frac{|\mathscr{M}_{j-n+1,j}|^2}{1 + \frac{1}{\theta}(j+1)}. \end{aligned} \tag{7.6}$$

Here we can notice that $\mathscr{M}_{n,n-1} = 0$, since Φ_{n-1}^* is a polynomial of degree, at most, $n-1$.

LEMMA 7.1. *For any $\theta > 0$, there exists a constant $C_\theta > 0$ such that the following holds. For $N \geq n \geq 1$, we have $\mathbb{E}|\mathscr{M}_{n,N}|^2 = \mathbb{E}(\mathfrak{B}_{n,N})$, and*

$$\mathbb{E}|\mathscr{M}_{n,N}|^2 \leq C_\theta \begin{cases} \dfrac{(N-n+1)^\theta}{n} & \text{if } \theta < 1, \\ \dfrac{(N-n+1)}{n}\Big(1+\max\Big(\log\Big(\dfrac{n}{N-n+1}\Big),0\Big)\Big) & \text{if } \theta = 1, \\ \dfrac{n^\theta}{n}\log\Big(1+\dfrac{(N-n+1)}{n}\Big) & \text{if } \theta > 1. \end{cases}$$

Moreover, for all $n, N_1, N_2 \in \mathbb{N}$ such that $1 \leq 3n/2 \leq N_1 < N_2$, we have

$$\mathbb{E}|\mathscr{M}_{n,N_2} - \mathscr{M}_{n,N_1}|^2 = \mathbb{E}(\mathfrak{B}_{n,N_2} - \mathfrak{B}_{n,N_1}) \leq C_\theta n^\theta \Big(\frac{1}{N_1} - \frac{1}{N_2}\Big).$$

This holds with $N_2 = \infty$ as well; in which case $\mathscr{M}_{n,N_2} = c_n$ and $1/N_2 := 0$.

PROOF. We have that $\mathscr{M}_{n,N_2}$ is bounded in L^p for all p by (7.3) and (7.4). Hence, the $N_2 = \infty$ case follows from uniform integrability and taking $N_2 \to \infty$. Hence, it suffices to show only the case of $N_2 < \infty$. Also, it follows the bracket process is uniformly bounded in L^p for all p from the Burkholder–Davis–Gundy inequalities. For $j = n-1$, we have

$$|\mathscr{M}_{j-n+1,j}|^2 = |\mathscr{M}_{0,n-1}|^2 = 1$$

since the constant term of Φ^*_{n-1} is equal to 1. Hence, for $N \geq n \geq 1$, we can write

$$\mathfrak{B}_{n,N} = \frac{1}{1+\frac{n}{\theta}} + \sum_{j=n}^{N-1} \frac{|\mathscr{M}_{j-n+1,j}|^2}{1+\frac{1}{\theta}(j+1)}.$$

Recall that, for $j \geq n$, by the definition of the bracket given in (7.6), $k \mapsto |\mathscr{M}_{j-n+1,k}|^2 - \mathfrak{B}_{j-n+1,k}$ is again a martingale, starting at zero for $k = j-n$, and hence

$$\mathbb{E}\mathfrak{B}_{n,N} = \frac{1}{1+\frac{n}{\theta}} + \sum_{j=n}^{N-1} \frac{\mathbb{E}\mathfrak{B}_{j-n+1,j}}{1+\frac{1}{\theta}(j+1)}. \tag{7.7}$$

We may develop this equation to produce

$$\begin{aligned} \mathbb{E}\mathfrak{B}_{n,N} &= \frac{1}{1+\frac{n}{\theta}} + \sum_{j=n}^{N-1} \frac{1}{1+\frac{1}{\theta}(j+1)}\Bigg(\frac{1}{1+\frac{j-n+1}{\theta}} + \sum_{k=j-n+1}^{j-1} \frac{\mathbb{E}\mathfrak{B}_{k-(j-n+1)+1,k}}{1+\frac{1}{\theta}(k+1)}\Bigg) \\ &= \frac{1}{1+\frac{n}{\theta}} + \sum_{j=n}^{N-1} \frac{1}{(1+\frac{1}{\theta}(j+1))(1+\frac{j-n+1}{\theta})} \\ &\quad + \sum_{j=n}^{N-1}\sum_{k=1}^{n-1} \frac{1}{1+\frac{1}{\theta}(j+1)} \frac{\mathbb{E}\mathfrak{B}_{k,k+j-n}}{1+\frac{1}{\theta}(k+j-n+1)}. \end{aligned} \tag{7.8}$$

Using uniform integrability of the bracket, we have, for any $k \geq 0$,

$$\mathbb{E}\mathfrak{B}_{k,\infty} = \mathbb{E}|c_k|^2.$$

Let us now prove the estimate of $\mathbb{E}|\mathscr{M}_{n,N}|^2 = \mathbb{E}\mathfrak{B}_{n,N}$ given in the lemma. We have

$$\begin{aligned} \mathbb{E}\mathfrak{B}_{n,N} &\leq \frac{1}{1+\frac{n}{\theta}} + \sum_{j=n}^{N-1} \frac{1}{(1+\frac{1}{\theta}(j+1))(1+\frac{j-n+1}{\theta})} \\ &\quad + \sum_{j=n}^{N-1}\sum_{k=1}^{n-1} \frac{1}{1+\frac{1}{\theta}(j+1)} \frac{\mathbb{E}|c_k|^2}{1+\frac{1}{\theta}(k+j-n+1)}. \end{aligned} \tag{7.9}$$

Since $j \geq n$ and $\mathbb{E}[|c_k|^2]$ is dominated by $k^{\theta-1}$, $\mathbb{E}\mathfrak{B}_{n,N}$ is bounded, up to a constant depending only on θ, by

$$\frac{1}{n}\left(1 + \sum_{j=n}^{N-1} \frac{1}{j-n+1} + \sum_{j=n}^{N-1}\sum_{k=1}^{n-1} \frac{k^{\theta-1}}{k+j-n+1}\right).$$

For $N = n$, we immediately deduce a bound of order $1/n$ (the sums are empty) which is enough for our purpose. We can then assume $N \geq n+1$, and in this case, we have

$$\frac{1}{n}\sum_{j=1}^{N-n}\sum_{k=0}^{n-1} \frac{(1+k)^{\theta-1}}{k+j} \leq \frac{1}{n}\sum_{j=1}^{N-n}\sum_{k=0}^{n-1} \frac{(1+k)^{\theta-1}}{\max\{j, 1+k\}}. \tag{7.10}$$

If $j \leq n$, the inner sum can be estimated as

$$\sum_{0 \leq k \leq j-1} \frac{(1+k)^{\theta-1}}{j} + \sum_{j-1<k\leq n-1} (1+k)^{\theta-2} \leq \begin{cases} C_\theta j^{\theta-1} & \text{if } \theta < 1, \\ 1 + \log(1+n/j) & \text{if } \theta = 1, \\ C_\theta n^{\theta-1} & \text{if } \theta > 1, \end{cases}$$

where $C_\theta > 0$ is some sufficiently large constant. If $j > n$, the inner sum of (7.10) is

$$\sum_{0\leq k\leq n-1} \frac{(1+k)^{\theta-1}}{j} \leq \frac{C_\theta n^\theta}{j}$$

for some $C_\theta > 0$ sufficiently large. Summing in j, we get from (7.10) and the bounds just established, that, for $\theta < 1$,

$$\frac{1}{n}\sum_{j=1}^{N-n}\sum_{k=0}^{n-1} \frac{(1+k)^{\theta-1}}{k+j} \leq \frac{C_\theta}{n}\left((N-n+1)^\theta + n^\theta \log\left(1 + \frac{N-n+1}{n}\right)\right)$$

which is dominated by $(N-n+1)^\theta/n$. For $\theta > 1$, we get

$$\frac{1}{n}\sum_{j=1}^{N-n}\sum_{k=0}^{n-1} \frac{(1+k)^{\theta-1}}{k+j} \leq \frac{C_\theta}{n}\left(n^{\theta-1}\big((N-n+1)\wedge n\big) + n^\theta \log\left(1 + \frac{N-n+1}{n}\right)\right),$$

a domination by $n^{\theta-1}\log(1+\frac{N-n+1}{n})$. For $\theta = 1$, we get a domination by

$$\frac{1}{n}\sum_{j=1}^{N-n}\sum_{k=0}^{n-1} \frac{(1+k)^{\theta-1}}{k+j} \leq \frac{C_\theta}{n}\sum_{j=1}^{N-n}\left(1 + \max\left(0, \log\frac{n}{j}\right)\right).$$

For $1 \leq N-n \leq n$, this quantity is, at most,

$$\begin{aligned} &n^{-1}\big((N-n)(1+\log n) - \log\big((N-n)!\big)\big) \\ &\leq n^{-1}\big((N-n)(1+\log n) - (N-n)\log(N-n) + (N-n)\big) \\ &\leq n^{-1}(N-n)\big(2 + \log\big(n/(N-n)\big)\big), \end{aligned}$$

and then we have, for all $N \geq n+1$, a domination by

$$n^{-1}(N-n)\big(1 + \max\big(0, \log\big(n/(N-n)\big)\big)\big).$$

We have now proven the first part of the lemma. For the second part, we observe that, under the assumptions of the lemma,

$$\begin{aligned} \mathbb{E}(\mathfrak{B}_{n,N_2} - \mathfrak{B}_{n,N_1}) \leq{}& \sum_{j=N_1}^{N_2-1}\sum_{k=1}^{n-1} \frac{1}{1+\frac{1}{\theta}(j+1)} \frac{\mathbb{E}(|c_k|^2)}{1+\frac{1}{\theta}(k+j-n+1)} \\ &+ \sum_{j=N_1}^{N_2-1} \frac{1}{1+\frac{1}{\theta}(j+1)} \frac{1}{1+\frac{j-n+1}{\theta}}. \end{aligned} \tag{7.11}$$

