

THE RIEMANN ZETA FUNCTION AND GAUSSIAN MULTIPLICATIVE CHAOS: STATISTICS ON THE CRITICAL LINE

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We prove that if ω is uniformly distributed on $[0, 1]$, then as $T \rightarrow \infty$, $t \mapsto \zeta(i\omega T + it + 1/2)$ converges to a nontrivial random generalized function, which in turn is identified as a product of a very well-behaved random smooth function and a random generalized function known as a complex Gaussian multiplicative chaos distribution. This demonstrates a novel rigorous connection between probabilistic number theory and the theory of multiplicative chaos—the latter is known to be connected to various branches of modern probability theory and mathematical physics.

We also investigate the statistical behavior of the zeta function on the mesoscopic scale. We prove that if we let δ_T approach zero slowly enough as $T \rightarrow \infty$, then $t \mapsto \zeta(1/2 + i\delta_T t + i\omega T)$ is asymptotically a product of a divergent scalar quantity suggested by Selberg’s central limit theorem and a strictly Gaussian multiplicative chaos. We also prove a similar result for the characteristic polynomial of a Haar distributed random unitary matrix, where the scalar quantity is slightly different but the multiplicative chaos part is identical. This says that up to scalar multiples, the zeta function and the characteristic polynomial of a Haar distributed random unitary matrix have an identical distribution on the mesoscopic scale.

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1. Introduction. In this article, we study a rather classical problem in probabilistic number theory, namely the statistical behavior of the Riemann zeta function on the critical line (for definitions, see Section 1.1), or what can be said about the behavior of the zeta function

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in the neighborhood of a random point on the critical line and far away from the origin. Before going into precise definitions and statements, we review briefly and informally our main contributions, their role compared to previous studies, and connections to other branches of modern probability theory and mathematical physics.

One of our main results, Theorem 1.1, states that if this neighborhood of a random point is of a fixed size compared to the distance to the origin, then the zeta function behaves like a random generalized function which can be viewed as a product of a random smooth function and a very special random generalized function known as a (complex) Gaussian multiplicative chaos distribution. The main contributions of this article continue with Theorem 1.2 and Theorem 1.3. Theorem 1.2 states that if the neighborhood of the random point shrinks slowly enough compared to the distance to the origin, one again has a limit theorem representing the zeta function in terms of Gaussian multiplicative chaos. Theorem 1.3 states that up to a constant random factor, we have identical behavior in a model which is believed to be very closely related to the zeta function, namely the characteristic polynomial of a large, Haar distributed random unitary matrix. Informally speaking, these results establish that on such a *mesoscopic* scale, the statistical behavior of the zeta function is indistinguishable from that of a characteristic polynomial of a large random unitary matrix. We now turn to the historical background and context of our problem and results.

The study of the statistical behavior of the zeta function began with the work of Bohr and Jessen in the 1930s [15, 16]. They studied pointwise statistics away from the critical line. Informally speaking, they found that if one considers a random point uniformly bounded away from the critical line, then the probability law of the zeta function at this random point converges to a nontrivial probability law as the distance of this random point from the origin tends to infinity. This left open the questions of what happens at a random point on the critical line and what are the functional statistics, or what happens in a neighborhood of this random point. The question of functional statistics was later studied by Bagchi ([6] see also [57], Chapter 5, and [57] in general for an extensive discussion about the statistical behavior of the zeta function), who found that in a fixed neighborhood of a random point far away from the origin and at a positive distance from the critical line, the zeta function behaves like a random analytic function.

The behavior on the critical line is more complicated. A classical result of Selberg's, [73], states that to have a meaningful limit theorem for a random point on the critical line, one must consider the logarithm of the absolute value of the zeta function and divide it by a suitable divergent quantity related to the distance of the random point from the origin—the limiting probability law being the standard normal one. For functional statistics, Laurinćikas proved in [56] that on the critical line (without a divergent normalization as in Selberg's limit theorem) one cannot have a functional limit theorem where the limit would be a continuous function. This means that there are various approaches one could take to obtain a functional limit theorem: one might try to normalize the zeta function as in Selberg's limit theorem, one might consider shrinking neighborhoods, or one might try to relax the regularity of the limit. In the spirit of Selberg's limit theorem, a partial result in the direction of a functional limit theorem for the logarithm of the zeta function on a mesoscopic scale was proven by Bourgade [17], the limiting object being a suitable Gaussian process (see also [45] for a related result). From the point of view of relaxing the regularity of the limit, our Theorem 1.1 can be seen as giving the first rigorous answer to the question of what can be said about the functional statistics on the critical line, and similarly Theorem 1.2 answers the same question in case we shrink the neighborhood at a mesoscopic rate—though we also refer to the works of Bourgade and Kuan [18] and Rodgers [70] which concern so-called linear statistics of zeros of the zeta function, but can be interpreted as stating that on a mesoscopic scale and assuming the truth of the Riemann hypothesis, the imaginary part of the logarithm of the zeta

function behaves like a Gaussian process whose covariance has a logarithmic singularity on the diagonal, and the precise interpretation of this involves viewing the Gaussian process as a random generalized function. The open and very hard problem remaining is to study the statistical behavior of the zeta function when one shrinks the neighborhood at a “*microscopic rate*.” Here (up to a divergent constant random factor), one expects to have convergence to a certain random analytic function related to the sine process arising in random matrix theory; see [23].

Since the 1970s, and increasingly during the last two decades, a slightly different direction in the study of the statistical behavior of the zeta function has been provided by a conjectural connection between random matrix theory and the zeta function; for a brief overview of this connection, see the book review of Conrey [24]. An influential conjecture in the development of random matrix theory, as well as certain branches of analytic number theory, has been the Montgomery–Dyson pair correlation conjecture, which states that after a suitable normalization, the nontrivial zeroes of the zeta function behave statistically speaking like the eigenvalues of a large random matrix. This conjectural connection was a motivation in the work of Keating and Snaith [53], who suggested that the statistical behavior of the zeta function should be modeled by the characteristic polynomial of a random unitary matrix. Moreover, they conjectured a connection between moments of the zeta function in the neighborhood of a random point on the critical line as well as moments of the characteristic polynomial of a random unitary matrix. Again the most interesting and most difficult problem would be establishing such a connection on the microscopic scale, where one can see individual zeroes of the zeta function. Nevertheless, our Theorem 1.2 and Theorem 1.3 can be seen as a rigorous justification of this Montgomery–Dyson–Keating–Snaith picture, when considering phenomena related to the mesoscopic scale.

As a final remark concerning the background and general context of our work, we point out rather recent work of Fyodorov, Hiary and Keating [36] and Fyodorov and Keating [37]. Perhaps the main message of these articles is that on the global and mesoscopic scale, one should think of the logarithm of the zeta function on the critical line behaving statistically like log-correlated field—a random generalized function which has a logarithmic singularity on the diagonal of its covariance kernel.¹ Log-correlated fields have emerged recently as an important class of objects arising naturally in many models of modern probability and mathematical physics (see, e.g., [3, 4, 20, 27, 33, 37, 44]). In addition to being connected to many important models, their interest also stems from the fact that despite being rough objects, one can make some sense of their fractal geometric properties, which typically end up being universal—common to all log-correlated fields. It turns out that a good way to describe and study these universal geometric properties is through objects which are formally exponentials of these log-correlated fields. The rigorous construction of these objects and the study of their properties goes under the name of the theory of multiplicative chaos, and its foundations were laid in the work of Kahane [52] (see also [69] for a recent review and [12] for a recent study). From this point of view, many of the conjectures of [36, 37] can be rephrased as stating that statistical properties of the zeta function can be expressed through the theory of multiplicative chaos. Some properties one would expect the zeta function to possess based on universality of properties of log-correlated fields have indeed been proven recently (without multiplicative chaos theory); see, for example, [2, 63] (see also [1, 22, 64] for related results in the setting of random matrices). From this point of view, our main results connecting the Riemann zeta function to multiplicative chaos can be seen as establishing a

¹In particular, they argue that known properties of maxima of log-correlated fields, properties of characteristic polynomials of random matrices and conjectures due to Montgomery and Farmer–Gonek–Hughes [35] fit rather well together. We refer to [40], Section 2, for a nice discussion of this issue.

novel connection between the zeta function and a class of stochastic objects arising in various models of great interest as well as further evidence for universal behavior in the class of log-correlated fields.

The methods we use to prove our main theorems involve mainly tools of probability theory and analysis; indeed the tools of analytic number theory we make use of are rather classical. From the point of view of number theory, our main results lie in the probabilistic part of analytic number theory. Our main contributions to probabilistic number theory should be viewed as introducing a new class of limit theorems and limiting objects that are shown to appear naturally in the theory. In the setting of random matrices, we provide a rigorous proof of a connection between the statistical behavior of the zeta function and random characteristic polynomials, while in the setting of the theory of log-correlated fields, our results can be seen as further evidence for log-correlated fields being an ubiquitous class of objects with many universal properties. In addition, from the point of view of analytic number theory, our contributions can be seen as providing rigorous support for a conceptual picture, describing what kind of behavior one should expect from the zeta function when studying questions on the global or mesoscopic scale. In the future, this connection to multiplicative chaos could also prove fruitful in studying fractal properties of the zeta function.

The remainder of this [Introduction](#) is devoted to the precise definitions of the objects we study, the precise statements of our main results (together with some further results), along with some conjectures. In this whole article, we have tried to be generous with details in our presentation to make the text accessible for readers with various backgrounds.

1.1. *Definitions, basic concepts and main result on the global scale.* In this section, we describe the main objects we study and state our main results concerning the study of the zeta function in a fixed size neighborhood of a random point on the critical line. As the study of multiplicative chaos is a less classical field of mathematics, we also discuss some of its background for the convenience of the reader.

We start by fixing our notation and recalling some basic facts about the *Riemann zeta function* defined for $\sigma > 1$ by the convergent series²

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

One may continue ζ as a meromorphic function to the whole complex plane, with only one pole. The pole is simple and located at $s = 1$ with residue 1. The *critical line* $\{\sigma = \frac{1}{2}\}$ is of particular importance in the study of the zeta function as the behavior of the zeta function on it is closely related to the distribution of the prime numbers, and is of course a subject of many fundamental conjectures in number theory. Many classical facts about the zeta function can be found, for example, in [7, 48, 77].

As stated at the beginning of our [Introduction](#), we are interested in the behavior of the zeta function in the neighborhood of a random point on the critical line. The precise object of our study is the following:

$$(1.1) \quad \mu_T(x) := \zeta(1/2 + ix + i\omega T) \quad \text{for } x \in \mathbb{R},$$

where ω is a random variable which is uniform on $[0, 1]$, $T \in (0, \infty)$, and we are interested in the behavior of this random function in the limit where $T \rightarrow \infty$. It is natural to expect that the precise distribution of the random variable ω is unimportant, but we will not discuss this further.

²We follow the custom in analytic number theory to denote by $s = \sigma + it$ the real and imaginary parts of the complex variable s .

As discussed at the beginning of this [Introduction](#), the other main player in our study is a random generalized function which is called a Gaussian multiplicative chaos distribution. The foundations of the mathematical theory of Gaussian multiplicative chaos were established in the 1980s by Kahane [52]. At that time, the main motivation was the desire to build mathematical models for Kolmogorov's statistical theory of turbulence by providing a continuous counterpart for multiplicative cascades that were originally introduced by Mandelbrot for the same purpose in the early 1970s. During the last 15 years, there has been a new wave of interest in multiplicative chaos, partly due to its important connections to stochastic Loewner evolution [4, 75], to quantum gravity and scaling limits of random planar maps [13, 27, 33, 59–62], as well as to models in finance and turbulence [69], Section 5.

In order to give a brief and informal description of multiplicative chaos, consider a sequence of a.s. continuous and centered real-valued Gaussian fields X_n , say on an interval $I \subset \mathbb{R}$. The elements of this sequence should be considered as suitable approximations of a (possibly distribution valued) Gaussian field X . For simplicity, assume that the increments $X_{n+1} - X_n$ are independent. One may then define the random measures λ_n on I by setting

$$\lambda_n(dx) := \exp\left(X_n(x) - \frac{1}{2}\mathbb{E}X_n(x)^2\right) dx.$$

In this situation, basic martingale theory implies that almost surely there exists a (random) limit measure $\lambda = \lim_{n \rightarrow \infty} \lambda_n$, where the convergence is understood in the weak*-sense. The measure λ is called the *multiplicative chaos measure* defined by X , and Kahane proved that under suitable conditions the limit does not depend on the choice of the approximating sequence (X_n) . However, a significant obstacle in defining a meaningful limiting object λ is that it may very well be the zero measure almost surely. The most important, and in some sense a borderline situation for defining meaningful limiting objects, is when the limit field X is log-correlated, that is, it has a covariance of the form

$$C_X(x, y) = -2\beta^2 \log|x - y| + G(x, y), \quad x, y \in I,$$

where G is a continuous and bounded function. Then Kahane's theory implies that the limit measure is almost surely nonzero for $0 < \beta < 1$, as we are in dimension one. The limiting random Borel measure $\lambda = \lambda_\beta$ on the interval I is almost surely singular and its basic properties like multifractal spectrum, tail of the total mass or scaling properties have been investigated.

At the threshold $\beta = \beta_c := 1$, one needs to add a deterministic nontrivial renormalization factor that depends on n in order to obtain the existence of a nontrivial object known as a *critical chaos measure*. This limit can also be achieved through a random normalization known as the derivative martingale.

Overall, the dependence of the chaos measure on the generating Gaussian field has many delicate features. For example, the universality property (how the law of the limiting object is independent of the precise details of the approximation scheme) is far from trivial for multiplicative chaos [50, 69, 74]. We refer to the nice survey [69] for the basic properties of these measures, to [12] for an elegant proof of the existence of subcritical chaos measures, and to [10, 31, 32] for the existence and basic properties of critical Gaussian chaos.

There is a further variant of multiplicative chaos that is important for the connection to the Riemann zeta function (and for random matrix theory as well), which is the concept of *complex multiplicative chaos*, where in the above one allows for complex Gaussian fields. Two basic cases have been studied in the literature. In the first variant, one allows the parameter β take complex values, and it turns out that one obtains analyticity in the parameter β for $\beta \in U$, where $U \subset \mathbb{C}$ is an open subset whose intersection with the real axis equals $(-1, 1)$ in the above normalization (see [8, 9] in the slightly simpler case of multiplicative cascades and, e.g., [4] in the case of multiplicative chaos). In the second case, one assumes that $X = \beta_1 X_1 + i\beta_2 X_2$ with X_1, X_2 independent copies of a log-correlated field and

$\beta_1, \beta_2 \in \mathbb{R}$. This case turns out to be more amenable to analysis, due to the independence of the real and imaginary parts, and many aspects of it have been studied thoroughly in [54]. However, the complex chaos we need to study here does not quite fit into either of these models. In our situation, we study a complex Gaussian field \mathcal{G} , for which there is a very special mutual dependence between the real and imaginary parts X_1 and X_2 , of the form

$$\mathbb{E}X_1(x)X_2(y) = -\frac{\pi}{4} \operatorname{sgn}(x - y) + \text{smooth},$$

where $\operatorname{sgn}(x)$ denotes the sign of x and the covariance is zero when $x = y$. Formally, this suggests that the 2-point function $\mathbb{E}e^{\mathcal{G}(x)+\overline{\mathcal{G}}(y)}$ is not absolutely integrable, which in general indicates that one cannot use L^2 -theory to construct a limiting object. Remarkably enough, it is exactly the above peculiar dependence of the real and imaginary part that produces the dominant part $(i(x - y))^{-1}$ to the two-point function, and hence the basic theory of one-dimensional singular integrals applies to resurrect the L^2 -theory. It turns out that the complex Gaussian chaos we study, which can be formally written as $\nu = e^{\mathcal{G}}$, has some unique features that arise from the fact that it can be considered as a boundary distribution of a random analytic function. For example, the finiteness of a moment $\mathbb{E}|\nu(\phi)|^p$ (where $\nu(\phi)$ stands for the action of the generalized function ν on the test function ϕ) with $p > 4$ can be shown to depend on the smoothness properties of the function ϕ , and thus their properties differ in some respects from the complex chaos considered in [54].

Before stating our result concerning the behavior of μ_T from (1.1), we need to recall some notions related to generalized functions to properly state our convergence results. We shall make extensive use of the classical L^2 -based Sobolev spaces $W^{\alpha,2}(\mathbb{R})$, where $\alpha \in \mathbb{R}$ is an arbitrary smoothness index. The space $W^{\alpha,2}(\mathbb{R})$ consists of all Schwartz generalized functions $f \in \mathcal{S}'(\mathbb{R})$ such that the Fourier transform \widehat{f} is locally L^2 -integrable and satisfies

$$\|f\|_{W^{\alpha,2}(\mathbb{R})}^2 := \int_{\mathbb{R}} (1 + \xi^2)^\alpha |\widehat{f}(\xi)|^2 d\xi < \infty,$$

where our convention for the normalization of the Fourier transform is given by

$$\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) dx$$

for $f \in \mathcal{S}(\mathbb{R})$. One has the duality $(W^{\alpha,2}(\mathbb{R}))' = W^{-\alpha,2}(\mathbb{R})$ with respect to the standard distributional pairing. If $\alpha > 1/2$, there exists a continuous embedding $W^{\alpha,2}(\mathbb{R}) \subset C(\mathbb{R})$, where the latter space is equipped with the sup-norm. In turn, if $\alpha < -1/2$, one has $\delta_a \in W^{\alpha,2}(\mathbb{R})$ for all $a \in \mathbb{R}$. We refer, for example, to [76] or [38] for basic facts on harmonic analysis and the relevant function spaces—we apply mostly only the L^2 -theory that can be often conveniently dealt with by basic Fourier analysis.

