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THE EXACT HAUSDORFF MEASURE OF THE ZERO SET OF CERTAIN STATIONARY GAUSSIAN PROCESSES

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It is shown that the exact measure function $\Psi(h)$ of a stationary Gaussian process with spectral density function $f(\lambda)$ proportional to $(\lambda^2 + a^2)^{-(\alpha+\frac{1}{2})}$, $0 < \alpha < \frac{1}{2}$, is given by $\Psi(h) = h^{1-\alpha}(\log|\log h|)^\alpha$.

1. Introduction and result.

1.1. Let $X(t, \omega)$ be a separable stationary Gaussian process defined on the probability space $(\Omega, \mathscr{F}, P)$. We assume that $X(t)$ has zero mean and spectral density function $f(\lambda)$ given by

$$f(\lambda) = a^{2\alpha} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\alpha)} (\lambda^2 + a^2)^{-(\alpha+\frac{1}{2})}, \qquad -\infty < \lambda < \infty, \tag{1}$$

where $a > 0$ and α, $0 < \alpha < \frac{1}{2}$, are constants. The normalizing factor in (1) ensures $E(X(t)^2) = 1$.

The standard deviation $\sigma(h)$ of the increment $X(t+h) - X(t)$ is given by

$$\sigma^2(h) = E((X(t+h) - X(t))^2) = 4 \int_{-\infty}^{\infty} \sin^2(\tfrac{1}{2}\lambda h) f(\lambda)\, d\lambda . \tag{2}$$

It follows from results of Berman [1, 2, 3, 4, 5], valid for a wider class of processes, that there exists a stochastic process $\varphi(x, t, \omega)$, the local time of $X(t)$, which is jointly continuous in x and t and for which

$$\int_0^t \chi_B(X(s))\, ds = \int_{-\infty}^{\infty} \chi_B(x)\varphi(x, t, \omega)\, dx \tag{3}$$

holds for every Borel set B and every t. χ_B denotes the indicator function of the set B.

1.2. In [6] the following iterated logarithm law for $\varphi(x, t)$ was proved:

$$0 < c_1 \leqq \lim_{h\downarrow 0} \sup \frac{\varphi(X(t), t+h) - \varphi(X(t), t)}{h^{1-\alpha}(\log(-\log h))^\alpha} \leqq c_2 < \infty \tag{4}$$

almost surely where c_1 and c_2 are fixed constants depending only on a and α. It was further shown in [6] that (4) holds for certain stopping times τ. This allowed a direct application of the method of Taylor and Wendel [16] to obtain a lower bound for the Hausdorff measure of the zero set of $X(t)$. To be more precise let

$$Q(t) = \{s : X(s, \omega) = 0, 0 \leqq s \leqq t\}, \tag{5}$$

$$\tau_t = \inf\{s : \varphi(0, s) = t, s \geqq 0\} \tag{6}$$

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and let $\Psi - m(E)$ denote the Hausdorff Ψ-measure of the set E. Theorem 3 of [6] states that with $\Psi(h) = h^{1-\alpha}(\log(-\log h))^{\alpha}$ and $t > 0$, $\Psi - m(Q(\tau_t))$ is almost surely strictly positive, perhaps infinite. The conjecture was made that $\Psi(h) = h^{1-\alpha}(\log(-\log h))^{\alpha}$ is the exact measure function for $Q(\tau_t)$ and it is the purpose of the present paper to prove this conjecture. We prove

THEOREM 1. *For the process $X(t)$ described above*

$$0 < \Psi - m(Q(\tau_t)) < \infty \tag{7}$$

for all $t > 0$ almost surely where $\Psi(h) = h^{1-\alpha}(\log(-\log h))^{\alpha}$.

1.3. The result of Taylor and Wendel goes further than the statement of Theorem 1. They showed that the stochastic process $Z(t) = \Psi - m(Q(t))$ is a constant multiple of the local time $\varphi(0, t)$. Their proof of this cannot be immediately carried over to the present situation and we make no attempt to identify the process $Z(t)$.

2. Notation and outline of proof.

2.1. As far as possible we will use the notation of [6]; any divergences will be explicitly mentioned. Constants, either absolute constants or ones whose value depends only on a and α will be denoted by $c_1, \cdots, c_{53}$. They will not in general be the same as the constants $c_1, \cdots, c_{19}$ of [6].

At this point we note the asymptotic equality

$$\sigma^2(h) \simeq \frac{2\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}(\tfrac{1}{2}a|h|)^{2\alpha} \tag{8}$$

which holds for small h. This follows from (1) and (2) by a theorem of Pitman [14].

2.2. The method of proof is the same as that of Taylor and Wendel, the problem being to obtain sufficiently accurate bounds for the relevant probabilities for the process $X(t)$. As the technical details tend to obscure the main idea we now give a brief outline of the proof.

We denote the collection of intervals $[(j-1)2^{-\nu}, (j+1)2^{-\nu})$, $j = 0, \pm 1, \pm 2, \cdots$ by Λ_ν, $\nu = 0, 1, 2, \cdots$. For a given $\varepsilon_1 > 0$ and a given integer $n_0 \geqq 1$ we choose an integer m_0 such that $2^{-m_0} < \min(\frac{1}{2}\varepsilon_1, h_{n_0})$ where

$$h_n = \exp(-n^{1+\varepsilon_0}), \qquad n = 1, 2, \cdots \tag{9}$$

as in [6].

Given $\delta_1 > 0$ (to be specified later) and $m \geqq m_0$, let N_m be the largest integer n such that

$$\tfrac{1}{2}\delta_1 h_{n+1}^{1-\alpha} \geqq 2^{-\alpha(1-\alpha)m/4}. \tag{10}$$

We assume that m is sufficiently large to ensure that $\frac{1}{2}N_m \geqq n_0$.

For fixed m the interval $[(j-1)2^{-m}, j2^{-m})$ will be denoted by $I_{j,m}$. An interval $I_{j,m}$ will be called bad for the sample point ω if (i) $X(t, \omega) = 0$ for some t

in $I_{j,m}$ and (ii) there is no interval $[a, b)$ of $\bigcup_{\nu=m_0}^{m} \Lambda_\nu$ containing $I_{j,m} \cap Q(1, \omega)$ for which

$$\varphi(0, b) - \varphi(0, a) \geqq \tfrac{1}{8}\delta_1(b - a)^{1-\alpha}(\log(-\log(b - a)))^\alpha . \tag{11}$$

All other intervals are said to be good.

If $I_{j,m}$ is good then either $I_{j,m} \cap Q(1, \omega)$ is empty or it can be covered by an interval $[a, b)$ of $\bigcup_{\nu=m_0}^{m} \Lambda_\nu$ satisfying (11). The covering of $Q(1, \omega)$ is then completed by taking $I_{j,m}$ itself to cover $I_{j,m} \cap Q(1, \omega)$ when $I_{j,m}$ is bad. Note that all the intervals of the covering have length less than ε_1.