We can estimate for some constant $C_\theta > 0$,

$$\mathbb{E}(\mathfrak{B}_{n,N_2} - \mathfrak{B}_{n,N_1}) \le C_\theta \sum_{j=N_1}^{N_2} \sum_{k=1}^{n-1} \frac{1}{1+\frac{1}{\theta}(j+1)} \frac{k^{\theta-1}}{1+\frac{1}{\theta}(k+j-n+1)} + \sum_{j=N_1}^{N_2-1} \frac{1}{1+\frac{1}{\theta}(j+1)} \frac{1}{1+\frac{j-n+1}{\theta}}.$$

If $N_1 \ge 3n/2$, we have that $j - n \ge j/3$, and then the sum of the terms corresponding to a given value of j is dominated by n^θ/j^2 which proves the second part of the lemma. □

Having an estimate on the secular coefficients of Φ^*_{N-1}, we may use (7.2) to relate $c_n^{(N)}$ to c_n. Recasting (7.2) in terms of coefficients,

$$c_n^{(N)} = \mathscr{M}_{n,N-1} - \eta \overline{\mathscr{M}_{N-n,N-1}}. \tag{7.12}$$

We deduce the following estimates relating the secular coefficients of CβE to the coefficients of the HMC.

LEMMA 7.2. *There is constant C_θ so that, for all $n, N \in \mathbb{N}$ with $1 \le n \le N/2$,*

$$\mathbb{E}|c_n - c_n^{(N)}|^2 \le \begin{cases} \dfrac{C_\theta n^\theta}{N} & \text{if } \theta < 1, \\ \dfrac{C_\theta n \log(N/n)}{N} & \text{if } \theta = 1, \\ \dfrac{C_\theta n}{N^{2-\theta}} & \text{if } \theta > 1. \end{cases}$$

PROOF. Using (7.12),

$$\mathbb{E}|c_n - c_n^{(N)}|^2 = \mathbb{E}|c_n - \mathscr{M}_{n,N-1}|^2 + \mathbb{E}|\mathscr{M}_{N-n,N-1}|^2.$$

We have $N \ge 2n$ and then $N - 1 \ge 3n/2$ (except for $n = 1$ and $N = 2$, in which case the lemma is obvious for a suitable value of C_θ). Hence, by Lemma 7.1 we have, after increasing the constant C_θ if needed,

$$\mathbb{E}|c_n - \mathscr{M}_{n,N-1}|^2 \le \frac{C_\theta n^\theta}{N},$$

and

$$\mathbb{E}|\mathscr{M}_{N-n,N-1}|^2 \le \begin{cases} \dfrac{C_\theta n^\theta}{N} & \text{if } \theta < 1, \\ \dfrac{C_\theta n \log(N/n)}{N} & \text{if } \theta = 1, \\ \dfrac{C_\theta n}{N^{2-\theta}} & \text{if } \theta > 1. \end{cases}$$

This latter part dominates the contribution $|\mathscr{M}_{N-n,N-1}|^2$ in all cases. □

From this point the proof of Theorem 1.10 is a simple corollary.

PROOF OF THEOREM 1.10. Under the assumption that $n/N \to 0$,

$$(\mathbb{E}|c_n - c_n^{(N)}|^2) n^{1-\theta} \to 0 \quad \text{as } n \to \infty,$$

and so we conclude from Slutsky's theorem, (1.34) and Theorem 1.8 that the desired distributional convergence holds. □

7.1. *Fractional moments.* In this section we prove Theorem 1.11. We define w_n as

$$w_n = n^{(\theta-1)/2}, \qquad w_n = \big(\log(1+n)\big)^{-\frac{1}{4}} \quad \text{or} \quad w_n = n^{\sqrt{\theta}-1}(\log n)^{-\frac{3}{4}\sqrt{\theta}}, \tag{7.13}$$

in the cases $\theta \in (0,1)$, $\theta = 1$ or $\theta > 1$, respectively.

We recall for convenience (3.11) and recall that $S^{(n)}$ is the shortest cycle in a Ewens distributed permutation of n symbols (c.f. Lemma 3.7),

$$\mathbb{E}\big(|c_n|^2 \mid \mathscr{G}_q\big) = |c_{n,q}|^2 + \sum_{r=0}^{n} |c_{r,q}|^2 \binom{n-r+\theta-1}{\theta-1} \mathbb{P}\big[S^{(n-r)} > q\big].$$

Using Lemma 3.7, we have a uniform bound

$$\mathbb{E}\big(|c_n|^2 \mid \mathscr{G}_q\big) \le |c_{n,q}|^2 + C_\theta n^{\theta-1} F_q \quad \text{where } F_q := \sum_{r=0}^{\infty} |c_{r,q}|^2 e^{-\theta h(q+1)}. \tag{7.14}$$

Using Parseval's identity and (5.2),

$$\sum_{r=0}^{\infty} |c_{r,q}|^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{\theta} G_q(e^{i\vartheta})}\, d\vartheta, \quad \text{where } G_q(z) = 2\Re \sum_{k=1}^{q} \frac{z^k \mathcal{N}_k}{\sqrt{k}}. \tag{7.15}$$

Moreover, the normalizing constant $e^{-\theta h(q+1)}$ is such that F_q has expectation 1, and F_q is thus an approximation to the mass of the chaos for $\theta < 1$.

LEMMA 7.3. *Let w_n, as in (7.13). If $\theta \in (0,1)$, then*

$$\mathbb{E}\big[n^{\theta-1} F_n / w_n^2\big] = 1$$

for all $n \ge 1$. If $\theta = 1$, then, for $p \in (0,1)$,

$$\sup\big\{\mathbb{E}\big[(F_n/w_n^2)^p\big] : n \in \mathbb{N}\big\} < \infty.$$

Finally, if $\theta > 1$, there exists $u_\theta > 0$, depending only on θ, such that

$$\big\{(\log n)^{-2v_\theta} n^{\theta-1} F_n / w_n^2 : n \in \mathbb{N}\big\}$$

is tight.

PROOF. For $\theta \in (0,1)$, the lemma means that F_n has expectation 1, as remarked before. For $\theta = 1$, the lemma follows from [34], Theorem 1.3. For $\theta > 1$, we use the same ingredients as those considered in the proof of regularity of the supercritical HMC. The convergence is obtained by bounding the expectation, after restricting it to the event that $G_n(e^{i\vartheta}) \le 2\log n + 10\log\log n$ for all $\vartheta \in 2\pi\mathbb{Z}/n$, whose probability goes to 1 when $n \to \infty$ (by a simple union bound). Using Girsanov's theorem, we get, for $n \ge 2$ and $|\vartheta - \vartheta'| = \mathcal{O}(1/n)$, that

$$\begin{aligned}
&\mathbb{E}\big[e^{\sqrt{\theta} G_n(e^{i\vartheta})} \mathbf{1}_{G_n(e^{i\vartheta'}) \le 2\log n + 10\log\log n}\big] \\
&\quad = e^{\theta(\log n + \mathcal{O}(1))} \mathbb{P}\big[G_n(e^{i\vartheta'}) + \mathrm{Cov}\big(G_n(e^{i\vartheta'}), \sqrt{\theta} G_n(e^{i\vartheta})\big) \le 2\log n + 10\log\log n\big] \\
&\quad = \big(\mathcal{O}(n^\theta)\big) \mathbb{P}\big[G_n(e^{i\vartheta'}) \le 2(1-\sqrt{\theta})\log n + 10\log\log n + c_\theta\big]
\end{aligned}$$

for some $c_\theta \in \mathbb{R}$ depending only on θ, since, from the fact that ϑ and ϑ' are close to each other, the covariance between $G_n(e^{i\vartheta'})$ and $G_n(e^{i\vartheta})$ is equal to $2\log n + \mathcal{O}(1)$. Hence, since $\theta > 1$,

$$\mathbb{E}\big[e^{\sqrt{\theta} G_n(e^{i\vartheta})} \mathbf{1}_{G_n(e^{i\vartheta'}) \le 2\log n + 10\log\log n}\big] \le C_\theta n^{\theta - (1-\sqrt{\theta})^2} (\log n)^{\mu_\theta} \tag{7.16}$$

for $C_\theta, \mu_\theta > 0$, depending only on θ. Taking $\vartheta' \in 2\pi\mathbb{Z}/n$, depending only of ϑ and n in such a way that $|\vartheta - \vartheta'| = \mathcal{O}(1/n)$, and integrating in ϑ, we deduce

$$\begin{aligned}&\mathbb{E}\big[n^{\theta-1} F_n w_n^{-2} \mathbf{1}_{\sup_{\vartheta' \in 2\pi\mathbb{Z}/n} G_n(e^{i\vartheta'}) \le 2\log n + 10 \log\log n}\big] \\ &\quad \le C'_\theta n^{\theta-1} n^{-\theta} n^{\theta-(1-\sqrt{\theta})^2} (\log n)^{\frac{3}{2}\theta+\mu_\theta} n^{-2\sqrt{\theta}+2} = C'_\theta (\log n)^{\frac{3}{2}\theta+\mu_\theta}\end{aligned}$$

for $C'_\theta > 0$. This proves the lemma with $2v_\theta = \frac{3}{2}\theta + \mu_\theta$. $\square$