We now turn to our first main result, which shows unconditionally,³ that the random shifts μ_T from (1.1) converge to a statistical limit that is a random generalized function whose nontrivial behavior is determined by a (complex) Gaussian multiplicative chaos distribution.

THEOREM 1.1.

(i) *There exists a nontrivial random variable $x \mapsto \zeta_{\text{rand}}(1/2 + ix)$ taking values in $\mathcal{S}'(\mathbb{R})$ —the space of tempered distributions—such that as $T \rightarrow \infty$,*

$$(1 + x^2)^{-1} \mu_T(x) \xrightarrow{d} (1 + x^2)^{-1} \zeta_{\text{rand}}(1/2 + ix),$$

where the convergence in law is with respect to the norm topology of the Sobolev space $W^{-\alpha,2}(\mathbb{R})$ for any $\alpha > 1/2$.

³That is, we do not assume the Riemann hypothesis.

(ii) Moreover, the law of the limit ζ_{rand} can be characterized in the following way: as random generalized functions,

$$\zeta_{\text{rand}}(1/2 + ix) = g(x)v(x),$$

where v is a random generalized function known as a Gaussian multiplicative chaos distribution, which can be formally written as

$$v(x) = "e^{\mathcal{G}(x)}",$$

where \mathcal{G} is a centered Gaussian field with the correlation structure

$$\mathbb{E}\mathcal{G}(x)\mathcal{G}(y) = 0 \quad \text{and} \quad \mathbb{E}\mathcal{G}(x)\overline{\mathcal{G}(y)} = \log(\zeta(1 + i(x - y))) \quad \text{for } x, y \in \mathbb{R}.$$

For a precise definition, see Lemma 4.1. The factor g is a random smooth function on \mathbb{R} , almost surely has no zeroes, and $\mathbb{E}(\|g(x)\|_{C^\ell(I)}^p + \|1/g(x)\|_{C^\ell(I)}^p)$ is finite for all $p \in \mathbb{R}$, any $\ell \geq 0$, and any finite interval $I \subset \mathbb{R}$. We emphasize here that g and v may not be independent.

Above, the identification of ζ_{rand} and gv can be understood to mean that as elements of $W^{-\alpha,2}(\mathbb{R})$, $(1 + x^2)^{-1}\zeta_{\text{rand}}(1/2 + ix) = (1 + x^2)^{-1}g(x)v(x)$ or then equality can be understood in the sense of tempered distributions. Note that we have convergence with respect to the norm of the Sobolev space, not just with respect to evaluations against test functions. Especially, one may define the action of $\zeta_{\text{rand}}(1/2 + ix)$ on a larger class of test functions than just Schwartz functions. For a definition of the norm, we use on C^ℓ ; see Lemma 3.1. We now turn to the mesoscopic behavior of the zeta function.

1.2. *The mesoscopic scale.* Recall that by mesoscopic, we mean that we study the zeta function in a suitably slowly shrinking neighborhood of the random point $1/2 + i\omega T$, or in the notation of (1.1), we are interested in the random function $\mu_T(\delta_T x)$ for some $\delta_T \rightarrow 0$ with $T \rightarrow \infty$ slowly enough. Before stating the description of the mesoscopic behavior, which comprises our second main result, we first make a couple of heuristic comments in an attempt to motivate and clarify the emerging picture. As one may expect from Theorem 1.1 that $\mu_T(\delta_T x)$ is close to $e^{\mathcal{G}(\delta_T x)}$, consider what happens to a log-correlated field under scaling; for simplicity, let us look at a real translation invariant log-correlated field X with covariance $C_X(x, y) = \log(1/|x - y|) + g(x - y)$, where g is smooth. If we scale the spatial variable by ε and let $\varepsilon \rightarrow 0$, the covariance will roughly be $\log \varepsilon^{-1} + g(0) - \log|x - y|$. So it is natural to expect that the precise details of the covariance are irrelevant: the field $X(\varepsilon x)$ should roughly consist of "zero mode" or constant Gaussian random variable whose variance explodes as $\varepsilon \rightarrow 0^+$, as well as of a log-correlated field with covariance $-\log|x - y| + \text{constant}$ (though this latter field may not strictly speaking exist on the whole axis; see a more precise formulation below).

For precise statements for our results in this setting, we need to recall the definition of the Sobolev space $W^{\alpha,2}(0, 1)$. For any $\alpha \in \mathbb{R}$, this space can be defined as the space of restrictions of elements in $W^{\alpha,2}(\mathbb{R})$ to the set $(0, 1)$. More formally,

$$W^{\alpha,2}(0, 1) := \{g|_{(0,1)} : g \in W^{\alpha,2}(\mathbb{R})\}$$

and

$$\|f\|_{W^{\alpha,2}(0,1)} := \inf\{\|g\|_{W^{\alpha,2}(\mathbb{R})} : f = g|_{(0,1)}\}.$$

The space $W^{\alpha,2}(0, 1)$ is again a separable Hilbert space. Moreover, it is easily verified that $\|(1 + x^2)^{-1}f\|_{W^{\alpha,2}(0,1)} \asymp \|f\|_{W^{\alpha,2}(0,1)}$ (where $a \asymp b$ means that a/b and b/a are bounded) for any $\alpha \in \mathbb{R}$, which lets us transfer our earlier convergence results on the whole line to restrictions to the interval. We also find it convenient to strengthen our notion of convergence

from convergence in law to convergence in the sense of the quadratic Wasserstein (or Kantorovich) metric \mathcal{W}_2 . Assume that (X, d_X) is a complete separable metric space and μ and ν are Borel probability distributions on X . Then

$$\mathcal{W}_2(\mu, \nu)_X := \left(\inf_{(U, V)} \mathbb{E}(d_X(U, V)^2) \right)^{1/2},$$

where U, V are random variables on a common probability space taking values in X so that $U \sim \mu$ and $V \sim \nu$. This defines a distance between probability distributions on X . In particular, convergence in the Wasserstein metric implies convergence in distribution and uniform boundedness of second moments. The monograph [78] is a good place to learn the basics about Wasserstein distances, although we hardly need more than the definition in this paper.

We are now ready to state our first mesoscopic result.

THEOREM 1.2. *There exists a deterministic $\delta_T > 0$ so that $\delta_T \rightarrow 0^+$ as $T \rightarrow \infty$ and*

$$\lim_{T \rightarrow \infty} \mathcal{W}_2(\zeta(1/2 + i\delta_T x + i\omega T), h_T(x)e^{Y_T} \eta(x))_{W^{-\alpha, 2}(0, 1)} = 0,$$

where $\alpha > 1/2$, h_T is a random smooth function satisfying $h_T \xrightarrow{d} 1$ in $C^1[0, 1]$ as $T \rightarrow \infty$,

$$Y_T \stackrel{d}{=} \sqrt{\log(1/\delta_T)}Z + R,$$

where Z is a standard complex Gaussian, R is a complex random variable independent of x and it satisfies $\mathbb{E}e^{\lambda|R|} < \infty$ for all $\lambda > 0$. Finally, $\eta(x)$ is a complex Gaussian multiplicative chaos distribution which can be formally written as

$$\eta(x) = \exp \left[\int_0^1 \frac{e^{-2\pi i x u} - 1}{\sqrt{u}} dB_u^{\mathbb{C}} + \int_1^\infty \frac{e^{-2\pi i x u}}{\sqrt{u}} dB_u^{\mathbb{C}} \right],$$

where $B_u^{\mathbb{C}}$ is a standard complex Brownian motion. For a precise definition, see Lemma 5.3.

Note that we do not claim any independence between η , R or Z .

Our next result (which we prove in Appendix C) is the claim that a nearly identical result holds for the characteristic polynomial of a random unitary matrix, or more precisely, for random unitary matrices whose probability law is the Haar measure on the unitary group.

THEOREM 1.3. *Let η be the multiplicative chaos distribution from Theorem 1.2 and let U_N be a Haar distributed $N \times N$ random unitary matrix. There exists a deterministic $\delta_N > 0$ so that $\delta_N \rightarrow 0^+$ as $N \rightarrow \infty$ and*

$$\lim_{N \rightarrow \infty} \mathcal{W}_2(\det(I - e^{i\delta_N x} U_N), e^{Y_N}(\eta(x) + o(1)))_{W^{-\alpha, 2}(0, 1)} = 0,$$

where $\alpha > 1/2$, and $o(1)$ is a $W^{-\alpha, 2}(0, 1)$ -valued random variable that tends to zero in probability, and

$$Y_N \stackrel{d}{=} \sqrt{\log(1/\delta_N)}Z + R,$$

where Z is a standard complex Gaussian, R is a complex Gaussian random variable independent of x with order one variance.

REMARK 1.4. According to Theorem 1.2, we also have the convergence of the form

$$\mathcal{W}_2(\zeta(1/2 + i\delta_T x + i\omega T), e^{Y_T}(\eta(x) + o(1)))_{W^{-\alpha, 2}(0, 1)} = 0,$$

analogous to that in Theorem 1.3. In other words, in the mesoscopic limit the functional statistics of random matrices and the Riemann zeta function coincide.

Finally, to obtain a proper limit theorem we formulate a result without the divergent term. For this, we define some further notation and concepts in order to consider almost surely nonzero generalized functions on $(0, 1)$ modulo multiplicative constants. More precisely, consider the space

$$W_{\text{mult}}^{-\alpha,2}(0, 1) := (W^{-\alpha,2}(0, 1) \setminus \{0\}) / \sim,$$

where two nonzero elements $f, g \in W^{-\alpha,2}(0, 1)$ are identified by \sim if $f = cg$ for some $c \in \mathbb{C} \setminus \{0\}$. To have the structure of a complete metric space, we equip $W_{\text{mult}}^{-\alpha,2}(0, 1)$ with the natural metric

$$d_{W_{\text{mult}}^{-\alpha,2}(0,1)}(f, g) := \inf_{\theta \in [0, 2\pi)} \|e^{i\theta} f\|_{W^{-\alpha,2}(0,1)}^{-1} - \|g\|_{W^{-\alpha,2}(0,1)}^{-1} \|W^{-\alpha,2}(0,1)\|.$$

In more geometric terms, we identify nonzero functions by their radial projections to the unit sphere of the Sobolev space. Functions on the unit sphere that differ by a unimodular constant factor are also identified.

Our second formulation of the behavior on the mesoscopic scale is the following one.

THEOREM 1.5. *Let δ_T and η be as in Theorem 1.2. Then as $T \rightarrow \infty$*

$$\zeta(1/2 + i\delta_T x + i\omega T) \xrightarrow{d} \eta(x)$$

in the topology of $W_{\text{mult}}^{-\alpha,2}(0, 1)$ for any $\alpha > 1/2$.

Above, η can be thought of as a representative (or in a sense, a regularization) of a Gaussian multiplicative chaos

$$\eta_{\text{mult}} := \text{“exp}(\mathcal{G}_{\text{meso}})\text{”},$$

that is defined only modulo multiplicative constants. Here, the field $\mathcal{G}_{\text{meso}}$ is defined up to additive constants, formally

$$\mathcal{G}_{\text{meso}}(x) = \int_0^\infty \frac{e^{-2\pi i x u}}{\sqrt{u}} dB_u^{\mathbb{C}},$$

that can be evaluated only against test functions f with vanishing integral: $\int_{\mathbb{R}} f(x) dx = 0$. It has rather interesting (formal) covariance structure

$$\mathbb{E}(\text{Re } \mathcal{G}_{\text{meso}}(x) \text{Re } \mathcal{G}_{\text{meso}}(y)) = \frac{1}{2} \log\left(\frac{1}{|x - y|}\right) = \mathbb{E}(\text{Im } \mathcal{G}_{\text{meso}}(x) \text{Im } \mathcal{G}_{\text{meso}}(y))$$

and

$$\mathbb{E}(\text{Re } \mathcal{G}_{\text{meso}}(x) \text{Im } \mathcal{G}_{\text{meso}}(y)) = -\frac{\pi}{4} \text{sgn}(x - y).$$

Naturally, such a result can be shown also on the RMT side, but we leave this to the reader.

We now turn to an outline of the proof of our main results Theorem 1.1, Theorem 1.2 and Theorem 1.3.

1.3. Outline of the proof and further results. We now describe roughly the proofs of our main results as well as state some further ones. We start with Theorem 1.1(i) and sketch the overall content of Section 2. For that end, recall that μ_T , as defined in (1.1) stands for the random shift of the zeta function. In order to identify the limit, we employ the truncated Dirichlet series for the zeta function: for $A \subset \mathbb{N} = \{1, 2, \dots\}$ finite and $s \in \mathbb{C}$, let $\zeta_A(s) =$

$\sum_{n \in A} n^{-s}$. Moreover, write $\mu_{T,A}(x) = \zeta_A(1/2 + ix + iT\omega)$. The proof uses in a crucial manner the explicit $T \rightarrow \infty$ limit of the two-point functions

$$\mathbb{E}\mu_T(x)\overline{\mu_T(y)}, \quad \mathbb{E}\mu_{T,A}(x)\overline{\mu_T(y)}, \quad \text{and} \quad \mathbb{E}\mu_{T,A}(x)\overline{\mu_{T,A}(y)}.$$

Large T asymptotics of the quantity $\mathbb{E}\mu_T(x)\overline{\mu_T(y)}$ are actually well known and essentially due to Ingham [47], though we rely on a more recent result due to Bettin [14], where the asymptotics are shown to be uniform in a suitable sense.

Interestingly enough, the main term in $\mathbb{E}\mu_T(x)\overline{\mu_T(y)}$ is given by the quantity $(i(x - y))^{-1}$, that is, the kernel of the Hilbert transform. Using this observation as a starting point, a careful analysis enables us to deduce suitable uniform estimates, which in turn show that the second moment $\mathbb{E}|\mu_T(f)|^2$ converges as soon as f is square-integrable with nice enough decay at infinity. From this, one may already fairly easily infer that $((1 + x^2)^{-1}\mu_T(x))_{T \geq 1}$ remains tight in $W^{-\alpha,2}(\mathbb{R})$ if $\alpha > 1/2$.

In order to get hold of the convergence as $T \rightarrow \infty$, we prove that the quantities $\mu_{T,A}(f)$ approximate well $\mu_T(f)$ in terms of variance if $A = A(f)$ is taken to be a large enough set, and one may then lift this to a good approximation on the level of negative index Sobolev functions. The final piece of information one needs is to note that as $T \rightarrow \infty$, and A chosen suitably, the random variable $\mu_{T,A}$ approximates the *randomized truncated Euler product*

$$(1.2) \quad \zeta_{N,\text{rand}}(s) := \prod_{k=1}^N \left(\frac{1}{1 - p_k^{-s} e^{2\pi i \theta_k}} \right),$$

where the θ_k 's are i.i.d. random variables, each uniformly distributed on $[0, 1]$. Finally, we will show that as $N \rightarrow \infty$, $\zeta_{N,\text{rand}}(1/2 + ix)$ converges almost surely (in the sense of generalized functions) to the *randomized Riemann zeta function* $\zeta_{\text{rand}}(1/2 + ix)$, where

$$\zeta_{\text{rand}}(s) := \prod_{k=1}^{\infty} \left(\frac{1}{1 - p_k^{-s} e^{2\pi i \theta_k}} \right).$$

In addition to being a limit of $\zeta_{N,\text{rand}}(1/2 + ix)$, $\zeta_{\text{rand}}(1/2 + ix)$ can be understood as the boundary values (in the sense of generalized functions) of the random analytic function $\zeta_{\text{rand}}(s)$ in the half-plane $\{\sigma > 1/2\}$. We mention that this object has appeared in the literature earlier, first in Bagchi's work [6] where the functional statistical limit of ζ on lines $\{\sigma = \sigma_0\}$ with $\sigma_0 > 1/2$ was identified. A truncated version of this object (motivated by Selberg's work) appears recently in the work of Arguin, Belius and Harper [3] that considers the distribution of maxima over short intervals.

As an aside, we mention that it is important to observe that the limiting statistical object $x \mapsto \zeta_{\text{rand}}(1/2 + ix)$ is truly a generalized function, and not a function or a complex measure. Note that this implies the same claim for the complex Gaussian multiplicative chaos $e^{\mathcal{G}}$. To our knowledge, this is the first proof of such a result in the setting of complex multiplicative chaos.

THEOREM 1.6. *Almost surely ζ_{rand} does not coincide with a (random complex) Borel measure on any open subinterval of the critical line $\{\sigma = 1/2\}$.*

It is also easy to verify that ζ_{rand} is not a Gaussian random generalized function. As discussed in [57], p. 250, and [56], it is known that the (localized) random shifts of the zeta function on the critical line do not converge in distribution in the space of continuous functions. Thus Theorem 1.6 gives an adequate explanation for this phenomenon: one is forced to seek for the limiting objects in a suitable space of generalized functions.

The proof of the second part of the Theorem 1.1 (which we give in Section 4) is based on the following result of independent interest, as it provides a direct functional Gaussian approximation in contrast to, for example, [3, 64].

THEOREM 1.7. *For each $N \geq 1$, and any $A > 0$ there exists a decomposition*

$$\log \zeta_{N,\text{rand}}(1/2 + ix) = \mathcal{G}_N(x) + \mathcal{E}_N(x),$$

where \mathcal{G}_N is a Gaussian process on $[-A, A]$ which can be written in the following way: let $(W_k^{(j)})_{k \in \mathbb{Z}_+, j \in \{0,1\}}$ be i.i.d. standard Gaussians, then

$$\mathcal{G}_N(x) = \sum_{k=1}^N \frac{1}{\sqrt{2p_k}} p_k^{-ix} (W_k^{(1)} + iW_k^{(2)}).$$

The function \mathcal{E}_N is smooth and as $N \rightarrow \infty$, it a.s. converges uniformly to a random smooth function $\mathcal{E} \in C^\infty[-A, A]$. Moreover, the maximal error and its derivatives in this decomposition have finite exponential moments:

$$\mathbb{E} \exp\left(\lambda \sup_{N \geq 1} \|\mathcal{E}_N(x)\|_{C^\ell[-A,A]}\right) < \infty \quad \text{for all } \lambda > 0 \text{ and } \ell \geq 0.$$

The proof of this is the content of Section 3, which relies on slightly technical Gaussian approximation arguments combined with classical estimates for the distribution of prime numbers. This result then allows us to use martingale techniques to establish Theorem 1.1(ii).

