The Annals of Probability
1977, Vol. 5, No. 2, 243-261

ON THE EXISTENCE, UNIQUENESS, CONVERGENCE AND EXPLOSIONS OF SOLUTIONS OF SYSTEMS OF STOCHASTIC INTEGRAL EQUATIONS

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A theory of stochastic integral equations is developed for the integrals of Kunita, Watanabe, and P. A. Meyer. Existence and uniqueness of solutions of systems of equations with semimartingale (or "quasi-martingale") differentials is proved, in which we include as particular cases the customary results as put forth by McKean and Gihman and Skorohod. Under weaker conditions we prove existence and uniqueness with explosions, and study the explosion times. We show that when the (random) coefficients or the differentials converge, the solutions converge to the solution of the limiting equation.

1. Introduction. About twenty-five years ago K. Itô developed a theory of stochastic integration with respect to Brownian motion. About 1967 Kunita and Watanabe proposed a generalization of what is now called the Itô integral: making use of P. A. Meyer's decomposition theorem for supermartingales, they proposed a stochastic integral with respect to square-integrable right continuous martingales. Meyer expanded their theory, and in particular a theory of stochastic integration for local martingales with continuous paths was developed, which includes Brownian motion as a special case. A theory of stochastic ordinary differential equations for the Itô integral is well known. McKean (1969) sets forth the basic theory, and Gihman and Skorohod (1972) consider somewhat different questions than McKean does. The integral of Kunita and Watanabe, however, admits any continuous local martingale as a differential, and a theory of stochastic differential equations for this integral has not been studied, except for the note of Kazamaki (1972).

The aim of this paper is to develop a theory of stochastic ordinary differential equations where continuous local martingales are admissible as differentials. We also allow random differentials whose paths are continuous and of bounded variation, and hence give rise to Riemann-Stieltjes integrals. As such we are essentially dealing with a continuous semimartingale (or "quasi-martingale") calculus, as was proposed by Doléans-Dade and Meyer (1970).

In the next section we develop a seminorm for the space $\mathscr{S}$ of continuous semimartingales. In Section 3 we utilize the seminorm to prove the existence and uniqueness in $\mathscr{S}$ of a solution of a system of equations.

The customary existence and uniqueness theorem with a Brownian differential

Received September 17, 1975; revised June 16, 1976.
AMS 1970 *subject classifications.* 60H10, 60H20.
Key words and phrases. Stochastic differential equations, stochastic integrals, Brownian motion, Wiener process, local martingales, semimartingales, quasi-martingales.

has two hypotheses on the (continuous) coefficients (see, for example, Gihman and Skorohod, 1972): they must satisfy a Lipschitz condition and a condition of the form:

$$|f(t, x)|^2 \leqq K^2(1 + |x|^2) \tag{1.1}$$

for some constant K. Using the technique of optional stopping, we are able to eliminate the growth condition (1.1) and still prove the existence and uniqueness of a continuous, nonexploding solution.

We then weaken the remaining Lipschitz condition (Section 4) and find that solutions do explode unless the Lipschitz condition is weakened in the time variable only.

In Section 5 we consider stochastic coefficients and extend Gihman and Skorohod's (1972) results. Here we weaken the Lipschitz condition so that it becomes a "Lipschitz process." In Section 6 we consider the question of convergence, allowing the coefficients to converge, the differentials to converge, and both to converge together. In all cases the solutions converge to the solution of the limiting equation. Last, we generalize to our situation the well-known result of Wong and Zakai (1965) concerning the convergence of solutions of ordinary differential equations to those of stochastic ones.

I wish to thank Professor R. K. Getoor for his advice and encouragement during the work on this paper.

2. Preliminaries and the seminorm. We will assume throughout a given underlying structure $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, a complete probability space together with an increasing, right-continuous family of σ-algebras $\mathscr{F}_t$ $(t \geqq 0)$ such that $\mathscr{F}_t \subseteqq \mathscr{F}$ and $\mathscr{F}_0$ contains all the P-null sets of $\mathscr{F}$. That is, $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ satisfies the "usual hypotheses." The reader is assumed to be familiar with the Kunita–Watanabe–Meyer treatment of stochastic integration relative to continuous local martingales as is set forth in Meyer (1967) or Doléans-Dade and Meyer (1970). For the reader's convenience, however, we will recall here some of the basic definitions and results.

A process X is called *previsible* if it is measurable with respect to the σ-algebra generated by the processes adapted to $(\mathscr{F}_t)$ with left-continuous paths. (Such a process is called *predictable* by many authors.) We will write $X = Y$ to denote that the process (X_t) is indistinguishable from the process (Y_t). It is in this sense that our uniqueness results are to be interpreted. All stopping times will be $(\mathscr{F}_t)$ stopping times, and a process X will be said to *stop* at a stopping time T if $X_{t \wedge T} = X_t$ for all t, a.s. For a stopping time T we write $X_t^T = X_{t \wedge T} 1_{\{T>0\}}$, where 1_A denotes the indicator function of a set $A \in \mathscr{F}$. We call X_t^T a *modified stopped process*. Of course, if $X_0 = 0$ a.s., then $X_t^T = X_{t \wedge T}$. An adapted process (L_t) is called a *local martingale* if it a.s. has left limits and is right continuous, and if there exists a sequence of stopping times (T^n) increasing to ∞ a.s. such that $L_t^{T^n}$ is a martingale for each n. The class of all local martingales L with continuous paths such that $L_0 = 0$ a.s. will be denoted $\mathscr{L}$.

We will let $\mathscr{A}^+$ denote the class of processes which have *continuous*, increasing, finite-valued paths, are adapted, and are such that $A_0 = 0$ a.s. We let $\mathscr{A} = \mathscr{A}^+ - \mathscr{A}^+$: that is, $A \in \mathscr{A}$ if it is the difference of two processes in $\mathscr{A}^+$. We also introduce the notation $\mathscr{F}_0 \oplus \mathscr{A} = \{(\eta_t): \eta_t = Z_0 + A_t$, where $Z_0 \in \mathscr{F}_0$, $Z_0 < \infty$ a.s., and $(A_t) \in \mathscr{A}\}$. We will call a process X a *continuous semimartingale* if it can be decomposed as $X_t = \eta_t + L_t$, with $(\eta_t) \in \mathscr{F}_0 \oplus \mathscr{A}$ and $(L_t) \in \mathscr{L}$. The collection of continuous semimartingales will be denoted by $\mathscr{S}$. Note that such a decomposition of a continuous semimartingale is unique due to the non-trivial fact that a continuous local martingale either a.s. has paths of unbounded variation or is constant (see Fisk, 1965). Examples of semimartingales include processes with stationary, independent increments, and Meyer's decomposition theorem assures us that any supermartingale (hence any submartingale) is a semimartingale. Fisk (1965) has given simple conditions under which a process has a unique decomposition into the sum of a continuous martingale and a process in $\mathscr{F}_0 \oplus \mathscr{A}$ which also has finite total expected variation. By requiring that Fisk's conditions only hold "locally" one obtains a corresponding (unique) decomposition into the sum of a continuous local martingale and an element of $\mathscr{F}_0 \oplus \mathscr{A}$. ("Locally" here means that there is a sequence (T^n) of stopping times increasing to ∞ such that the property in question holds for the modified stopped process corresponding to each T^n.)

For any $L \in \mathscr{L}$, we let $\langle L, L\rangle$ denote the unique process in $\mathscr{A}^+$ such that $L^2 - \langle L, L\rangle$ is in $\mathscr{L}$. Note that $E\{L_t^2\} = E\{\langle L, L\rangle_t\}$. For $L, M \in \mathscr{L}$, $\langle L, M\rangle$ denotes the unique process in $\mathscr{A}$ such that $LM - \langle L, M\rangle \in \mathscr{L}$. For any stopping time T, $\langle L, M\rangle_t^T = \langle L^T, M\rangle_t = \langle L, M^T\rangle_t$.

For $M \in \mathscr{L}$, we let $L^2(M)$ denote the collection of all real valued previsible processes C such that $E \int_0^\infty C_s^2 \, d\langle M, M\rangle_s < \infty$. Also, $L^2_{\text{loc}}(M)$ will denote the processes C which are locally in $L^2(M)$. As shown (for example) in Meyer (1967), for any $M \in \mathscr{L}$ and $C \in L^2_{\text{loc}}(M)$ we can define the *stochastic integral* $C \cdot M$ as the unique element of $\mathscr{L}$ such that $\langle C \cdot M, N\rangle_t = \int_0^t C_s \, d\langle M, N\rangle_s$ for any $N \in \mathscr{L}$. (The preceding is a Lebesgue–Stieltjes integral for a.a. fixed ω.) We also write $C \cdot M_t = \int_0^t C_s \, dM_s$. The reader is referred to Meyer (1967) for further properties of this integral, including a change of variable formula, or "Itô's lemma." We will also use the notation $[[R, S]] = \{(t, w): R(w) \leqq t \leqq S(w)$, and $t < \infty\}$ to denote stochastic intervals, where R and S are nonnegative random variables.

We recall a stopping time is said to be *previsible* (or *predictable*) if there exists an increasing sequence (S^k) of stopping times such that $\lim_{k\to\infty} S^k = R$, and $S^k < R$ on $\{R > 0\}$ for each k. Such a sequence *announces* R. Previsible stopping times are treated in detail in, for example, Dellacherie (1972).

We now begin the development of the seminorm. For a process $Z \in \mathscr{S}$ and a stopping time T, we define

$$\nu_T(Z) = Z_0^2 1_{\{T>0\}} + \sup_{t \leqq T} L_t^2 + \left(\int_0^T |dA_t|\right)^2 \tag{2.1}$$

where $Z_t = Z_0 + L_t + A_t$, with $(Z_0 + A_t) \in \mathscr{F}_0 \oplus \mathscr{A}$, and $(L_t) \in \mathscr{L}$. We also write $|A|_t = \int_0^t |dA_s|$, the total variation up to t of the signed measure induced by A. For any $Z \in \mathscr{S}$ and stopping time T, we let

$$||Z||_T = (E\{\nu_T(Z)\})^{\frac{1}{2}}$$

denote a seminorm on $\mathscr{S}$ which depends on T.