Following Taylor and Wendel the problem is to show that the contribution to the covering from the bad intervals is small for suitable ε_0 in (9) and δ_1 in (10). To do this we proceed as follows. Using a refinement of the techniques of [6] we show that $P(B_{j,m}^{(n)} \mid \mathscr{F}_{(j-1)2^{-m}})$ is small for large n where

$$\begin{aligned} B_{j,m}^{(n)} = \{\omega : \varphi(X((j-1)2^{-m}), (j-1)2^{-m} + h_\nu) - \varphi(X((j-1)2^{-m}), \\ (j-1)2^{-m}) \leqq \delta_1 h_\nu^{1-\alpha}(\log(-\log h_\nu))^\alpha, n \leqq \nu \leqq 2n\} . \end{aligned} \tag{12}$$

It turns out that the conditional probability $P(B_{j,m}^{(n)} \mid \mathscr{F}_{(n-1)2^{-m}})$, depending as it does only on the increments of the process $X(t)$, is largely independent of $X((j-1)2^{-m})$. This is expressed by Lemma 3. The continuity of the paths of $X(t)$ implies that $X((j-1)2^{-m})$ and the event $I_{j,m} \cap Q(1, \omega) \neq \emptyset$ are highly mutually dependent. It follows that $P(B_{j,m}^{(n)} \mid \mathscr{F}_{(j-1)2^{-m}})$ is largely independent of whether the process $X(t)$ has a zero in $I_{j,m}$ or not. The continuity of the local time process $\varphi(x, t)$ allows the event

$$\begin{aligned} A_{j,m}^{(n)} = \{\omega : \varphi(X(\tau_j), \tau_j + h_\nu) - \varphi(X(\tau_j), \tau_j) \\ \leqq \tfrac{1}{2}\delta_1 h_\nu^{1-\alpha}(\log(-\log h_\nu))^\alpha, n \leqq \nu \leqq 2n\} \end{aligned} \tag{13}$$

to be closely approximated by the event $B_{j,m}^{(n)}$. Here τ_j denotes the time of the first zero of $X(t, \omega)$ in the interval $I_{j,m}$. Putting all this together we conclude that the conditional probability of $A_{j,m}^{(n)}$ given τ_j is small for large n.

To finish the proof we require an upper bound for the probability of the event $I_{j,m} \cap Q(1, \omega) \neq \emptyset$. This is given by Lemma 9.

3. Preliminary results.

3.1. Apart from Lemma 3, the lemmas in this section contain certain technical results whose proofs are not particularly interesting. The first three lemmas are general whilst the last three give a more precise lower bound for $P(C_{j,m}^\nu \mid \mathscr{F}_{(j-1)2^{-m}})$ than that given in [6], where

$$\begin{aligned} C_{j,m}^\nu = \{\omega : \varphi(X((j-1)2^{-m}), (j-1)2^{-m} + h_\nu) \\ - \varphi(X((j-1)2^{-m}), (j-1)2^{-m}) \\ \geqq \delta_1 h_\nu^{1-\alpha}(\log(-\log h_\nu))^\alpha\} . \end{aligned} \tag{14}$$

3.2. The following lemma is required at several stages of the proof of Theorem 1.

LEMMA 1. *Let $\xi(t, \omega)$ be a zero mean separable Gaussian process which satisfies*

$$\sup_{0 \leqq t \leqq h} E(\xi(t)^2) \leqq c_3 h^{2\alpha} \tag{15}$$

and

$$E((\xi(t) - \xi(s))^2) \leqq c_3 |t - s|^{2\alpha}, \qquad 0 \leqq s, t \leqq h. \tag{16}$$

Then there exist positive constants c_4 and c_5 such that for all h, $0 < h < e^{-2}$,

$$P\left(\sup_{0<t<h} \frac{|\xi(t)|}{t^\alpha(\log(-\log t))^{\frac{1}{2}}} \geqq \lambda\right) \leqq c_4 \exp(-c_5 \lambda^2 \log(-\log h)) \tag{17}$$

for all $\lambda > 0$.

PROOF. The proof is a straightforward application of Fernique's lemma (see [7] and [11]) and is consequently omitted.

LEMMA 2. *There exist positive constants c_6 and c_7 such that*

$$P\left(\sup_{0 \leqq t < \infty} \frac{|X(s-t)|}{(1 + \log^+ t)^{\frac{1}{2}}} \geqq \lambda\right) \leqq c_6 \exp(-c_7 \lambda^2) \tag{18}$$

for all $\lambda > 0$.

PROOF. The process $X(s - t)/(1 + \log^+ t)^{\frac{1}{2}}$, $0 \leqq t < \infty$, is bounded with probability one ([12], page 522). It therefore follows from a theorem of Marcus and Shepp and others ([13]) that

$$\lim_{\lambda \to \infty} \lambda^{-2} \log\left(P\left(\sup_{0 \leqq t < \infty} \frac{|X(s-t)|}{(1 + \log^+ t)^{\frac{1}{2}}} \geqq \lambda\right)\right) = -\tfrac{1}{2}\theta^{-2}$$

where $\theta^2 = \sup_{0 \leqq t < \infty} E(X(s-t)^2/(1 + \log^+ t)) \leqq 1$. The lemma follows from this and the stationarity of the process.

The next lemma expresses the fact mentioned in 2.2 that the increments $X(t) - X(s)$, $0 \leqq s \leqq t$, are largely independent of $X(0)$. Now and in future we shall denote the σ-algebra generated by the random variable $X(t)$ by $\mathscr{F}_{tt}$.

The definitions of $\mathscr{F}_t$, $X_p(t) = X_p(t \mid 0)$ and $X_e(t) = X_e(t \mid 0)$ which appear in the statement and proof of the next lemma are given in 2.1 of [6].

LEMMA 3. *For $t \geqq 0$ let the random variables $Y(t)$ and $Y_p(t)$ be given by*

$$Y(t) = X(-t) - (1 - \tfrac{1}{2}\sigma^2(t))X(0) \tag{19}$$

and

$$Y_p(t) = X_p(t) - (1 - \tfrac{1}{2}\sigma^2(t))X(0). \tag{20}$$

Then $Y(t)$ and $Y_p(t)$ are $\mathscr{F}_0$-measurable and independent of $\mathscr{F}_{00}$. Furthermore we have

$$E(Y(t)) = E(Y_p(t)) = 0, \tag{21}$$

$$Y(0) = Y_p(0) = 0 \tag{22}$$

and

$$E((Y_p(t) - Y_p(s))^2) \leqq E((Y(t) - Y(s))^2) \leqq c_8|t - s|^{2\alpha} \tag{23}$$

for $0 \leqq t,\ s \leqq 2$.

PROOF. The proof is a simple application of the fact that in the Gaussian case orthogonality implies independence. The last inequality of (23) follows from (8).

3.3. The next section is concerned with improving Lemma 12 of [6]. We require

LEMMA 4. *Suppose that* $0 < \delta < e^{-2}$. *Then there exists a constant* c_9 *and an* $n_1 = n_1(\varepsilon_0)$ *such that*

$$\sup_{1-\delta\leqq s\leqq 1}\left|\frac{d}{ds}X_p(sh_n \mid h_{n+1})\right| \leqq c_9 h_n{}^\alpha(\log(-\log h_n))^{\frac{1}{2}}(\Delta_{1,n} + \Delta_{2,n}) \tag{24}$$

where $\Delta_{1,n}$ *is an* $\mathscr{F}_0$*-measurable random variable which is independent of* $\mathscr{F}_{00}$.
Further, there exist constants c_{10} *and* $c_{11} > 0$ *such that for all* $\lambda > 0$

$$P(\Delta_{1,n} \geqq \lambda) \leqq c_{10} n^{-c_{11}\lambda^2} \tag{25}$$

and

$$P(\Delta_{2,n} \geqq \lambda) \leqq c_{10}\exp(-c_{11}n^4\lambda^2)\,. \tag{26}$$