After this, we move onto Section 5, where we prove our mesoscopic results for zeta, namely Theorem 1.2 and Theorem 1.5. The proof of these follow essentially from a Gaussian approximation result of a similar flavor as Theorem 1.7, but stronger. The proof of this combines slightly lengthy arguments from basic number theory, harmonic analysis and probability theory.

As we discuss later, it is natural to expect that the zeta function would give rise also to real multiplicative chaos measures in the sense that $|\mu_T(x)|^\beta$ normalized by its mean would converge to nontrivial random measure on \mathbb{R} , and these measures could be used to analyze some of the fractal structure of μ_T . Moreover, in view of Theorem 1.1, it is natural to expect that these measures could be constructed from $\zeta_{N,\text{rand}}$ as well. As the existence of the measures constructed from $\zeta_{N,\text{rand}}$ can be deduced from Theorem 1.7 and slight modifications of it, we prove results concerning them in this article as well. The precise results are the following.

THEOREM 1.8. *As $N \rightarrow \infty$, the random measure*

$$\frac{|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta}{\mathbb{E}|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta} dx, \quad x \in [0, 1],$$

converges almost surely with respect to the weak topology of measures to a random measure $\eta_\beta(dx)$ on $[0, 1]$. For $\beta \geq \beta_c = 2$, this limit is almost surely the zero measure. For $\beta < \beta_c$, it can be written as

$$\eta_\beta(dx) = f_\beta(x)\lambda_\beta(dx),$$

where $\lambda_\beta(dx)$ is a Gaussian multiplicative chaos measure (for a precise definition, see Theorem 6.1) and f_β is a random continuous positive function such that for any $\ell \geq 1$ the norms $\|f_\beta\|_{C^\ell[0,1]}$ and $\|1/f_\beta\|_{C^\ell[0,1]}$ possess moments of all orders. Moreover, for $\beta < \beta_c$ and $p < 4/\beta^2$,

$$\mathbb{E}\eta_\beta[0, 1]^p < \infty.$$

The previous result is formulated on a finite interval $[0, 1]$ for the sake of simplicity. From the theory of Gaussian multiplicative chaos and log-correlated fields, it is known that even for $\beta \geq \beta_c$, there is a way of constructing nontrivial random limiting measures. This involves a

more complicated normalization procedure. The “critical measure” corresponding to $\beta = \beta_c$, that was already discussed in Section 1.1, is particularly important as it plays a significant role in the study of the maximum of the field, and it essentially determines the distribution of the limiting (atomic) measures for $\beta > \beta_c$. We also prove a result concerning this critical measure.

THEOREM 1.9. *As $N \rightarrow \infty$, the random measure*

$$\sqrt{\log \log N} \frac{|\zeta_{N,\text{rand}}(1/2 + ix)|^{\beta_c}}{\mathbb{E}|\zeta_{N,\text{rand}}(1/2 + ix)|^{\beta_c}} dx$$

converges in distribution (with respect to the topology of weak convergence) to a nontrivial random measure $\xi_{\beta_c}(dx)$ which can be written as $f_{\beta_c}(x)\lambda_{\beta_c}(dx)$, where again f_{β_c} is a positive random continuous function such that the norms $\|f_{\beta_c}\|_{C^\ell[0,1]}$ and $\|1/f_{\beta_c}\|_{C^\ell[0,1]}$ possess moments of all orders, and $\lambda_{\beta_c}(dx)$ is a critical Gaussian multiplicative chaos measure. For a precise definition, see Section 7.

While we expect that Theorem 1.7 could be used to describe how to construct nontrivial limiting measures for $\beta > \beta_c$ and to describe the maximum of $|\zeta_{N,\text{rand}}|$ as in [3], this would require significant analysis of the Gaussian field \mathcal{G}_N and we choose to not go into this here. The proofs of these two results are given in Section 6 and Section 7. Finally in Appendix A, we prove a Gaussian approximation result, in Appendix B, we prove an analogue of Theorem 1.1 for random unitary matrices, and in Appendix C, we prove Theorem 1.3.

To conclude this Introduction, we offer some further discussion on the background of the problems we study and state some natural conjectures raised by our main results.

1.4. *Further comments, conjectures and questions.* There are several open questions that are raised by or are related to the present paper, and we mention here a few of them. First of all, one should study the properties of the relevant complex Gaussian chaos distribution. For example, one may ask which finite moments does it possess? What kind of universality does it possess, that is, under which conditions do different approximations for the complex Gaussian field lead to the same chaos? What are the a.s. exact smoothness properties of the realizations as generalized functions? How quickly can $\delta_T \rightarrow 0^+$ in the mesoscopic scaling result? Finally, to what extent do this type of limit theorems hold for more general functions such as L-functions.

Some preliminary steps related to some of the above questions are taken in [43], where one considers a notion of ‘Hardy’ chaos, which roughly speaking means multiplicative chaos obtained as boundary values of a class of random analytic functions. This allows one to, for example, consider statistical limits of the powers $(\zeta_{\text{rand},N})^k$ as $N \rightarrow \infty$. Compared to the theory of real chaos this may sound surprising since for the Hardy chaos there seems not to appear a critical point where the normalization changes.

Further, and likely far more difficult questions, lie in studying fractal properties of the (statistical behavior) of the zeta function. A key tool here would be establishing a connection between $|\mu_T(x)|^\beta$ and real multiplicative chaos. More precisely, one would expect that the measures

$$\frac{|\mu_T(x)|^\beta}{\mathbb{E}|\mu_T(x)|^\beta} dx$$

would converge to the measures of Theorem 1.8 for $0 < \beta < 2$, and a similar statement for $\beta = 2$ concerning the measure of Theorem 1.9. Such results are also likely to be of key importance in settling some of the conjectures of Fyodorov and Keating. Such results would of

course not be very surprising in view of Theorem 1.1, but there are still significant obstacles. First of all, in proving such results, one typically needs good asymptotics for the normalizing constant $a(T) := \mathbb{E}|\mu_T(x)|^\beta$. Note that asymptotics for $a(T)$ would also be of obvious interest for understanding the typical size of the zeta function on an interval of fixed size. A guess for the asymptotics of this quantity can be obtained from Selberg’s central limit theorem (see [66, 73]), namely the fact that

$$\left(\frac{1}{2} \log \log(T)\right)^{-1/2} \log|\zeta(1/2 + iT + i\omega T)| \xrightarrow{d} N(0, 1)$$

as $T \rightarrow \infty$. In view of Selberg’s CLT, one would expect that for some $c(\beta) > 0$

$$(1.3) \quad a(T) \sim c(\beta)e^{\frac{1}{4}\beta^2 \log \log T},$$

which is exactly the long-standing prediction for the Riemann zeta function. Unfortunately, (1.3) is known unconditionally only for $\beta = 2, 4$ due to Hardy–Littlewood and Ingham. Also a lower bound of the desired type is known unconditionally [65]. A conditional (assuming the RH) upper bound of the same type was given in [41, 67, 68] for $\beta \leq 4$, and for $\beta \geq 4$ in [39]. In addition, some fairly sharp conditional estimates for the shifted moments such as the two-point function are given in [21], but bounds of the correct order in $\log T$ are still unknown. The asymptotics of $a(T)$ are actually just the tip of the iceberg. To prove the type of convergence results one would naturally expect for the relevant measures, one would expect to need precise asymptotics for quantities such as $\mathbb{E}|\mu_T(x)|^\beta |\mu_T(y)|^\beta$. There are in fact very precise conjectures concerning various kinds of asymptotics of correlation functions of the zeta function (see, e.g., [25, 26]) but it seems that with current knowledge, such results still remain out of reach. Nevertheless, we formulate as precise conjectures what we expect the connection between the statistical behavior of the zeta function and real multiplicative chaos to be.

CONJECTURE 1.10. *For $\beta \in (0, 2)$, the random densities*

$$(\log T)^{-\frac{1}{4}\beta^2} |\zeta(1/2 + ix + iT)|^\beta, \quad x \in [0, 1]$$

converge in distribution to a constant multiple of the multiplicative chaos measure described in Theorem 1.8.

CONJECTURE 1.11. *The previous conjecture holds for $\beta = \beta_c = 2$ as soon as one adds the normalizing factor $(\log \log T)^{1/2}$.*

CONJECTURE 1.12. *There are mesoscopic analogues of the above conjectures and they can be formulated in a similar way as in Section 5.*

These conjectures all can be viewed as (very strong versions of) statements that the real part of $\log \zeta(1/2 + i\omega T + ix)$ behaves like a log-correlated field. In view of Theorem 1.1, it is of course natural to expect similar statements for the imaginary part of $\log \zeta(1/2 + i\omega T + ix)$. This seems even further out of reach currently as the imaginary part of $\log \zeta$ is closely related to the location of the zeros of ζ . This being said, there do exist some results of the flavor of the imaginary part of $\log \zeta$ being log-correlated. In particular, this is closely related to the issue of linear statistics of zeroes of the zeta function, which has been studied by Bourgade and Kuan [18], Rodgers [70], as well as Maples and Rodgers [58], who prove under varying assumptions Gaussian approximation results for linear statistics of the zeros, which—assuming the Riemann hypothesis—can be interpreted as the fluctuations of the zeros being log-correlated. Much stronger results exist in the setting of random matrix theory; for a connection between multiplicative chaos and the corresponding object in random matrix theory see, for example, [55, 79]. We will not formulate any precise conjectures concerning the imaginary part of $\log \zeta$ and multiplicative chaos, but simply note that this is also an interesting direction of study.

2. Existence of the limit $\mu_T \rightarrow \mu$: Proof of Theorem 1.1(i). The structure of this section is as follows: we start by proving several auxiliary results that rely on basic Fourier analysis. Then we prove a result allowing us to control of the second moment of $\|\mu_T - \mu_{T,A}\|$ in a suitable weighted Sobolev norm. These auxiliary results are formulated in such a manner that after they are established the proof of the first part of Theorem 1.1 will follow effortlessly.

We begin by defining a suitable space of test functions for our random distributions.

DEFINITION 2.1. The space V consists of locally integrable functions $g : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$\|g\|_V^2 := \int_{\mathbb{R}} \frac{|g(x)|^2}{1+x^2} dx < \infty.$$

Obviously, $L^\infty(\mathbb{R}) \subset V$. Next, we define another norm that plays a central role in what follows.

DEFINITION 2.2. Assume that $f \in L^1(\mathbb{R})$, or more generally, that $f \in \mathcal{S}'(\mathbb{R})$ is such that $\widehat{f} \in C(\mathbb{R})$. Then we set

$$\|f\|_Z := \left(\sum_{n=1}^{\infty} \frac{|\widehat{f}(\frac{1}{2\pi} \log(n))|^2}{n} \right)^{1/2}.$$

If $\|f\|_Z < \infty$ and $\|g\|_Z < \infty$, we write

$$\langle f, g \rangle_Z := \sum_{n=1}^{\infty} \frac{\widehat{f}(\frac{1}{2\pi} \log(n)) \overline{\widehat{g}(\frac{1}{2\pi} \log(n))}}{n}.$$

Given a set $A \subset \mathbb{N}$, we write analogously

$$\begin{aligned} \|f\|_{Z,A} &:= \left(\sum_{n \in A} \frac{|\widehat{f}(\frac{1}{2\pi} \log(n))|^2}{n} \right)^{1/2} \quad \text{and} \\ \langle f, g \rangle_{Z,A} &:= \sum_{n \in A} \frac{\widehat{f}(\frac{1}{2\pi} \log(n)) \overline{\widehat{g}(\frac{1}{2\pi} \log(n))}}{n}. \end{aligned}$$

Above the assumption that \widehat{f} is continuous makes sure that point evaluations $\widehat{f}(a)$ for $a \in \mathbb{R}$ are well-defined, and hence the quantity $\|f\|_Z$ is well-defined, but of course it may still take the value ∞ . It is useful to note that for any such f one has

$$(2.1) \quad \|f\|_Z = \lim_{A \rightarrow \mathbb{N}} \|f\|_{Z,A},$$

where the notation can be interpreted as saying that if we take (A_k) to be an increasing sequence of finite sets, which increase to \mathbb{N} as $k \rightarrow \infty$, then $\|f\|_{Z,A_k} \rightarrow \|f\|_Z$.

We shall make use of the following embedding result.

LEMMA 2.3. *There exists a positive constant C such that for all $f \in L^1$*

$$\|f\|_Z^2 \leq C \int_{\mathbb{R}} (1+x^2)|f(x)|^2 dx.$$

PROOF. Let us write $g := \widehat{f}$. By the definition of the Sobolev space $W^{1,2}(\mathbb{R})$ it is equivalent to prove the inequality

$$\|f\|_Z^2 = \sum_{n=1}^{\infty} \frac{|g(\frac{1}{2\pi} \log(n))|^2}{n} \lesssim \|g\|_{W^{1,2}(\mathbb{R})}^2.$$

Approximating by smooth functions, we may also assume that $g \in \mathcal{S}(\mathbb{R})$. In this case, $\|f\|_Z$ is finite and by using the dual representation of the ℓ_2 norm, we see that

$$\begin{aligned} \|f\|_Z &= \sup_{\|(b_n)\|_{\ell^2} \leq 1} \left| \sum_{n=1}^{\infty} b_n n^{-1/2} g\left(\frac{1}{2\pi} \log(n)\right) \right| \\ &= \sup_{\|(b_n)\|_{\ell^2} \leq 1} \left| \left\langle \sum_{n=1}^{\infty} b_n n^{-1/2} \delta_{\frac{1}{2\pi} \log n}, g \right\rangle \right|, \end{aligned}$$

so it is enough to show that

$$(2.2) \quad \left\| \sum_{n=1}^{\infty} b_n n^{-1/2} \delta_{\frac{1}{2\pi} \log n} \right\|_{W^{-1,2}(\mathbb{R})} \lesssim \|(b_n)\|_{\ell^2}.$$

We need the auxiliary estimate

$$(2.3) \quad \|\delta_u - h^{-1} \chi_{(u, u+h]}\|_{W^{-1,2}(\mathbb{R})} \lesssim h^{1/2} \quad \text{for } u \in \mathbb{R}, h > 0,$$

where χ_I denotes the indicator function of the set I . For this end, we may translate to $u = 0$ and simply compute that the square of the left-hand side equals

$$\int_{\mathbb{R}} \left| 1 - \frac{1 - e^{-2\pi i h \xi}}{2\pi i h \xi} \right|^2 \frac{d\xi}{1 + \xi^2} \lesssim \int_{|\xi| \leq h^{-1}} h^2 d\xi + \int_{|\xi| > h^{-1}} \frac{d\xi}{1 + \xi^2} \lesssim h,$$

where we noted that by Taylor’s formula $|1 - \frac{1 - e^{-2\pi i h \xi}}{2\pi i h \xi}|^2 \lesssim |h\xi|^2$ for $|\xi| \leq h^{-1}$.

To return to the proof of (2.2), write $\tilde{b}_n := b_n n^{-1/2} (\frac{1}{2\pi}(\log(n + 1) - \log(n)))^{-1}$ so that $|\tilde{b}_n| \sim 2\pi n^{1/2} |b_n|$, and observe that (2.3) yields the estimate

$$(2.4) \quad \begin{aligned} &\left\| \sum_{n=1}^{\infty} b_n n^{-1/2} \delta_{\frac{1}{2\pi} \log n} - \sum_{n=1}^{\infty} \tilde{b}_n \chi_{(\frac{1}{2\pi} \log(n), \frac{1}{2\pi} \log(n+1)]} \right\|_{W^{-1,2}(\mathbb{R})} \\ &\lesssim \sum |b_n| n^{-1/2} \left(\frac{1}{2\pi} \log(n + 1) - \frac{1}{2\pi} \log(n) \right)^{1/2} \lesssim \sum_{n=1}^{\infty} |b_n| n^{-1} \lesssim \|(b_n)\|_{\ell^2}. \end{aligned}$$

On the other hand, as the characteristic functions have disjoint supports we obtain

$$(2.5) \quad \begin{aligned} \left\| \sum_{n=1}^{\infty} \tilde{b}_n \chi_{(\frac{1}{2\pi} \log(n), \frac{1}{2\pi} \log(n+1)]} \right\|_{W^{-1,2}(\mathbb{R})} &\leq \left\| \sum_{n=1}^{\infty} \tilde{b}_n \chi_{(\frac{1}{2\pi} \log(n), \frac{1}{2\pi} \log(n+1)]} \right\|_{L^2(\mathbb{R})} \\ &\lesssim \|(b_n)\|_{\ell^2}, \end{aligned}$$

Now (2.2) is an immediate consequence of this estimate in combination with (2.4). \square

REMARK 2.4. By slightly modifying the above argument, one obtains the sharper result $\|f\|_Z^2 \lesssim \int_{\mathbb{R}} (1 + x^2)^a |f(x)|^2 dx$ for $a > 1/2$, which is optimal as one can check that the estimate fails for the choice $a = 1/2$.

The following computation of a specific Fourier transform is used to establish the slightly delicate equality (2.8) which is crucial for our purposes.