The following sequence of lemmas develops the desired properties of the seminorm $||\cdot||_T$.

(2.2) LEMMA. *Suppose T is a stopping time, and $(Z^m) \subset \mathscr{S}$ such that $||Z^m||_T < \infty$ for each m, and (Z^m) is Cauchy in $||\cdot||_T$. Then there exists a $Z \in \mathscr{S}$ such that Z stops at T and $\lim_{m\to\infty} ||Z^m - Z||_T = 0$. Also, there exists a subsequence m' such that $(Z_t^{m'})$ stopped at T converges uniformly in t to Z_t^T.*

PROOF. We will show the existence of the three parts of Z: Z_0, L, A.

Part 1. Clearly $Z_0^m 1_{\{T>0\}}$ is Cauchy in $L^2(dP)$ and we let Z_0 denote its limit. We let m' denote a subsequence such that $Z^{m'} 1_{\{T>0\}} \to Z_0$ a.s. We take here $Z_t^m = Z_0^m + L_t^m + A_t^m$.

Part 2. The class of all centered square-integrable martingales is a Hilbert space under the norm $||M||_2 = E\{M_\infty^2\}^{\frac{1}{2}}$, and all continuous squareintegrable martingales is a closed subspace. Assuming without loss of generality that L_t^m stops at T for each m, (Z_m) Cauchy in $||\cdot||_T$ implies L^m Cauchy in $||\cdot||_2$, hence there is a continuous L^2 martingale L which is its limit. Also $\sup_t |L_t^m|$ Cauchy in $L^2(dP)$ implies there exists a subsequence m' such that $\lim_{m'} \sup_t |L_t^{m'} - L_t| = 0$ a.s.

Part 3. We assume A_t^m stops at T for each m. We can choose a subsequence of (m), which we call (k), such that $E\{\sup_t |A_t^{k+1} - A_t^k|^2\}^{\frac{1}{2}} \leqq 2^{-k}$. Then A_t^k is dominated uniformly in k by an L^2 function, and A_t^k converges a.s., uniformly in t. We let $A_t = \lim_k A_t^k$ a.s. Then A_t is the uniform limit of continuous functions, so a.s. $t \to A_t(\omega)$ is continuous, and is adapted. It remains to show $|A|_T < \infty$ and that A^m converges to A in expected squared total variation. For any $\varepsilon > 0$, there exists an N such that for $n, m \geqq N$,

$$\begin{aligned} \varepsilon &> E(|A^n - A^m|_T)^2 \\ &\geqq E(\textstyle\sum_{t_i \in P_j} |(A^n_{t_i+1} - A^m_{t_i+1}) - (A^n_{t_i} - A^m_{t_i})|)^2 \end{aligned}$$

where P_j is a sequence of refining partitions of $[0, 1]$ with mesh $(P_j) \to 0$. Letting $m \to \infty$ through the subsequence k and using Fatou's lemma one obtains, for $n \geqq N$, $E(|A^n - A|_T)^2 \leqq \varepsilon$. $|A|_T$ is clearly in $L^2(dP)$. This completes the proof of Lemma 2.2.

(2.3) LEMMA. *Let $(Z^n) \subset \mathscr{S}$ and (R_k) be a sequence of stopping times* a.s. *increasing to ∞. Suppose also that $||Z^n||_{R_k} < \infty$ for all n and k, and that (Z^n) is a Cauchy sequence in $||\cdot||_{R_k}$ for each k. Then there exists a $Z \in \mathscr{S}$ such that $\lim_{n\to\infty} ||Z - Z^n||_{R_k} = 0$ for each k, and there exists a subsequence m' such that* a.s. *$\lim_{m'\to\infty} Z_t^{m'} = Z_t$ for $t \in (0, \infty)$. Also, $Z_t^{m'}$ converges to Z_t uniformly in t* a.s. *on $[[0, R_k]]$ for each k.*

PROOF. Let $k = 1$. Then by Lemma 2.2 we know there exists a $Z^{(1)} \in \mathscr{S}$ such that $Z^{(1)}$ stops at R_1 and $\lim_{n\to\infty} \|Z^n - Z^{(1)}\|_{R_1} = 0$. Further, except on a P-null set Λ_1, there exists a subsequence $n(1)$ of n such that $Z_t^{n(1)}$ converges to $Z_t^{(1)}$ uniformly in t on $[[0, R_1]]$. Inductively we get a P-null set Λ_k, a subsequence $n(k)$ of $n(k-1)$, and a $Z^{(k)} \in \mathscr{S}$ stopping at R_k such that $\lim_{n\to\infty} \|Z^n - Z^{(k)}\| = 0$, and for $\omega \notin \Lambda_k$, $Z_t^{n(k)}$ converges to $Z_t^{(k)}$ uniformly in t on $[[0, R_k]]$. Let $\Lambda = \bigcup_{k=1}^{\infty} \Lambda_k$, so that $P(\Lambda) = 0$. If $k' > k$ and $\omega \notin \Lambda$, then on $((0, R_k]]$, $\lim Z_t^{n(k')} = Z_t^{(k')} = Z_t^{(k)}$, so a.s. $Z_t^{(k')} = Z_t^{(k)}$ on $((0, R_k]]$. Since $[[0, R_k]]$ increases to $\mathbf{R}_+ \times \Omega$ as k increases to ∞, we can now well define Z by $Z_t = Z_t^{(k)}$ on $((0, R_k]]$, and $Z_0 = Z_0^{(k)}$ on $\{R_k > 0\}$. Our one subsequence of (n) consists of the sequence having its kth term be the kth term of the sequence $n(k)$. This completes the proof of Lemma 2.3.

3. A simple stochastic differential equation. In this section we establish a theorem whose hypotheses are simple. This prevents the notation from obscuring the proof, and we will refer to this proof when proving Theorem 5.3, which includes this theorem as a special case. Note that since the Wiener process is in $\mathscr{L}$ and $V_t(\omega) \equiv t$ (nonrandom) is in $\mathscr{A}$, the following theorem includes as a special case the standard existence and uniqueness theorem as presented, for example, in Gihman and Skorohod (1972).

(3.1) THEOREM. *Fix an* $M \in \mathscr{L}$, *a* $V \in \mathscr{A}$, *and a* $Y_0 \in \mathscr{F}_0$ *such that* a.s. $|Y_0| < \infty$. *Let* $f(t, x)$ *and* $g(t, x)$ *mapping* $\mathbf{R}_+ \times \mathbf{R}$ *to* $\mathbf{R}$ *be jointly continuous and satisfy*

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leqq K|x - y| \tag{3.2}$$

for all x, y *and* t, *and for some constant* K. *Then there exists a unique* $Y \in \mathscr{S}$ *such that*

$$Y_t = Y_0 + \int_0^t f(s, Y_s)\, dM_s + \int_0^t g(s, Y_s)\, dV_s\,. \tag{3.3}$$

PROOF. Let C_k be a sequence of constants increasing to ∞. We define the following stopping times:

$$T_1^k = \inf\{t > 0 : |M_t| \geqq C_k\} \qquad T_2^k = \inf\{t > 0 : |V|_t \geqq C_k\}$$
$$T_3^k = k 1_{\{|Y_0| \leqq C_k\}} \qquad T_4^k = \inf\{t > 0 : \langle M, M\rangle_t \geqq C_k\}$$
$$T^k = \min_{1\leqq i\leqq 4} T_i^k\,.$$

Note that $\lim_{k\to\infty} T^k = \infty$ a.s.

We now fix an arbitrary k and let $T = T^k$; we let $N_t = M_t^T$, $W_t = V_t^T$, and $Z_0 = Y_0 1_{\{T>0\}}$. Since $X \in \mathscr{S}$ implies $t \to f(t, X_t)$ is continuous, $\int_0^t f(s, X_s)\, dN_s$ and $\int_0^t g(s, X_s)\, dW_s$ both exist, and the first defines an element of $\mathscr{L}$ for $X \in \mathscr{S}$.

We define $Y_t^0 = Z_0$. Inductively, we define

$$Y_t^{n+1} = Z_0 + \int_0^t f(s, Y_s^n)\, dN_s + \int_0^t g(s, Y_s^n)\, dW_s\,. \tag{3.4}$$

Suppose $Y^n \in \mathscr{S}$. Since $s \to f(s, Y_s^n)$ is continuous, it is in $L^2_{\mathrm{loc}}(N)$, so $\int_0^t f(s, Y_s^n)\, dN_s \in \mathscr{L}$ and $\int_0^t g(s, Y_s^n)\, dW_s \in \mathscr{A}$.

Now let R be an arbitrary stopping time. Working first with the local martingale part of $Y^{n+1} - Y^n$, by approximating R by bounded stopping times and using Doob's maximal quadratic inequality ([9], page 94) for continuous local martingales we get:

$$\begin{aligned} E\{\sup_{t\leq R}(\textstyle\int_0^t f(s, Y_s^n)\,dN_s - \int_0^t f(s, Y_s^{n-1})\,dN_s)^2\} \\ \leqq 4K^2E\{\textstyle\int_0^R |Y_s^n - Y_s^{n-1}|^2\,d\langle N, N\rangle_s\} \\ \leqq 8K^2E\{\textstyle\int_0^R \nu_s(Y^n - Y^{n-1})\,d\langle N, N\rangle_s\} \end{aligned} \tag{3.5}$$

using the Lipschitz condition, $(a+b)^2 \leqq 2a^2 + 2b^2$ for positive a, b, and where ν_s is as defined in (2.1).