PROOF. As in [6] we may write

$$X_p(sh_n \mid h_{n+1}) - X(0)$$
$$= c_{12}(sh_n - h_{n+1})^{\alpha+\frac{1}{2}} \int_0^\infty \frac{\exp(-a(sh_n - h_{n+1} + p))X(h_{n+1} - p) - X(0)}{p^{\alpha+\frac{1}{2}}(sh_n - h_{n+1} + p)}\,dp$$

and on differentiating we obtain after calculations similar to those in the proof of Lemma 8 of [6]

$$\left|\frac{d}{ds}X_p(sh_n \mid h_{n+1})\right| \leqq c_{14}h_n^{\alpha+\frac{1}{2}} \int_0^\infty \frac{|X(h_{n+1} - p) - X(0)|}{p^{\alpha+\frac{1}{2}}(sh_n + p)}\,dp + c_{14}h_n^{\alpha+\frac{1}{2}}\Delta_{3,n} \tag{27}$$

where

$$\Delta_{3,n} = \sup_{0\leqq p<\infty} \frac{|X(h_{n+1} - p)|}{(1 + \log^+ p)^{\frac{1}{2}}}\,. \tag{28}$$

Again following the proof of Lemma 8 of [6] we have

$$\int_0^\infty \frac{|X(h_{n+1} - p) - X(0)|}{p^{\alpha+\frac{1}{2}}(h_n + p)}\,dp \leqq c_{15}(h_n{}^{-\frac{1}{2}}(\log h_n))^{\frac{1}{2}}\Delta_{1,n}$$
$$+ h_n^{\alpha-\frac{1}{2}}|X(0)| + h_n{}^{-(\alpha+\frac{1}{2})/2}\Delta_{3,n}$$
$$+ h_{n+1}^{\frac{1}{2}}h_n{}^{-1}(\log(-\log h_n))^{\frac{1}{2}}\Delta_{4,n})$$

where

$$\Delta_{4,n} = \sup_{0\leqq t\leqq h_{n+1}} \frac{|X(t) - X(0)|}{t^\alpha(\log(-\log t))^{\frac{1}{2}}}$$

and

$$\Delta_{1,n} = \sup_{0 \le t \le h_n^{\frac{1}{2}}} \frac{|Y(t)|}{t^\alpha(\log(-\log t))^{\frac{1}{2}}} \tag{29}$$

with $Y(t)$ as defined in Lemma 3.

On substituting into (27) we obtain (24) with

$$\Delta_{2,n} = h_n^\alpha|X(0)| + h_n^{(\frac{1}{2}-\alpha)/2}\Delta_{3,n} + (h_{n+1}/h_n)^{\frac{1}{2}}(\log(-\log h_n))^{\frac{1}{2}}\Delta_{4,n}\,.$$

It follows from Lemma 2 that $\Delta_{1,n}$ is independent of $\mathscr{F}_{00}$ and (25) follows from (9), (23) and Lemma 1. The inequality (26) follows, on using crude inequalities, from the definition of the sequence $(h_n)_1^\infty$, the tail of the normal distribution, Lemma 2, (8) and Lemma 1.

LEMMA 5. *For all δ, $0 < \delta < e^{-2}$, there exists an $n_3 = n_3(\varepsilon_0, \delta)$ such that for all $n \geqq n_3$ and all s, $1 - \delta \leqq s \leqq 1$,*

$$|X_p(sh_n \mid h_{n+1}) - X(0) - Y_p(h_n)| \leqq c_{16}\delta h_n^\alpha(\log(-\log h_n))^{\frac{1}{2}}(\Delta_{1,n} + \Delta_{5,n}) \tag{30}$$

where $Y_p(t)$ is as in Lemma 3, $\Delta_{1,n}$ as in Lemma 4 and where $\Delta_{5,n}$ satisfies

$$P(\Delta_{5,n} \geqq \lambda) \leqq c_{17}\exp(-c_{18}n^4\lambda^2) \tag{31}$$

for all $\lambda > 0$.

PROOF. We have

$$\begin{aligned} &|X_p(sh_n \mid h_{n+1}) - X(0) - Y_p(h_n)| \\ &\quad \leqq c_9\delta h_n^\alpha(\log(-\log h_n^{\frac{1}{2}}))(\Delta_{1,n} + \Delta_{2,n}) \\ &\quad\quad + \tfrac{1}{2}\sigma^2(h_n)|X(0)| + |X_p(h_n \mid h_{n+1}) - X_p(h_n)| \end{aligned} \tag{32}$$

by Lemma 4 where $X_p(h_n)$ is $X_p(h_n \mid 0)$.

Now

$$X_p(h_n \mid h_{n+1}) - X_p(h_n) = \textstyle\int_0^{h_{n+1}} \hat{g}(h_n - u)\, d\xi(u, \omega)$$

where $\xi(u, \omega)$ is Brownian motion and $\hat{g}(x) = (2a)^\alpha x^{\alpha-\frac{1}{2}}e^{-ax}/(\Gamma(2\alpha))^{\frac{1}{2}}$ for $x > 0$ (see [6]). This yields

$$E((X_p(h_n \mid h_{n+1}) - X_p(h_n))^2) \leqq c_{19}h_n^{2\alpha}(h_{n+1}/h_n)$$

which implies

$$P(\Delta_{6,n} \geqq \lambda) \leqq c_{20}\exp(-c_{21}n^4\lambda^2) \tag{33}$$

for all $n \geqq n_4(\varepsilon_0)$ where

$$\Delta_{6,n} = (h_n/h_{n+1})^{\frac{1}{2}}|X_p(h_n \mid h_{n+1}) - X_p(h_n)|/(h_n^\alpha(\log(-\log h_n))^{\frac{1}{2}})\,.$$

On substituting this into (32) we obtain (30) with

$$\begin{aligned} \Delta_{5,n} = \Delta_{2,n} &+ \delta_1^{-1}\sigma^2(h_n)|X(0)|/(h_n^\alpha(\log(-\log h_n))^{\frac{1}{2}}) \\ &+ \delta_1^{-1}(h_{n+1}/h_n)^{\frac{1}{2}}\Delta_{6,n}\,. \end{aligned}$$

The inequality (31) follows from (26), (33) and the tail of the normal distribution.

The reader is referred to [6] for the definitions of the matrix $\boldsymbol{\mu} = (\mu_{\nu\nu'})$ with inverse $\boldsymbol{\mu}^{-1} = ((\boldsymbol{\mu}^{-1})_{\nu\nu'})$ ([6], 3.1), the random variables Z_ν ([6], Lemma 5), the row vector $\mathbf{s}_{2m} = (s_1, \cdots, s_{2m})$ ([6], 2.2) and the set $\mathscr{S}_{2m}(\delta)$ ([6], 2.2) which appear in the statement and proof of the next lemma.