REMARK 7.4. While beyond the scope of what we do here, the optimal power is $u_\theta = 0$; in which case the statement would be

$$\{n^{\theta-1} F_n / w_n^2 : n \in \mathbb{N}\}$$

is a tight family. Indeed, as a consequence of [21], the maximum of $G_n(e^{i\vartheta})$ on the lattice $2\pi\mathbb{Z}/n$ is, at most, $2\log n - \frac{3}{2}\log\log n + y$ with a probability controlled uniformly in n by y. Moreover, one can establish (and herein lies the technical work) that the random walk $(G_{2^k}(e^{i\vartheta}) : k < \log_2 n)$ stays below $2(k\log 2 - \frac{3}{4}\log k) + (k(\log_2 n - k)/\log_2 n)^{1/10} + y$ for all lattice ϑ again, except with a probability that can be made small in y uniformly in n. On this good event, $\mathcal{G}(n, y)$, it is now possible to show, using the appropriate ballot theorem, that for $\theta > 1$, for $|\vartheta - \vartheta'| = \mathcal{O}(1/n)$ and for $k \in [0, \sqrt{\log n}]$

$$\begin{aligned}&\mathbb{E}\bigg[e^{\sqrt{\theta} G_n(e^{i\vartheta})} \mathbf{1}\bigg\{G_n(e^{i\vartheta'}) - \bigg(2\log n - \frac{3}{4}\log\log n\bigg) - y \in [-k, -k-1]\bigg\} \mathbf{1}\{\mathcal{G}(n, y)\}\bigg] \\ &\quad \le C_{\theta,y} e^{\sqrt{\theta}(2\log n - \frac{3}{2}\log\log n - k)} \times \frac{k e^k}{n},\end{aligned}$$

the first term being the contribution of $e^{\sqrt{\theta} G_n(e^{i\vartheta})}$, which is essentially deterministic on the event, and the second term being the probability of the Gaussian ends in the claimed window. Note that the ballot theorem is needed to kill the factor of $(\log n)^{3/2}$ that would otherwise result from the Gaussian tail. As $\theta > 1$, we can sum this contribution in k. We also need to control the contribution of ϑ at which $G_n(e^{i\vartheta}) \le 2\log n - \sqrt{\log n}$, which can be done using the same Girsanov argument as in (7.16), and we arrive at

$$\mathbb{E}\bigg[\int_0^{2\pi} e^{\sqrt{\theta} G_n(e^{i\vartheta})}\, d\vartheta\, \mathbf{1}\{\mathcal{G}(n, y)\}\bigg] \le \frac{C_{\theta,y} e^{\sqrt{\theta}(2\log n - \frac{3}{2}\log\log n)}}{n}.$$

This would lead to $u_\theta = 0$.

Note this is optimal, as can be seen from the contribution of a $\mathcal{O}(1/n)$ window of the global maximum of $G_n(e^{i\vartheta})$.

LEMMA 7.5. *If* $\theta \in (0, 1)$, *then*

$$\sup\{\mathbb{E}(|c_n/w_n|^2) : n \in \mathbb{N}\} < \infty.$$

If $\theta = 1$, *and* $p \in (0, 1)$, *then*

$$\sup\{\mathbb{E}(|c_n/w_n|^{2p}) : n \in \mathbb{N}\} < \infty.$$

Finally, if $\theta > 1$, *there exists* $v_\theta > 0$, *depending only on* θ, *such that*

$$\{(c_n/w_n)(\log n)^{-v_\theta} : n \in \mathbb{N}\}$$

is tight.

PROOF. The case $\theta \in (0,1)$ is already proven before. Moreover, using (4.39) and Lemma 3.6, we know that, for $n \geq 3$,

$$\max_{1 \leq q < n/\log n} \mathbb{E}|c_{n,q}|^2 \leq \max_{1 \leq q < n/\log n} C_\theta n^{\theta-1} \mathbb{P}(L^{(n-q)} \leq q) \leq n^{-\omega(n)}, \tag{7.17}$$

where $\omega(n) \to \infty$, as $n \to \infty$.

We now use (7.14) for $q_n = \lfloor n/\log n \rfloor$ when $n \geq 3$,

$$\mathbb{E}(|c_n|^2 \mid \mathcal{G}_{q_n}) \leq |c_{n,q_n}|^2 + C_\theta n^{\theta-1} F_{q_n}.$$

For $\theta = 1$, we take $p \in (0,1)$ and get, using Hölder's inequality for exponent $1/p > 1$ and subadditivity of the pth power,

$$\mathbb{E}(|c_n|^{2p} \mid \mathcal{G}_{q_n}) \leq [\mathbb{E}(|c_n|^2 \mid \mathcal{G}_{q_n})]^p \leq |c_{n,q_n}|^{2p} + (C_1 F_{q_n})^p,$$

which implies

$$\mathbb{E}(|c_n|^{2p}) \leq \mathbb{E}(|c_{n,q_n}|^{2p}) + \mathbb{E}(|C_1 F_{q_n}|^p)$$

from (7.17), and conclude

$$\mathbb{E}(|c_n/w_n|^{2p}) \leq \mathcal{O}(n^{-p\omega(n)}/w_n^2) + \mathbb{E}(|C_1 F_{q_n}/w_n^2|^p).$$

We then use Lemma 7.3 to complete the proof, taking into account the fact that $w_n = (\log(1+n))^{-1/4}$ is equivalent to w_{q_n} when n goes to infinity. When $\theta > 1$, we write

$$\begin{aligned}\mathbb{E}(\min(1, |c_n w_n^{-1}(\log n)^{-v_\theta}|^2) \mid \mathcal{G}_{q_n}) &\leq w_n^{-2}(\log n)^{-2v_\theta}|c_{n,q_n}|^2 \\ &\quad + \min(1, C_\theta w_n^{-2}(\log n)^{-2v_\theta} n^{\theta-1} F_{q_n}).\end{aligned}$$

Since w_n/w_{q_n} and $n^{\theta-1}/q_n^{\theta-1}$ are equivalent up to powers of $\log n$ (depending on θ) when $n \to \infty$ and $\log n$ is equivalent to $\log q_n$, we deduce from Lemma 7.3 that

$$w_n^{-2}(\log n)^{-2v_\theta} n^{\theta-1} F_{q_n} \xrightarrow[r\to\infty]{p} 0$$

for any v_θ sufficiently large. Then

$$\mathbb{E}[\min(1, C_\theta w_n^{-2}(\log n)^{-2v_\theta} n^{\theta-1} F_{q_n})] \xrightarrow[n\to\infty]{} 0,$$

as soon as v_θ is large enough. Since $\mathbb{E}[|c_{n,q_n}|^2]$ decreases to 0 faster than any power of n, we have

$$\mathbb{E}[w_n^{-2}(\log n)^{-2v_\theta}|c_{n,q_n}|^2] \xrightarrow[n\to\infty]{} 0,$$

and then

$$\mathbb{E}[\min(1, |c_n w_n^{-1}(\log n)^{-v_\theta}|^2)] \xrightarrow[n\to\infty]{} 0$$

which implies that

$$|c_n w_n^{-1}(\log n)^{-v_\theta}| \xrightarrow[r\to\infty]{p} 0. \qquad \square$$

We deduce the following result on the secular coefficients of $C\beta E$.

THEOREM 7.6. *If* $\theta \in (0,1)$, *then*

$$\sup\{\mathbb{E}(|c_n^{(N)}/w_n|^2) : n \in \mathbb{N}, N \geq 2n\} < \infty.$$

If $\theta = 1$, *and* $p \in (0,1)$, *then*

$$\sup\{\mathbb{E}(|c_n^{(N)}/w_n|^{2p}) : n \in \mathbb{N}, N \geq 2n\} < \infty.$$

Finally, if $\theta \in (1,2)$, *there exists* $v_\theta, v'_\theta > 0$ *depending only on* θ *so that with* $N_0(n) = n^{\frac{3-2\sqrt{\theta}}{2-\theta}}(\log n)^{v'_\theta}$, *for all* $\delta > 0$,

$$\sup_{N \geq N_0(n)} \mathbb{P}[(c_n^{(N)}/w_n)(\log n)^{-v_\theta} \geq \delta] \underset{n\to\infty}{\longrightarrow} 0.$$

PROOF. Let us first suppose that $\theta \leq 1$: we may assume $p \in (1/2, 1)$, when $\theta = 1$, by Hölder's inequality. For $r \geq 1$ and any $k, n \in \mathbb{N}$, we have from Jensen's inequality

$$\mathbb{E}|\mathscr{M}_{n,k}|^r = \mathbb{E}|\mathbb{E}(c_n \mid \mathscr{F}_k)|^r \leq \mathbb{E}(|c_n|^r).$$

Hence, recalling (7.12), for any $p \geq 1$,

$$(\mathbb{E}|c_n^{(N)}|^r)^{1/r} \leq (\mathbb{E}(|c_n|^r))^{1/r} + (\mathbb{E}(|c_{N-n}|^r))^{1/r}.$$

We can now conclude the proof from Lemma 7.5, taking $r = 2$ for $\theta \in (0,1)$ and $r = 2p \in (1,2)$ for $\theta = 1$: notice that $w_{N-n} \leq w_n$ since $\theta \leq 1$ and $N \geq 2n$.