We next consider the part of $Y^{n+1} - Y^n$ that is in $\mathscr{A}$. We have

$$\begin{aligned} E\{(\textstyle\int_0^R |d(\int_0^t g(s, Y_s^n)\,dW_s - \int_0^t g(s, Y_s^{n-1})\,dW_s)|)^2\} \\ \leqq E\{(\textstyle\int_0^R |g(s, Y_s^n) - g(s, Y_s^{n-1})|\,|dW_s|)^2\} \\ \leqq K^2E\{|W|_R \textstyle\int_0^R |Y_s^n - Y_s^{n-1}|^2|dW_s|\}\,, \end{aligned}$$

by the Cauchy-Schwarz inequality and (3.2). Since $|W|_R$ is uniformly bounded, we have for some new constant C,

$$E\{(\textstyle\int_0^R |d(\int_0^t g(s, Y_s^n) - g(s, Y_s^{n-1})\,dW_s)|)^2\} \leqq CE\{\textstyle\int_0^R \nu_s(Y^n - Y^{n-1})|dW_s|\}\,. \tag{3.6}$$

We now let $A_t = \langle N, N\rangle_t + |W|_t$. Combining equations (3.5) and (3.6) we have for some new constant K,

$$E\{\nu_R(Y^{n+1} - Y^n)\} \leqq CE\{\textstyle\int_0^R \nu_s(Y^n - Y^{n-1})\,dA_s\}\,. \tag{3.7}$$

The inequality (3.7) suggests a Gronwall-type procedure, and since we use it repeatedly in this paper, we state the next step in the proof as a lemma.

(3.8) LEMMA. (i) *Let $(A_t) \in \mathscr{A}^+$ be such that $A_\infty \leqq a$ a.s., where a is a finite constant. Let Q be a stopping time such that $(X^n) \subset \mathscr{S}$ satisfies $\|X^n\|_Q < \infty$, for each n. Suppose for any stopping time R, we have*

$$E\{\nu_R(X^n - X^{n-1})\} \leqq CE\{\textstyle\int_0^R \nu_s(X^{n-1} - X^{n-2}\}\,dA_s\}\,. \tag{3.9}$$

Then (X^n) is Cauchy in $\|\cdot\|_Q$.

(ii) *Further, if the hypotheses of* (i) *are satisfied for a sequence of stopping times (Q_k) increasing to ∞, then there exists an $X \in \mathscr{S}$ such that $\lim_{n\to\infty} \|X^n - X\|_{Q_k} = 0$ for each k.*

PROOF. Define $\tau_t = \inf\{s > 0: A_s > t\}$, the right-continuous inverse of the increasing process (A_t). For each fixed t, $\{\tau_t < s\} = \bigcup_n \{A_{s-1/n} > t\} \in \mathscr{F}_s$, since (A_s) is adapted. So τ_t is a stopping time for each t. By (3.9) we have

$$\begin{aligned} E\{\nu_{\tau_t\wedge Q}(X^n - X^{n-1})\} &\leqq CE\{\textstyle\int_0^{\tau_t\wedge Q} \nu_s(X^{n-1} - X^{n-2})\,dA_s\} \\ &\leqq CE\{\textstyle\int_0^{A_\infty\wedge t} \nu_{\tau_s\wedge Q}(X^{n-1} - X^{n-2})\,ds\} \\ &\leqq C\textstyle\int_0^{a\wedge t} E\{\nu_{\tau_s\wedge Q}(X^{n-1} - X^{n-2})\}\,ds \end{aligned}$$

by Lebesgue's change of time formula (see, for example, page 91 of Dellacherie

(1972)), and by Fubini's theorem. For notational simplicity, let $g_n(t) = E\{\nu_{\tau_t \wedge Q}(X^n - X^{n-1})\}$. For $t \leqq a$, the above yields $g_n(t) \leqq C \int_0^t g_{n-1}(s)\, ds$. Using Fubini's theorem repeatedly we have

$$\begin{aligned} g_n(t) &\leqq C^2 \int_0^t \int_0^s g_{n-2}(u)\, du\, ds \\ &= C^2 \int_0^t (t-u) g_{n-2}(u)\, du \\ &\leqq C^3 \int_0^t \frac{(t-u)^2}{2!}\, g_{n-3}(u)\, du \end{aligned}$$

and continuing inductively yields

$$g_n(t) \leqq C^{n-1} \int_0^t \frac{(t-u)^{n-2}}{(n-2)!}\, g_1(u)\, du\,. \tag{3.10}$$

But $g_1(u)$ is bounded, since $||X^n||_Q < \infty$ for each n, in particular for $n = 0, 1$. Therefore (3.10) implies, for $t \leqq a$,

$$g_n(t) = ||X^n - X^{n-1}||^2_{\tau_t \wedge Q} \leqq \frac{K(Ca)^{n-1}}{(n-1)!}$$

which is the nth term of a convergent series. Letting $t = a$, we observe that $\tau_a \equiv \infty$ a.s., since $A_\infty \leqq a$ a.s. We conclude that

$$||X^n - X^{n-1}||_Q{}^2 \leqq \frac{K(Ca)^{n-1}}{(n-1)!}$$

and so (X^n) is a Cauchy sequence $||\cdot||_Q$.

The proof of (ii) is of course nothing more than an application of Lemma 2.3, given the truth of (i). This completes the proof of Lemma 3.8.

Returning to the proof of Theorem 3.1, we wish to show the sequence (Y^n) as defined in (3.4) converges to a process $Z \in \mathscr{S}$. By (3.7) and Lemma 3.8, it will suffice to show that there exists a sequence of stopping times (R_m) increasing to ∞ such that $||Y^n||_{R_m} < \infty$ for all n and m. However, $||Y^n||_{R_m} < \infty$ for all n provided that R_m is bounded: clearly $||Y^0||_R < \infty$, since $|Y^0|$ is bounded; assuming $||Y^n||_R < \infty$, we have

$$\nu_R(Y^{n+1}) \leqq C_1 + 4E(\int_0^R f(s, Y_s^n)^2\, d\langle N, N\rangle_s) + E\{|W|_R \int_0^R g(s, Y_s^n)^2 |dW_s|\}$$

and since R is bounded, f Lipschitz and continuous implies $|f(t, x)|^2 \leqq K(1 + |x|^2)$ on $[0, R(\omega)]$, so

$$\begin{aligned} \nu_R(Y^{n+1}) &\leqq C_1 + 4C_2(1 + E\{\sup_{s\leqq R} |Y_s^n|^2\}) + C_3(1 + E\{\sup_{s\leqq R} |Y_s^n|^2\}) \\ &\leqq C(1 + 4||Y^n||_R{}^2)\,. \end{aligned}$$

Therefore $\nu_{R_m}(Y^{n+1}) < \infty$ and so $||Y^{n+1}||_{R_m} < \infty$. Thus we may let $Z \in \mathscr{S}$ denote the limit of the sequence (Y^n): that is, for any sequence of bounded stopping times (R_m) increasing to ∞, we have $\lim_{n\to\infty} ||Y^n - Z||_{R_m} = 0$, for each m.

Such a process $Z \in \mathscr{S}$ stops at the stopping time $T = T^k$ chosen at the beginning of the proof. Let us denote this limit by $Z = Z^k$. Then by Lemma 2.3 we may well define a new process Y on $[[0, \infty))$ by:

$$(3.11)\qquad \begin{aligned} Y_t(w) &= Z_t^k(\omega) \quad \text{if} \quad 0 < t < T^k(\omega) \\ &= Z_0^k(\omega) \quad \text{if} \quad t = 0 \quad \text{and} \quad T^k(\omega) > 0 . \end{aligned}$$

The process Y given in (3.11) is our candidate for the solution to equation (3.3). It remains to show that Y actually satisfies (3.3) and is the unique process that does so. Since limits are unique, it suffices to show that for each k,

$$(3.12)\qquad \lim_{n\to\infty} ||Y^{n+1} - Y_0 - \textstyle\int_0^t f(s, Y_s)\, dM_s - \int_0^t g(s, Y_s)\, dV_s||_{T^k} = 0 .$$

We fix an arbitrary k and let $T = T^k$. Using (3.4) and letting $A_t = \langle M, M\rangle_{t\wedge T} + |V|_{t\wedge T}$, we have

$$(3.13)\qquad \begin{aligned} &||Y_0 + \textstyle\int_0^t f(s, Y_s)\, dM_s + \int_0^t g(s, Y_s)\, dV_s - Y_t^{n+1}||_T \\ &\qquad \leqq CE\{\textstyle\int_0^T |Y_s - Y_s^n|^2\, dA_s\} \\ &\qquad \leqq C'E\{\nu_T(Y - Y^n)\} \\ &\qquad = C'||Y - Y^n||_T^2 , \end{aligned}$$

since T may be assumed bounded, without loss of generality. Since $\{(t, \omega): 0 \leqq t \leqq T^k(\omega); T^k(\omega) > 0\}$ increases to $\mathbf{R}_+ \times \Omega$, we deduce Y satisfies (3.3).

Suppose there exists a $Z \in \mathscr{S}$ such that Z also satisfies (3.3). By analogous reasoning as the preceding, one obtains $||Y - Z||_{T^k} = 0$ for each k; hence $Y = Z$. This completes the proof of Theorem 3.1.

For simplicity we have stated Theorem 3.1 with only one local martingale differential, and only one differential of bounded variation. The proof of the theorem with a finite number of martingales and/or processes of bounded variation, while notationally more cumbersome, is exactly analogous.

Similarly, the extension of Theorem 3.1 to systems of equations poses no new difficulties. We let d be the (implicit) number of equations. We let $\mathbf{S} = (\mathscr{S}, \mathscr{S}, \cdots, \mathscr{S})$ denote the class of (d-dimensional) vector semimartingales. For a vector $\mathbf{x} = (x^1, \cdots, x^d)$ in $\mathbf{R}^d$ we let $|\mathbf{x}| = \sum_{i=1}^d |x^i|$. For a $\mathbf{Z} \in \mathbf{S}$ we define the vector norm by

$$(3.14)\qquad ||\mathbf{Z}||_T = \textstyle\sum_{i=1}^d ||Z^i||_T$$

for some stopping time T. Lemmas 2.2 and 2.3 are true for the obvious generalizations (Protter, 1975). With these definitions, we may now state the analogue of Theorem 3.1 that holds for systems of equations. We omit the proof, which is a straightforward adaptation of the proof of Theorem 3.1.