LEMMA 6. *For all $\varepsilon > 0$ there exists a $\delta_2 = \delta_2(\varepsilon_0, \varepsilon)$, $0 < \delta_2 < e^{-2}$, and an $n_5 = n_5(\varepsilon_0, \varepsilon)$ such that for all $n \geqq n_5$ there exist exceptional sets $\Omega_{1,n}$ and $\Omega_{2,n}$ in $\mathscr{F}$ with the following properties*:

(a) *for all $\omega \in \Omega\backslash(\Omega_{1,n} \cup \Omega_{2,n})$, all δ, $0 < \delta < \delta_2$, and all $\mathbf{s}_{2m} \in \mathscr{S}_{2m}(\delta)$ the inequality*

$$c_{22} h_k^{-2} Y_p(h_k)^2 - \varepsilon \log(-\log h_k) \leqq \sum_{1 \leqq \nu,\nu' \leqq 2m} Z_\nu Z_{\nu'} (\boldsymbol{\mu}^{-1})_{\nu\nu'}(2m) \leqq c_{22} h_k^{-2\alpha} Y_p(h_k)^2 + \varepsilon \log(-\log h_k)$$

holds for all $m \geqq 1$ and $k \geqq n$ where $c_{22} = 2\alpha\Gamma(2\alpha)/(2a)^{2\alpha}$,

(b) *$\Omega_{1,n} \in \mathscr{F}_0$ and is independent of $\mathscr{F}_{00}$,*

(c) *$P(\Omega_{1,n}) \leqq n^{-6}$ and $P(\Omega_{2,n}) \leqq \exp(-n^2)$.*

PROOF. As in the proof of Lemma 10 of [6] we can apply Lemma 4 of the present paper to obtain the following. For all $\varepsilon_2 > 0$ there exists a $\delta_3 = \delta_3(\varepsilon_0, \varepsilon_2)$ and an $n_6 = n_6(\varepsilon_0, \varepsilon_2)$ such that for all $m \geqq 1$, $n \geqq n_6$, all δ, $0 < \delta < \delta_3$, and for all $\mathbf{s}_{2m} \in \mathscr{S}_{2m}(\delta)$ we have

$$\begin{aligned}(1-\varepsilon_2)c_{22}(s_1 h_n)^{-2\alpha} Z_1^2 &\\ \leqq \sum_{1 \leqq \nu,\nu' \leqq 2m} Z_\nu Z_{\nu'} (\boldsymbol{\mu}^{-1})_{\nu\nu'}(2m) &\\ \leqq (1+\varepsilon_2)c_{22}(s_1 h_n)^{-2\alpha} Z_1^2 + \delta c_{23}(\log(-\log h_n))(\Delta_{1,n}^2 + \Delta_{2,n}^2)\,. &\end{aligned} \tag{34}$$

Elementary inequalities yield

$$|(X_p(s_1 h_n \mid h_{n+1}) - X(0))^2 - Y_p(h_n)^2| \leqq c_{24}\delta_3 h_n^{2\alpha}(\log(-\log h_n))(\Delta_{7,n}^2 + \Delta_{5,n}^2) \tag{35}$$

where

$$\Delta_{7,n} = \Delta_{1,n} + |Y_p(h_n)|/(h_n^\alpha(\log(-\log h_n))^{\frac12})\,.$$

It follows from Lemmas 3 and 4 that $\Delta_{7,n}$ is $\mathscr{F}_0$-measurable and independent of $\mathscr{F}_{00}$. Further, on applying (23), Lemma 1 and finally (25) we conclude that

$$P(\Delta_{7,n} \geqq \lambda) \leqq c_{25} n^{-c_{26}\lambda^2} \tag{36}$$

for $n \geqq n_7(\varepsilon_0)$ and $\lambda > 0$.

The inequalities (34) and (35) together with the fact that $s_1^{-2} \leqq 1 + c_{27}\delta$ yield

$$\begin{aligned}c_{22} h_n^{-2\alpha} Y_p(h_n)^2 - c_{28}(\log(-\log h_n))(\Delta_{7,n}^2 + \Delta_{8,n}^2)(\varepsilon_2 + \delta_3) &\\ \leqq \sum_{1 \leqq \nu,\nu' \leqq 2m} Z_\nu Z_{\nu'} (\boldsymbol{\mu}^{-1})_{\nu\nu'}(2m) &\\ \leqq c_{22} h_n^{-2\alpha} Y_p(h_n)^2 + c_{28}(\log(-\log h_n))(\Delta_{7,n}^2 + \Delta_{8,n}^2)(\varepsilon_2 + \delta_3) &\end{aligned} \tag{37}$$

where $\Delta_{8,n} = \Delta_{5,n} + \Delta_{2,n}$. From (26) and (31) we conclude that

$$P(\Delta_{8,n} \geqq \lambda) \leqq c_{29} \exp(-c_{30} n^4 \lambda^2)$$

for all $\lambda > 0$.

From this it is clear that with

$$\Omega_{3,n} = \{\omega : \Delta_{7,n} \geqq \lambda_0\}$$

and

$$\Omega_{4,n} = \{\omega : \Delta_{8,n} \geqq \lambda_0\},$$

ε_2, $\delta_3 = \delta_3(\varepsilon_0, \varepsilon_2)$ and λ_0 can be so chosen that (a) and (c) of the lemma hold where

$$\Omega_{1,n} = \bigcup_{k=n}^{\infty} \Omega_{3,k} \qquad \text{and} \qquad \Omega_{2,n} = \bigcup_{k=n}^{\infty} \Omega_{4,k}.$$

To prove (b) we note that $\Delta_{7,n}$ is $\mathscr{F}_0$-measurable and independent of $\mathscr{F}_{00}$ which implies that $\Omega_{1,n}$ belongs to $\mathscr{F}_0$ and is independent of $\mathscr{F}_{00}$. This completes the proof of the lemma.

To ease the notation we shall from now on denote $\varphi(X(0), t)$ by $\varphi(t)$.

LEMMA 7. *For all $\varepsilon > 0$ there exists a $\delta_4 = \delta_4(\varepsilon_0, \varepsilon)$, $0 < \delta_4 < e^{-2}$, and an $n_{10} = n_{10}(\varepsilon_0, \varepsilon)$ such that for all $n \geqq n_{10}$ there exist exceptional sets $\Omega_{1,n}$ and $\Omega_{2,n}$ with the following properties;*

(a) *for all $\omega \in \Omega \backslash (\Omega_{1,n} \cup \Omega_{2,n})$ and all $k \geqq n \geqq n_{10}$*

$$P(\varphi(h_k) \geqq \delta h_k^{1-\alpha}(\log(-\log h_k))^\alpha \mid \mathscr{F}_{h_{k+1}}) \geqq \exp(-\tfrac{1}{2}c_{22}h_k^{-2\alpha}Y_p(h_k)^2 - \varepsilon \log(-\log h_k))$$

for all δ, $0 < \delta < \delta_4$.

(b) *$\Omega_{1,n} \in \mathscr{F}_0$ and is independent of $\mathscr{F}_{00}$.*

(c) *$P(\Omega_{1,n}) \leqq n^{-6}$ and $P(\Omega_{2,n}) \leqq \exp(-n^2)$.*

PROOF. This follows from Lemma 6 using the methods of [6].

4. Proof of main lemmas.

4.1. We consider first the problem of obtaining an upper bound for the conditional probability $P(B_{j,m}^{(n)} \mid \mathscr{F}_{(j-1)2^{-m}+h})$ where $h < h_{2n+1}$. Because of the stationarity of the process it is sufficient to consider the conditional probability $P(B^{(n)} \mid \mathscr{F}_h)$ where

$$B^{(n)} = B^{(n)}(\delta) = \{\omega : \varphi(h_\nu) \leqq \delta h_\nu^{1-\alpha}(\log(-\log h_\nu))^\alpha,\ n \leqq \nu \leqq 2n\}.$$

We denote the indicator function of a set F in $\mathscr{F}$ by $\chi(F)$.