Let us now suppose that $\theta \in (1,2)$. In this case, because of Lemma 7.5, it is sufficient to prove that

$$\sup_{N \geq N_0(n)} \mathbb{E}[|c_n - c_n^{(N)}|^2 w_n^{-2}(\log n)^{-2v_\theta}] \underset{n\to\infty}{\longrightarrow} 0.$$

For suitable v'_θ, this is a consequence of the bound

$$\mathbb{E}[|c_n - c_n^{(N)}|^2] \leq C_\theta n/N^{2-\theta}$$

given by Lemma 7.2. □

REMARK 7.7. In the supercritical phase, we compare c_n with $c_n^{(N)}$ by using the L^2 norm, which is expected not to be optimal, since the L^2 norm of c_n does not gives its correct order of magnitude. Hence, we expect that the exponent $(3 - 2\sqrt{\theta})/(2-\theta)$ can be improved.

8. Sharpness of the tightness. We have seen that $(c_n(\log n)^{-v_\theta}/w_n)_{n\geq 1}$ is a tight family for random variables for $\theta > 0$ for some $v_\theta > 0$, depending only on θ, which can be taken equal to 0 for $\theta \in (0,1]$. Moreover informally, c_n has order, at most, w_n, up to a logarithmic factor. It is natural to ask if this bound is optimal or if c_n has a smaller order of magnitude. In the following section, we show that c_n is no smaller than w_n in order of magnitude (i.e., that w_n/c_n is tight) when $\theta \in (0,1]$.

THEOREM 8.1. *For all* $\theta \in (0,1]$ *and all* $u \in \mathbb{C}$ *such that* $|u| = 1$, *one has*

$$\sup_{n\geq 1} \mathbb{P}[|\Re(uc_n)|/w_n \leq \delta] \underset{\delta\to 0}{\longrightarrow} 0.$$

PROOF. Since the law of c_n is rotational invariant in distribution, we can assume $u = 1$. We have the following equality:

$$c_n = c_{n,\lfloor n/2\rfloor} + \sum_{q=0}^{\lceil \frac{n}{2}\rceil - 1} \frac{\mathcal{N}_{n-q}\theta^{1/2}}{\sqrt{n-q}} c_q,$$

and then, since $(\mathcal{N}_{n-q})_{0\le q<n/2}$ are independent of $c_{n,\lfloor n/2\rfloor}$ and $(c_q)_{0\le q<n/2}$,

$$c_n \overset{d}{=} c_{n,\lfloor n/2\rfloor} + \mathcal{N}\left(\sum_{q=0}^{\lceil \frac{n}{2}\rceil -1} \frac{|c_q|^2\theta}{n-q}\right)^{1/2},$$

where $\mathcal{N}$ is a standard complex Gaussian independent of the other variables which are involved in the formula. Hence, by the fact that the law of $\Re(\mathcal{N})$ has a bounded density with respect to the Lebesgue measure,

$$\mathbb{P}\big[\big|\Re(c_n)\big|/w_n \le \delta \mid \mathcal{G}_{\lfloor n/2\rfloor}\big] \le C\delta w_n\left(\sum_{q=0}^{\lceil \frac{n}{2}\rceil -1} \frac{|c_q|^2\theta}{n-q}\right)^{-1/2},$$

for some constant $C>0$, which implies

$$\mathbb{P}\big[\big|\Re(c_n)\big|/w_n \le \delta\big] \le \mathbb{E}\left[\min\left(1, C\delta w_n\left(\sum_{q=0}^{\lceil \frac{n}{2}\rceil -1} \frac{|c_q|^2\theta}{n-q}\right)^{-1/2}\right)\right]. \tag{8.1}$$

Recalling (1.31), there is a constant $c_\theta>0$ so that, for all $n>2$ and with $n_0=\lceil \frac{n}{2}\rceil -1$,

$$\sum_{q=0}^{n_0} \frac{|c_q|^2\theta}{n-q} \ge \frac{\theta}{n}\sum_{q=0}^{n_0}|c_q|^2 \ge c_\theta(\sqrt{\log n})^{-\mathbf{1}\{\theta=1\}} n^\theta \mathcal{M}_{\theta,n_0}.$$

The expression $\mathcal{M}_{\theta,n_0}$ is an approximation to the total mass. By Lemma 1.32 this converges in probability as $n\to\infty$ to a multiple of $\mathcal{M}_\theta$. In particular,

$$\left\{\left(\sum_{q=0}^{\lceil \frac{n}{2}\rceil -1} \frac{|c_q|^2\theta}{n-q}\right)^{-1/2} w_n : n\in\mathbb{N}\right\}$$

is tight (recall when $\theta\in(0,1)$, $w_n=n^{\theta-1}$ and when $\theta=1$, $w_n=(\log(1+n))^{-1/4}$). Using (8.1),

$$\mathbb{P}\big[\big|\Re(c_n)\big|/w_n \le \delta\big] \le C\delta^{1/2} + \mathbb{P}\left(\left(\sum_{q=0}^{\lceil \frac{n}{2}\rceil -1} \frac{|c_q|^2\theta}{n-q}\right)^{-1/2} w_n \ge \delta^{-1/2}\right).$$

The tightness property above shows that

$$\sup_{n\ge 1}\mathbb{P}\big[\big|\Re(c_n)\big|/w_n \le \delta\big] \underset{\delta\to 0}{\longrightarrow} 0$$

which completes the proof. □

We deduce a similar result for the secular coefficients $c_n^{(N)}$ when N is sufficiently large with respect to n.

THEOREM 8.2. *Let $\theta\in(0,1]$, $u\in\mathbb{C}$ on the unit circle, and let φ be any function from $\mathbb{N}$ to $\mathbb{R}$ such that $\varphi(n)\ge 2$ for all $n\ge 1$, $\varphi(n)$ tends to infinity with n, and for $\theta=1$, $\varphi(n)/(\sqrt{\log n}(\log\log n))$ tends to infinity with n. Then*

$$\sup_{n\ge 1, N\ge n\varphi(n)} \mathbb{P}\big[\big|\Re\big(uc_n^{(N)}\big)\big|/w_n \le \delta\big] \underset{\delta\to 0}{\longrightarrow} 0.$$

PROOF. We assume $u = 1$. We know that, for all $\epsilon > 0$, there exists $\delta > 0$ depending only on ϵ and θ, such that, for $n \geq 1$,

$$\mathbb{P}\big[\big|\Re(c_n)\big|/w_n \leq 2\delta\big] \leq \epsilon.$$

Now,

$$\mathbb{P}\big[\big|\Re\big(c_n^{(N)}\big)\big|/w_n \leq \delta\big] \leq \mathbb{P}\big[\big|\Re(c_n)\big|/w_n \leq 2\delta\big] + \mathbb{P}\big[\big|c_n - c_n^{(N)}\big|/w_n \geq \delta\big],$$

and then, using Lemma 7.2 and Markov's inequality,

$$\mathbb{P}\big[\big|\Re\big(c_n^{(N)}\big)\big|/w_n \leq \delta\big] \leq \epsilon + \delta^{-2} C_\theta w_n^{-2} n^\theta / N$$

for $\theta \in (0, 1)$ and

$$\mathbb{P}\big[\big|\Re\big(c_n^{(N)}\big)\big|/w_n \leq \delta\big] \leq \epsilon + \delta^{-2} C_1 w_n^{-2} n \log(N/n)/N$$

for $\theta = 1$. Hence,

$$\mathbb{P}\big[\big|\Re\big(c_n^{(N)}\big)\big|/w_n \leq \delta\big] \leq \epsilon + \delta^{-2} C_\theta n/N \leq \epsilon + \delta^{-2} C_\theta/\varphi(n)$$

for $\theta \in (0, 1)$ and $N \geq n\varphi(n)$, and

$$\begin{aligned}\mathbb{P}\big[\big|\Re\big(c_n^{(N)}\big)\big|/w_n \leq \delta\big] &\leq \epsilon + \delta^{-2} C_1 \big(\log(1+n)\big)^{1/2} \log(N/n)(n/N) \\ &\leq \epsilon + 2\delta^{-2} C_1 \big(\log(1+n)\big)^{1/2} \big(\log\big(\varphi(n)\big)\big)/\varphi(n)\end{aligned}$$

for $\theta = 1$ and $N \geq n\varphi(n)$. The factor 2 comes from the fact that $\varphi(n) \geq 2$ by assumption and $(\log a)/a \leq 2(\log b)/b$ for $a \geq b \geq 2$. For $\theta \in (0, 1)$, we have

$$\delta^{-2} C_\theta/\varphi(n) \underset{n\to\infty}{\longrightarrow} 0$$

since $\varphi(n) \to \infty$. For $\theta = 1$, we have

$$\min\big(\varphi(n), 2 + \log n\big)/\big(\sqrt{\log n}(\log\log n)\big) \underset{n\to\infty}{\longrightarrow} \infty,$$

and then

$$\begin{aligned}&\big(\log(1+n)\big)^{1/2}\big(\log\big(\varphi(n)\big)\big)/\varphi(n) \\ &\quad \leq 2\big(\log(1+n)\big)^{1/2} \log\big(\min\big(\varphi(n), 2 + \log n\big)\big)/\min\big(\varphi(n), 2 + \log n\big), \\ &\quad \leq 2\big(\log(1+n)\big)^{1/2} \log(2 + \log n)/\min\big(\varphi(n), 2 + \log n\big)\end{aligned}$$

tends to zero, when n goes to infinity, which implies

$$2\delta^{-2} C_1 \big(\log(1+n)\big)^{1/2}\big(\log\big(\varphi(n)\big)\big)/\varphi(n) \underset{n\to\infty}{\longrightarrow} 0.$$