(3.15) THEOREM. *Fix* $M^1, \cdots, M^p \in \mathscr{L}$; $V^1, \cdots, V^q \in \mathscr{A}$; *and* $\mathbf{Y}_0$, *where* $Y_0^i \in \mathscr{F}_0$ *such that* $|Y_0^i| < \infty$ a.s. *for* $1 \leqq i \leqq d$. *For* $1 \leqq j \leqq p$, $1 \leqq k \leqq q$, *let* $\mathbf{f}_j(t, \mathbf{x}) : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}^d$ *and* $\mathbf{g}_k(t, \mathbf{x}) : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}^d$ *be such that* $f_j^i(t, \mathbf{x})$ *and* $g_k^i(t, \mathbf{x})$ *are jointly continuous for* $1 \leqq i \leqq d$. *Further, suppose they satisfy*

$$(3.16)\qquad |\mathbf{f}_j(t, \mathbf{x}) - \mathbf{f}_j(t, \mathbf{y})| \leqq K|\mathbf{x} - \mathbf{y}| , \qquad |\mathbf{g}_k(t, \mathbf{x}) - \mathbf{g}_k(t, \mathbf{y})| \leqq K|\mathbf{x} - \mathbf{y}|$$

for some constant K, *and all* t, $\mathbf{x}$ *and* $\mathbf{y}$. *Then there exists a unique* $\mathbf{Y} \in \mathbf{S}$ *that* *satisfies*

$$\mathbf{Y}_t = \mathbf{Y}_0 + \sum_{j=1}^{p} \int_0^t \mathbf{f}_j(s, \mathbf{Y}_s)\, dM_s^j + \sum_{k=1}^{q} \int_0^t \mathbf{g}_k(s, \mathbf{Y}_s)\, dV_s^k .$$

4. Exploding solutions. As discussed in the introduction, Theorems 3.1 and 3.15 improve even upon the standard result with the Wiener process and Lebesgue measure as differentials, since they eliminate the growth condition on the coefficients while obtaining nonexploding solutions. One then wonders if also the Lipschitz conditions (3.2) (respectively (3.16)) can be weakened. The answer is that they can, but one must be careful: in Theorem 5.3 if we take $K_t(\omega) = K_m$ for $t \in (m-1, m]$, we see that the hypotheses of Theorems 3.1 and 3.15 are stronger than they need to be, but Theorem 4.3 shows that a straightforward weakening of the Lipschitz condition introduces "explosions" into the solutions.

We begin with the definition of an explosion time.

(4.1) DEFINITION. A stopping time R is called an *explosion time* for a process $\mathbf{Z}$ if R is a previsible stopping time such that for any sequence of times (S^k) announcing R, $\mathbf{Z}$ is defined on $[[0, S^k]]$, and $\limsup_{t \uparrow R} |\mathbf{Z}_t| = \infty$.

(4.2) DEFINITION. For any stopping time R we let $\mathscr{S}_R$ (respectively $\mathbf{S}_R$) be the family of $\mathbf{R}$-valued (resp. $\mathbf{R}^d$-valued) processes such that $Z \in \mathscr{S}_R$ (resp. $\mathbf{Z} \in \mathbf{S}_R$) if there exists a sequence (S^k) of stopping times, $S^k \leqq R$ and increasing to R, such that $Z_t^{S^k} \in \mathscr{S}$ (resp. $\mathbf{Z}_t^{S^k} \in \mathbf{S}$) for each k.

Note that R need not be previsible in Definition 4.2; hence we only require that $[[0, R)) \subset \bigcup_{k=1}^{\infty} [[0, S^k]] \subset [[0, R]]$. Functions that satisfy the condition below we will call *weak Lipschitz*.

(4.3) THEOREM. *Let the hypotheses of Theorem* 3.15 *be satisfied, except that in place of* (3.16) *we require only that* $(|f_j(t,x) - f_j(t,y)| + |g_k(t,x) - g_k(t,y)|)/|x-y|$ *as a function of* (t, x, y) *be bounded on compact sets, where* $1 \leqq j \leqq p$, $1 \leqq k \leqq q$. *Then there exists a unique process* $\mathbf{Y}$ *in* $\mathbf{S}_R$ *with explosion time* R *such that on* $[[0, R))$ $\mathbf{Y}$ *satisfies*

$$\mathbf{Y}_t = \mathbf{Y}_0 + \sum_{j=1}^{p} \int_0^t \mathbf{f}_j(s, \mathbf{Y}_s)\, dM_s^j + \sum_{k=1}^{q} \int_0^t \mathbf{g}_k(s, \mathbf{Y}_s)\, dV_s^k . \tag{4.5}$$

PROOF. For notational simplicity, we establish only the case where $d = p = q = 1$, and hence denote the coefficients by $f(t, x)$ and $g(t, x)$. The proof for general d, p and q is an obvious generalization. The method of proof will be to extend f and g off of $U^m = [0, m] \times [-m, m] \subset \mathbf{R}_+ \times \mathbf{R}$ in such a way that we may use Theorem 3.15. We then have a solution on each U^m such that if $m' > m$, the solution for m' agrees with the one for m on U^m. We begin by defining

$$\begin{aligned} f^m(t, x) &= f(t, x) \qquad \text{if} \quad (t, x) \in U^m \\ &= f(u, y) \qquad \text{if} \quad (t, x) \notin U^m \end{aligned} \tag{4.6}$$

where (u, y) is the point on ∂U^m that minimizes $\{|(u, y) - (t, x)| : (u, y) \in \partial U^m\}$. We define $g^m(t, x)$ in the analogous manner. Both f^m and g^m are Lipschitz on

$\mathbf{R}_+ \times \mathbf{R}$ and jointly continuous. So by Theorem 3.1 (or Theorem 3.15 for the vector case) there exists a unique process $Y^m \in \mathscr{S}$ such that

$$Y_t^m = Y_0 + \int_0^t f^m(s, Y_s^m)\, dM_s + \int_0^t g^m(s, Y_s^m)\, dV_s\,. \tag{4.7}$$

We define the stopping times $S^m = \inf\{t > 0 : (t, Y_t^m) \in (U^m)^c\}$.

We fix an m and let $S = S^m$. Using (4.6) and (4.7), we have

$$\begin{aligned}(Y^m)^S &= Y_0 1_{\{S>0\}} + \int_0^{t\wedge S} f^m(s, Y_s^m)\, dM_s + \int_0^{t\wedge S} g^m(s, Y_s^m)\, dV_s \\ &= Y_0 1_{\{S>0\}} + \int_0^{t\wedge S} f(s, Y_s^m)\, dM_s + \int_0^{t\wedge S} g(s, Y_s^m)\, dV_s \\ &= Y_0 1_{\{S>0\}} + \int_0^{t\wedge S} f^{m+1}(s, Y_s^m)\, dM_s + \int_0^{t\wedge S} g^{m+1}(s, Y_s^m)\, dV_s\end{aligned}$$

since $f^m = f = f^{m+1}$ on U^m, and $(s \wedge S, Y^m(s \wedge S)) \in U^m$. We have used the simple result that for a process $C \in L^2_{\mathrm{loc}}(M)$ and a stopping time S, $(C \cdot M)_{t\wedge S} = (C^S \cdot M)_{t \wedge S}$, where $C \cdot M_t$ is the stochastic integral $\int_0^t C_s\, dM_s$. Using the uniqueness of each Y^m as a solution to (4.7), we may conclude from the above that

$$Y_t^m = Y_t^{m+1} \quad \text{on} \quad [[0, S]]\,. \tag{4.8}$$

Since each Y^m is continuous, $S^m < S^{m+1}$ on $\{S^m < \infty\}$, so $R = \lim S^m$ is a previsible stopping time. Indeed, R is the explosion time referred to in the statement of the theorem. By (4.8) we define

$$Y_t = Y_t^m \quad \text{on} \quad [[0, S^m]]\,. \tag{4.9}$$

It is easy to verify that for any sequence of stopping times (T^k) increasing to R we have $Y^{T^k} \in \mathscr{S}$, for each k.

It is straightforward to verify that, since Y satisfies (4.7), and hence (4.5) on $[[0, S^m]]$, it must satisfy (4.5) on the whole interval $[[0, R))$.

Suppose $Z \in \mathscr{S}_R$ also satisfies (4.5). Then $(Z_t^{S^m})$ will satisfy equation (4.7), and so by the uniqueness result of Theorem 3.1 we have $Z_t 1_{\{S^m>0\}} = Y_t 1_{\{S^m>0\}}$ on $[[0, S^m]]$, for each m. Thus $Z_t 1_{[[0,R))} = Y_t 1_{[[0,R))}$. This completes the proof of Theorem 4.3.

Theorem 4.3 raises the possibility that the solution will explode if the coefficients are permitted to grow too fast. If an explosion time R is such that $P(\{R = \infty\}) = 1$, then of course $\mathscr{S}_R = \mathscr{S}$ (resp. $\mathbf{S}_R = \mathbf{S}$), and Theorem 4.3 emerges as stronger than Theorem 3.15. This is not true in general, but one may put additional conditions on $\mathbf{f}_j(t, \mathbf{x})$ and $\mathbf{g}_k(t, \mathbf{x})$ which make $P(\{R = \infty\}) = 1$; that is, they make explosions impossible. For a result of this type, the reader is referred to Protter (1975).

McKean (1969) states and proves what he calls "Feller's test for explosions," where he considers the equation

$$X_t = x + \int_0^t f(X_s)\, dB_s + \int_0^t g(X_s)\, ds$$

where (B_t) is the Brownian motion martingale with $B_0 = 0$ a.s., and x represents a sure initial state. Assuming $f\,(\neq 0)$ and g are continuously differentiable (hence weak Lipschitz), McKean obtains a condition for explosions to be a.s. impossible. The following theorem replaces (B_t) with an arbitrary $M \in \mathscr{L}$ and allows a stochastic initial state. For the proof, which is adapted from McKean's, we refer the reader to Protter (1975).