LEMMA 8. *For all $\varepsilon > 0$ there exists a $\delta_5 = \delta_5(\varepsilon_0, \varepsilon)$, $0 < \delta_5 < e^{-2}$, and an $n_{11} = n_{11}(\varepsilon_0, \varepsilon)$ such that for all $n \geqq n_{11}$ and all h, $0 < h < h_{2n+1}$,*

$$E(\chi(B^{(n)}) \mid \mathscr{F}_h) \leqq \exp(-\textstyle\sum_{\nu=n}^{2n} \exp(-V_\nu^2 - \varepsilon \log(-\log h_\nu))) + nE(\chi(\Omega_n') \mid \mathscr{F}_h)$$

where

(i) *$\Omega_n = \Omega_{1,n} \cup \Omega_{2,n}$ with $\Omega_{1,n}$ and $\Omega_{2,n}$ as in Lemma 7, and*

(ii) *$V_\nu = (\tfrac{1}{2}c_{22})^{\frac{1}{2}} h_\nu^{-\alpha} Y_p(h_\nu)$.*

PROOF. Let

$$B^\nu = \{\omega : \varphi(h_\nu) \leqq \delta h^{1-\alpha}(\log(-\log h_\nu))^\alpha\}$$

so that $B^\nu \in F_{h_\nu}$.

As $h_n > h_{n+1} > \cdots > h_{2n+1} > h > 0$ it follows that $\mathscr{F}_{h_n} \supset \mathscr{F}_{h_{n+1}} \supset \cdots \supset \mathscr{F}_h \supset \mathscr{F}_0$. The random variables V_ν are $\mathscr{F}_0$-measurable and hence $\mathscr{F}_{h_\nu}$-measurable, $\nu = n, \cdots, 2n$, and also $\mathscr{F}_h$-measurable.

We have

$$E(\chi(B^{(n)}) \mid \mathscr{F}_h) = E(\textstyle\prod_{\nu=n}^{2n} \chi(B^\nu) \mid \mathscr{F}_h) = E(E(\prod_{\nu=n}^{2n} \chi(B^\nu) \mid \mathscr{F}_{h_{n+1}}) \mid \mathscr{F}_h)$$

as $\mathscr{F}_h \subset \mathscr{F}_{h_{n+1}}$. Thus

$$E(\chi(B^{(n)}) \mid \mathscr{F}_h) = E(\textstyle\prod_{\nu=n+1}^{2n} \chi(B^\nu) E(\chi(B^n) \mid \mathscr{F}_{h_{n+1}}) \mid \mathscr{F}_h)$$

as $B^{n+1}, \cdots, B^{2n}$ are $\mathscr{F}_{h_{n+1}}$-measurable. If $n \geqq n_{10}$ and $0 < \delta < \delta_4$ Lemma 7 yields

$$\begin{aligned} E(\chi(B^{(n)}) \mid \mathscr{F}_h) &= E(\textstyle\prod_{\nu=n+1}^{2n} \chi(B^\nu)\chi(\Omega\backslash\Omega_n) E(\chi(B^n) \mid \mathscr{F}_{h_{n+1}}) \mid \mathscr{F}_h) \\ &\quad + E(\textstyle\prod_{\nu=n+1}^{2n} \chi(B^\nu)\chi(\Omega_n) E(\chi(B^n) \mid \mathscr{F}_{h_{n+1}}) \mid \mathscr{F}_h) \\ &\leqq E(\textstyle\prod_{\nu=n+1}^{2n} \chi(B^\nu)\chi(\Omega\backslash\Omega_n) \exp(-V_n^2 - \varepsilon \log(-\log h_n)) \mid \mathscr{F}_h) \\ &\quad + E(\chi(\Omega_n) \mid \mathscr{F}_h) \end{aligned}$$

where we have used the inequality $1 - x \leqq \exp(-x)$. This yields

$$\begin{aligned} &E(\chi(B^{(n)}) \mid \mathscr{F}_h) \\ &\quad \leqq E(\textstyle\prod_{\nu=n+1}^{2n} \chi(B^\nu) \exp(-\exp(-V_n^2 - \varepsilon \log(-\log h_n))) \mid \mathscr{F}_h) \\ &\quad + E(\chi(\Omega_n) \mid \mathscr{F}_h) . \end{aligned} \tag{38}$$

On repeating this process n times we obtain

$$\begin{aligned} E(\chi(B^{(n)}) \mid \mathscr{F}_h) &\leqq E(\exp(-\textstyle\sum_{\nu=n}^{2n} \exp(-V_\nu^2 - \varepsilon \log(\log h_\nu))) \mid \mathscr{F}_h) \\ &\quad + nE(\chi(\Omega_n) \mid \mathscr{F}_h) \\ &= \exp(-\textstyle\sum_{\nu=n}^{2n} \exp(-V_\nu^2 - \varepsilon \log(-\log h_\nu))) \\ &\quad + nE(\chi(\Omega_n) \mid \mathscr{F}_n) \end{aligned}$$

which proves the lemma.

4.2. The next lemma gives an upper bound for the probability that the process $X(t)$ has at least one zero in the small time interval $[t, t + h)$. We require the following notation:

$$D_m = \{\omega : X(t, \omega) = 0 \text{ for some } t \text{ in } [0, 2^{-m})\} , \tag{39}$$

$$M_p(m) = \sup_{0 \leqq t \leqq 2^{-m}} |Y_p(t)| \tag{40}$$

and

$$E_{\nu,m} = \{\omega : \nu M_p(m) \leqq \tfrac{1}{2}|X(0)| < (\nu + 1)M_p(m)\} , \qquad \nu = 0, 1, 2 \cdots . \tag{41}$$

LEMMA 9. *For all* $m \geqq m_1$

$$E(\chi(D_m) \mid \mathscr{F}_0) \leqq \chi(E_{0,m}) + c_{31} \textstyle\sum_{\nu=1}^{\infty} \chi(E_{\nu,m}) \exp(-c_{32}(\nu - 1)^2 M_p(m)^2 2^{2\alpha m}) . \tag{42}$$

PROOF. Suppose that $X(t, \omega) = 0$ for some t in $[0, 2^{-m})$. For this t we have

$X_e(t) + X_p(t) = 0$ which implies

$$|X_e(t)| = |X_p(t)| \geqq \tfrac{1}{2}|X(0)| - M_p(m)$$

for $m \geqq m_1$ on using (8).

This yields

$$M_e(m) \geqq \tfrac{1}{2}|X(0)| \quad M_p(m)$$

where

$$M_e(m) = \sup_{0 \leqq t \leqq 2^{-m}} |X_e(t)| .$$

Thus if

$$F_{\nu,m} = \{\omega : (\nu - 1)M_p(m) \leqq M_e(m)\}$$

we have

$$\chi(D_m) \leqq \chi(E_{0,m}) + \textstyle\sum_{\nu=1}^{\infty} \chi(E_{\nu,m})\chi(F_{\nu,m}) .$$

The random variable $X_e(t)$ is independent of $\mathscr{F}_0$ and hence so is $M_e(m)$. Furthermore, $E_{\nu,m} \in \mathscr{F}_0$ as $Y_p(t)$ is $\mathscr{F}_0$-measurable. This implies

$$\text{(43)} \qquad \begin{aligned} E(\chi(D_m) \,|\, \mathscr{F}_0) &\leqq E(\chi(E_{0,m}) \,|\, \mathscr{F}_0) + \textstyle\sum_{\nu=1}^{\infty} E(\chi(E_{\nu,m})\chi(F_{\nu,m}) \,|\, \mathscr{F}_0) \\ &= \chi(E_{0,m}) + \textstyle\sum_{\nu=1}^{\infty} \chi(E_{\nu,m})E(\chi(F_{\nu,m}) \,|\, \mathscr{F}_0) . \end{aligned}$$