LEMMA 2.5. Assume that $a > 0$ and define the function $h_a : \mathbb{R} \rightarrow \mathbb{C}$ by setting $h_a(u) := \zeta(1 + iu) - \frac{e^{-iau}}{iu}$, where the value at $u = 0$ is defined as the limit as $u \rightarrow 0$. Then

$$\hat{h}_a = \sum_{n=1}^{\infty} n^{-1} \delta_{-\frac{1}{2\pi} \log(n)} - 2\pi \chi_{(-\infty, -\frac{1}{2\pi} a]}.$$

PROOF. Observe that $g(z) := \zeta(1+z) - z^{-1}e^{-az}$ is entire and as $z \rightarrow \infty$, $g(z)$ grows at most logarithmically in the half-plane $\{\operatorname{Re} z \geq 0\}$. This implies that its Fourier-transform over the imaginary axis is obtained as the limit of the Fourier transform over the line $\{\operatorname{Re} z = \varepsilon\}$, in the limit $\varepsilon \rightarrow 0^+$. In other words,

$$\mathcal{F}(h_a) = \lim_{\varepsilon \rightarrow 0^+} \mathcal{F}(\zeta(1 + \varepsilon + i \cdot)) - \lim_{\varepsilon \rightarrow 0^+} \mathcal{F}((\varepsilon + i \cdot)^{-1}(e^{-a(\varepsilon + i \cdot)})) =: \eta_1 - \eta_2,$$

with convergence in \mathcal{S}' , as soon as we verify that both limits exist separately. This follows from the continuity of the Fourier transform $\mathcal{F} : \mathcal{S}' \rightarrow \mathcal{S}'$ with respect to the weak topology. Since $\zeta(1 + \varepsilon + iu) = \sum_{n=1}^\infty n^{-1-\varepsilon} e^{-i \log(n)u}$, with uniform convergence, we immediately see that

$$\eta_1 = \lim_{\varepsilon \rightarrow 0^+} \sum_{n=1}^\infty n^{-1-\varepsilon} \delta_{-\frac{1}{2\pi} \log(n)} = \sum_{n=1}^\infty n^{-1} \delta_{-\frac{1}{2\pi} \log(n)}.$$

In turn, we observe that for any $\varepsilon > 0$ the function $2\pi e^{2\pi\varepsilon\xi} \chi_{(-\infty, -\frac{1}{2\pi}a]}(\xi)$ belongs to $L^1(\mathbb{R})$ and

$$2\pi \int_{-\infty}^{-\frac{1}{2\pi}a} e^{2\pi i u \xi} e^{2\pi \varepsilon \xi} d\xi = \int_{-\infty}^{-a} e^{i u \xi'} e^{\varepsilon \xi'} d\xi' = \frac{e^{-a(\varepsilon + iu)}}{\varepsilon + iu}.$$

In other words, the Fourier transform of $\frac{e^{-a(\varepsilon + iu)}}{\varepsilon + iu}$ equals $2\pi e^{2\pi\varepsilon\xi} \chi_{(-\infty, -\frac{1}{2\pi}a]}(\xi)$, and letting $\varepsilon \rightarrow 0^+$ we see that $\eta_2 = 2\pi \chi_{(-\infty, -\frac{1}{2\pi}a]}$. \square

Our next lemma connects the norms $\|\cdot\|_Z$ and $\|\cdot\|_{Z,A}$ to the Riemann zeta function and the truncated Dirichlet series of the zeta function. The principal claims are formulas (2.8) and (2.9). Recall that the truncated Dirichlet series of the zeta function was defined by

$$\zeta_A(s) := \sum_{n \in A} n^{-s}.$$

LEMMA 2.6. Assume that $f, g \in L^1$ and $\int_{\mathbb{R}} (1+x^2)(|f(x)|^2 + |g(x)|^2) dx < \infty$. Then for any $A \subset \mathbb{N}$,

$$(2.6) \quad \langle f, g \rangle_{Z,A} = \int_{\mathbb{R}^2} f(x) \zeta_A(1 + i(x-y)) \overline{g(y)} dx dy$$

and

$$(2.7) \quad \begin{aligned} \langle f, g \rangle_Z &= \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^2} f(x) \zeta(1 + \varepsilon + i(x-y)) \overline{g(y)} dx dy \\ &= \lim_{A \rightarrow \mathbb{N}} \langle f, g \rangle_{Z,A}, \end{aligned}$$

where the interpretation of the limit is as in (2.1).

We also have

$$(2.8) \quad \langle f, g \rangle_Z = \lim_{T \rightarrow \infty} K_T(f, g),$$

where

$$K_T(f, g) = \int_{\mathbb{R}^2} f(x) \left(\zeta(1 + i(x-y)) + \frac{\zeta(1 - i(x-y))}{1 - i(x-y)} T^{-i(x-y)} \right) \overline{g(y)} dx dy$$

Moreover, there is a constant C such that for all $T \geq 1$,

$$(2.9) \quad |K_T(f, g)| \leq C \left(\int_{\mathbb{R}} (1+x^2) |f(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} (1+y^2) |g(y)|^2 dy \right)^{1/2}.$$

PROOF. Note that under our assumption on f (and g), we have by Cauchy–Schwarz that for some $\varepsilon > 0$,

$$(2.10) \quad \int_{\mathbb{R}} |f(x)|(1 + |x|)^\varepsilon < \infty.$$

In particular, this entails that all the integrals in the statements are well-defined, due to the boundedness of $x \mapsto \zeta_A(1 + ix)$ (which follows from the definition of ζ_A) and then at most logarithmic growth of ζ over the line $\sigma = 1$; see, for example, [77], Theorem 3.5.

To prove (2.6), we note (recalling that $\zeta_A(s) = \sum_{n \in A} n^{-s}$ and A is finite) that

$$\begin{aligned} \int_{\mathbb{R}^2} f(x)\zeta_A(1 + i(x - y))\overline{g(y)} dx dy &= \sum_{n \in A} n^{-1} \int_{\mathbb{R}} f(x)n^{-ix} dx \int_{\mathbb{R}} \overline{g(y)}n^{iy} dy \\ &= \sum_{n \in A} \frac{\widehat{f}\left(\frac{1}{2\pi} \log n\right)\overline{\widehat{g}\left(\frac{1}{2\pi} \log n\right)}}{n}, \end{aligned}$$

which is precisely (2.6).

Next, the fact that $\sum_{n=1}^\infty n^{-1-\varepsilon} < \infty$ allows us to deduce in an analogous way that

$$\int_{\mathbb{R}^2} f(x)\zeta(1 + \varepsilon + i(x - y))\overline{g(y)} dx dy = \sum_{n=1}^\infty n^{-1-\varepsilon} \widehat{f}\left(\frac{1}{2\pi} \log n\right)\overline{\widehat{g}\left(\frac{1}{2\pi} \log n\right)},$$

and by letting $\varepsilon \rightarrow 0^+$ we deduce (2.7) as according to Lemma 2.3 and the definition of the inner product $\langle f, g \rangle_Z$, the right-hand series above has a convergent majorant series of the form $\sum_{n=1}^\infty n^{-1} |\widehat{f}\left(\frac{1}{2\pi} \log n\right)| |\widehat{g}\left(\frac{1}{2\pi} \log n\right)|$, so here (as in many other computations we make in this proof) we may apply the dominated convergence theorem.

In order to prove (2.8), we note first that the function $s \mapsto (1 - s)^{-1}\zeta(1 - s) + s^{-1}$ is analytic and bounded over the imaginary axis, whence (2.10) implies that

$$f(x)\overline{g(y)} \left(\frac{\zeta(1 - i(x - y))}{1 - i(x - y)} + (i(x - y))^{-1} \right)$$

is integrable over \mathbb{R}^2 . A fortiori,

$$\lim_{T \rightarrow \infty} \int_{\mathbb{R}^2} \left[f(x)\overline{g(y)} \left(\frac{\zeta(1 - i(x - y))}{1 - i(x - y)} + (i(x - y))^{-1} \right) \right] T^{-i(x-y)} dx dy = 0$$

by the Riemann–Lebesgue lemma. In the notation of Lemma 2.5, it remains to show that

$$\langle f, g \rangle_Z = \lim_{T \rightarrow \infty} \int_{\mathbb{R}^2} h_{\log T}(x - y) f(x)\overline{g(y)} dx dy$$

Assume first that $f, g \in C_0^\infty(\mathbb{R})$. Then we may compute directly

$$\int_{\mathbb{R}^2} f(x)\overline{g(y)} h_{\log T}(x - y) dx dy = \int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \widehat{h}_{\log T}(-\xi) d\xi,$$

where the integral on the right-hand side has to be understood as the distributional pairing between the Schwartz test function $\widehat{f}(\xi)\overline{\widehat{g}(\xi)}$ and the Schwartz distribution $\widehat{h}_{\log T}(-\cdot)$. Lemma 2.5 verifies that the right-hand side equals

$$(2.11) \quad \sum_{n=1}^\infty n^{-1} \widehat{f}\left(\frac{1}{2\pi} \log(n)\right)\overline{\widehat{g}\left(\frac{1}{2\pi} \log(n)\right)} - 2\pi \int_{\frac{1}{2\pi} \log T}^\infty \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi.$$

For general f, g , we note that in particular, our assumptions imply that $f, g \in L^2$. Then by Lemma 2.3 and the L^2 -continuity of the Fourier transform, we may approximate general f

and g by test functions and take limits in (2.11) to verify its validity in full generality. Finally, (2.8) and (2.9) are immediate consequences of (2.11) and Lemma 2.3. \square

In what follows, it is notationally convenient to introduce the “weight regularization” of a given function $f : \mathbb{R} \rightarrow \mathbb{C}$ by setting

$$f^{\mathbf{r}}(x) := (1 + x^2)^{-1} f(x).$$

Directly from the definitions we note that if $f \in V$ (recall Definition 2.1) we have for $T \geq 1$

$$\begin{aligned} \mu_T^{\mathbf{r}}(f) &= \mu_T(f^{\mathbf{r}}) = \int_{\mathbb{R}} \zeta(1/2 + ix + iT\omega) f^{\mathbf{r}}(x) dx \quad \text{and} \\ \mathbb{E}\mu_T^{\mathbf{r}}(f) &= \mathbb{E}\mu_T(f^{\mathbf{r}}) = \frac{1}{T} \int_0^T \int_{\mathbb{R}} \zeta(1/2 + ix + it) f^{\mathbf{r}}(x) dx dt. \end{aligned}$$

To see that these quantities are well-defined, note first the classical growth bound (see, e.g., [77], Chapter V, or [19] for the currently best known bound $\mu_0 = 13/84 + \varepsilon$, due to Bourgain): for $\sigma \geq \frac{1}{2}$,

$$(2.12) \quad \zeta(\sigma + it) = \mathcal{O}(t^{\mu_0}) \quad \text{with } \mu_0 < 1/6,$$

and the implied constant is independent of $\sigma \geq \frac{1}{2}$. Using Cauchy–Schwarz, we deduce that for some $C_T, \tilde{C}_T > 0$,

$$\begin{aligned} |\mu_T(f^{\mathbf{r}})| &\leq C_T \int_{\mathbb{R}} |f^{\mathbf{r}}(x)| (1 + x^2)^{1/12} dx \\ &= C_T \int_{\mathbb{R}} (|f(x)| (1 + x^2)^{-1/2}) (1 + x^2)^{1/2+1/12-1} dx \leq \tilde{C}_T \|f\|_V. \end{aligned}$$

Analogous formulas hold for $\mu_{T,A}(f)$ with ζ replaced by ζ_A .

We next turn to the crucial estimates for the two-point functions. Let us first give names to the relevant objects.

DEFINITION 2.7. Let ω be uniform on $[0, 1]$, $T > 0$, $A \subset \mathbb{N}$ be finite, and $x, y \in \mathbb{R}$. Then we set

$$\begin{aligned} V_T^{(1)}(x, y) &= \mathbb{E}[\zeta(1/2 + i\omega T + ix) \overline{\zeta(1/2 + i\omega T + iy)}], \\ V_T^{(A,1)}(x, y) &= \mathbb{E}[\zeta_A(1/2 + i\omega T + ix) \overline{\zeta(1/2 + i\omega T + iy)}], \\ V_T^{(A,2)}(x, y) &= \mathbb{E}[\zeta(1/2 + i\omega T + ix) \overline{\zeta_A(1/2 + i\omega T + iy)}], \\ V_T^{(A,3)}(x, y) &= \mathbb{E}[\zeta_A(1/2 + i\omega T + ix) \overline{\zeta_A(1/2 + i\omega T + iy)}], \end{aligned}$$

and finally

$$V_T^{(A)}(x, y) = V_T^{(A,1)}(x, y) + V_T^{(A,2)}(x, y).$$

A direct application of Fubini shows that

$$(2.13) \quad \mathbb{E}[\mu_T(f^{\mathbf{r}}) \overline{\mu_T(g^{\mathbf{r}})}] = \int_{\mathbb{R}^2} f^{\mathbf{r}}(x) \overline{g^{\mathbf{r}}(y)} V_T^{(1)}(x, y) dx dy.$$

Analogous formulas hold for

$$\mathbb{E}[\mu_{T,A}(f^{\mathbf{r}}) \overline{\mu_T(g^{\mathbf{r}})}], \quad \mathbb{E}[\mu_T(f^{\mathbf{r}}) \overline{\mu_{T,A}(g^{\mathbf{r}})}], \quad \text{and} \quad \mathbb{E}[\mu_{T,A}(f^{\mathbf{r}}) \overline{\mu_{T,A}(g^{\mathbf{r}})}],$$

respectively, where one replaces $V_T^{(1)}(x, y)$ in (2.13) by $V_T^{(A,1)}(x, y)$, $V_T^{(A,2)}(x, y)$, and $V_T^{(A,3)}(x, y)$, respectively. Especially, one should observe that for real-valued test functions f one has

$$(2.14) \quad \mathbb{E}[\mu_T(f^{\mathbf{r}})\overline{\mu_{T,A}(f^{\mathbf{r}})} + \mu_{T,A}(f^{\mathbf{r}})\overline{\mu_T(f^{\mathbf{r}})}] = \int_{\mathbb{R}^2} f^{\mathbf{r}}(x)f^{\mathbf{r}}(y)V_T^{(A)}(x, y) dx dy.$$

We next establish asymptotics for the kernels $V_T^{(\cdot)}(x, y)$. In the case of $V_T^{(1)}(x, y)$, this essentially reduces to a classical result on second moments of the shifted zeta function due already to Ingham [47], though the precise form of the result that we make use of is due to Bettin [14], and we record the result⁴ here.

PROPOSITION 2.8. As $T \rightarrow \infty$,

$$(2.15) \quad V_T^{(1)}(x, y) = \zeta(1 + i(x - y)) + \frac{\zeta(1 - i(x - y))}{1 - i(x - y)} \left(\frac{T}{2\pi}\right)^{-i(x-y)} + E_1(x, y, T),$$

where

$$E_1(x, y, T) = T^{-\varepsilon}(1 + |x|^{1/2} + |y|^{1/2})$$

with a suitable small enough $\varepsilon > 0$, which is independent of T, x, y .

PROOF. Fix $\varepsilon \in (0, 1/6)$ and consider first the case where $|x|, |y| \leq T^\varepsilon$. In this situation, Bettin’s [14] result states that

$$(2.16) \quad \begin{aligned} \text{TV}_T^{(1)}(x, y) &= \int_0^T \zeta(1/2 + it + ix)\overline{\zeta(1/2 + it + iy)} dt \\ &= \int_0^T (\zeta(1 + i(x - y)) \\ &\quad + \zeta(1 - i(x - y))\chi(1/2 + i(x + t))\chi(1/2 - i(y + t))) dt \\ &\quad + \mathcal{O}((T^{1/2} + |x|^{1/2} + |y|^{1/2}) \log^2 T). \end{aligned}$$

Here, $\chi(s) = 2(2\pi)^{s-1}\Gamma(1 - s)\sin(\pi s/2)$ is the factor in the functional equation of ζ , that is, $\zeta(s) = \chi(s)\zeta(1 - s)$. Obviously, χ is unimodular on the critical line, and we may write $\chi(1/2 + iu) = \exp(i\psi(u))$, where the continuous (actually smooth) branch of $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is uniquely determined by letting $\psi(0) = 0$. Stirling’s formula for Γ'/Γ yields the following asymptotics:

$$(2.17) \quad \psi'(u) = -\log u + \log(2\pi) + \mathcal{O}(u^{-1}) \quad \text{as } u \rightarrow \infty.$$

The counterpart for $u \rightarrow -\infty$ follows from the relation $\psi(-u) = -\psi(u)$. Especially, we note that $|\psi(u_1) - \psi(u_2)| \leq 2\log(A)|u_1 - u_2|$ if $u_1, u_2 \in [-A, A]$ with large enough A . By (2.17), we see that in the range $T^{1/2} \leq t \leq T$ and $|x|, |y| \leq T^\varepsilon$ it holds that

$$\begin{aligned} &\psi(t + x) - \psi(t + y) \\ &= \log(2\pi)(x - y) - \int_{t+y}^{t+x} \log(u) du + \mathcal{O}(|x - y|T^{-1/2}) \\ &= \log(2\pi)(x - y) - \int_{t+y}^{t+x} (\log(t) + \mathcal{O}((|x| + |y|)T^{-1/2})) du + \mathcal{O}(|x - y|T^{-1/2}) \\ &= (\log(2\pi) + \log(1/t))(x - y) + \mathcal{O}(|x - y|T^{\varepsilon-1/2}). \end{aligned}$$

⁴Actually Bettin’s result covers a larger range of x, y in relation to T , but we only state the result in a form that is enough for our purposes.