(4.13) THEOREM. *Let $M \in \mathscr{L}$ and $f, g: \mathbf{R} \to \mathbf{R}$ be continuous and weak Lipschitz such that $f \neq 0$. Let $X_0 \in \mathscr{F}_0$ be finite* a.s. *and let $(X_t) \in \mathscr{S}_R$ satisfy*

$$X_t = X_0 + \int_0^t f(X_s)\,dM_s + \int_0^t g(X_s)\,d\langle M, M\rangle_s$$

where R is the explosion time for X. Let

$$\hat{x} = j(x) = \int_0^x \exp(-2\int_0^u g(v)[f(v)]^{-2}\,dv)\,du\,,$$

and

$$(a) = \int_{-\infty}^0 [j(x) - j(-\infty)]\hat{f}(j(x))^{-2}j(dx)$$
$$(b) = \int_0^\infty [j(\infty) - j(x)]\hat{f}(j(x))^{-2}j(dx)$$

where $\hat{f}(j(x)) = j'(x)f(x)$. We have

(i) *if both $(a) = (b) = \infty$, then $P(\{R = \infty\}) = 1$;*

(ii) *if $\lim_{t\to\infty} \langle M, M\rangle_t = \infty$* a.s. *and either $P(\{X_0 > 0\}) = 1$ and $(b) < \infty$, or $P(\{X_0 < 0\}) = 1$ and $(a) < \infty$, then $P(\{R = \infty\}) < 1$.*

5. Existence and uniqueness for stochastic coefficients. We next consider the possibility of coefficients which are themselves random. This has been considered previously by Gihman and Skorohod ((1972), page 50) for Brownian motion and Lebesgue measure differentials, and also by McShane ((1974), page 154). McShane considers a more general equation than Gihman and Skorohod, using the definition of stochastic integral which he developed. He does not, however, permit the Lipschitz constants to depend on time, and in his existence theorems he requires the differentials to satisfy several Lipschitz conditions which Brownian motion satisfies, but which do not, in general, hold for elements of $\mathscr{L}$.

Due to the nature of stochastic coefficients, we must relax the condition that an explosion time must be previsible, as required in Definition 4.1.

(5.1) DEFINITION. A stopping time R is an *explosion time* for a progressively measurable process (K_t) if $R = \lim_{m\to\infty} Q^m$, where

$$(5.2) \qquad Q^m = \min(\inf\{t > 0: |K_t| > m\}, m1_{\{K_0 < m\}})\,.$$

We require (K_t) to be progressively measurable so that Q^m are stopping times (see Dellacherie (1972)). A previsibly measurable process is progressively measurable.

(5.3) THEOREM. *Fix $M^1, \cdots, M^p \in \mathscr{L}$; $V^1, \cdots, V^q \in \mathscr{A}$; $\eta^1, \cdots, \eta^d \in \mathscr{F}_0 \oplus \mathscr{A}$. Let $\mathbf{F}_j(t, \mathbf{x}) = (F_j^1(t, \mathbf{x}), \cdots, F_j^d(t, \mathbf{x}))$, $1 \leqq j \leqq p$; and $\mathbf{G}_k(t, \mathbf{x})$, $1 \leqq k \leqq q$, be random $\mathbf{R}^d$-valued functions such that* a.s. *$(t, \mathbf{x}) \to \mathbf{F}_j(t, \mathbf{x})$ and $(t, \mathbf{x}) \to \mathbf{G}_k(t, \mathbf{x})$ are jointly continuous, and $F_j^i(t, \mathbf{x})$, $G_k^i(t, \mathbf{x})$ are $\mathscr{F}_t$-measurable for each fixed t and $\mathbf{x}$, $1 \leqq i \leqq d$. We also require that there exist a finite-valued process (K_t) defined on $[[0, R))$ such that $\mathbf{F}_j$ and $\mathbf{G}_k$ satisfy, on $[[0, R))$:*

$$(5.4) \qquad \max_{1\leqq j\leqq p; 1\leqq k\leqq q}(|\mathbf{F}_j(t, \mathbf{x}, \omega) - \mathbf{F}_j(t, \mathbf{y}, \omega)|, |\mathbf{G}_k(t, \mathbf{x}, \omega) - \mathbf{G}_k(t, \mathbf{y}, \omega)|) \leqq K_t(\omega)|\mathbf{x} - \mathbf{y}|\,.$$

Let R be the explosion time of (K_t). Then there exists a unique process $(\mathbf{X}_t) \in \mathbf{S}_E$ such that for $1 \leqq i \leqq d$ on $[[0, R))$, $\mathbf{X}$ satisfies

$$(5.5) \qquad X_t^i = \eta_t^i + \sum_{j=1}^p \int_0^t F_j^i(s, \mathbf{X}_s)\, dM_s^j + \sum_{k=1}^q \int_0^t G_k^i(s, \mathbf{X}_s)\, dV_s^k .$$

PROOF. Without loss of generality, we may assume that

$$(5.6) \qquad K_t = \sup_{x^*, y^*} \max_{1\leqq j \leqq p; 1 \leqq k \leqq q} (|\mathbf{F}_j(t, \mathbf{x}^*) - \mathbf{F}_j(t, \mathbf{y}^*)|/|\mathbf{x}^* - \mathbf{y}^*|, \\ |\mathbf{G}_k(t, \mathbf{x}^*) - \mathbf{G}_k(t, \mathbf{y}^*)|/|\mathbf{x}^* - \mathbf{y}^*|)$$

where $\mathbf{x}^*, \mathbf{y}^* \in \mathbf{Q}^d$, $\mathbf{Q}$ being the rational numbers. Then (K_t) is the supremum of countably many previsible processes and so is previsible. Also, (5.6) implies that a.s. $t \to K_t$ is lower semicontinuous.

Since (K_t) is previsible, (Q^m) as defined in (5.2) are stopping times, and so also is R. Further, the lower semicontinuity of (K_t) assures us that $|K_t^{Q^m}| = K_t^{Q^m} \leqq m$: suppose ω_0 is not in the exceptional set where (K_t) is not lower semicontinuous. Let $Q^m(\omega_0) = t_0$ and $K_{t_0}(\omega_0) > m$. Then there will exist a neighborhood N_0 of t_0 such that $K_t(\omega_0) > m$ for $t \in N_0$. Since $N_0 \cap [0, t_0) \neq \emptyset$, we have $Q^m(\omega_0) < t_0$, which is a contradiction.

From the above and as in the proof of Theorem 3.1 we know we can find a sequence (T^m) of stopping times increasing to R such that for $T = T^m$, $|(\eta^i)_t^T| \leqq m$, $\langle M^j, M^j\rangle_{t\wedge T} \leqq m$, $|(V^k)|_{t\wedge T} \leqq m$, and $K_t^T \leqq m$ for $1 \leqq i \leqq d$, $1 \leqq j \leqq p$, $1 \leqq k \leqq q$ and each m. We fix an arbitrary m and denote: $\phi^i = (\eta^i)^T$; $N_t^j = M_{t\wedge T}^j$; $W_t^k = V_{t\wedge T}^k$; and $C_t = K_t^T$.

We let $Y_t^0 = \phi_t = (\phi_t^1, \cdots, \phi_t^d)$, and inductively define

$$(5.7) \qquad \mathbf{Y}_t^{n+1} = \phi_t + \sum_{j=1}^p \int_0^t \mathbf{F}_j(s, \mathbf{Y}_s^n)\, dN_s^j + \sum_{k=1}^q \int_0^t \mathbf{G}_k(s, \mathbf{Y}_s^n)\, dW_s^k .$$

Each $\mathbf{Y}^n$ so defined is in $\mathbf{S}$: clearly $\mathbf{Y}^0 \in \mathbf{S}$; assuming $\mathbf{Y}^n \in \mathbf{S}$, it is adapted and continuous, so $\mathbf{F}_j(t, \mathbf{Y}_t^n)$ and $\mathbf{G}_k(t, \mathbf{Y}_t^n)$ are also, hence they are previsible. They are also locally bounded, so $F_j^i(t, \mathbf{Y}_t^n) \in L^2_{\text{loc}}(N^j)$, and $\int_0^t F_j^i(s, \mathbf{Y}_s^n)\, dN_s^j \in \mathscr{L}$ for $1 \leqq i \leqq d$, $1 \leqq j \leqq p$. Analogously, $|G_k^i(s, \mathbf{Y}_s^n)|$ is locally bounded, so $\int_0^t G_k^i(s, \mathbf{Y}_s^n)\, dW_s^k$ is of bounded variation since it is a (finite) Riemann–Stieltjes integral. It is adapted and in $\mathscr{A}$ as a consequence of the definition of the Stieltjes integral.

We will now show that $(\mathbf{Y}^n)$ is a Cauchy sequence in the vector norm $\|\cdot\|_T$. Let S be an arbitrary stopping time. Let

$$H(i, j, n, t) = \int_0^t F_j^i(s, \mathbf{Y}_s^n)\, dN_s^j - \int_0^t F_j^i(s, \mathbf{Y}_s^{n-1})\, dN_s^j .$$

Then by Doob's maximal inequality (Meyer, 1966) extended to local martingales, using $(a + b)^k \leqq 2^k(a^k + b^k)$ and using (5.4), we have

$$\begin{aligned} E\{\sup_{t\leqq S} (\textstyle\sum_{j=1}^p H(i, j, n, t))^2\} &\leqq 4E\{(\textstyle\sum_{j=1}^p H(i, j, n, S))^2\} \\ &\leqq 4 \cdot 2^p \textstyle\sum_{j=1}^p E\{\langle H(i, j, n, \cdot), H(i, j, n, \cdot)\rangle_S\} \\ &\leqq 2^{p+2} \textstyle\sum_{j=1}^p E\{\int_0^S C_t^2 |\mathbf{Y}_t^n - \mathbf{Y}_t^{n-1}|^2\, d\langle N^j, N^j\rangle_t\} \end{aligned}$$

and so

$$(5.8) \qquad E\{\sup_{t\leqq S} (\textstyle\sum_{j=1}^p H(i, j, n, t))^2\} \\ \leqq 2^{p+2} 2^{d+1} m^2 \textstyle\sum_{j=1}^p E\{\int_0^S \nu_t(\mathbf{Y}^n - \mathbf{Y}^{n-1})\, d\langle N^j, N^j\rangle_t\} .$$