Since we take δ depending only on ϵ and θ, we deduce that there exists $n(\epsilon, \theta) \geq 1$ such that, for all $n \geq n(\epsilon, \theta)$,

$$\delta^{-2} C_\theta/\varphi(n) \leq \epsilon$$

if $\theta \in (0, 1)$, and

$$2\delta^{-2} C_1 \big(\log(1+n)\big)^{1/2}\big(\log\big(\varphi(n)\big)\big)/\varphi(n) \leq \epsilon$$

if $\theta = 1$. This gives, in any case,

$$\mathbb{P}\big[\big|\Re\big(c_n^{(N)}\big)\big|/w_n \leq \delta\big] \leq 2\epsilon$$

for $n \geq n(\epsilon, \theta)$, $N \geq \varphi(n)$. Now, for $N \geq n \geq 1$, $c_n^{(N)}$ is an elementary symmetric function of random points on the unit circle whose joint distribution is absolutely continuous with respect to the distribution of N i.i.d., uniform points on the unit circle. For $N = n$, $|c_n^{(N)}|$ is equal to 1 and then different from 0. For $N > n$, if we fix $N - 1$ of the N points on the circle, we have that $c_n^{(N)}$ is an affine function of the last point, and then the conditional probability that $c_n^{(N)} = 0$ vanishes as soon as one of the two coefficients of this affine function is nonzero. Taking the constant coefficient, which is the nth symmetric function of the $N - 1$ points which have been fixed, we deduce, by induction on N, that $\mathbb{P}[c_n^{(N)} = 0] = 0$ for all $N \geq n \geq 1$. Since the law of $c_n^{(N)}$ is rotationally invariant, we have that $\mathbb{P}[\Re(c_n^{(N)}) = 0 | |c_n^{(N)}|] = 0$ when $|c_n^{(N)}| \neq 0$, and then $\mathbb{P}[\Re(c_n^{(N)}) = 0] = 0$ since $|c_n^{(N)}|$ is almost surely different from zero. Moreover, we know that $\Re(c_n) \neq 0$ almost surely: since $c_n^{(N)}$ converges to c_n in law for n fixed and $N \to \infty$, we have that, for each $n \geq 1$, $(w_n / |\Re(c_n^{(N)})|)_{N \geq n}$ is a tight family of real-valued random variables.

Hence, for each $n \geq 1$, there exists $\delta_n > 0$, depending only on n, ϵ and θ, such that

$$\sup_{N \geq n} \mathbb{P}\big[\big|\Re\big(c_n^{(N)}\big)\big| / w_n \leq \delta_n\big] \leq 2\epsilon.$$

Let us define

$$\delta_0 := \min\Big(\delta, \min_{1 \leq n < n(\epsilon,\theta)} \delta_n\Big).$$

We have that $\delta_0 > 0$ depends only on ϵ and θ, and that

$$\sup_{n \geq 1, N \geq n\varphi(n)} \mathbb{P}\big[\big|\Re\big(c_n^{(N)}\big)\big| / w_n \leq \delta_0\big] \leq 2\epsilon,$$

which implies

$$\limsup_{\delta \to 0} \sup_{n \geq 1, N \geq n\varphi(n)} \mathbb{P}\big[\big|\Re\big(c_n^{(N)}\big)\big| / w_n \leq \delta\big] \leq 2\epsilon.$$

By letting $\epsilon \to 0$, we are done. $\square$

APPENDIX A: TAUBERIAN THEORY

Recall that a function $L : [0, \infty) \to \mathbb{R}$ is called *slowly varying* if it is measurable, eventually positive and satisfies

$$L(\lambda u)/L(u) \to 1 \quad \text{as } u \to \infty \text{ for all } \lambda > 0.$$

The principal examples of such functions are logarithms, iterated logarithms and powers thereof. These functions have many useful analytic properties (see [39], Section IV.2, for details). Such functions appear frequently in Tauberian theory and, in particular, in the following.

THEOREM A.1 (Karamata–Hardy–Littlewood Tauberian Theorem [39], IV.1.1). *Let $\sum_{k=0}^{\infty} a_k z^k$ converge for $|z| < 1$. Suppose that, for some number $\theta \geq 0$ and some slowly varying function L, we have the limit from below*

$$\lim_{z \to 1^-} L\Big(\frac{1}{1-z}\Big)(1-z)^{\theta} \sum_{k=0}^{\infty} a_k z^k = M. \tag{A.1}$$

Then, subject to the condition

$$L(k) k a_k \geq -C k^{\theta}, \quad k \geq 1, \tag{A.2}$$

it follows that

$$\lim_{n \to \infty} \frac{L(n)\Gamma(\theta+1)}{n^{\theta}} \sum_{k=0}^{n} a_k = M. \tag{A.3}$$

We need to justify this in a probabilistic situation.

THEOREM A.2. *Fix $\theta > 0$ and a slowly varying function L, and suppose for simplicity that $\{a_k\}_{k=0}^{\infty}$ are a family of nonnegative random variables such that the series $\sum_{k=0}^{\infty} a_k z^k$ converges almost surely for $|z| < 1$. Suppose further that we have the convergence in probability*

$$L\left(\frac{1}{1-z}\right)(1-z)^{\theta} \sum_{k=0}^{\infty} a_k z^k \xrightarrow[z \to 1_-]{p} M \tag{A.4}$$

with $M < \infty$ almost surely. Then it follows that

$$\frac{L(n)\Gamma(\theta+1)}{n^{\theta}} \sum_{k=0}^{n} a_k \xrightarrow[n\to\infty]{p} M. \tag{A.5}$$

PROOF. Let us define

$$X_n = L\left(\frac{1}{1-e^{-1/n}}\right)\left(1-e^{-1/n}\right)^{\theta} \sum_{k=0}^{\infty} a_k g\left(e^{-k/n}\right), \tag{A.6}$$

where

$$g(x) = \begin{cases} 0 & 0 \le x < 1/e, \\ 1 & 1/e \le x \le 1. \end{cases} \tag{A.7}$$

If we can show that $X_n \xrightarrow{p}_{n\to\infty} M/\Gamma(\theta+1)$, we will have proved the result, because $X_n \Gamma(\theta+1)$ is equivalent to the left-hand side of (A.5) when $n \to \infty$. To do this, we construct polynomial approximations of g and bound from above and below. It is important to note that the construction of the polynomials is deterministic and does not depend on the random coefficients a_k.

For any polynomial P without constant term, it is a direct computation using (A.4), using additivity of convergence in probability and using the definition of slow variation, that

$$L\left(\frac{1}{1-e^{-1/n}}\right)\left(1-e^{-1/n}\right)^{\theta} \sum_{k=0}^{\infty} a_k P\left(e^{-k/n}\right) \xrightarrow[n\to\infty]{p} \frac{M}{\Gamma(\theta)} \int_0^{\infty} t^{\theta-1} P\left(e^{-t}\right) dt. \tag{A.8}$$

We claim that, for any $\epsilon > 0$, we can find polynomials $P_{\epsilon}^{\pm}(x)$ on $[0, 1]$ without constant term such that $g(x) \le P_{\epsilon}^{+}(x)$ (and $P_{\epsilon}^{-}(x) \le g(x)$) and that $P_{\epsilon}^{+}(x) - g(x)$ (and $g(x) - P_{\epsilon}^{-}(x)$) are both small. Since the coefficients a_k are nonnegative, this allows us to sandwich X_n via

$$X_{n,\epsilon}^{-} \le X_n \le X_{n,\epsilon}^{+}, \tag{A.9}$$

where

$$X_{n,\epsilon}^{\pm} := L\left(\frac{1}{1-e^{-1/n}}\right)\left(1-e^{-1/n}\right)^{\theta} \sum_{k=0}^{\infty} a_k P_{\epsilon}^{\pm}\left(e^{-k/n}\right). \tag{A.10}$$