Since $u \mapsto e^{iu}$ is 1-Lipschitz, this implies that for the same range of t, x, y we have

$$\chi(1/2 + i(x + t))\chi(1/2 - i(y + t)) = (t/2\pi)^{-i(x-y)} + \mathcal{O}(|x - y|T^{\varepsilon-1/2}).$$

On the other hand, for same range of x, y but with $0 \leq t \leq T^{1/2}$ Lipschitz continuity yields the simple bound

$$|\chi(1/2 + i(x + t))\chi(1/2 - i(y + t)) - (t/2\pi)^{-i(x-y)}| = \mathcal{O}(|x - y| \log T).$$

Substituting these two estimates into Bettin’s result, observing that

$$\int_0^T (t/2\pi)^{-i(x-y)} dt = T(1 - i(x - y))^{-1} (T/2\pi)^{-i(x-y)},$$

and using the logarithmic growth of zeta on the 1-line in the form $|(x - y)\zeta(1 - i(x - y))| \leq (|x| + |y|) \log(e + |x| + |y|)$ we obtain the statement of the proposition with the acceptable estimate

$$\begin{aligned} E_1(x, y, T) &= \mathcal{O}((|x| + |y|) \log(e + |x| + |y|)(T^{-1}T^{1/2} \log T + T^{-1}TT^{\varepsilon-1/2}) \\ &\quad + T^{-1}(T^{1/2} + |x|^{1/2} + |y|^{1/2}) \log^2 T) \\ &= \mathcal{O}(T^{3\varepsilon-1/2}) \end{aligned}$$

Finally, in case $\max(|x|, |y|) \geq T^\varepsilon$, we use throughout $t \in [0, T]$ a crude upper bound for the Lipschitz constants (for $x - y$ small) and the logarithmic growth of ζ on the 1-line (for $|x - y| \geq 1$) to deduce

$$\begin{aligned} &|\zeta(1 - i(x - y))((T/2\pi)^{-i(x-y)} - \chi(1/2 + i(x + t))\chi(1/2 - i(y + t)))| \\ &= \mathcal{O}(\log(|x| + |y| + T)) \end{aligned}$$

This time we obtain an upper bound for the error E_1 in the form

$$\begin{aligned} &\mathcal{O}(T^{-1}(T^{1/2} + |x|^{1/2} + |y|^{1/2}) \log^2 T + \log(e + |x| + |y| + T)) \\ &= \mathcal{O}(T^{-\varepsilon/2}(1 + |x|^{1/2} + |y|^{1/2})). \end{aligned} \quad \square$$

We now turn to terms involving ζ_A .

LEMMA 2.9. *Assume that $A \subset \mathbb{N}$ is a finite subset. Then it holds that*

$$(2.18) \quad V_T^{(A,1)}(x, y) = \zeta_A(1 + i(x - y)) + \mathcal{O}(T^{-1/6}(1 + |y|^{2/6})),$$

$$(2.19) \quad V_T^{(A,2)}(x, y) = \zeta_A(1 + i(x - y)) + \mathcal{O}(T^{-1/6}(1 + |x|^{2/6})),$$

and

$$(2.20) \quad V_T^{(A,3)}(x, y) = \zeta_A(1 + i(x - y)) + \mathcal{O}(T^{-1}),$$

where all error estimates may depend on A , but the implied constants are independent of x, y .

PROOF. We will first prove the second statement (2.19). As we do not care on the dependence of the error on the finite set A , it is enough to treat the case of a singleton, say $A = \{m\}$. Let us first assume that $|x| \leq T/2$, write

$$(2.21) \quad V_T^{(A,2)}(x, y) = \frac{1}{iT} \int_{1/2}^{1/2+iT} m^{s-1+iy} \zeta(s + ix) ds,$$

and use the residue theorem to move the integration to the line $[3/2, 3/2 + iT]$, so that

$$V_T^{(A,2)}(x, y) = m^{-1-i(x-y)} + T^{-1} \sum_{n=1, n \neq m}^{\infty} m^{1/2+iy} n^{-3/2-ix} \frac{(m/n)^{iT} - 1}{\log(m/n)} + E(x, y, T, A).$$

Here, the error term consists of the horizontal integrals and the (possible) residue at $s = 1 - ix$ (with obvious modification for $x = 0$). The contribution of the residue is $T^{-1}\mathcal{O}(1)$ and by (2.12) the horizontal terms contribute at most $T^{-1}\mathcal{O}(T^{1/6} + |x|^{1/6}) = \mathcal{O}(T^{-1/6}(1 + |x|^{2/6}))$. The infinite sum term is of order $\mathcal{O}(T^{-1})$ uniformly in x, y (note though that almost all of our error terms depend on m). In case $|x| \geq T/2$ we note that size the whole integral is $\mathcal{O}(T^{1/6} + |x|^{1/6}) = \mathcal{O}(T^{-1/6}(1 + |x|^{2/6}))$. Putting everything together, the estimate (2.19) follows.

Equation (2.18) follows by symmetry, so it remains to prove (2.20). Again, as we do not care about uniformity in A , it is in fact sufficient to consider two singletons:

$$\frac{n^{-1/2+iy}m^{-1/2-ix}}{T} \int_0^T n^{it}m^{-it} dt = \begin{cases} \frac{n^{-1/2+iy}m^{-1/2-ix}}{iT \log \frac{n}{m}} (e^{iT \log \frac{n}{m}} - 1), & n \neq m, \\ n^{-1-i(x-y)}, & n = m. \end{cases}$$

The $n \neq m$ -term gives rise to the $\mathcal{O}(T^{-1})$ -error, while the $n = m$ term yields precisely $\zeta_A(1 + i(x - y))$. This concludes the proof. \square

Having these estimates in hand, we are now able to begin constructing our proof of Theorem 1.1(i). We begin with the following moment bound for μ_T .

LEMMA 2.10. *Assume that f is real valued and $f \in V$. Then*

$$(2.22) \quad \mathbb{E}|\mu_T(f^{\mathbf{r}})|^2 \leq C \|f^{\mathbf{r}}\|_Z^2 \lesssim \|f\|_V^2,$$

where C is independent of both $T \geq 1$ and $f \in V$. Moreover,

$$(2.23) \quad \lim_{T \rightarrow \infty} \mathbb{E}|\mu_T(f^{\mathbf{r}})|^2 = \|f^{\mathbf{r}}\|_Z^2 \lesssim \|f\|_V^2.$$

PROOF. Let us write

$$V_T^{(1)}(x, y) = \zeta(1 + i(x - y)) + \frac{\zeta(1 - i(x - y))}{1 - i(x - y)} \left(\frac{T}{2\pi}\right)^{-i(x-y)} + E_1(x, y, T).$$

Due to the error estimate (2.15) and Cauchy–Schwarz, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} E_1(x, y, T) f^{\mathbf{r}}(x) \overline{f^{\mathbf{r}}(y)} dx dy \right| \\ & \lesssim T^{-\varepsilon} \int_{\mathbb{R}^2} |f^{\mathbf{r}}(x)| |\overline{f^{\mathbf{r}}(y)}| (1 + |x| + |y|) dx dy \\ & \leq CT^{-\varepsilon} \int_{\mathbb{R}^2} |f(x)| |f(y)| (1 + x^2)^{1/2-1} (1 + y^2)^{1/2-1} dx dy \\ & \leq CT^{-\varepsilon} \left(\int_{\mathbb{R}} |f(x)|^2 (1 + x^2)^{-1} dx \right), \end{aligned}$$

which shows that the contribution from the error term vanishes in the limit $T \rightarrow \infty$. Finally, by recalling the notation of Lemma 2.6 we see that the contribution of the main term equals $K_{\log \frac{T}{2\pi}}(f^{\mathbf{r}}, f^{\mathbf{r}})$, and the uniform boundedness along with the statement about the limit of this

term follows immediately from Lemma 2.6 as soon as we observe that $\int |f^{\mathbf{r}}(x)|^2(1+x^2) = \|f\|_V^2$. \square

We have analogous estimates for similar quantities arising from the truncated Dirichlet series and for the cross terms involved.

LEMMA 2.11. *Assume that $A \subset \mathbb{N}$ is finite, f is real-valued and $f \in V$. Then*

$$(2.24) \quad \mathbb{E}|\mu_{T,A}(f^{\mathbf{r}})|^2 \leq C(A) \|f^{\mathbf{r}}\|_Z^2 \leq C(A) \|f\|_V^2,$$

where $C(A)$ is independent of $T \geq 1$ and $f \in V$. Moreover,

$$(2.25) \quad \lim_{T \rightarrow \infty} \mathbb{E}|\mu_{T,A}(f^{\mathbf{r}})|^2 = \|f^{\mathbf{r}}\|_{Z,A}^2 \lesssim \|f\|_V^2.$$

In a similar vein, for all $T \geq 1$ one has

$$(2.26) \quad |\mathbb{E}(\mu_{T,A}(f^{\mathbf{r}})\overline{\mu_T(f^{\mathbf{r}})} + \mu_T(f^{\mathbf{r}})\overline{\mu_{T,A}(f^{\mathbf{r}})})| \leq C(A) \|f\|_V^2,$$

and

$$(2.27) \quad \lim_{T \rightarrow \infty} \mathbb{E}(\mu_{T,A}(f^{\mathbf{r}})\overline{\mu_T(f^{\mathbf{r}})} + \mu_T(f^{\mathbf{r}})\overline{\mu_{T,A}(f^{\mathbf{r}})}) = 2\|f^{\mathbf{r}}\|_{Z,A}^2 \leq C(A) \|f\|_V^2.$$

PROOF. The statements (2.24) and (2.25) are proven exactly as the corresponding statements in the previous lemma by invoking the error estimate (2.20). In turn, (2.26) is obtained by Cauchy–Schwarz from (2.22) and (2.24), and the proof of (2.27) is also analogous in view of (2.18), (2.19) and (2.14). \square

As an immediate consequence of the previous lemmata, we will now deduce an interesting intermediate result, which already gives quantitative estimates for our approximation when considering a fixed test function. We also point out here a fact that can be easily checked by adapting the rest of our proof in a simple way, namely that when considering a fixed test function, we do not actually need any smoothness from it—it is enough for it to be locally square integrable and have nice enough decay at infinity. Thus the action of our limiting object can be defined for some test functions that are not in the Sobolev space $W^{\alpha,2}(\mathbb{R})$. This is a common phenomenon when studying random generalized functions.

PROPOSITION 2.12. *For each finite $A \subset \mathbb{N}$ there exists a constant $C(A) < \infty$ such that for any (possibly complex-valued) $f \in V$*

$$(2.28) \quad \mathbb{E}|\mu_T(f^{\mathbf{r}}) - \mu_{T,A}(f^{\mathbf{r}})|^2 \leq C(A) \|f^{\mathbf{r}}\|_Z^2 \leq C(A) \|f\|_V^2.$$

Moreover, we have

$$(2.29) \quad \limsup_{T \rightarrow \infty} \mathbb{E}|\mu_T(f^{\mathbf{r}}) - \mu_{T,A}(f^{\mathbf{r}})|^2 \leq 4(\|f^{\mathbf{r}}\|_Z^2 - \|f^{\mathbf{r}}\|_{Z,A}^2) \leq C' \|f\|_V^2,$$

where C' is independent of f and A .

PROOF. Let us first assume in addition that f is real-valued. Now (2.28) is an immediate consequence of the previous two lemmata. In addition, by combining (2.23), (2.25) and (2.27) we find

$$\mathbb{E}|\mu_T(f^{\mathbf{r}}) - \mu_{T,A}(f^{\mathbf{r}})|^2 \rightarrow \|f^{\mathbf{r}}\|_Z^2 - \|f^{\mathbf{r}}\|_{Z,A}^2 \quad \text{as } T \rightarrow \infty.$$

For complex-valued f , the claims follow by considering separately the corresponding quantities for real and imaginary parts of f . \square

One should note that (2.29) shows that for a fixed test function f , choosing *first* A “large” (depending on f), and *then* T large enough (depending on f and A), the second moment of $\mu_T(f^{\mathbf{r}}) - \mu_{T,A}(f^{\mathbf{r}})$ can be made arbitrarily small.

We fix $\alpha > 1/2$ for the rest of this section and consider both $\mu_T^{\mathbf{r}}$ and $\mu_{T,A}^{\mathbf{r}}$ as $W^{-\alpha,2}(\mathbb{R})$ -valued random variables. Actually, by the growth estimates of ζ , namely (2.12) and the fact that $\mu_{T,A}$ is bounded, they are even $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ -valued random variables. Measurability is obvious since they are continuous mappings from $\Omega := [0, 1]$ to $L^2(\mathbb{R})$ (and $W^{-\alpha,2}(\mathbb{R})$). Our next step is to extend the approximation hinted at in the previous proposition to the level of mean square approximation in the Sobolev space.

PROPOSITION 2.13. *Assume that $\alpha > 1/2$. Then*

$$\lim_{A \rightarrow \mathbb{N}} \limsup_{T \rightarrow \infty} \mathbb{E} \|\mu_T^{\mathbf{r}} - \mu_{T,A}^{\mathbf{r}}\|_{W^{-\alpha,2}(\mathbb{R})}^2 = 0$$

and consequently

$$\lim_{A \rightarrow \mathbb{N}} \limsup_{T \rightarrow \infty} \mathcal{W}_2(\mu_T^{\mathbf{r}}, \mu_{T,A}^{\mathbf{r}})_{W^{-\alpha,2}(\mathbb{R})} = 0.$$

PROOF. We first estimate the limsup in the statement for a fixed finite $A \subset \mathbb{N}$. As noted above, the random variables actually take values also in $L^1(\mathbb{R})$, and hence their Fourier-transform can be considered pointwise. Denote the complex exponential function $x \rightarrow e^{-2\pi i \xi x}$ by e_{ξ} and note that we have the estimate $\|e_{\xi}\|_V \leq c_0$ for all $\xi \in \mathbb{R}$. Hence we may apply Fubini’s theorem to compute

$$\mathbb{E} \|\mu_T^{\mathbf{r}} - \mu_{T,A}^{\mathbf{r}}\|_{W^{-\alpha,2}(\mathbb{R})}^2 = \int_{\mathbb{R}} (1 + \xi^2)^{-\alpha} \mathbb{E} |\mu_T(e_{\xi}^{\mathbf{r}}) - \mu_{T,A}(e_{\xi}^{\mathbf{r}})|^2 d\xi.$$

Proposition 2.12 verifies that the integrand has an integrable majorant of the form $C'(A)(1 + \xi^2)^{-\alpha}$ and that

$$\limsup_{T \rightarrow \infty} \mathbb{E} |\mu_T(e_{\xi}^{\mathbf{r}}) - \mu_{T,A}(e_{\xi}^{\mathbf{r}})|^2 \leq C'' (\|e_{\xi}^{\mathbf{r}}\|_Z^2 - \|e_{\xi}^{\mathbf{r}}\|_{Z,A}^2).$$

Clearly, the integrand is continuous jointly in T and ξ , and hence a simple application of the dominated convergence theorem shows that

$$\limsup_{T \rightarrow \infty} \mathbb{E} \|\mu_T^{\mathbf{r}} - \mu_{T,A}^{\mathbf{r}}\|_{W^{-\alpha,2}}^2 \leq C'' \int_{\mathbb{R}} (1 + \xi^2)^{-\alpha} (\|e_{\xi}^{\mathbf{r}}\|_Z^2 - \|e_{\xi}^{\mathbf{r}}\|_{Z,A}^2) d\xi.$$

The proposition now follows by applying (2.1). \square

We next record a simple fact whose validity is seen easily by approximating the full sum by a partial sum.

LEMMA 2.14. *Assume that $X_n, n \geq 1$, are random variables and assume that $Y_{T,n}$ are uniformly bounded random variables, $|Y_{T,n}| \leq 1$ for all $T \in [1, \infty) n \geq 1$. Assume also that for all $\ell \geq 1$,*

$$(Y_{T,1}, Y_{T,2}, \dots, Y_{T,\ell}) \xrightarrow{d} (X_1, X_2, \dots, X_{\ell}) \quad \text{as } T \rightarrow \infty.$$

Then, if (u_j) is a sequence of elements in a Banach space E such that $\sum_{n=1}^{\infty} \|u_n\|_E < \infty$, one has

$$\mathcal{W}_2 \left(\sum_{n=1}^{\infty} Y_{T,n} u_n, \sum_{n=1}^{\infty} X_n u_n \right)_E \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

For the rest of this section, we shall denote by $\theta_k, k \geq 1$, i.i.d. random variables, uniformly distributed on $[0, 1]$. Then the variables

$$e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, \dots$$

are i.i.d. and uniformly distributed on the unit circle. Recall from (1.2) the notation $\zeta_{N,\text{rand}}$ for the truncated randomized Euler product, and we will in fact often use the shorthand notation $\zeta_{N,\text{rand}}$ for $\zeta_{N,\text{rand}}(\frac{1}{2} + ix)$.

The next proposition constructs the final limiting element ζ_{rand} and shows that it is well approximated in distribution by suitably chosen $\mu_{T,A}$.

PROPOSITION 2.15.

(i) Assume that $\alpha > 1/2$. For each $N \geq 1$, let $A_N \subset \mathbb{N}$, be the finite set of positive integers of the form $p_1^{\alpha_1} \cdots p_N^{\alpha_N}$, where $p_1 \leq \dots \leq p_N$ are the first N primes and $0 \leq \alpha_i \leq N$ are nonnegative integers. Then

$$(2.30) \quad \lim_{N \rightarrow \infty} \limsup_{T \rightarrow \infty} \mathcal{W}_2(\mu_{T,A_N}^{\mathbf{r}}, \zeta_{N,\text{rand}}^{\mathbf{r}})_{W^{-\alpha,2}(\mathbb{R})} = 0.$$

(ii) Almost surely there exists a $W^{-\alpha,2}(\mathbb{R})$ -valued limiting random variable

$$(2.31) \quad \zeta_{\text{rand}}^{\mathbf{r}} := \lim_{N \rightarrow \infty} \zeta_{N,\text{rand}}^{\mathbf{r}}.$$

In addition,

$$\mathcal{W}_2(\zeta_{\text{rand}}^{\mathbf{r}}, \zeta_{N,\text{rand}}^{\mathbf{r}})_{W^{-\alpha,2}(\mathbb{R})} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

PROOF. We shall make use of the well-known fact that for any any $N \geq 1$ as $T \rightarrow \infty$ one has that

$$(p_1^{-i\omega T}, \dots, p_N^{-i\omega T}) \xrightarrow{d} (e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_N})$$

This follows simply by observing that if r_1, \dots, r_N are integers which are not all zero, then all the mixed moments

$$\mathbb{E} \prod_{k=1}^N p_k^{ir_k\omega T} = \int_0^1 e^{iuT \sum_{k=1}^N r_k \log p_k} du,$$

tend to zero as we let $T \rightarrow \infty$ since $\sum_{k=1}^N r_k \log p_k \neq 0$ by the uniqueness of the prime number decomposition of integers. As a consequence, for any $\ell \geq 1$,

$$(2.32) \quad (1^{-i\omega T}, \dots, \ell^{-i\omega T}) \xrightarrow{d} (e^{2\pi i\theta \cdot \alpha(1)}, \dots, e^{2\pi i\theta \cdot \alpha(\ell)}) \quad \text{as } T \rightarrow \infty,$$

where $\theta := (\theta_1, \theta_2, \dots)$ and the sequence $\alpha(\ell) = (\alpha_1(\ell), \alpha_2(\ell), \dots)$ is defined via $\ell = \prod_{k=1}^{\infty} p_k^{\alpha_k(\ell)}$.