Again using $(a+b)^k \leqq 2^k(a^k+b^k)$ as well as the Cauchy–Schwarz inequality, we have

$$E\{(\textstyle\sum_{k=1}^q \int_0^S |G_k^i(t, \mathbf{Y}_t^n) - G_k^i(t, \mathbf{Y}_t^{n-1})|\,|dW_t^k|)^2\}$$
$$\leqq 2^q m \textstyle\sum_{k=1}^q E\{\int_0^S C_t^2 |\mathbf{Y}_t^n - \mathbf{Y}_t^{n-1}|^2 |dW_t^k|\}$$
$$\leqq 2^q m^3 2^{d+1} \textstyle\sum_{k=1}^q E\{\int_0^S \nu_t(\mathbf{Y}^n - \mathbf{Y}^{n-1})|dW_t^k|\}\,,$$

and combining this with (5.8) we have:

$$E\{\nu_S(\mathbf{Y}^{n+1} - \mathbf{Y}^n)\} \leqq dm^3 2^{p+q+d+4} E\{\textstyle\int_0^S \nu_t(\mathbf{Y}^n - \mathbf{Y}^{n-1})\, dL_t\} \tag{5.9}$$

where $L_t = \sum_{j=1}^p \langle N^j, N^j\rangle_t + \sum_{k=1}^q |W^k|_t$. Note that $L_\infty \leqq (p+q)m$ a.s. Thus by (5.9) we may apply a vector version of Lemma 3.8 provided there exists a sequence (R_l) of stopping times increasing to R such that $||Y^n||_{R_l} < \infty$ for each n, l. Taking

$$R_l = \min_{1\leqq j\leqq p;1\leqq k\leqq q} (\inf\{t > 0 : |\mathbf{F}_j(t, \mathbf{0})| > l\}, \inf\{t > 0 : |\mathbf{G}_k(t, \mathbf{0})| > l\})$$

we have $\max(|\mathbf{F}_j(t, \mathbf{x})|^2, |\mathbf{G}_k(t, \mathbf{x})|^2) \leqq K(1 + |\mathbf{x}|^2)$ for some constant K, on $[[0, R_l \wedge T]]$. An induction on n as in the proof of Theorem 3.1 shows that $||\mathbf{Y}^n||_{R_l} < \infty$ for each n, l.

Since $\mathbf{Y}^n$ as defined by (5.7) depends implicitly on m, we write $\mathbf{Y}^{n,m} = \mathbf{Y}^n$. Since $\lim_{l\to\infty} R_l = \infty$ a.s. and $\lim_{m\to\infty} T^m = R$ a.s., we may choose one sequence of stopping times (Q^m) with $Q^m \leqq T^m$ and $||Y^{n,m}||_{Q^m} < \infty$ for all n; then by Lemma 3.8 there exists an $\mathbf{X} \in \mathbf{S}_R$ such that

$$\lim_{n\to\infty} ||\mathbf{Y}^{n,m} - \mathbf{X}||_{Q^m} = 0 \tag{5.10}$$

for each m.

It remains to show that the above process $\mathbf{X}$ satisfies equation (5.5) and is the unique process in $\mathbf{S}_R$ that does so. By (5.10) it suffices to show that

$$\lim_{n\to\infty} ||\mathbf{Y}^{n,m} - \eta_t - \textstyle\sum_{j=1}^p \int_0^t \mathbf{F}_j(s, \mathbf{X}_s)\, dM_s^j - \sum_{k=1}^q \int_0^t \mathbf{G}_k(s, \mathbf{X}_s)\, dV_s^k||_{Q^m} = 0\,.$$

By arguments analogous to those leading to (5.9), we have

$$||\mathbf{Y}^{n,m} - \eta_t - \textstyle\sum_{j=1}^p \int_0^t \mathbf{F}_j(s, \mathbf{X}_s)\, dM_s^j - \sum_{k=1}^q \int_0^t \mathbf{G}_k(s, \mathbf{X}_s)\, dV_s^k||_{Q^m}$$
$$\leqq D||\mathbf{Y}^{n,m} - \mathbf{X}||_{Q^m}$$

for some constant D, and the right-hand side tends to 0 as $n \to \infty$, each m.

If $\mathbf{Z} \in \mathbf{S}_R$ also satisfies (5.5), we have for any stopping time S and each $Q = Q^m$:

$$E\{\nu_{Q\wedge S}(\mathbf{Z} - \mathbf{X})\} \leqq DE\{\textstyle\int_0^{Q\wedge S} \nu_t(\mathbf{Z} - \mathbf{X})\, dL_s\}$$

for some constant D. This implies that $||\mathbf{X} - \mathbf{Z}||_{Q^m} = 0$ for each m, hence $\mathbf{X} = \mathbf{Z}$ on $[[0, R))$. This completes the proof of Theorem 5.3.

6. Convergence theorems. Several different questions may, and have, been posed concerning the convergence of solutions of stochastic integral equations under varying hypotheses. Wong and Zakai (1965) investigated the convergence of solutions of ordinary integral equations to the solutions of stochastic integral equations. This was generalized to systems by McShane (1974), who used his own integrals, and allowed more general differentials than just Brownian motion. Theorem 6.7 extends Wong and Zakai's theorem to include general local martingales.

Gihman and Skorohod (1972) consider a convergent sequence of stochastic coefficients and show the solutions converge to the solution of the limit equation. Theorem 6.4 is a somewhat stronger and more general version of their result.

The following theorem shows that when the local martingales $({}^nM^j)$ converge to a local martingale (M^j), and the processes $({}^nV^k)$ of bounded variation converge to V^k, the solutions converge to the solution of the limiting equation. We believe this is the first time this question has been raised.

(6.1) THEOREM. *For all* $n \geqq 1$, ${}^nZ^j = {}^nM^j + {}^nV^j$ *with* $({}^nM^j) \in \mathscr{L}$, $({}^nV^j) \in \mathscr{A}$, *and* $(\eta^n) \in \mathscr{F}_0 \oplus \mathscr{A}$, $1 \leqq j \leqq p$. *Let* $\mathbf{F}_j(t, \mathbf{x})$, $\mathbf{G}_j(t, \mathbf{x})$ *be as in Theorem* 5.3. *Let* R *be the explosion time for the Lipschitz process* (K_t). *Suppose further there exists a sequence of stopping times* (T_m) *increasing to* ∞; *semimartingales* $Z^j = M^j + V^j$ *and processes* $\eta^j \in \mathscr{F} \oplus \mathscr{A}$ *where* (for each j, $1 \leqq j \leqq p$):

(a) $\lim_{n\to\infty} \|{}^nZ^j - Z^j\|_{T_m} = 0$ *for each* m;
(b) $\lim_{n\to\infty} E\{\sup_{s\leqq T_m}|\eta_s^n - \eta_s|^2\} = 0$ *for each* m;
(c) $\lim_{n\to\infty} E\{|\eta^n - \eta|_{T_m}^2\} = 0$ *for each* m;

and let $\mathbf{X}^n$ *and* $\mathbf{X}$ *satisfy on* $[[0, R))$:

$$\mathbf{X}_t^n = \eta_t^n + \sum_{j=1}^p \left(\int_0^t \mathbf{F}_j(s, \mathbf{X}_s^n)\, d^nM_s^j + \int_0^t \mathbf{G}_j(s, \mathbf{X}_s^n)\, d^nV_s^j\right)$$
$$\mathbf{X}_t = \eta_t + \sum_{j=1}^p \left(\int_0^t \mathbf{F}_j(s, \mathbf{X}_s)\, dM_s^j + \int_0^t \mathbf{G}_j(s, \mathbf{X}_s)\, dV_s^j\right) .$$

If (a) *and* (b) *hold, then* $\mathbf{X}^n$ *converges locally to* $\mathbf{X}$ *in maximal quadratic mean. If* (a) *and* (c) *hold, then* $\mathbf{X}^n$ *converges to* $\mathbf{X}$ *in* $\|\cdot\|_{T_m}$, *for each* m.

PROOF. We only give the proof for the case where (a) and (b) hold, since the modification of the proof for when (a) and (c) hold is clear. Also, we only consider the case of one equation, and take $p = 1$. The proof for general d and p follows easily. We need to show only that the hypotheses imply the existence of a convergent subsequence, since the limit is identified. We choose a sequence of stopping times increasing to R, denoted (T^m), that simultaneously satisfies the hypotheses (a), (b), (c).

We shall show that by passing to a subsequence if necessary we may assume without loss of generality that $|V^n|_{t\wedge T^m} \leqq m$, $|V|_{t\wedge T^m} \leqq m$, $\langle M^n, M^n\rangle_{t\wedge T^m} \leqq m$, and $\langle M, M\rangle_{t\wedge T^m} \leqq m$, all a.s. By redefining T^m to be $T^{m'}$ given by

$$T^{m'} = \min\left(T^m, \inf\left\{t > 0 \colon |V|_t > \frac{m}{2}\right\}, \inf\left\{t > 0 \colon \langle M, M\rangle_t > \frac{m}{2}\right\}\right),$$

we may assume $|V|_{t\wedge T^m} \leqq m/2$ and $\langle M, M\rangle_{t\wedge T^m} \leqq m/2$. By (b), $\lim_{n\to\infty} |V^n - V|_{T^m} = 0$ in $L^1(dP)$. Since P has finite mass, we know there exists a subsequence n' such that $\lim_{n'\to\infty} \sup_{t\leqq T^m} |V_t^{n'} - V_t| = 0$ a.s. That is, a.s. $V_{t\wedge T^m}^{n'}$ converges to $V_{t\wedge T^m}$ uniformly in t, and so for large n', $|V^{n'}|_{t\wedge T^m} \leqq m$. By Proposition 3 on page 77 of Meyer (1967), we know that for bounded H and K, and $L, N \in \mathscr{L}$,

$$E\{\textstyle\int_0^t |H_s|\,|K_s|\,|d\langle L, N\rangle_s|\} \leqq E\{\int_0^t H_s^2\, d\langle L, L\rangle_s\}^{\frac{1}{2}} E\{\int_0^t K_s^2\, d\langle N, N\rangle_s\}^{\frac{1}{2}} .$$