It remains to obtain an upper bound for $E(\chi(F_{\nu,m}) \,|\, \mathscr{F}_0)$. As $E(X_e(t)) = 0$, $E(X_e(t)^2) \leqq \sigma^2(t) = O(|t|^{2\alpha})$ and $E((X_e(t) - X_e(s))^2) \leqq \sigma^2(t - s) = O(|t - s|^{2\alpha})$ for $0 \leqq s,\ t \leqq 2$ we conclude from Fernique's lemma that

$$E(\chi(\{\omega : M_e(m) \geqq \lambda 2^{-\alpha m}\})) \leqq c_{31} \exp(-c_{32}\lambda^2)$$

for all $\lambda > 0$. The random variable $M_p(m)$ is $\mathscr{F}_0$-measurable and as $X_e(t)$ (and hence $M_e(m)$) is independent of $\mathscr{F}_0$ this implies

$$E(\chi(F_{\nu,m}) \,|\, \mathscr{F}_0) \leqq c_{31} \exp(-c_{32}(\nu - 1)^2 M_p(m)^2 2^{2\alpha m}) .$$

The lemma follows on substituting this into (43).

4.3. In the next lemmas we combine Lemmas 8 and 9 to obtain an upper bound for $P(B^{(n)} \cap D_m)$.

Lemma 10. *For all $\varepsilon > 0$ there exists a $\delta_6 = \delta_6(\varepsilon_0, \varepsilon)$, $0 < \delta_6 < e^{-2}$, and an $n_{12} = n_{12}(\varepsilon_0, \varepsilon)$ such that for all δ, $0 < \delta < \delta_6$, all $n \geqq n_{12}$ and all $m \geqq m_2$ with $2^{-m} < h_{2n+1}$ we have*

$$\text{(44)} \qquad \begin{aligned} E(\chi(B^{(n)})\chi(D_m)) \leqq n \exp(-n^2) + c_{33} 2^{-\alpha m}(E(\exp(-2 \textstyle\sum_{\nu=n}^{2n} \exp(-V_\nu^2 \\ - \varepsilon \log(-\log h_\nu))))^{\frac{1}{2}} + n^{-2}) . \end{aligned}$$

Proof. We have

$$\begin{aligned} E(\chi(B^{(n)})\chi(D_m)) &= E(E(\chi(B^{(n)})\chi(D_m) \,|\, \mathscr{F}_{2^{-m}})) \\ &= E(\chi(D_m)E(\chi(B^{(n)}) \,|\, \mathscr{F}_{2^{-m}})) \end{aligned}$$

as $D_m \in \mathscr{F}_{2^{-m}}$. As $2^{-m} < h_{2n+1}$ we may apply Lemma 8 to obtain

$$E(\chi(B^{(n)})\chi(D_m)) \leqq E(\chi(D_m)W_n + n\chi(D_m)E(\chi(\Omega_n) \,|\, F_{2^{-m}}))$$

where $W_n = \exp(-\sum_{\nu=n}^{2n} \exp(-V_\nu^2 - \varepsilon \log(-\log h_\nu)))$. We note that W_n is $\mathscr{F}_0$-measurable and independent of $\mathscr{F}_{00}$. Thus

$$\begin{aligned} E(\chi(B^{(n)})\chi(D_m)) &\leqq E(E(\chi(D_m)W_n \mid \mathscr{F}_0)) \\ &\quad + nE(E(\chi(D_m)E(\chi(\Omega_n) \mid \mathscr{F}_{2-m})) \mid \mathscr{F}_0) \\ &= E(W_n E(\chi(D_m) \mid \mathscr{F}_0)) \\ &\quad + nE(E(\chi(D_m)\chi(\Omega_n) \mid \mathscr{F}_0)) \end{aligned}$$

and on applying Lemma 9 we obtain

$$\begin{aligned} &E(\chi(B^{(n)})\chi(D_m)) \\ &\quad \leqq E(W_n(\chi(E_{0,m}) + c_{31} \textstyle\sum_{\nu=1}^{\infty} \chi(E_{\nu,m}) \exp(-c_{32}(\nu-1)^2 M_p(m)^2 2^{2\alpha m}))) \\ &\qquad + nE(\chi(\Omega_{n,1})E(\chi(D_m) \mid \mathscr{F}_0)) + nE(\chi(\Omega_{n,2})) \end{aligned}$$

as $\Omega_{n,1} \in \mathscr{F}_0$. Thus

$$\begin{aligned} &E(\chi(B^{(n)})\chi(D_m)) \\ &\quad \leqq E((W_n + n\chi(\Omega_{1,n}))(\chi(E_{0,m}) \\ &\qquad + c_{31} \textstyle\sum_{\nu=1}^{\infty} \chi(E_{\nu,m}) \exp(-c_{32}(\nu-1)^2 M_p(m)^2 2^{2\alpha m}))) + n\exp(-n^2) \end{aligned}$$

by Lemma 9.

The random variables W_n, $\chi(\Omega_{1,n})$ and $M_p(m)$ are independent of $\mathscr{F}_{00}$. We denote by $\mathscr{F}_{00}^0$ the σ-algebra which is generated by W_n, $\chi(\Omega_{1,n})$ and $M_p(m)$. It follows that $\mathscr{F}_{00}^0$ is independent of $\mathscr{F}_{00}$. Thus

$$\begin{aligned} &E((W_n + n\chi(\Omega_{1,n}))(\chi(E_{0,m}) + c_{31} \textstyle\sum_{\nu=1}^{\infty} \chi(E_{\nu,m}) \exp(-c_{32}(\nu-1)^2 M_p(m)^2 2^{2\alpha m}))) \\ &\quad = E(W_n + n\chi(\Omega_{1,n}))(E(\chi(E_{0,m}) \mid \mathscr{F}_{00}^0) \\ &\qquad + c_{32} \textstyle\sum_{\nu=1}^{\infty} E(\chi(E_{\nu,m}) \mid \mathscr{F}_{00}^0) \exp(-c_{32}(\nu-1)^2 M_p(m)^2 2^{2\alpha m}))) \end{aligned}$$

so that

$$\begin{aligned} E(\chi(B^{(n)})\chi(D_m)) &\leqq E((W_n + n\chi(\Omega_{1,n}))(E(\chi(E_{0,m}) \mid \mathscr{F}_{00}^0) \\ &\quad + c_{31} \textstyle\sum_{\nu=1}^{\infty} E(\chi(E_{\nu,m}) \mid \mathscr{F}_{00}^0) \\ &\quad \times \exp(-c_{32}(\nu-1)^2 M_p(m^2) 2^{2\alpha m}))) + n\exp(-n^2)\,. \end{aligned} \tag{45}$$

We now obtain an upper bound for $E(\chi(E_{\nu,m}) \mid \mathscr{F}_{00}^0)$. As $X(0)$ is independent of $\mathscr{F}_{00}^0$ and $M_p(m)$ is $\mathscr{F}_{00}^0$-measurable we have

$$\begin{aligned} E(\chi(E_{\nu,m}) \mid \mathscr{F}_{00}^0) &= (2/\pi)^{\frac{1}{2}} \textstyle\int_{2\nu M_p(m)}^{2(\nu+1)M_p(m)} \exp(-\frac{1}{2}x^2)\,dx \\ &\leqq c_{33} M_p(m)\,. \end{aligned}$$