To proceed, we introduce some further notation, let \mathbb{N}_N denote the set of positive integers whose largest prime factor is p_N . Moreover, slightly overloading our notation, let us write

$$\mu_{T,N}^{\mathbf{r}}(x) := \frac{1}{1+x^2} \prod_{k=1}^N (1 - p_k^{-1/2-ix-i\omega T})^{-1},$$

for the (weighted and randomly shifted) truncated Euler product of the zeta function. Noting that this product is bounded in T, x , one readily checks that we may write

$$\mu_{T,N}^{\mathbf{r}}(x) = \sum_{n \in \mathbb{N}_N} n^{-1/2-ix} \frac{n^{-i\omega T}}{1+x^2},$$

where $\sum_{n \in \mathbb{N}_N} n^{-1/2} < \infty$. Moreover, we see that

$$|\mu_{T,N}^{\mathbf{r}}(x) - \mu_{T,A_N}^{\mathbf{r}}(x)| \leq \frac{1}{1+x^2} \sum_{n \in \mathbb{N}_N \setminus A_N} n^{-1/2}$$

and $\sum_{n \in \mathbb{N}_N \setminus A_N} n^{-1/2} \rightarrow 0$ as $N \rightarrow \infty$. Hence Lemma 2.14 yields (2.30) in view of (2.32) even with respect to the 2-Wasserstein distance in L^2 instead of $W^{-\alpha,2}(\mathbb{R})$. This proves part (i).

In order to treat (ii), we observe that by the Gauss mean value theorem or expansion to a geometric series

$$\mathbb{E}\left(\frac{1}{1 - p_k^{-s} e^{2\pi i \theta_k}}\right) = 1$$

for any $s \in \{\sigma > 0\}$. Hence, by the very definition and independence, the sequence of random variables $(\zeta_{N,\text{rand}})_{N \geq 1}$ forms a $W^{-\alpha,2}(\mathbb{R})$ -valued martingale sequence, with respect to the natural filtration $\mathcal{F}_N := \sigma(\theta_1, \dots, \theta_N)$. Moreover, we claim that this martingale is bounded in L^2 :

$$\mathbb{E}(\|\zeta_{N,\text{rand}}\|_{W^{-\alpha,2}(\mathbb{R})}^2) \leq C < \infty,$$

where C does not depend on N . In order to verify the above uniform bound we compute as in the proof of Proposition 2.13: for any $T \geq 1$,

$$(2.33) \quad \mathbb{E}\|\mu_{T,A_N}^{\mathbf{r}}\|_{W^{-\alpha,2}(\mathbb{R})}^2 = \int_{\mathbb{R}} (1 + \xi^2)^{-\alpha} \mathbb{E}|\mu_{T,A_N}(e_{\xi}^{\mathbf{r}})|^2 d\xi.$$

Since obviously $\sup_{T \geq 1} \|\mu_{T,A_N}\|_{L^{\infty}(\mathbb{R})}$ is bounded in the probability space variable, also $\mathbb{E}|\mu_{T,A_N}(e_{\xi}^{\mathbf{r}})|^2$ is bounded in ξ, T . We deduce that the integrand in (2.33) has an integrable majorant. In view of (2.25), we see that

$$\lim_{T \rightarrow \infty} \mathbb{E}|\mu_{T,A_N}(e_{\xi}^{\mathbf{r}})|^2 = \|e_{\xi}^{\mathbf{r}}\|_{Z,A_N}^2 \leq c_0$$

and by recalling (2.30) and taking first the limit $T \rightarrow \infty$ and then $N \rightarrow \infty$ inside the integral (2.33), it finally follows that

$$\limsup_{N \rightarrow \infty} \mathbb{E}(\|\zeta_{N,\text{rand}}\|_{W^{-\alpha,2}(\mathbb{R})}^2) \leq c_0 \int_{\mathbb{R}} (1 + \xi^2)^{-\alpha} d\xi.$$

At this stage, both claims of part (ii) are a direct consequence of basic Banach space valued martingale theory (see, e.g., [46], Theorem 3.61, Theorem 1.95, or [30], Corollary V.2.4, Corollary III.2.13, the basic fact being that $W^{-\alpha,2}(\mathbb{R})$ is a separable Hilbert space, and thus possesses the Radon–Nikodým property). \square

We are then ready to prove the first part of our main result.

PROOF OF THEOREM 1.1(i). Assume that $\varepsilon > 0$ is given. We first apply Proposition 2.15(ii) to pick N_0 large enough that

$$\mathcal{W}_2(\zeta_{\text{rand}}^{\mathbf{r}}, \zeta_{N,\text{rand}}^{\mathbf{r}})_{W^{-\alpha,2}(\mathbb{R})} \leq \varepsilon \quad \text{for } N \geq N_0.$$

Next, according to Proposition 2.13 and Proposition 2.15(i) we may select $N_1 \geq N_0$ and $T_0 \geq 1$ so large that (in the notation of Proposition 2.15)

$$\mathcal{W}_2(\mu_T^{\mathbf{r}}, \mu_{T,A_{N_1}}^{\mathbf{r}})_{W^{-\alpha,2}(\mathbb{R})} \leq \varepsilon \quad \text{and} \quad \mathcal{W}_2(\zeta_{N_1,\text{rand}}^{\mathbf{r}}, \mu_{T,A_{N_1}}^{\mathbf{r}})_{W^{-\alpha,2}(\mathbb{R})} \leq \varepsilon$$

if $T \geq T_0$.

By setting $N = N_1$ in the first inequality, and combining all the three estimates it follows that

$$\mathcal{W}_2(\mu_T^{\mathbf{r}}, \zeta_{\text{rand}}^{\mathbf{r}})_{W^{-\alpha,2}(\mathbb{R})} \leq 3\varepsilon \quad \text{whenever } T \geq T_0,$$

and Theorem 1.1(i) follows. \square

REMARK 2.16. It is natural to ask whether the class of test functions for which this convergence holds can be enlarged—or if one can regularize our objects in some weaker way than by $f \mapsto f^{\mathbf{r}}$. We do not consider this question further here, but just point out that the answer to both of these questions is positive in view of Remark 2.4 and the the fact that we have not striven for optimality in this respect in the proof of Theorem 1.1(i).

REMARK 2.17. In the proof of Theorem 1.1(i), it is possible to bypass the approximation by finite partial sums $\mu_{T,A}^{\mathbf{r}}$ and instead consider directly the partial Euler products $\zeta_{N,\text{rand}}$ and their approximations $\mu_{T,N}$, defined in an obvious manner. In this approach, quantities like

$$\mathbb{E} \left[\prod_{k=1}^N (1 - p_k^{-\frac{1}{2} - i\omega T - ix})^{-1} \overline{\zeta(1/2 + i\omega T + iy)} \right],$$

have to be analyzed. In particular, one needs to prove an analogue of Lemma 2.9 for them. This can be done by following the lines of Ingham’s proof of the asymptotics for the second moment. Details can be found in a preprint version of this article [72], Appendix 2, which also contains an alternative approach to Proposition 2.8.

We next consider briefly ζ_{rand} as a random analytic function in the half-plane $\{\sigma > 1/2\}$. First of all, in our convergence statements above we may equally well switch from the factor $(1 + x^2)^{-1}$ to $(1/2 + ix)^{-2}$. Namely, if we write for $x \in \mathbb{R}$,

$$m(x) := (1/2 + ix)^{-2}(1 + x^2)$$

we clearly have that $\|(\frac{d}{dx})^k m\|_{L^\infty(\mathbb{R})} \leq C_k$ for all $k \geq 1$. Thus m is a multiplier in the Sobolev spaces $W^{u,2}(\mathbb{R})$, that is, $\|mf\|_{W^{u,2}(\mathbb{R})} \leq C\|f\|_{W^{u,2}(\mathbb{R})}$ for all $f \in W^{u,2}(\mathbb{R})$. This well-known fact is easily verified for $u \in \mathbb{N}$, it follows by interpolation for noninteger $u \geq 0$, and finally by duality for negative u . The convergence statement in (2.31) can thus be rewritten as

$$(2.34) \quad s^{-2}\zeta_{\text{rand}}(s) := \lim_{N \rightarrow \infty} s^{-2}\zeta_{N,\text{rand}}(s) \quad \text{on the line } \{\sigma = 1/2\},$$

with convergence in $W^{-\alpha,2}(\mathbb{R})$ for, say $\alpha \in (1/2, 1]$. It is classical⁵ that the Poisson extension of a (generalized) function $f \in W^{-\alpha,2}(\{\sigma = 1/2\})$ is well-defined and yields a harmonic function in $\{\sigma > 1/2\}$ and f is obtained as the distributional boundary limit of its harmonic extension. Moreover, the point evaluations in a compact subset $K \subset \{\sigma > 1/2\}$ are uniformly bounded with respect to the Sobolev-norm on the boundary. Hence, if a sequence of boundary functions (f_k) satisfies $\|f - f_k\|_{W^{-\alpha,2}(\mathbb{R})} \rightarrow 0$ as $k \rightarrow \infty$, then the Poisson extensions of f_k

⁵To sketch a short argument for these facts, let K be a compact set in the right half-plane $\{\sigma > 1/2\}$. One easily checks with a direct computation that the Poisson kernel $P(\cdot, \cdot)$ satisfies for any integer $k \geq 0$ $\sup_{z \in K} \int_{(\sigma=1/2)} (\sum_{0 \leq j \leq k} (D^j P(\cdot, z))^2) < \infty$. In other words, $\sup_{z \in K} \|P(\cdot, z)\|_{W^{\alpha,2}(\sigma=1/2)} < \infty$. for any α . This implies that the extension of elements in $W^{-\alpha,2}(\sigma = 1/2)$ are well-defined and uniformly bounded on K . By using the density of C_0^∞ in the Sobolev spaces, we deduce that the extension is also harmonic. Finally, the statement of the distributional boundary values follows by noting that for any $\lambda \in W^{-\alpha,2}(\sigma = 1/2)$ and $\phi \in C_0^\infty(\sigma = 1/2)$ we obtain $\langle P(\cdot, r) * \lambda, \phi \rangle = \langle \lambda, P(\cdot, r) * \phi \rangle \rightarrow \langle \lambda, \phi \rangle$ as $r \rightarrow 1/2^+$.

tend to that of f locally uniformly in $\{\sigma > 1/2\}$. In our situation, the Poisson extension of $s^{-2}\zeta_{N,\text{rand}}(s)$ obviously equals

$$s^{-2}\zeta_{N,\text{rand}}(s) := s^{-2} \prod_{k=1}^N \left(\frac{1}{1 - p_k^{-s} e^{2\pi i \theta_k}} \right), \quad \sigma > 1/2.$$

Moreover, as $N \rightarrow \infty$, the product in the right-hand side obviously converges locally uniformly in $\{\sigma > 1\}$ to

$$(2.35) \quad s^{-2}\zeta_{\text{rand}}(s) := s^{-2} \prod_{k=1}^{\infty} \left(\frac{1}{1 - p_k^{-s} e^{2\pi i \theta_k}} \right).$$

All these facts together with the convergence (2.34) enable us to deduce that, almost surely, the product in (2.35) actually converges locally uniformly in $\{\sigma > 1/2\}$ to a random analytic function, with distributional boundary limit $s^{-2}\zeta_{\text{rand}}$. Then of course, the same convergence statement holds for

$$\zeta_{\text{rand}}(s) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - p_k^{-s} e^{2\pi i \theta_k}} \right) \quad \text{in } \sigma > 1/2.$$

A fortiori, as a consequence of Theorem 1.1 we obtain the basic known results of the functional statistical convergence of the zeta function in the open strip $\{1/2 < \sigma < 1\}$ (or in the half-plane $\{\sigma > 1/2\}$).

One should note that by the local uniform convergence and Hurwitz’s theorem, the randomized zeta function $\zeta_{\text{rand}}(s)$ is nonzero for $\sigma > 1/2$. This has been noticed before; see, for example, [42], Theorem 4.6., where the different realizations of $\zeta_{\text{rand}}(s)$ were obtained as possible vertical limit functions of the Riemann zeta function.

We finish this section with the following.

PROOF OF THEOREM 1.6. Let us first establish the following auxiliary result.

LEMMA 2.18. *Almost surely one has for almost every $t \in \mathbb{R}$,*

$$\limsup_{\sigma \rightarrow 1/2^+} |\zeta_{\text{rand}}(\sigma + it)| = \infty.$$

PROOF. In order to prove the lemma, we cover \mathbb{R} by the intervals $[m, m + 1)$, and note that by translation invariance it is enough to prove that the blow-up takes place almost surely for almost every $t \in [0, 1)$. By joint measurability, translation invariance and Fubini’s theorem, it actually suffices to show that

$$\limsup_{\sigma \rightarrow 1/2^+} |\zeta_{\text{rand}}(\sigma)| = \infty.$$

For that end, we note that by the Euler product we may infer

$$\log |\zeta_{\text{rand}}(\sigma)| = \sum_{k=1}^{\infty} \frac{\cos(2\pi \theta_k)}{p_k^\sigma} + E(\sigma),$$

where for $\sigma > 1/2$

$$E(\sigma) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\cos(4\pi \theta_k)}{p_k^{2\sigma}} + \mathcal{O}(1),$$

and the $\mathcal{O}(1)$ -term is uniform in $\sigma \geq 1/2$. We now skip ahead slightly and make use of Lemma 3.1 to argue that since $\sum_{k=1}^{\infty} p_k^{-4\sigma} < \infty$, almost surely the whole error term $E(\sigma)$

is uniformly $\mathcal{O}(1)$. Here, one uses the fact that a standard Dirichlet series is bounded on the half-line $[\sigma_0, \infty)$ assuming that it converges at σ_0 .

It remains to check that almost surely

$$\limsup_{\sigma \rightarrow 1/2^+} \sum_{k=1}^{\infty} \frac{\cos(2\pi \theta_k)}{p_k^\sigma} =: \limsup_{\sigma \rightarrow 1/2^+} g(\sigma) = \infty.$$

Let us note that for any $\sigma \in (1/2, 1)$ we may simply compute

$$\begin{aligned} \mathbb{E}|g(\sigma)|^4 &= 3 \sum_{k \neq j}^{\infty} p_k^{-2\sigma} p_j^{-2\sigma} (\mathbb{E} \cos^2(2\pi \theta_1))^2 + \sum_{k=1}^{\infty} p_k^{-4\sigma} \mathbb{E} \cos^4(2\pi \theta_1) \\ &\leq 3(\mathbb{E}|g(\theta)|^2)^2, \end{aligned}$$

since $\mathbb{E} \cos^2(2\pi \theta_1) = 1/2$ and $\mathbb{E} \cos^4(2\pi \theta_1) = 3/8$. The Paley–Zygmund inequality now verifies that

$$\mathbb{P}(|g(\sigma)| \geq (\mathbb{E}|g(\sigma)|^2)^{1/2}) \geq c_0 > 0 \quad \text{for all } \sigma \in (1/2, 1).$$

Especially, as $\lim_{\sigma \rightarrow 1/2^+} (\mathbb{E}|g(\sigma)|^2)^{1/2} = \infty$ we gather that

$$\mathbb{P}\left(\limsup_{\sigma \rightarrow 1/2^+} |g(\sigma)| = \infty\right) \geq c_0,$$

and the claim follows by symmetry and Kolmogorov’s 0–1 law. \square

To return to the proof of the original statement, it is thus enough to show that the Poisson extension of an element $g \in W^{-\alpha, 2}(\mathbb{R})$ has radial boundary limits in a set of positive measure assuming that it coincides with a measure on an interval $I \subset \mathbb{R}$. However, we may then decompose g by a suitable cut-off function into the sum $u_1 + u_2$, where u_1 is the Poisson extension of a measure supported on I , and u_2 is a $W^{-\alpha, 2}(\mathbb{R})$ -function, supported on $\mathbb{R} \setminus I'$, where $I' \subset I$ is an open subinterval. Then u_2 has zero boundary values on I' and by basic harmonic analysis (see, e.g., [71], Theorem 11.24, for a variant on the unit circle), u_1 has finite radial limits at almost every point on I' .

Lemma 2.18 then implies that $\zeta_{\text{rand}}(1/2 + it)$ cannot be a measure on any open interval. \square

3. A Gaussian approximation for the field: Proof of Theorem 1.7. The goal of this section is to prove Theorem 1.7, namely that on any interval $[-A, A]$ we can indeed write $\log \zeta_{N, \text{rand}}(1/2 + ix) = \mathcal{G}_N(x) + \mathcal{E}_N(x)$, where \mathcal{G}_N converges to a (complex) log-correlated Gaussian field, and \mathcal{E}_N converges to a smooth function. Since this has no consequence for the proof, we shall simplify notation slightly and replace the interval $[-A, A]$ by $[0, 1]$.