Therefore for the subsequence n' obtained above, letting $T = T^m$, and using hypothesis (a),

$$\begin{aligned} E\{&\sup_{t \leq T} |\langle M^{n'}, M^{n'}\rangle_t - \langle M, M\rangle_t|\}^2 \\ &= E\{\sup_{t\leq T} |\langle M^{n'} + M, M^{n'} - M\rangle_t|\}^2 \\ &\leqq E\{\langle M^{n'} + M, M^{n'} + M\rangle_T\} E\{\langle M^{n'} - M, M^{n'} - M\rangle_T\} \\ &= E\{(M^{n'} + M)_T{}^2\} E\{(M^{n'} - M)_T{}^2\} \end{aligned}$$

which converges to 0 as n' tends to ∞, since

$$E\{(M^{n'} + M)_T{}^2\} \leqq 2E\{(M^{n'} - M)_T{}^2\} + 2E\{(M + M\}_T{}^2\} ,$$

which is bounded. Therefore there exists a subsequence n'' such that

$$\lim_{n''\to\infty} \sup_{t\leq T^m} |\langle M^{n''}, M^{n''}\rangle_t - \langle M, M\rangle_t| = 0 ,$$

so a.s. $\langle M^{n''}, M^{n''}\rangle_{t\wedge T^m}$ converges uniformly in t to $\langle M, M\rangle_{t\wedge T^m}$, and so for large n'', $\langle M^{n''}, M^{n''}\rangle_{t\wedge T^m} \leqq m$ a.s.

Without loss of generality we may take $K_t = \max(K_t', |F(t, 0)|, |G(t, 0)|)$, where

$$K_t' = \sup_{x^*, y^*} \max\left(\frac{|F(t, x^*) - F(t, y^*)|}{|x^* - y^*|}, \frac{|G(t, x^*) - G(t, y^*)|}{|x^* - y^*|}\right)$$

where x^*, y^* are in $\mathbf{Q}$. Thus we may also assume without loss of generality that $K_t^{T^m} \leqq m$. Finally, it is also convenient to assume $|X_t^{T^m}| \leqq m$, which of course we may also do without loss of generality.

We now fix an m and let $T = T^m$. Let S be an arbitrary stopping time. Repeatedly using the inequality $(a + b)^2 \leqq 2a^2 + 2b^2$, and then by Doob's quadratic inequality, the Cauchy–Schwarz inequality, and the fact that $\max(|F(t, X_t)^T|, |G(t, X_t)^T|) \leqq m + m^2$, we have:

$$\begin{aligned} E\{&\sup_{t\leq S\wedge T} (X_t^n - X_t)^2\} \\ &\leqq 4E\{\sup_{t\leq S\wedge T} (\eta_t^n - \eta_t)^2\} + 8E\{\sup_{t\leq S\wedge T} |\textstyle\int_0^t F(s, X_s)\, d(M_s - M_s^n)|^2\} \\ &\qquad + 8E\{\sup_{t\leq S\wedge T} |\textstyle\int_0^t F(s, X_s) - F(s, X_s^n)\, dM_s^n|^2\} \\ &\qquad + 8E\{(\textstyle\int_0^{S\wedge T} |G(s, X_s)|\,|d(V - V^n)_s|)^2\} \\ &\qquad + 8E\{(\textstyle\int_0^{S\wedge T} |G(s, X_s) - G(s, X_s^n)|\,|dV_s^n|)^2\} \\ &\leqq 4E\{\sup_{t\leq S\wedge T} (\eta_t^n - \eta_t)^2\} + 32(m + m^2)^2 E\{(M - M^n)_T{}^2\} \\ &\qquad + 16(m^2 + m^3) E\{|V - V^n|_{S\wedge T}\} \\ &\qquad + 32m^2 E\{\textstyle\int_0^{S\wedge T} |X_s - X_s^n|^2\, d\langle M^n, M^n\rangle_s\} \\ &\qquad + 8m^3 E\{\textstyle\int_0^{S\wedge T} |X_s - X_s^n|^2 |dV_s^n|\} . \end{aligned}$$

Letting $h(n, m)$ denote the sum of the first three terms in the preceding expression, we have shown

(6.2) $$E\{\sup_{t\leq S\wedge T} (X_t^n - X_t)^2\} \leqq h(n, m) + 32m^3 E\{\textstyle\int_0^{S\wedge T} (X_s - X_s^n)^2\, dL_s\}$$

where $L_t = L_t^{n,m} = |V^n|_{t\wedge T^m} + \langle M^n, M^n\rangle_{t\wedge T^m}$. Note that $L_\infty^{n,m} \leqq 2m$ for each n. We let $\tau_t = \tau_t^{n,m} = \inf\{s > 0: L_s^{n,m} > t\}$ be the right-continuous inverse of (L_t). Since (L_t) is previsible, τ_t is a stopping time for each t. For integrability reasons, we define $R = R^{n,l} = \min(\inf\{t > 0: |X_t^n| > l\}, l\cdot 1_{\{|X_0^n|\leqq l\}})$, a stopping time. We denote $f_{n,l}(t) = E\{\sup_{s\leqq\tau_t\wedge R}[(X_s - X_s^n)^T]^2\}$ and by (6.2) and Lebesgue's change of time formula (see, for example, Dellacherie (1972), page 91):

$$f_{n,l}(t) \leqq h(n, m) + 32m^3 E\{\textstyle\int_0^t 1_\Gamma(\tau_s)(X_{\tau_s} - X_{\tau_s}^n)\, ds\}\,,$$

where $\Gamma = [0, T\wedge R]$. This yields

$$f_{n,l}(t) \leqq h(n, m) + 32m^3 \textstyle\int_0^t f_{n,l}(s)\, ds\,.$$

Since $s \to f_{n,l}(s)$ is positive, increasing and finite for each l, $\int_0^t f_{n,l}(s)\,ds \leqq tf_{n,l}(t) < \infty$, so we may apply the Bellman–Gronwall lemma (see, for example, Gihman and Skorohod (1969)) to conclude $f_{n,l}(t) \leqq h(n, m)e^{32m^3t}$. By the monotone convergence theorem this yields

$$E\{\sup\nolimits_{s\leqq\tau_t\wedge T}(X_s - X_s^n)^2\} \leqq h(n, m)e^{32m^3t}\,. \tag{6.3}$$

But $L_t^{n,m} \leqq 2m$ implies $\tau_{2m} = \tau_{2m}^{n,m} = \infty$, so (6.3) becomes

$$E\{\sup\nolimits_{s\leqq T^m}(X_s - X_s^n)^2\} \leqq h(n, m)e^{64m^4}\,.$$

Since $\lim_{n\to\infty} h(n, m) = 0$ for each m, we conclude $\lim_{n\to\infty} E\{\sup_{s\leqq T^m}(X_s - X_s^n)^2\} = 0$ for each m, and Theorem 6.1 is established.

Our next theorem shows that the solutions converge to the solution of the limiting equation if the coefficients converge. We note that for any $A \in \mathscr{A}^+$ we may define a measure μ_A on the previsible sets by

$$\mu_A(\Gamma) = E\textstyle\int_0^\infty 1_\Gamma(t, \omega)\, dA_t(\omega)\,.$$

(6.4) THEOREM. *Fix* $(M^j) \in \mathscr{L}$, $1 \leqq j \leqq p$; $(V^k) \in \mathscr{A}$, $1 \leqq k \leqq q$; $({}^n\eta^i)$, $\eta^i \subset \mathscr{F}_0 \oplus \mathscr{A}$, $1 \leqq i \leqq d$, $n \geqq 1$. *Let* $\mathbf{F}_j^n(t, \mathbf{x})$, $\mathbf{F}_j(t, \mathbf{x})$, $1 \leqq j \leqq p$, *and* $\mathbf{G}_k^n(t, \mathbf{x})$, $\mathbf{G}_k(t, \mathbf{x})$, $1 \leqq k \leqq q$ *be as in Theorem* 5.3, *all* $n \geqq 1$, *with Lipschitz constant process* (K_t) *and explosion time* R. *Let stopping times* (T^m) *increase to* R a.s. *such that for each* m *the following conditions are satisfied, where* $T = T^m$:

(a) $\lim_{n\to\infty} P\{\sup_{|\mathbf{x}|\leqq m} |(\mathbf{F}_j^n(s, \mathbf{x}) - \mathbf{F}_j(s, \mathbf{x}))^T| > \varepsilon\} = 0$ *and* $\sup_{|\mathbf{x}|\leqq m} |(\mathbf{F}_j^n(s, \mathbf{x})^T)| \leqq H_m(s, \omega)$ *for* $1 \leqq j \leqq p$, $n \geqq 0$;
(b) $\lim_{n\to\infty} P\{\sup_{|\mathbf{x}|\leqq m} |(\mathbf{G}_k^n(s, \mathbf{x}) - \mathbf{G}_k(s, \mathbf{x}))^T| > \varepsilon\} = 0$ *and* $\sup_{|\mathbf{x}|\leqq m} |(\mathbf{G}_k^n(s, \mathbf{x}))^T| \leqq J_m(s, \omega)$ *for* $1 \leqq k \leqq q$, $n \geqq 0$;
(c) (η_t^n) *converges locally to* (η_t) *in maximal quadratic mean.*

The processes H_m *and* J_m *above satisfy*

$$E\{\textstyle\int_0^T H_m(s)^2\, d\langle M, M\rangle_s\} < \infty \qquad \text{and} \qquad E\{\textstyle\int_0^T J_m(s)^2|dV_s|\} < \infty\,.$$

We let $\mathbf{X}^n$ *and* $\mathbf{X}$ *denote the respective solutions of*

$$\mathbf{X}_t^n = \eta_t^n + \textstyle\sum_{j=1}^p \int_0^t \mathbf{F}_j^n(s, \mathbf{X}_s^n)\, dM_s^j + \sum_{k=1}^q \int_0^t \mathbf{G}_k^n(s, \mathbf{X}_s^n)\, dV_s^k$$
$$\mathbf{X}_t = \eta_t + \textstyle\sum_{j=1}^p \int_0^t \mathbf{F}_j(s, \mathbf{X}_s)\, dM_s^j + \sum_{k=1}^q \int_0^t \mathbf{G}_k(s, \mathbf{X}_s)\, dV_s^k\,.$$