On substituting this into (45) we obtain

$$\begin{aligned} E(\chi(B^{(n)})\chi(D_m)) &\leqq c_{34} E((W_n + n\chi(\Omega_{1,n}))(M_p(m) \\ &\quad + \textstyle\sum_{\nu=1}^{\infty} M_p(m) \exp(-c_{32}(\nu-1)^2 M_p(m^2) 2^{2\alpha m}))) \\ &\quad + n\exp(-n^2)\,. \end{aligned}$$

The inequality

$$\textstyle\sum_{\nu=1}^{\infty} \lambda \exp(-\frac{1}{2}\nu^2\lambda^2) \leqq \sum_{\nu=0}^{\infty} \int_{\nu\lambda}^{(\nu+1)\lambda} \exp(-\frac{1}{2}x^2)\,dx = (2/\pi)^{\frac{1}{2}}$$

for all $\lambda > 0$ yields

$$E(\chi(B^{(n)})\chi(D_m)) \tag{46}$$
$$\leqq c_{36}2^{-\alpha m}(E(W_n^2)^{\frac{1}{2}} + nE(\chi(\Omega_{1,n}))^{\frac{1}{2}})(1 + 2^{\alpha m}E(M_p(m)^2)^{\frac{1}{2}}) + n\exp(-n^2)\,.$$

Let H_m denote the distribution function of $M_p(m)$. It follows from Fernique's lemma that $1 - H_m(\lambda) \leqq c_{37}\exp(-c_{38}\lambda^2 2^{2\alpha m})$ and hence

$$\begin{aligned} E(M_p(m)^2) &= \textstyle\int_0^\infty x^2\,dH_m(x) = 2\int_0^\infty x(1 - H_m(x))\,dx \\ &\leqq 2c_{37}\textstyle\int_0^\infty x\exp(-c_{38}x^2 2^{2\alpha m})\,dx \\ &\leqq c_{39}2^{-2\alpha m}\,. \end{aligned}$$

On applying this inequality in (46) and using Lemma 7 (c) we obtain (44) and the proof of the Lemma 10 is complete.

LEMMA 11. *Let $(Z_\nu)_1^\infty$ be a sequence of normally distributed random variables with zero means and variances $(\sigma_\nu^2)_1$ and let $(\lambda_\nu)_1^\infty$ be a sequence of positive numbers satisfying* $\lim_{n\to\infty} n\Lambda_n = \infty$ *where $\Lambda_n = \min_{1\leqq\nu\leqq n}\lambda_\nu$. Then if* $\sup_{1\leqq\nu<\infty}\sigma_\nu^2 \leqq \Sigma^2 < \infty$ *there exists a constant K depending on Σ such that*

$$E(\exp(-\textstyle\sum_{\nu=1}^n \lambda_\nu\exp(-Z_\nu^2))) \leqq K(n\Lambda_n)^{-1/(2\Sigma^2)}$$

for all $n \geqq 1$.

PROOF. Hölder's inequality and elementary considerations yield

$$E(\textstyle\prod_{\nu=1}^n \exp(-\lambda_\nu\exp(-Z_\nu^2))) \leqq E(\exp(-n\Lambda_n\exp(-\Sigma^2 Z^2)))$$

where Z is normally distributed with zero mean and unit variance. This implies

$$\begin{aligned} E(\textstyle\prod_{\nu=1}^n \exp(-\lambda_\nu\exp(-Z_\nu^2))) &\leqq \frac{(n\Lambda_n)^{-1/(2\Sigma^2)}}{(2\pi)^{\frac{1}{2}}\Sigma}\int_0^{n\Lambda_n}\frac{\exp(-x)x^{-1+1/(2\Sigma^2)}}{(-\log(x/(n\Lambda_n)))^{\frac{1}{2}}}\,dx \\ &\leqq K(n\Lambda_n)^{-1/(2\Sigma^2)} \end{aligned}$$

after elementary calculations, which proves the lemma.

LEMMA 12. *If $0 < \varepsilon_0 < 1$ then there exists a $\delta_7 = \delta_7(\varepsilon_0)$, $0 < \delta_7 < e^{-2}$, and an $n_{13} = n_{13}(\varepsilon_0)$ such that for all δ, $0 < \delta < \delta_7$, all $n \geqq n_{13}$ and all m with $2^{-m} < h_{2n+1}$ we have*

$$E(\chi(B^{(n)})\chi(D_m)) \leqq c_{40}(2^{-\alpha m})n^{-(1+c_{41})} + n\exp(-n^2) \tag{46}$$

where $c_{41} > 0$.

PROOF. It follows from (8), Lemma 6 (a) and Lemma 8 (ii) that

$$\begin{aligned} E(V_\nu^2) &= \tfrac{1}{2}c_{22}h_\nu^{-2\alpha}E(Y_p(h_\nu)^2) \\ &\cong \frac{\alpha\Gamma(2\alpha)}{(2a)^\alpha}\left(\frac{2\Gamma(1-\alpha)a^{2\alpha}}{\alpha\Gamma(\alpha)2^{2\alpha}} - \frac{(2a)^\alpha}{2\alpha\Gamma(2\alpha)}\right) \\ &= \tfrac{1}{2}\left(\frac{4\Gamma(2\alpha)\Gamma(1-\alpha)}{2^{4\alpha}\Gamma(\alpha)} - 1\right) = c_{41} < \tfrac{1}{2}\,. \end{aligned}$$

An application of Lemmas 10 and 11 yields

$$\begin{aligned}E(\chi(B^{(n)})\chi(D_m)) &\leqq c_{42}(2^{-\alpha m}(n(2n)^{-\varepsilon(1+\varepsilon_0)})^{1/(2c_{41})}) + n\exp(-n^2)\\ &\leqq c_{43}(2^{-\alpha m}(n(2n)^{-2\varepsilon})^{1+c_{44}}) + n\exp(-n^2)\end{aligned}$$

where $c_{44} > 0$. We now choose ε, $0 < \varepsilon < \frac{1}{2}$, so small that $(1 - 2\varepsilon)(1 + c_{44}) > 1 + \frac{1}{2}c_{44}$. With this choice of ε we have

$$E(\chi(B^{(n)})\chi(D_m)) < c_{45}2^{-\alpha m}n^{-(1+c_{46})} + n\exp(-n^2)$$

where $c_{46} > 0$. This proves the lemma.

5. Proof of theorem.

5.1. In order to prove the theorem we must first show that we can approximate the events $A_{j,m}^{(n)}$ of (13) by the events $B_{j,m}^{(n)}$ of (12). To this end we require the following lemma.

LEMMA 13. *Given $\varepsilon > 0$ there exists an exceptional set $\Omega_\varepsilon \in \mathscr{F}$ with $P(\Omega_\varepsilon) < \varepsilon$, an integer $M_{1\varepsilon}$ and a positive number η_ε such that the following holds. For all $\omega \in \Omega\backslash\Omega_\varepsilon$*

$$\begin{aligned}(47)\qquad \varphi(X(t), t+h) - \varphi(X(t), t)\\ \leqq \varphi(X(s), s+h) - \varphi(X(s), s) + 2^{1-\alpha(1-\alpha)m/4}\end{aligned}$$

for all s, t, h and m satisfying

$$0 \leqq s, t, \quad t+h \leqq 2, \quad t \leqq s \leqq t + 2^{-m}, \quad 0 \leqq h \leqq \eta_\varepsilon \quad \textit{and}\quad m \geqq M_{1\varepsilon}.$$

PROOF. The proof of this lemma is a straightforward application of the Hölder condition

$$|\varphi(x+y, t+h) - \varphi(x+y, t) - \varphi(x, t+h) + \varphi(x, t)| \leqq |hy|^{(1-\alpha)/2}$$

given by Berman in [4] (proof of Theorem 3.1) and of the (crude) Hölder condition

$$|X(s) - X(t)| \leqq |s - t|^{\alpha/2}$$

for $X(s)$ which follows from (1) (see [10], Theorem 4).