Now, by (A.8) we have $X_{n,\epsilon}^{\pm} \xrightarrow{p} X_{\epsilon}^{\pm}$, where

$$\begin{aligned} X_{\epsilon}^{\pm} &= \frac{M}{\Gamma(\theta)}\left(\int_0^{\infty} t^{\theta-1} g\left(e^{-t}\right) dt \pm c^{\pm}(\epsilon)\right) \\ &= \frac{M}{\Gamma(\theta+1)}\left(1 \pm \theta c^{\pm}(\epsilon)\right) \end{aligned} \tag{A.11}$$

and

$$c^{\pm}(\epsilon) = \int_0^\infty t^{\theta-1} \left| P_\epsilon^{\pm}(e^{-t}) - g(e^{-t}) \right| dt. \tag{A.12}$$

We claim that $P_\epsilon^{\pm}(x)$ can be chosen so that $0 \le c^{\pm}(\epsilon) \le c_\theta \epsilon$ for some absolute constant c_θ and so $\lim_{\epsilon \to 0} c^{\pm}(\epsilon) = 0$. This is enough to show convergence in probability of X_n to the limit $X = M/\Gamma(\theta+1)$, as we now demonstrate. By (A.9) we have, for any $\delta > 0$,

$$\begin{aligned}
\mathbb{P}(|X_n - X| > \delta) &\le \mathbb{P}(|X_{n,\epsilon}^{+} - X| > \delta) + \mathbb{P}(|X_{n,\epsilon}^{-} - X| > \delta) \\
&\le \mathbb{P}(|X_{n,\epsilon}^{+} - X_\epsilon^{+}| + |X_\epsilon^{+} - X| > \delta) \\
&\quad + \mathbb{P}(|X_{n,\epsilon}^{-} - X_\epsilon^{-}| + |X_\epsilon^{-} - X| > \delta) \\
&\le \mathbb{P}(|X_{n,\epsilon}^{+} - X_\epsilon^{+}| > \delta/2) + \mathbb{P}(|X_\epsilon^{+} - X| > \delta/2) \\
&\quad + \mathbb{P}(|X_{n,\epsilon}^{-} - X_\epsilon^{-}| > \delta/2) + \mathbb{P}(|X_\epsilon^{-} - X| > \delta/2).
\end{aligned} \tag{A.13}$$

Since $X_{n,\epsilon}^{\pm} \xrightarrow{p}_{n\to\infty} X_\epsilon^{\pm}$, two of the terms in the final inequality above vanish in the limit $n \to \infty$ for any fixed ϵ. For the remaining terms, note that by definition

$$|X_\epsilon^{\pm} - X| = c^{\pm}(\epsilon) M/\Gamma(\theta) \tag{A.14}$$

so that in the limit $\epsilon \to 0$ we have, using the almost sure finiteness of M,

$$\mathbb{P}(|X_\epsilon^{\pm} - X| > \delta/2) = \mathbb{P}(M > \delta\Gamma(\theta)/(2c^{\pm}(\epsilon)) \to 0. \tag{A.15}$$

The existence of the approximating polynomials is classical, but for completeness we give the argument. We begin by sandwiching $g(x)$ by the continuous function $h_\epsilon^{\pm}(x)$ where $h_\epsilon^{\pm}(x)$ is equal to $g(x)$ outside $[1/e \mp \epsilon, 1/e]$ and is linear inside the interval. Then by the Weierstrass approximation theorem, we get polynomials $P_\epsilon^{\pm}(x)/x$ such that

$$\left| \frac{h_\epsilon^{\pm}(x)}{x} \pm \epsilon - \frac{P_\epsilon^{\pm}(x)}{x} \right| \le \epsilon, \quad 0 \le x \le 1. \tag{A.16}$$

Then $P_\epsilon^{+}(x) \ge h^{+}(x) \ge g(x)$ and $P_\epsilon^{-}(x) \le h^{-}(x) \le g(x)$. Now, we can bound $c^{\pm}(\epsilon)$ using

$$c^{\pm}(\epsilon) \le \int_0^\infty t^{\theta-1} \left| P_\epsilon^{\pm}(e^{-t}) - h^{\pm}(e^{-t}) \right| dt + \int_0^\infty t^{\theta-1} \left| h^{\pm}(e^{-t}) - g(e^{-t}) \right| dt. \tag{A.17}$$

The substitution $x = e^{-t}$ shows that

$$\begin{aligned}
c^{\pm}(\epsilon) &\le \int_0^1 (-\log(x))^{\theta-1} \frac{|P_\epsilon^{\pm}(x) - h^{\pm}(x)|}{x} dx \\
&\quad + \left| \int_{1/e \pm \epsilon}^{1/e} (-\log(x))^{\theta-1} \frac{|h^{\pm}(x) - g(x)|}{x} dx \right|.
\end{aligned} \tag{A.18}$$

The integrand in the second term of (A.18) is uniformly bounded, so the integral is bounded by a constant times the length of the integration interval which is ϵ. By (A.16) the first term in (A.18) is bounded by

$$2\epsilon \int_0^1 (-\log(x))^{\theta-1} dx = 2\epsilon\Gamma(\theta), \tag{A.19}$$

and so $|c^{\pm}(\epsilon)| \le c_\theta \epsilon$, as desired. $\square$

LEMMA A.3. *Let $Z_1 = X_1 + iY_1$ and $Z_2 = X_2 + iY_2$ be complex valued random variables with characteristic functions*

$$\begin{aligned}\Phi_{Z_1}(s,t) &= \mathbb{E}(e^{isX_1+itY_1}),\\ \Phi_{Z_2}(s,t) &= \mathbb{E}(e^{isX_2+itY_2}).\end{aligned} \tag{A.20}$$

Then

$$\left|\Phi_{Z_1}(s,t) - \Phi_{Z_2}(s,t)\right| \le \left(|s|+|t|\right)\sqrt{\mathbb{E}\left(|Z_1-Z_2|^2\right)}. \tag{A.21}$$

PROOF. This follows from the standard inequality $|e^{ix}-1| \le |x|$ valid for any real number x. We have

$$\begin{aligned}\left|\Phi_{Z_1}(s,t) - \Phi_{Z_2}(s,t)\right| &\le \mathbb{E}\left(\left|e^{isX_1+itY_1} - e^{isX_2+iY_2}\right|\right)\\ &= \mathbb{E}\left(\left|e^{is(X_1-X_2)+it(Y_1-Y_2)} - 1\right|\right)\\ &\le \mathbb{E}\left(\left|s(X_1-X_2)+t(Y_1-Y_2)\right|\right)\\ &\le |s|\mathbb{E}\left(|X_1-X_2|\right) + |t|\mathbb{E}\left(|Y_1-Y_2|\right)\\ &\le |s|\sqrt{\mathbb{E}\left(|X_1-X_2|^2\right)} + |t|\sqrt{\mathbb{E}\left(|Y_1-Y_2|^2\right)}\\ &\le \left(|s|+|t|\right)\sqrt{\mathbb{E}\left(|Z_1-Z_2|^2\right)}.\end{aligned}$$

□

APPENDIX B: THE CONVERGENCE RESULT OF JUNNILA AND SAKSMAN

The goal of this Appendix is to show that the convergence results (1.24) and (1.26) are simple consequences of those obtained in [34].

COROLLARY B.1. *Let $0 < \theta \le 1$. The following limit exists in L^q for any $0 < q < 1$,*

$$\mathrm{GMC}_\theta(d\vartheta) := \lim_{r\to 1} L_\theta(r)\left(1-r^2\right)^\theta e^{\sqrt{\theta}G(re^{i\vartheta})}\,d\vartheta, \tag{B.1}$$

where

$$L_\theta(r) = \begin{cases} 1 & 0 < \theta < 1,\\ \sqrt{\log\left(\frac{1}{1-r^2}\right)} & \theta = 1.\end{cases} \tag{B.2}$$

PROOF. The proof of this convergence was established in [34] but for a slightly modified field containing an additional constant Gaussian, say $G_0(e^{i\vartheta}) = A_0 + G(e^{i\vartheta})$ where A_0 is a centered real-valued Gaussian with fixed variance σ^2 independent of the rest of the field $G(e^{i\vartheta})$. Now, consider approximating measures

$$\mu_0^{(r)}(d\vartheta) = L_\theta(r)e^{\sqrt{\theta}G_0(re^{i\vartheta})-\frac{\theta}{2}\mathrm{Var}(G_0(re^{i\vartheta}))}(d\vartheta), \tag{B.3}$$

and similarly, construct $\mu^{(r)}(d\vartheta)$ from $G(re^{i\vartheta})$. Then by construction and using (1.10), the two measures differ by a random multiplicative constant independent of the regularization,

$$\mu^{(r)}(d\vartheta) = K\mu_0^{(r)}(d\vartheta), \tag{B.4}$$

where $K = e^{-(\sqrt{\theta}A_0-\theta\frac{\sigma^2}{2})}$. By [34] we have that $\mu_0^{(r)}(d\vartheta)$ converges to a limit $\mu_0(d\vartheta)$ in L^s for any $0 < s < 1$.