We shall make repeated use of the following auxiliary technical result that can also be deduced from known estimates due to, for example, Kahane and Kwapien for random series in Banach spaces. However, for the reader’s convenience we provide a proof below.

LEMMA 3.1. For $f \in C^\ell[0, 1]$, write $\|f\|_{C^\ell[0, 1]} = \sum_{j=0}^\ell \|f^{(j)}\|_{L^\infty[0, 1]}$, and let us define $F_0 = 0$.

(i) For $n \geq 1$, consider the random functions F_n on the interval $[0, 1]$ defined by the series

$$F_n(x) := \sum_{k=1}^n A_k f_k(x).$$

Here, the A_k 's are i.i.d., centered and possibly complex valued random variables that are either standard normal variables or they are symmetric and uniformly bounded: $|A_1| \leq C < \infty$ almost surely. The functions f_k are assumed to be continuously differentiable on $[0, 1]$ with

$$(3.1) \quad \sum_{k=1}^{\infty} \|f_k\|_{C^1[0,1]}^2 < \infty.$$

Then almost surely the series

$$F(x) := \sum_{k=1}^{\infty} A_k f_k(x),$$

converges uniformly on $[0, 1]$ to a continuous limit function F . Moreover, one has

$$(3.2) \quad \mathbb{E} \exp(\lambda \|F\|_{L^\infty[0,1]}) < \infty \quad \text{for all } \lambda > 0 \quad \text{and}$$

$$(3.3) \quad \mathbb{E} \exp\left(\lambda \sup_{0 \leq n' < n} \|F_n - F_{n'}\|_{L^\infty[0,1]}\right) < \infty \quad \text{for all } \lambda > 0.$$

(ii) Assume, in addition, that the functions f_k are smooth and that

$$\sum_{k=1}^{\infty} \|f_k\|_{C^\ell[0,1]}^2 < \infty \quad \text{for all } \ell \geq 0.$$

Then almost surely $F \in C^\infty[0, 1]$ and for all $\ell \geq 1$,

$$\mathbb{E} \exp(\lambda \|F\|_{C^\ell[0,1]}) < \infty \quad \text{for all } \lambda > 0.$$

Moreover, for every index $\ell \geq 0$ it holds that

$$\mathbb{E} \exp\left(\lambda \sup_{0 \leq n' < n} \|F_n - F_{n'}\|_{C^\ell[0,1]}\right) < \infty \quad \text{for all } \lambda > 0.$$

PROOF. (i) We may assume that the A_k 's and f_k 's are real-valued, since the general case is then obtained by considering the four series obtained by multiplying the real or imaginary part of A_k 's with the real or imaginary parts of f_k 's (note that for (3.2) and (3.3) one applies Hölder's inequality). We recall the standard Sobolev embedding

$$(3.4) \quad \|f\|_{C[0,1]} \leq 2 \|f\|_{W^{1,2}(0,1)} := 2 \left(\int_0^1 (|f(x)|^2 + |f'(x)|^2) dx \right)^{1/2}.$$

Since the variables A_k are independent and centered, we find for $M < N$,

$$\mathbb{E} \|F_N - F_M\|_{W^{1,2}(0,1)}^2 \leq 2 \sum_{k=M+1}^N [\|f_k\|_{L^\infty[0,1]}^2 + \|f'_k\|_{L^\infty[0,1]}^2].$$

Then Lévy's inequality (see [51], Lemma 2.3.1), applied here to our $W^{1,2}(0, 1)$ -valued symmetric random variables) yields that

$$(3.5) \quad \mathbb{E} \sup_{M \leq r \leq N} \|F_r - F_M\|_{W^{1,2}(0,1)}^2 \leq 4 \sum_{k=M+1}^N [\|f_k\|_{L^\infty[0,1]}^2 + \|f'_k\|_{L^\infty[0,1]}^2].$$

By our assumption (3.1), we may pick a subsequence (N_j) so that

$$4 \sum_{k=N_j+1}^{N_{j+1}} [\|f_k\|_{L^\infty[0,1]}^2 + \|f'_k\|_{L^\infty[0,1]}^2] < j^{-6},$$

for $j \geq 1$. Borel–Cantelli along with (3.5) then ensures that there exists a random threshold j_0 , which is almost surely finite, such that for $j \geq j_0$,

$$\sup_{N_j \leq u \leq N_{j+1}} \|F_u - F_{N_{j+1}}\|_{W^{1,2}(0,1)} \leq j^{-2}$$

for $j \geq j_0$. Thus summing over j implies that the subsequence F_{N_j} almost surely converges absolutely in the space $W^{1,2}(0, 1)$, and by (3.4) also in $C([0, 1], \mathbb{C})$. Also, at the same time the above inequality implies that the full sequence F_N converges uniformly to a random continuous function.

It remains to check the claim about the exponential moments. For that purpose, an application of Azuma’s inequality and (3.1) implies the existence of a finite number C such that for small enough $a > 0$,

$$(3.6) \quad \mathbb{E}e^{a|F(x)|^2} \leq C \quad \text{and} \quad \mathbb{E}e^{a|F'(x)|^2} \leq C \quad \text{for all } x \in [0, 1].$$

Making use of Jensen’s inequality (applied to integration with respect to x) and the observation that $t \mapsto e^{at}$ is convex on $[0, \infty)$, we obtain by (3.6) and Fubini’s theorem for small enough $a > 0$,

$$\mathbb{E}e^{a\|F\|_{W^{1,2}(0,1)}^2} = \mathbb{E}e^{a \int_0^1 (|F(x)|^2 + |F'(x)|^2) dx} \leq \mathbb{E} \int_0^1 e^{a(|F(x)|^2 + |F'(x)|^2)} dx < \infty.$$

Lévy’s inequality then strengthens this for small enough $a > 0$ to

$$\mathbb{E}e^{a \sup_{N \geq 1} \|F_N\|_{L^\infty[0,1]}^2} < \infty.$$

The obtained estimates clearly imply (3.2) and (3.3).

The case where the variables A_k are i.i.d. standard normal random variables can be proven with exactly the same proof or more directly by considering F as a $W^{1,2}(0, 1)$ -valued random variable that is well-defined by the assumption (3.1). Then (3.3) is obtained directly from Fernique’s theorem [51], Theorem 12.7.2.

(ii) The statement is a direct consequence of part (i) applied successively to the derivative series defining $F^{(\ell)}(x)$, $\ell \geq 0$. \square

We will now start the proper consideration of the truncated randomised Euler products by expanding

$$\log \zeta_{N,\text{rand}}(1/2 + ix) = \sum_{j=1}^{\infty} \sum_{k=1}^N \frac{1}{j} e^{2\pi i j \theta_k} p_k^{-j(\frac{1}{2} + ix)}$$

and first verifying that the sum over the $j \geq 2$ terms yields a negligible contribution in the sense that it is almost surely smooth over $x \in [0, 1]$.

LEMMA 3.2. *Let*

$$L_N(x) = \sum_{j=2}^{\infty} \sum_{k=1}^N \frac{1}{j} e^{2\pi i j \theta_k} p_k^{-j(\frac{1}{2} + ix)}.$$

Almost surely, as $N \rightarrow \infty$, L_N converges uniformly on $[0, 1]$ to a random C^∞ -function L . The derivatives $L_N^{(\ell)}$ also converge uniformly, and for any $\ell \geq 0$,

$$\mathbb{E}e^{\lambda \sup_{N \geq 1} \|L_N(x)\|_{C^\ell[0,1]}} < \infty \quad \text{for all } \lambda \in \mathbb{R}.$$

PROOF. Let us separate L_N into the $j = 2$ part and the $j \geq 3$ part. As the series

$$\sum_{k=1}^{\infty} \sum_{j=3}^{\infty} \frac{1}{j} p_k^{-j/2} (\log p_k)^\ell$$

converges for any $\ell \geq 0$, we see that the series

$$\sum_{k=1}^{\infty} \sum_{j=3}^{\infty} \frac{1}{j} e^{2\pi i j \theta_k} p_k^{-\frac{j}{2} - i j x}$$

together with the arbitrarily many times differentiated series converges absolutely and uniformly to a (deterministically) bounded quantity. Thus this part of the sum certainly satisfies the statements of the lemma.

In turn, the ℓ th derivative of the k th term of the $j = 2$ sum has the deterministic upper bound $p_k^{-1} (\log p_k)^\ell$ and the claims for the $j = 2$ sum follow immediately from Lemma 3.1(ii) together with the fact $\sum_{k=1}^{\infty} p_k^{-2} (\log p_k)^{2\ell} < \infty$. \square

To study the main term, that is, the part of the sum with $j = 1$, we will split the field into a sum over blocks, where within the blocks, the quantities $\log p$ are roughly constant, and perform a Gaussian approximation on each block separately. To make this formal, let $(r_k)_{k=1}^{\infty}$ be a sequence of strictly increasing positive integers with $r_1 = 1$ and then $\{p_{r_k}, \dots, p_{r_{k+1}-1}\}$ will be the set of primes appearing in the block we have mentioned. We will make some preliminary requirements on the blocks. We assume that $1 < r_{k+1} - r_k \rightarrow \infty$ as $k \rightarrow \infty$, but on the other hand $r_{k+1}/r_k \rightarrow 1$ as $k \rightarrow \infty$. Also it is useful to assume that $p_{r_{m+1}-1}/p_{r_m} \leq 2$, for all $m \geq 1$.

We then define the blocks of the field as well as a “freezing approximation,” where we approximate p_k^{-ix} by $p_{r_m}^{-ix}$ within a block.

DEFINITION 3.3. For $(r_m)_{m=1}^{\infty}$ as above, define for $x \in [0, 1]$ and $m \geq 1$:

$$Y_m(x) = \sum_{k=r_m}^{r_{m+1}-1} \frac{1}{\sqrt{p_k}} e^{2\pi i \theta_k} p_k^{-ix}.$$

Consider also the approximation to this where the x -dependent terms within each block are “frozen”:

$$\begin{aligned} \tilde{Y}_m(x) &= p_{r_m}^{-ix} \sum_{k=r_m}^{r_{m+1}-1} \frac{1}{\sqrt{p_k}} e^{2\pi i \theta_k} \\ (3.7) \qquad &=: p_{r_m}^{-ix} (C_m + i S_m), \end{aligned}$$

where C_m is the real part of the sum, and S_m the imaginary part.

The required Gaussian approximation uses the following fairly standard result. We state it in a slightly more general form than we actually need here, since we expect it might be of use in further study of certain more general non-Gaussian chaos models. Some initial steps in this direction are contained in [49].

PROPOSITION 3.4.

(i) Assume that $d \geq 2$ and $H_j = (H_j^{(1)}, \dots, H_j^{(d)})$, $j \in \{1, \dots, n\}$ are independent and symmetric \mathbb{R}^d -valued random variables with

$$b_0^{-1} \leq c_j := d^{-1} \text{Tr}(\text{Cov}(H_j)) \leq b_0$$

for all $j \in \{1, \dots, n\}$, where $b_0 > 0$. Assume also that the following uniform exponential bound holds for some $b_1, b_2 > 0$:

$$(3.8) \quad \mathbb{E} \exp(b_1 |H_k|) \leq b_2 \quad \text{for all } k = 1, \dots, n.$$

Then there is a d -dimensional Gaussian random variable U with

$$\text{Cov}(U) = \left(\sum_{j=1}^n c_j \right)^{-1} \left(\sum_{j=1}^n \text{Cov}(H_j) \right), \quad \text{Tr}(\text{Cov}(U)) = d,$$

and such that the difference

$$V := U - \left(\sum_{j=1}^n c_j \right)^{-1/2} \left(\sum_{j=1}^n H_j \right)$$

satisfies

$$(3.9) \quad \mathbb{E}|V| \leq a_1 n^{-\beta}.$$

Above $\beta = \beta(d) > 0$ depends only on the dimension and a_1 on d, b_0, b_1, b_2 . Moreover, U can be chosen to be measurable with respect to $\sigma(G, H_1, \dots, H_n)$, where G is a d -dimensional standard Gaussian independent of the H_j 's. In addition, there is the exponential estimate

$$(3.10) \quad \mathbb{E} \exp(\lambda|V|) \leq 1 + a_2 e^{a_3 \lambda^2} n^{-a_4} \quad \text{for } 0 \leq \lambda \leq a_5 n^{1/2},$$

where the constants $a_2, a_3, a_4, a_5 > 0$ depend only on b_0, b_1, b_2 and the dimension d .

In the case where the variables H_k are uniformly bounded, say $|H_k| \leq b_3$ for all k , then (3.10) holds true for all $\lambda > 0$, where now the constants a_2, a_3, a_4 may also depend on b_3 , and there are constants $a_6, a_7, a_8 > 0$ that depend only on b_0, b_1, b_2, b_3, d so that

$$\mathbb{E} \exp(a_6 |V|^2) \leq 1 + a_7 n^{-a_8},$$

(ii) If we assume that $\text{Cov}(H_j) = c_j d^{-1} I$, where I the the $d \times d$ identity matrix, and the dimension $d \geq 1$ is arbitrary, then the conclusion (3.9) can be strengthened to

$$(3.11) \quad \mathbb{E}|V| \leq a_1 \log(n)^{d+1} n^{-1/2}.$$

We will postpone the proof of this result to Appendix A since its ingredients are basically well known; however, the above formulation is tailored for our purposes.

Our aim is to apply Proposition 3.4 to approximate (C_m, S_m) from (3.7) by a \mathbb{R}^2 -valued Gaussian random variable. To do this, we need to scale things a bit differently. Define the following sequence of \mathbb{R}^2 -valued random variables (so in the setting of Proposition 3.4, $d = 2$): for $j = j = 1, \dots, r_{m+1} - r_m$, let

$$H_{j,m} = \left(\frac{\sqrt{p_{r_{m+1}-1}}}{\sqrt{p_{r_m-1+j}}} \cos(2\pi \theta_{p_{r_m-1+j}}), \frac{\sqrt{p_{r_{m+1}-1}}}{\sqrt{p_{r_m-1+j}}} \sin(2\pi \theta_{p_{r_m-1+j}}) \right).$$

We then have

$$|H_{j,m}|^2 \leq \frac{p_{r_{m+1}-1}}{p_{r_m}} \leq 2$$

and

$$\text{Cov}(H_{j,m}) = \frac{1}{2} \frac{p_{r_{m+1}-1}}{p_{r_m-1+j}} I = c_{j,m} I,$$

where $1/2 \leq c_{j,m} \leq 1$. In this notation, we have (recalling (3.7))

$$(C_m, S_m) = \frac{1}{\sqrt{p_{r_{m+1}-1}}} \sum_{j=1}^{r_{m+1}-r_m} H_{j,m} = \frac{b_m}{\sqrt{\sum_{j=1}^{r_{m+1}-r_m} c_{j,m}}} \sum_{j=1}^{r_{m+1}-r_m} H_{j,m},$$

where

$$b_m = \sqrt{\frac{1}{2} \sum_{j=1}^{r_{m+1}-r_m} \frac{1}{p_{r_m-1+j}}}.$$

Proposition 3.4(ii) thus yields a sequence of independent standard two-dimensional normal variables $(V_m^{(1)}, V_m^{(2)})$, $m = 1, 2, \dots$, so that the distance between (C_m, S_m) and $b_m(V_m^{(1)}, V_m^{(2)})$ is controlled as in the statement of Proposition 3.4(ii).

We may assume that our probability space is large enough for us to write for each $m \geq 1$ and $i \in \{1, 2\}$,

$$\sqrt{\frac{1}{2} \sum_{j=1}^{r_{m+1}-r_m} \frac{1}{p_{r_m-1+j}}} V_m^{(i)} = \sum_{j=r_m}^{r_{m+1}-1} \frac{1}{\sqrt{2p_j}} W_j^{(i)},$$

where the $W_j^{(i)}$'s are independent standard normal random variables for all $j \geq 1$ and $i \in \{1, 2\}$. Finally, we can write down our Gaussian approximation to the field, its blocks and frozen versions of the blocks.

DEFINITION 3.5. Let $(W_k^{(j)})_{k \geq 1, j \in \{1,2\}}$ be the i.i.d. standard Gaussians constructed above. For any $N \geq 1$ and $x \in [0, 1]$, the Gaussian approximation of the “ $j = 1$ part” of $\log \zeta_{N,\text{rand}}$ is given by the Gaussian field

$$\mathcal{G}_N(x) := \sum_{k=1}^N \frac{1}{\sqrt{2p_k}} (W_k^{(1)} + i W_k^{(2)}) p_k^{-ix}.$$

Moreover, we define the blocks of \mathcal{G}_N as

$$Z_m(x) = \sum_{k=r_m}^{r_{m+1}-1} \frac{1}{\sqrt{2p_k}} (W_k^{(1)} + i W_k^{(2)}) p_k^{-ix}$$

and a “frozen” version of the block as

$$\tilde{Z}_m(x) = b_m p_{r_m}^{-ix} (V_m^{(1)} + i V_m^{(2)}).$$

We then start the analysis of the error produced by our Gaussian approximation. This is first performed only for sums over full blocks. We introduce some notation for the errors. Let us call the error we make by approximating our frozen field by the frozen Gaussian one by

$$\tilde{E}_{1,n}(x) := \sum_{m=1}^n (\tilde{Y}_m(x) - \tilde{Z}_m(x)), \quad x \in [0, 1].$$

In a similar vein, the error obtained from the freezing procedure is denoted by

$$\tilde{E}_{2,n}(x) := \sum_{m=1}^n (Y_m(x) - \tilde{Y}_m(x) + \tilde{Z}_m(x) - Z_m(x)), \quad x \in [0, 1]$$

whence the total error can be written as

$$\tilde{E}_n(x) := \tilde{E}_{1,n}(x) + \tilde{E}_{2,n}(x).$$

We study first the size of the error $\tilde{E}_{1,n}$.