5.2. We are now in a position to obtain an upper bound for the probability that the interval $I_{j,m}$ is bad in the sense described in 2.2. We set the δ_1 in (10), (11), (12) and (13) equal to $\frac{1}{2}\delta_7$ where δ_7 is as in Lemma 12. We set the n_0 of 2.2 equal to $n_{13}(\varepsilon_0)$ where n_{13} is as in Lemma 12 and ε_0 is chosen so that

$$(1 + c_{41})/(1 + \varepsilon_0) > 1 + c_{41}/2 = 1 + c_{47} > 1$$

where c_{41} is as in (46).

We denote the event that the interval $I_{j,m}$ is bad by $G_{j,m}$.

LEMMA 14. *For all $\varepsilon > 0$ there exists an exceptional set Ω_ε of probability less than ε and an integer $M_{2\varepsilon}$ such that for all $m \geqq M_{2\varepsilon}$ and all j, $1 \leqq j \leqq 2^m$,*

$$E(\chi(G_{j,m})\chi(\Omega\backslash\Omega_\varepsilon)) \leqq c_{48}2^{-\alpha m}m^{-(1+c_{47})}.$$

PROOF. Let $D_{j,m}$ denote the event that the process $X(t)$ has a zero in the interval $I_{j,m}$. With N_m as defined by (10) it follows from Lemma 13 that for all $m \geqq M_{1\varepsilon}$

$$A_{j,m} \cap D_{j,m} \cap (\Omega \backslash \Omega_\varepsilon) \subset B_{j,m} \cap D_{j,m} \cap (\Omega \backslash \Omega_\varepsilon)$$

where we have written $A_{j,m}$ and $B_{j,m}$ for $A_{j,m}^{(n)}$ and $B_{j,m}^{(n)}$ respectively with $n = \frac{1}{2}N_m$.

As $\frac{1}{2}\delta_1 h_{N_{m+1}}^{1-\alpha} \geqq 2^{-\alpha(1-\alpha)m/4}$ and $0 < \alpha < \frac{1}{2}$ it follows that for $m \geqq M_3$, $2^{-m} < h_{N_{m+1}}$. We may therefore apply Lemma 12 to obtain

$$\begin{aligned} E(\chi(A_{j,m})\chi(D_{j,m})\chi(\Omega\backslash\Omega_\varepsilon)) &\leqq E(\chi(B_{j,m})\chi(D_{j,m})) \\ &\leqq c_{49}(2^{-\alpha m}N_m{}^{-(1+c_{41})} + N_m \exp(-N_m{}^2/4)) \end{aligned}$$

where we have used the stationarity of the process. It is easily seen that $N_m > c_{50}m^{1/(1+\varepsilon_0)}$ and hence

$$E(\chi(A_{j,m})\chi(D_{j,m})\chi(\Omega\backslash\Omega_\varepsilon)) \leqq c_{51}2^{-\alpha m}m^{-(1+c_{47})} .$$

Suppose now that $\omega \in \Omega\backslash(A_{j,m} \cap D_{j,m})$. Then either $X(t, \omega)$ has no zero in the interval $I_{j,m}$ or there exists an n, $\frac{1}{2}N_m \leqq n \leqq N_m$ such that

$$\varphi(X(\tau_j), \tau_j + h_n) - \varphi(X(\tau_j), \tau_j) > \tfrac{1}{2}\delta_1 h_n{}^{1-\alpha}(\log(-\log h_n))^\alpha .$$

The number h_n lies in the interval $[\delta_1{}^{-1/(1-\alpha)}2^{1/(1-\alpha)-\alpha m/4}, 2^{-m_0}) \subset [2^{-m}, 2^{-m_0})$ and hence there exists an interval $[a, b)$ of $\bigcup_{\nu=m_0}^{m} \Lambda_\nu$ which covers $[\tau_j, \tau_j + h_n)$ and is such that $b - a \leqq 4h_n$ (see [16]). For this interval we have

$$\begin{aligned} \frac{\varphi(0, b) - \varphi(0, a)}{(b-a)^{1-\alpha}(\log(-\log(b-a)))^\alpha} &\geqq \frac{\varphi(0, \tau_j + h_n) - \varphi(0, \tau_j)}{(4h_n)^{1-\alpha}(\log(-\log 4h_n))^\alpha} \\ &\geqq \tfrac{1}{8}\delta_1 . \end{aligned}$$

The interval $I_{j,m}$ is therefore good. This implies

$$\begin{aligned} E(\chi(G_{j,m})\chi(\Omega\backslash\Omega_\varepsilon)) &\leqq E(\chi(A_{j,m})\chi(D_{j,m})\chi(\Omega\backslash\Omega_\varepsilon)) \\ &\leqq c_{51}2^{-\alpha m}m^{-1(1+c_{47})} \end{aligned}$$

which completes the proof of the lemma.

5.3. The final step in the proof of the theorem consists of showing that the contribution of the bad intervals to the covering is small. We denote by T_m the number of bad intervals $I_{j,m}$, $1 \leqq j \leqq 2^m$. We have

$$\begin{aligned} E(T_m\chi(\Omega\backslash\Omega_\varepsilon)) &= E(\textstyle\sum_{j=1}^{2m} \chi(G_{j,m})\chi(\Omega\backslash\Omega_\varepsilon)) \\ &= \textstyle\sum_{j=1}^{2m} E(\chi(G_{j,m})\chi(\Omega\backslash\Omega_\varepsilon)) \\ &\leqq \textstyle\sum_{j=1}^{2m} c_{51}2^{-\alpha m}m^{-(1+c_{47})} \\ &= c_{51}2^{(1-\alpha)m}m^{-(1+c_{47})} . \end{aligned}$$

This implies that the contribution $\sum_m'$ of the bad interval to the covering satisfies

$$E(\sum_m' \chi(\Omega \backslash \Omega_\varepsilon)) = E(T_m \chi(\Omega \backslash \Omega_\varepsilon) 2^{-m(1-\alpha)} (\log (\log 2^m))^\alpha)$$
$$\leqq c_{52} (\log m)^\alpha m^{-(1+c_{47})}$$

from which follows that

$$P(\{\sum_m' (\log m)^{-1}\} \cap (\Omega \backslash \Omega_\varepsilon)) \leqq c_{52} (\log m)^{1+\alpha} m^{-(1+c_{47})} .$$

As $c_{47} > 0$ we have $\sum_{m=m_0}^\infty (\log m)^{1+\alpha} m^{-(1+c_{47})} < \infty$ and the Borel–Cantelli lemma implies that for almost all ω in $\Omega \backslash \Omega_\varepsilon$

$$\sum_m' < (\log m)^{-1}$$

for all sufficiently large m.

Arguing now as in [16], page 179, we obtain

$$\Psi - m(Q(1)) \leqq c_{53} \varphi(0, 1)$$

for almost all ω in $\Omega \backslash \Omega_\varepsilon$. As $P(\Omega_\varepsilon) < \varepsilon$ and we may take ε arbitrarily small this implies

$$\Psi - m(Q(1)) \leqq c_{53} \varphi(0, 1)$$

for almost all ω. Theorem 1 is a trivial consequence of this last inequality and Theorem 3 of [6].

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