Fix any $0 < \delta < 1$. Then by Hölder's inequality,

$$\mathbb{E}(|\mu^{(r)}(d\vartheta) - K\mu_0(d\vartheta)|^{\delta}) \tag{B.5}$$

$$= \mathbb{E}((K|\mu_0^{(r)}(d\vartheta) - \mu_0(d\vartheta)|)^{\delta}) \tag{B.6}$$

$$\leq \mathbb{E}(K^{\delta p})^{\frac{1}{p}} \mathbb{E}(|\mu_0^{(r)}(d\vartheta) - \mu_0(d\vartheta)|^{\delta q})^{\frac{1}{q}}, \tag{B.7}$$

where the exponents p and q are chosen so that $p > \frac{1}{1-\delta}$, since K has all moments, and $1 < q = \frac{1}{1-\frac{1}{p}} < \frac{1}{\delta}$, in particular, $\delta q < 1$ and so letting $r \to 1$,

$$\mathbb{E}(|\mu_0^{(r)}(d\vartheta) - \mu_0(d\vartheta)|^{\delta q})^{\frac{1}{q}} \to 0. \tag{B.8}$$

This means that our measure $\mu^{(r)}(d\vartheta)$ converges as $r \to 1$ to $\mathrm{GMC}_\theta(d\vartheta) = K\mu_0(d\vartheta)$ in L^δ for any $0 < \delta < 1$. $\square$

APPENDIX C: BINOMIAL SUMS

LEMMA C.1. *For any $0 < \theta < \frac{1}{2}$, we have*

$$\sum_{k=0}^{\infty} \binom{k+\theta-1}{\theta-1}^2 = \frac{\Gamma(1-2\theta)}{\Gamma(1-\theta)^2}. \tag{C.1}$$

PROOF. We have the generating function

$$\sum_{k=0}^{\infty} \binom{k+\theta-1}{\theta-1} z^k = (1-z)^{-\theta}, \qquad |z| < 1 \tag{C.2}$$

so that by Parseval's theorem

$$\sum_{k=0}^{\infty} \binom{k+\theta-1}{\theta-1}^2 = \frac{1}{2\pi} \int_0^{2\pi} |1 - e^{i\nu_1}|^{-2\theta} \, d\nu_1. \tag{C.3}$$

Now, the right-hand side of (C.3) coincides with that of (C.1) by taking $k = 2$ in the Morris integral (1.23) and using rotational invariance in the ν_2 coordinate. $\square$

LEMMA C.2. *Let a and b be nonnegative integers such that $b \geq a$, c and d complex numbers such that*

$$d \notin \mathbb{C} \setminus \{-1, -2, -3, \dots\} \quad \textit{and} \quad a + c - d \notin \mathbb{C} \setminus \{-1, -2, -3, \dots\}.$$

Then

$$\sum_{q=a}^{b} \binom{q+c}{d} = \binom{b+c+1}{d+1} - \mathbf{1}\{a+c-d \neq 0\} \binom{a+c}{d+1}. \tag{C.4}$$

PROOF. This follows from summing the binomial coefficient identity

$$\binom{q+c}{d} = \binom{q+c+1}{d+1} - \binom{q+c}{d+1}, \tag{C.5}$$

from $q = a$ to $q = b$, so that the sum telescopes and gives the right-hand side of (C.4).

We justify (C.5), as we consider nonintegral parameters. Recall that the Gamma function $\Gamma(x)$ has poles in the complex plane at $\{0, -1, -2, -3, \dots\}$. The identity (C.5) holds, provided none of the Gamma functions (c.f. (1.18)) in the expansions

$$\binom{q+c}{d} = \frac{\Gamma(q+c+1)}{\Gamma(d+1)\Gamma(q+c-d+1)} \quad \text{and} \quad \binom{q+c}{d+1} = \frac{\Gamma(q+c+1)}{\Gamma(d+2)\Gamma(q+c-d)}$$

have a pole. Note that, as q ranges over $\{a, a+1, \dots, b\}$, the conditions in statement of the lemma are necessary and sufficient.

Finally, if $a+c=d$, then the $q=a$ term in the sum is exceptional in that

$$\binom{a+c}{d} = 1 = \binom{a+c+1}{d+1}.$$

Thus, we use this in place of (C.5) so that the lower boundary term from the telescoping series vanishes. □

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REFERENCES

[1] ARGUIN, L.-P., BELIUS, D. and BOURGADE, P. (2017). Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.* **349** 703–751. MR3594368 https://doi.org/10.1007/s00220-016-2740-6

[2] ARGUIN, L.-P., BELIUS, D., BOURGADE, P., RADZIWIŁŁ, M. and SOUNDARARAJAN, K. (2019). Maximum of the Riemann zeta function on a short interval of the critical line. *Comm. Pure Appl. Math.* **72** 500–535. MR3911893 https://doi.org/10.1002/cpa.21791

[3] ARGUIN, L.-P., BOURGADE, P. and RADZIWIŁŁ, M. (2020). The Fyodorov–Hiary–Keating conjecture. I. Available at arXiv:2007.00988.

[4] ARRATIA, R., BARBOUR, A. D. and TAVARÉ, S. (1992). Poisson process approximations for the Ewens sampling formula. *Ann. Appl. Probab.* **2** 519–535. MR1177897

[5] ARRATIA, R., BARBOUR, A. D. and TAVARÉ, S. (2003). *Logarithmic Combinatorial Structures: A Probabilistic Approach. EMS Monographs in Mathematics*. European Mathematical Society (EMS), Zürich. MR2032426 https://doi.org/10.4171/000

[6] ARU, J., JEGO, A. and JUNNILA, J. (2022). Density of imaginary multiplicative chaos via Malliavin calculus. *Probab. Theory Related Fields* **184** 749–803. MR4507934 https://doi.org/10.1007/s00440-022-01135-y

[7] BARRAL, J. and JIN, X. (2010). Multifractal analysis of complex random cascades. *Comm. Math. Phys.* **297** 129–168. MR2645749 https://doi.org/10.1007/s00220-010-1030-y

[8] BARRAL, J., JIN, X. and MANDELBROT, B. (2010). Convergence of complex multiplicative cascades. *Ann. Appl. Probab.* **20** 1219–1252. MR2676938 https://doi.org/10.1214/09-AAP665

[9] BARRAL, J., JIN, X. and MANDELBROT, B. (2010). Uniform convergence for complex [0, 1]-martingales. *Ann. Appl. Probab.* **20** 1205–1218. MR2676937 https://doi.org/10.1214/09-AAP664

[10] BECK, M. and PIXTON, D. (2003). The Ehrhart polynomial of the Birkhoff polytope. *Discrete Comput. Geom.* **30** 623–637. MR2013976 https://doi.org/10.1007/s00454-003-2850-8

[11] BERESTYCKI, N., WEBB, C. and WONG, M. D. (2018). Random Hermitian matrices and Gaussian multiplicative chaos. *Probab. Theory Related Fields* **172** 103–189. MR3851831 https://doi.org/10.1007/s00440-017-0806-9

[12] CANFIELD, E. R. and MCKAY, B. D. (2009). The asymptotic volume of the Birkhoff polytope. *Online J. Anal. Comb.* **4** 4. MR2575172

[13] CARPENTIER, D. and LE DOUSSAL, P. (2001). Glass transition of a particle in a random potential, front selection in nonlinear renormalization group, and entropic phenomena in Liouville and sinh-Gordon models. *Phys. Rev. E* **63**.

[14] CHHAIBI, R., MADAULE, T. and NAJNUDEL, J. (2018). On the maximum of the $C\beta E$ field. *Duke Math. J.* **167** 2243–2345. MR3848391 https://doi.org/10.1215/00127094-2018-0016

[15] CHHAIBI, R. and NAJNUDEL, J. (2019). On the circle, $GMC^{\gamma} = \varprojlim C\beta E_n$ for $\gamma = \sqrt{\frac{2}{\beta}}$, $(\gamma \leq 1)$. Available at arXiv:1904.00578.

[16] CLAEYS, T., FAHS, B., LAMBERT, G. and WEBB, C. (2021). How much can the eigenvalues of a random Hermitian matrix fluctuate? *Duke Math. J.* **170** 2085–2235. MR4278668 https://doi.org/10.1215/00127094-2020-0070

[17] CONREY, B. and GAMBURD, A. (2006). Pseudomoments of the Riemann zeta-function and pseudomagic squares. *J. Number Theory* **117** 263–278. MR2213764 https://doi.org/10.1016/j.jnt.2005.01.006

[18] DE LOERA, J. A., LIU, F. and YOSHIDA, R. (2009). A generating function for all semi-magic squares and the volume of the Birkhoff polytope. *J. Algebraic Combin.* **30** 113–139. MR2519852 https://doi.org/10.1007/s10801-008-0155-y

[19] DIACONIS, P. and GAMBURD, A. (2004/06). Random matrices, magic squares and matching polynomials. *Electron. J. Combin.* **11** 2. MR2120097

[20] DIACONIS, P. and SHAHSHAHANI, M. (1994). On the eigenvalues of random matrices. Studies in applied probability. *J. Appl. Probab.* **31A** 49–62. MR1274717 https://doi.org/10.2307/3214948

[21] DING, J., ROY, R. and ZEITOUNI, O. (2017). Convergence of the centered maximum of log-correlated Gaussian fields. *Ann. Probab.* **45** 3886–3928. MR3729618 https://doi.org/10.1214/16-AOP1152

[22] DUPLANTIER, B., RHODES, R., SHEFFIELD, S. and VARGAS, V. (2017). Log-correlated Gaussian fields: An overview. In *Geometry, Analysis and Probability*. *Progr. Math.* **310** 191–216. Birkhäuser/Springer, Cham. MR3821928 https://doi.org/10.1007/978-3-319-49638-2_9