Then $\lim_{n\to\infty} E\{\sup_{S\leq T^m} |\mathbf{X}_s^n - \mathbf{X}_s|^2\} = 0$; *that is*, $\mathbf{X}^n$ *converges locally to* $\mathbf{X}$ *in maximal quadratic mean.*

PROOF. We treat the case $d = p = q = 1$ for notational simplicity. As in the proof of Theorem 6.1 we may assume without loss of generality that the sequence of stopping times (T^m) increasing to R a.s. of hypotheses (a), (b) and (c) is also such that for each m and $T = T^m$; $K_t^T \leq m$; $|X_t^T| \leq m$; $\langle M, M\rangle_{t\wedge T} \leq m$; and $|V|_{t\wedge T} \leq m$. Using the hypotheses, the inequality $(a+b)^2 \leq 2(a^2+b^2)$, Doob's quadratic inequality for continuous local martingales, and the Cauchy–Schwarz inequality, we have for an arbitrary stopping time S,

$$
\begin{aligned}
&E\{\sup_{t\leq S\wedge T} |X_t^n - X_t|^2\} \\
&\quad \leq 4E\{\sup_{t\leq T} (\eta_t^n - \eta_t)^2\} \\
&\qquad + 32E\{\textstyle\int_0^{S\wedge T} \sup_{|x|\leq m} (F^n(s,x) - F(s,x))^2\, d\langle M, M\rangle_s\} \\
&\qquad + 8E\{m \textstyle\int_0^{S\wedge T} \sup_{|x|\leq m} (G^n(s,x) - G(s,x))^2 |dV_s|\} \\
&\qquad + 32E\{\textstyle\int_0^{S\wedge T} (F^n(s, X_s^n) - F^n(s, X_s))^2\, d\langle M, M\rangle_s\} \\
&\qquad + 8E\{|V|_{S\wedge T} \textstyle\int_0^{S\wedge T} (G^n(s, X_s^n) - G^n(s, X_s))^2 |dV_s|\}\,.
\end{aligned}
$$

Denoting the first three terms on the right-hand side of the above by $h(n, m)$, the above yields.

$$
(6.5) \qquad E\{\sup_{t\leq S\wedge T} |X_t^n - X_t|^2\} \leq h(n,m) + 32m^3 E\{\textstyle\int_0^{S\wedge T} (X_s^n - X_s)^2\, dL_s\}
$$

where $L_t = L_t^m = \langle M, M\rangle_{t\wedge T} + |V|_{t\wedge T}$. Note that $L_\infty^m \leq 2m$.

Hypotheses (a), (b) and (c) imply that $\lim_{n\to\infty} h(n,m) = 0$ for each m, since, for example, $\sup_{|x|\leq m} (F^n(s,x) - F(s,x))^2$ tends to 0 as n tends to ∞ in λ-measure while dominated by H_m which is in $L^2(d\lambda)$. Here λ is the measure given by $\lambda(Y) = E\int_0^T Y_s\, d\langle M, M\rangle_s$, for any previsibly measurable positive process (Y_s). The desired result follows by an argument analogous to the one following the inequality (6.2) in the proof of Theorem 6.1. This completes the proof of Theorem 6.4.

We may also allow both the differentials and the coefficients to converge. Our next theorem shows that then the solutions converge to the solution of the limiting equation. The techniques used in the proofs of Theorems 6.1 and 6.4 may be utilized to prove Theorem 6.6, which we state here without proof.

(6.6) THEOREM. *For all* $n \geq 1$ *let*: $({}^nM^j), (M^j) \subset \mathscr{L}$, $1 \leq j \leq p$; $({}^nV^k)$, $(V^k) \subset \mathscr{A}$, $1 \leq k \leq q$; $({}^n\eta^i), (\eta^i) \subset \mathscr{F}_0 \oplus \mathscr{A}$, $1 \leq i \leq d$. *For all* $n \geq 1$ *let* $\mathbf{F}_j^n(t,\mathbf{x})$, $\mathbf{F}_j(t,\mathbf{x})$, $\mathbf{G}_k^n(t,\mathbf{x})$, $\mathbf{G}_k(t,\mathbf{x})$, $1 \leq j \leq p$, $1 \leq k \leq q$ *be as in Theorem* 5.3 *with Lipschitz constant process* (K_t) *and explosion time* R. *Further, suppose*:

(a) $({}^n\eta^i)$ *converges locally to* η^i *in maximal quadratic mean for* $1 \leq i \leq d$;

(b) $({}^nM^j)$ *converges locally to* M^j *in quadratic mean*, $1 \leq j \leq p$;

(c) $({}^nV^k)$ *converges locally to* V^k *in expected total variation*, $1 \leq k \leq q$;

(d) *for any* $\varepsilon > 0$, $1 \leq j \leq p$, $\lim_{n\to\infty} P\{\sup_{|x|\leq m} |\mathbf{F}_j(s,\mathbf{x}) - \mathbf{F}_j^n(s,\mathbf{x})| > \varepsilon\} = 0$ *and* $|(\mathbf{F}_j^n(\cdot,\mathbf{x}))_s^{T^m}| \leq H_m(s,\omega)$, $n \geq 1$;

(e) *for any* $\varepsilon > 0$, $1 \leqq k \leqq q$, $\lim_{n\to\infty} P\{\sup_{|\mathbf{x}|\leqq m} |\mathbf{G}_k(s, \mathbf{x}) - \mathbf{G}_k{}^n(s, \mathbf{x})| > \varepsilon\} = 0$ *and* $|(\mathbf{G}_k{}^n(\cdot, \mathbf{x}))_s{}^{T^m}| \leqq J_m(s, \omega)$, $n \geqq 1$.

The processes H_m *and* J_m *above satisfy*:

$$E \int_0^{T^m} H_m(s)^2 d\langle M^j, M^j\rangle_s < \infty \,; \qquad E \int_0^{T^m} H_m(s)^4 d\langle M^j - {}^nM^j, M^j - {}^nM^j\rangle_s < \infty \,;$$
$$E \int_0^{T^m} J_m(s)^2 |dV_s{}^k| < \infty \,; \qquad \int_0^{T^m} J_m(s)^2 |d(V^k - {}^nV^k)_s| \leqq C_m \quad \text{a.s.};$$

for constants C_m; *all* m, n; $1 \leqq j \leqq p$; $1 \leqq k \leqq q$. *On* $[[0, R))$, *let* $\mathbf{X}^n$ *and* $\mathbf{X}$ *respectively be solutions of*

$$\mathbf{X}_t{}^n = \eta_t{}^n + \sum_{j=1}^p \int_0^t \mathbf{F}_j{}^n(s, \mathbf{X}_s{}^n)\, d^nM_s{}^j + \sum_{k=1}^q \int_0^t \mathbf{G}_k{}^n(s, \mathbf{X}_s{}^n)\, d^nV_s{}^k$$
$$\mathbf{X}_t = \eta_t + \sum_{j=1}^p \int_0^t \mathbf{F}_j(s, \mathbf{X}_s)\, dM_s{}^j + \sum_{k=1}^q \int_0^t \mathbf{G}_k(s, \mathbf{X}_s)\, dV_s{}^k \,.$$

Then $(\mathbf{X}_t{}^n)$ *converges locally to* $(\mathbf{X}_t)$ *in maximal quadratic mean.*

The proof of the preceding theorem as well as the proof of the next theorem is in Protter (1975). The next theorem generalizes the well-known convergence result of Wong and Zakai (1965).

(6.7) THEOREM. *Let* $\eta \in \mathscr{F}_0 \oplus \mathscr{A}$, $V \in \mathscr{A}$, *and* $M \in \mathscr{L}$. *Let* (η^n), (V^n), *and* (H^n) *have piecewise continuous derivatives and be such that* a.s. $\eta_t{}^n \to \eta_t$, $V_t{}^n \to V_t$ *and* $H_t{}^n \to M_t$ *for all* t. *Let* $F(t, x)$, $G(t, x)$ *be random continuous functions as in Theorem* 5.3, *with Lipschitz constant process* (K_t) *and explosion time* R. *Suppose also that* $|F(t, x)| \geqq \beta > 0$ a.s., *where* β *is random. Let* $F_1(t, x) = \partial F(t, x)/\partial t$, $F_2(t, x) = \partial F(t, x)/\partial x$ *be continuous, and* $|F_2(t, x)| \leqq KF(t, x)^2$, *for random* K. *Let* (T^m) *be the announcing sequence of stopping times associated with* R, *and suppose on each stochastic interval* $[[0, T^m]]$, *we have either*

(a) $|V^n|_t$, $|\eta_t{}^n|$, *and* $H_t{}^n$ *are all uniformly bounded*, $n \geqq 1$; *or*

(b) $(\eta_t{}^n)$, $(V_t{}^n)$ *and* $(H_t{}^n)$ a.s. *converge uniformly in* t *to* (η_t), (V_t), *and* (M_t) *respectively.*

Let X^n *and* X *respectively satisfy*

$$X_t{}^n = \eta_t{}^n + \int_0^t F(s, X_s{}^n)\, dH_s{}^n + \int_0^t G(s, X_s{}^n)\, dV_s{}^n$$
$$X_t = \eta_t + \int_0^t F(s, X_s)\, dM_s + \int_0^t G(s, X_s)\, dV_s + \tfrac{1}{2} \int_0^t F(s, X_s) F_2(s, X_s)\, d\langle M, M\rangle_s \,.$$

Then under hypothesis (a), a.s. $\lim_{n\to\infty} X_t{}^n = X_t$; *under hypotheses* (b), a.s. $\lim_{n\to\infty} X_t{}^n = X_t$, *and the convergence is uniform in* t *on* $[[0, T^m]]$, *for each* m.

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