[23] FYODOROV, Y. V. and BOUCHAUD, J.-P. (2008). Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential. *J. Phys. A* **41** 372001. MR2430565 https://doi.org/10.1088/1751-8113/41/37/372001

[24] FYODOROV, Y. V., HIARY, G. A. and KEATING, J. P. (2012). Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function. *Phys. Rev. Lett.* **108** 170601. https://doi.org/10.1103/PhysRevLett.108.170601

[25] FYODOROV, Y. V. and KEATING, J. P. (2014). Freezing transitions and extreme values: Random matrix theory, and disordered landscapes. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372** 20120503. MR3151088 https://doi.org/10.1098/rsta.2012.0503

[26] FYODOROV, Y. V., KHORUZHENKO, B. A. and SIMM, N. J. (2016). Fractional Brownian motion with Hurst index $H = 0$ and the Gaussian unitary ensemble. *Ann. Probab.* **44** 2980–3031. MR3531684 https://doi.org/10.1214/15-AOP1039

[27] HAAKE, F., KUŚ, M., SOMMERS, H.-J., SCHOMERUS, H. and ŻYCZKOWSKI, K. (1996). Secular determinants of random unitary matrices. *J. Phys. A* **29** 3641–3658. MR1400168 https://doi.org/10.1088/0305-4470/29/13/029

[28] HALL, P. and HEYDE, C. C. (1980). *Martingale Limit Theory and Its Application*. *Probability and Mathematical Statistics*. Academic Press, New York. MR0624435

[29] HARPER, A. J. (2020). Moments of random multiplicative functions, I: Low moments, better than squareroot cancellation, and critical multiplicative chaos. *Forum Math. Pi* **8** e1. MR4061962 https://doi.org/10.1017/fmp.2019.7

[30] HARTUNG, L. and KLIMOVSKY, A. (2015). The glassy phase of the complex branching Brownian motion energy model. *Electron. Commun. Probab.* **20** 78. MR3417450 https://doi.org/10.1214/ECP.v20-4360

[31] HARTUNG, L. and KLIMOVSKY, A. (2018). The phase diagram of the complex branching Brownian motion energy model. *Electron. J. Probab.* **23** 127. MR3896864 https://doi.org/10.1214/18-EJP245

[32] HUGHES, C. P., KEATING, J. P. and O'CONNELL, N. (2001). On the characteristic polynomial of a random unitary matrix. *Comm. Math. Phys.* **220** 429–451. MR1844632 https://doi.org/10.1007/s002200100453

[33] JIANG, T. and MATSUMOTO, S. (2015). Moments of traces of circular beta-ensembles. *Ann. Probab.* **43** 3279–3336. MR3433582 https://doi.org/10.1214/14-AOP960

[34] JUNNILA, J. and SAKSMAN, E. (2017). Uniqueness of critical Gaussian chaos. *Electron. J. Probab.* **22** 11. MR3613704 https://doi.org/10.1214/17-EJP28

[35] JUNNILA, J., SAKSMAN, E. and VIITASAARI, L. (2019). On the regularity of complex multiplicative chaos. Available at arXiv:1905.12027.

[36] JUNNILA, J., SAKSMAN, E. and WEBB, C. (2020). Imaginary multiplicative chaos: Moments, regularity and connections to the Ising model. *Ann. Appl. Probab.* **30** 2099–2164. MR4149524 https://doi.org/10.1214/19-AAP1553

[37] KEATING, J. P., RODGERS, B., RODITTY-GERSHON, E. and RUDNICK, Z. (2018). Sums of divisor functions in $\mathbb{F}_q[t]$ and matrix integrals. *Math. Z.* **288** 167–198. MR3774409 https://doi.org/10.1007/s00209-017-1884-1

[38] KILLIP, R. and NENCIU, I. (2004). Matrix models for circular ensembles. *Int. Math. Res. Not.* **50** 2665–2701. MR2127367 https://doi.org/10.1155/S1073792804141597

[39] KOREVAAR, J. (2004). *Tauberian Theory*: *A Century of Developments*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **329**. Springer, Berlin. MR2073637 https://doi.org/10.1007/978-3-662-10225-1

[40] LACOIN, H. (2022). A universality result for subcritical complex Gaussian multiplicative chaos. *Ann*. *Appl*. *Probab*. **32** 269–293. MR4386527 https://doi.org/10.1214/21-aap1677

[41] LACOIN, H., RHODES, R. and VARGAS, V. (2015). Complex Gaussian multiplicative chaos. *Comm*. *Math*. *Phys*. **337** 569–632. MR3339158 https://doi.org/10.1007/s00220-015-2362-4

[42] LAMBERT, G. (2021). Mesoscopic central limit theorem for the circular β-ensembles and applications. *Electron*. *J*. *Probab*. **26** 7. MR4216520 https://doi.org/10.1214/20-ejp559

[43] LAMBERT, G., OSTROVSKY, D. and SIMM, N. (2018). Subcritical multiplicative chaos for regularized counting statistics from random matrix theory. *Comm*. *Math*. *Phys*. **360** 1–54. MR3795187 https://doi.org/10.1007/s00220-018-3130-z

[44] LEDOUX, M. (1999). Concentration of measure and logarithmic Sobolev inequalities. In *Séminaire de Probabilités*, *XXXIII*. *Lecture Notes in Math*. **1709** 120–216. Springer, Berlin. MR1767995 https://doi.org/10.1007/BFb0096511

[45] MACDONALD, I. G. (2015). *Symmetric Functions and Hall Polynomials*, 2nd ed. *Oxford Classic Texts in the Physical Sciences*. The Clarendon Press, Oxford University Press, New York. With contribution by A. V. Zelevinsky and a foreword by Richard Stanley, Reprint of the 2008 paperback edition [MR1354144]. MR3443860

[46] MADAULE, T., RHODES, R. and VARGAS, V. (2016). Glassy phase and freezing of log-correlated Gaussian potentials. *Ann*. *Appl*. *Probab*. **26** 643–690. MR3476621 https://doi.org/10.1214/14-AAP1071

[47] MEZZADRI, F., REYNOLDS, A. K. and WINN, B. (2017). Moments of the eigenvalue densities and of the secular coefficients of β-ensembles. *Nonlinearity* **30** 1034–1057. MR3622276 https://doi.org/10.1088/1361-6544/aa518c

[48] NAJNUDEL, J. (2018). On the extreme values of the Riemann zeta function on random intervals of the critical line. *Probab*. *Theory Related Fields* **172** 387–452. MR3851835 https://doi.org/10.1007/s00440-017-0812-y

[49] NIKULA, M., SAKSMAN, E. and WEBB, C. (2020). Multiplicative chaos and the characteristic polynomial of the CUE: The L^1-phase. *Trans*. *Amer*. *Math*. *Soc*. **373** 3905–3965. MR4105514 https://doi.org/10.1090/tran/8020

[50] PAQUETTE, E. and ZEITOUNI, O. (2018). The maximum of the CUE field. *Int*. *Math*. *Res*. *Not*. *IMRN* **16** 5028–5119. MR3848227 https://doi.org/10.1093/imrn/rnx033

[51] REMY, G. (2020). The Fyodorov-Bouchaud formula and Liouville conformal field theory. *Duke Math*. *J*. **169** 177–211. MR4047550 https://doi.org/10.1215/00127094-2019-0045

[52] RHODES, R. and VARGAS, V. (2014). Gaussian multiplicative chaos and applications: A review. *Probab*. *Surv*. **11** 315–392. MR3274356 https://doi.org/10.1214/13-PS218

[53] SAKSMAN, E. and WEBB, C. (2020). The Riemann zeta function and Gaussian multiplicative chaos: Statistics on the critical line. *Ann*. *Probab*. **48** 2680–2754. MR4164452 https://doi.org/10.1214/20-AOP1433

[54] SIMON, B. (2005). *Orthogonal Polynomials on the Unit Circle*. *Part* 1. *American Mathematical Society Colloquium Publications* **54**. Amer. Math. Soc., Providence, RI. Classical theory. MR2105088 https://doi.org/10.1090/coll054.1

[55] STANLEY, R. P. (1989). Some combinatorial properties of Jack symmetric functions. *Adv*. *Math*. **77** 76–115. MR1014073 https://doi.org/10.1016/0001-8708(89)90015-7

[56] STANLEY, R. P. (1999). *Enumerative Combinatorics*. *Vol*. 2. *Cambridge Studies in Advanced Mathematics* **62**. Cambridge Univ. Press, Cambridge. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR1676282 https://doi.org/10.1017/CBO9780511609589

[57] VALKÓ, B. and VIRÁG, B. (2022). The many faces of the stochastic zeta function. *Geom*. *Funct*. *Anal*. **32** 1160–1231. MR4498842 https://doi.org/10.1007/s00039-022-00613-8

[58] WEBB, C. (2015). The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos—the L^2-phase. *Electron*. *J*. *Probab*. **20** 104. MR3407221 https://doi.org/10.1214/EJP.v20-4296

[59] WU, L. (2000). A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. *Probab*. *Theory Related Fields* **118** 427–438. MR1800540 https://doi.org/10.1007/PL00008749