LEMMA 3.6. *Assume (in addition to our previous constraints on (r_m)) that*

$$(3.12) \quad \sum_{m=1}^{\infty} (r_{m+1} - r_m)^{-a_4} < \infty \quad \text{and} \quad \sup_{m \geq 1} \frac{r_{m+1} - r_m}{r_m} (\log r_m)^\ell < \infty$$

for all $\ell \geq 1$, where a_4 is the constant from Proposition 3.4. Then, almost surely there exists a C^∞ -smooth limit function

$$\tilde{E}_1(x) := \lim_{n \rightarrow \infty} \tilde{E}_{1,n}(x),$$

where the convergence is in the norm $\|\cdot\|_{C^\ell[0,1]}$ for any $\ell \geq 0$. Moreover, one has

$$(3.13) \quad \mathbb{E} \exp\left(\lambda \sup_{0 \leq n' < n} \|\tilde{E}_{1,n} - \tilde{E}_{1,n'}\|_{C^\ell[0,1]}\right) < \infty \quad \text{for all } \lambda > 0,$$

where one applies the convention $\tilde{E}_{1,0}(x) \equiv 0$, and, in particular,

$$(3.14) \quad \mathbb{E} \exp(\lambda \|\tilde{E}_1\|_{C^\ell[0,1]}) < \infty \quad \text{for all } \lambda > 0$$

for any $\ell \geq 0$.

PROOF. To prove convergence of $\tilde{E}_{1,m}$, we fix an integer $\ell \geq 0$ and observe that

$$\begin{aligned} \|\tilde{E}_{1,m-1} - \tilde{E}_{1,m}\|_{C^\ell[0,1]} &\leq |(C_m, S_m) - b_m(V_m^{(1)}, V_m^{(2)})| (1 + \log r_m)^\ell \\ &= b_m \left| \frac{1}{\sqrt{\sum_{j=1}^{r_{m+1}-r_m} c_{j,m}}} \sum_{j=1}^{r_{m+1}-r_m} H_{j,m} - (V_m^{(1)}, V_m^{(2)}) \right| \\ &\quad \times (1 + \log r_m)^\ell. \end{aligned}$$

We then recall that we assumed that $r_{m+1}/r_m \rightarrow 1$ so we see from the prime number theorem (and a crude estimate on the sum) that

$$b_m^2 \lesssim \frac{r_{m+1} - r_m}{r_m \max(\log r_m, 1)}.$$

Thus by the second condition in the assumption (3.12)

$$b_m^2 (1 + \log r_m)^{2\ell} \lesssim 1.$$

Proposition 3.4 (more precisely (3.10)) then implies that we have

$$\begin{aligned} \mathbb{E} \|\tilde{E}_{1,m-1} - \tilde{E}_{1,m}\|_{C^\ell[0,1]} &\lesssim \mathbb{E} (e^{|\sum_{j=1}^{r_{m+1}-r_m} c_{j,m}|^{-1/2} \sum_{j=1}^{r_{m+1}-r_m} H_{j,m} - (V_m^{(1)}, V_m^{(2)})|} - 1) \\ &\lesssim (r_{m+1} - r_m)^{-a_4}. \end{aligned}$$

Thus by our assumption on (r_m) , the series $\tilde{E}_1 = \sum_{m=1}^{\infty} (\tilde{E}_{1,m} - \tilde{E}_{1,m-1})$ almost surely converges absolutely in $C^\ell[0, 1]$.

We next use the crude estimate

$$\sup_{0 \leq n' < n} \|\tilde{E}_{1,n} - \tilde{E}_{1,n'}\|_{C^\ell[0,1]} \leq \sum_{m=1}^\infty |(C_m, S_m) - b_m(V_m^{(1)}, V_m^{(2)})|(1 + \log r_m)^\ell,$$

so that by independence and (3.10)

$$\begin{aligned} \mathbb{E} \exp\left(\lambda \sup_{0 \leq n' < n} \|\tilde{E}_{1,n} - \tilde{E}_{1,n'}\|_{C^\ell[0,1]}\right) &\leq \prod_{m=1}^\infty \mathbb{E} e^{\lambda |(C_m, S_m) - b_m(V_m^{(1)}, V_m^{(2)})|(1 + \log r_m)^\ell} \\ &\leq \prod_{m=1}^\infty (1 + a_2 e^{a_3 \lambda^2 b_m^2 (1 + \log r_m)^{2\ell}} (r_{m+1} - r_m)^{-a_4}). \end{aligned}$$

As we saw that $b_m(1 + \log r_m)^{2\ell}$ is bounded, we find for some constant C (depending on λ) that

$$\begin{aligned} \mathbb{E} \exp\left(\lambda \sup_{0 \leq n' < n} \|\tilde{E}_{1,n} - \tilde{E}_{1,n'}\|_{C^\ell[0,1]}\right) &\leq \prod_{m=1}^\infty (1 + C(r_{m+1} - r_m)^{-a_4}) \\ &\leq e^{C \sum_{m=1}^\infty (r_{m+1} - r_m)^{-a_4}}, \end{aligned}$$

and (3.13) follows. Finally, (3.14) is an obvious consequence of (3.13). \square

Let us then estimate the error due to the freezing procedure.

LEMMA 3.7. *Assume that the sequence (r_m) is chosen so that*

$$(3.15) \quad \sum_{m=1}^\infty \frac{(r_{m+1} - r_m)(p_{r_{m+1}} - p_{r_m})^2 \log^\ell r_m}{r_m^3} < \infty$$

for any $\ell \geq 1$. Then almost surely there exists a C^∞ -smooth limit function

$$\tilde{E}_2(x) := \lim_{n \rightarrow \infty} \tilde{E}_{2,n}(x),$$

where the convergence is in the sup-norm over $[0, 1]$. Moreover, for any $\ell \geq 1$ we have both

$$\begin{aligned} \mathbb{E} \exp(\lambda \|\tilde{E}_2\|_{C^\ell[0,1]}) &< \infty \quad \text{for all } \lambda > 0 \quad \text{and} \\ \mathbb{E} \exp\left(\lambda \sup_{0 \leq n' < n} \|\tilde{E}_{2,n} - \tilde{E}_{2,n'}\|_{C^\ell[0,1]}\right) &< \infty \quad \text{for all } \lambda > 0. \end{aligned}$$

PROOF. The proof is again based on Lemma 3.1. It follows immediately from the definitions that

$$\begin{aligned} Y_m(x) - \tilde{Y}_m(x) + \tilde{Z}_m(x) - Z_m(x) &= \sum_{k=r_m}^{r_{m+1}-1} \frac{1}{\sqrt{p_k}} (p_k^{-ix} - p_{r_m}^{-ix}) \left(e^{2\pi i \theta_k} - \frac{1}{\sqrt{2}} [W_k^{(1)} + i W_k^{(2)}] \right) \\ &:= \sum_{k=r_m}^{r_{m+1}-1} f_k(x) \left(e^{2\pi i \theta_k} - \frac{1}{\sqrt{2}} [W_k^{(1)} + i W_k^{(2)}] \right), \end{aligned}$$

where $f_k(x) := \frac{1}{\sqrt{p_k}} (p_k^{-ix} - p_{r_m}^{-ix})$. Given any integer $\ell \geq 0$, we may use the properties of the sequence (r_m) and the 1-Lipschitz property of $u \rightarrow e^{iu}$ to estimate for any $x \in [0, 1]$ and

$$r_m \leq k \leq r_{m+1} - 1,$$

$$\begin{aligned} p_k^{1/2} |f_k^{(\ell)}(x)| &= |(-i)^\ell (p_k^{-ix} \log^\ell p_k - p_{r_m}^{-ix} \log^\ell p_{r_m})| \\ &\leq |p_k^{-ix} - p_{r_m}^{-ix}| \log^\ell p_k + |\log^\ell p_k - \log^\ell p_{r_m}| \lesssim \log^\ell(p_{r_m})(\log p_k - \log p_{r_m}) \\ &\lesssim \log^\ell(p_{r_m}) \frac{p_k - p_{r_m}}{p_{r_m}}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{k=1}^\infty \|f_k^{(\ell)}\|_{L^\infty[0,1]}^2 &\lesssim \sum_{m=1}^\infty \frac{r_{m+1} - r_m}{p_{r_m}} \left(\frac{p_{r_{m+1}} - p_{r_m}}{p_{r_m}}\right)^2 \log^{2\ell}(p_{r_m}) \\ &\lesssim \sum_{m=1}^\infty (r_{m+1} - r_m) \frac{(p_{r_{m+1}} - p_{r_m})^2}{p_{r_m}^3} \log^{2\ell}(p_{r_m}) < \infty \end{aligned}$$

by our assumption. The claim now follows by a two-fold application of Lemma 3.1(ii). \square

We next combine the error estimates proven so far and make the final choice for the subsequence (r_m) . For that purpose, we need the following well-known lemma, whose proof we include for the reader’s convenience.

LEMMA 3.8. *Let $\text{Li}(x) = \int_2^x \frac{dt}{\log t}$ denote the logarithmic integral function and Li^{-1} its inverse. For large enough n , it holds that*

$$-ne^{-\sqrt{\log n}} \lesssim p_n - \text{Li}^{-1}(n) \lesssim ne^{-\sqrt{\log n}}.$$

PROOF. We note first that the inverse Li^{-1} is convex since Li itself is concave. Furthermore, we have $(\text{Li}^{-1})'(x) = \log(\text{Li}^{-1}(x)) \leq \log(2x \log(x)) \leq 2 \log(x)$ for large enough x . Hence, as a suitable quantitative version of the prime number theorem verifies that for any $c \geq 1$ there is the error estimate $|\pi(x) - \text{Li}(x)| = \mathcal{O}(x \exp(-c\sqrt{\log x}))$, we have $n = \pi(p_n) \leq \text{Li}(p_n) + ne^{-2\sqrt{\log n}}$. In particular, for large enough n ,

$$\begin{aligned} p_n &\geq \text{Li}^{-1}(n - ne^{-2\sqrt{\log n}}) \geq \text{Li}^{-1}(n) - ne^{-2\sqrt{\log n}} (\text{Li}^{-1})'(n) \\ &\geq \text{Li}^{-1}(n) - ne^{-\sqrt{\log n}}. \end{aligned}$$

The proof of the other direction is analogous. \square

PROPOSITION 3.9. *Choose (for the rest of the paper) $r_m = \lfloor \exp(3 \log^2 m) \rfloor$. Then the combined error $\tilde{E}_n(x) = \tilde{E}_{n,1}(x) + \tilde{E}_{n,2}(x)$ a.s. converges for any $\ell \geq 0$ in $C^\ell[0, 1]$ to a C^∞ -smooth limit*

$$E(x) := \lim_{n \rightarrow \infty} (\tilde{E}_{n,1}(x) + \tilde{E}_{n,2}(x)).$$

Moreover, for all $\lambda > 0$ and $\ell \geq 0$,

$$(3.16) \quad \mathbb{E} \exp(\lambda \|E\|_{C^\ell[0,1]}) < \infty$$

and

$$\mathbb{E} \exp\left(\lambda \sup_{0 \leq n' < n} \|\tilde{E}_n - \tilde{E}_{n'}\|_{C^\ell[0,1]}\right) < \infty.$$

PROOF. We first recall the condition of Lemma 3.6, namely that the first error term converges as soon as

$$(3.17) \quad \sum_{m=1}^{\infty} (r_{m+1} - r_m)^{-a_4} < \infty \quad \text{and} \quad \sup_{m \geq 1} \frac{r_{m+1} - r_m}{r_m} (\log r_m)^\ell < \infty$$

for any $\ell \geq 0$. Lemma 3.8 yields for our sequences that $p_{r_{m+1}} - p_{r_m} \lesssim (r_{m+1} - r_m) \log r_m + r_m e^{-\sqrt{\log r_m}}$. By plugging this into condition (3.15), we see that a sufficient condition to apply Lemma 3.7 in order to control the second error term is given by the pair of conditions

$$(3.18) \quad \sum_{m=1}^{\infty} \left(\frac{r_{m+1} - r_m}{r_m} \right)^3 \log^L(r_m) < \infty \quad \text{and} \quad \sum_{m=1}^{\infty} e^{-2\sqrt{\log r_m}} \log^L(r_m) < \infty$$

for all $L \geq 1$. Finally, it remains to observe that the choice $r_m = \lfloor \exp(3 \log^2 m) \rfloor$ satisfies both (3.17) and (3.18), and satisfies the initial properties postulated for (r_m) after the proof of Lemma 3.2. \square

To complete the approximation procedure, we finally verify that the fields \mathcal{G}_N are good approximations also for indices N inside the intervals $r_m \leq N < r_{m+1}$.

PROPOSITION 3.10. *Denote the total error of the Gaussian approximation by setting*

$$E_N(x) := \sum_{k=1}^N \frac{1}{\sqrt{p_k}} p_k^{-ix} e^{2\pi i \theta_k} - \mathcal{G}_N(x) \quad \text{for } N \geq 1 \text{ and } x \in [0, 1].$$

Then for each $\ell \geq 0$, almost surely, $E_N(x)$ converges in $C^\ell[0, 1]$ to a C^∞ -smooth limit function

$$E(x) := \lim_{N \rightarrow \infty} E_N(x),$$

where the obtained limit is of course the same as in Proposition 3.9. Moreover, for all $\lambda > 0$ and any $\ell \geq 0$

$$\mathbb{E} \exp(\lambda \|E\|_{C^\ell[0,1]}) < \infty$$

and

$$(3.19) \quad \mathbb{E} \exp\left(\lambda \sup_{N \geq 1} \|E_N\|_{C^\ell[0,1]}\right) < \infty.$$

PROOF. After Proposition 3.9, it is enough to show that any given partial sum of the original series is in fact well approximated by the sum of the blocks below it, and that a similar statement holds also true for the Gaussian approximation series. Let us fix $m \geq 1$ and recall our notation

$$Y_m(x) = \sum_{k=r_m}^{r_{m+1}-1} \frac{1}{\sqrt{p_k}} p_k^{-ix} e^{2\pi i \theta_k} =: \sum_{k=r_m}^{r_{m+1}-1} A_k(x),$$

which is just the partial sum of our original field corresponding to the m th block. Observing first that

$$\begin{aligned} \sum_{k=r_m}^{r_{m+1}-1} \frac{\log^{2\ell} p_k}{p_k} &\lesssim \sum_{k=\lfloor e^{\log^2 m} \rfloor}^{\lfloor e^{\log^2(m+1)} \rfloor} \frac{\log^{2\ell} k}{k \log k} \lesssim \log m^{4\ell-2} (\log^2(m+1) - \log^2 m) \\ &\lesssim \log m^{4\ell-1} m^{-1} \\ &\lesssim m^{-1/2}, \end{aligned}$$

Azuma’s inequality yields⁶

$$\mathbb{P}(|Y_m^{(\ell)}(x)| \geq \lambda) \lesssim \exp\left(-c\lambda^2 \left(\sum_{k=r_m}^{r_{m+1}-1} \frac{\log^{2\ell} p_k}{p_k}\right)^{-1}\right) \lesssim \exp(-c'\lambda^2 m^{1/2}).$$

In particular, we obtain that for some constants c, c' that work for all $x \in [0, 1]$,

$$\mathbb{E} \exp(cm^{1/2} |Y_m^{(\ell)}(x)|^2) \leq c'.$$

This holds true for all $\ell \geq 0$. As at the end of the proof of Lemma 3.1, we deduce that $\mathbb{E} \exp(cm^{1/2} \|Y_m(x)\|_{C^\ell[0,1]}^2) \leq c$, and again Lévy’s inequality enables us to gather that

$$\mathbb{P}\left(\max_{r_m \leq u \leq r_{m+1}-1} \left\| \sum_{k=r_m}^u A_k \right\|_{C^\ell[0,1]} > \lambda\right) \lesssim \exp(-cm^{1/2}\lambda^2).$$

Summing over m yields for $\lambda \geq 1$,

$$(3.20) \quad \mathbb{P}\left(\sup_{m \geq 1} \max_{r_m \leq u \leq r_{m+1}-1} \left\| \sum_{k=r_m}^u A_k \right\|_{C^\ell[0,1]} > \lambda\right) \lesssim \sum_{m=1}^\infty \exp(-cm^{1/2}\lambda^2) \lesssim \exp(-c\lambda^2).$$

Exactly the same proof where Azuma is replaced by elementary estimates for Gaussian variables yields the corresponding estimate for our Gaussian approximation fields. An easy Borel–Cantelli argument that uses estimates like (3.20) in combination with Proposition 3.9 then shows the existence of the uniform limit $E(x) = \lim_{N \rightarrow \infty} E_N(x)$. Finally, combining (3.20) with (3.16) yields (3.19). Together with our previous considerations, this concludes the proof of the proposition. \square

Finally, putting things together we obtain the following.

PROOF OF THEOREM 1.7. Noting that $\log \zeta_{N,\text{rand}} = \mathcal{G}_N + E_N + L_N$ and writing $\mathcal{E}_N = E_N + L_N$, we see that Theorem 1.7 follows by combining Lemma 3.2 with Proposition 3.10. \square

4. The relationship to complex Gaussian multiplicative chaos: Proof of Theorem 1.1(ii). In this section, we prove the second part of our main result which states that ζ_{rand} can be expressed as a product of a complex Gaussian multiplicative chaos distribution and a smooth function with good regularity properties. We will first start by proving the existence of the complex Gaussian chaos needed. Recall from Section 2 that we write $f^{\mathbf{r}}(x)$ for $(1 + x^2)^{-1} f(x)$.

LEMMA 4.1. Let $(W_k^{(j)})_{k \geq 1, j \in \{1,2\}}$ be as in Definition 3.5 and denote

$$v_N(x) := \prod_{j=1}^N e^{\frac{1}{\sqrt{2p_j}} p_j^{-ix} (W_j^{(1)} + iW_j^{(2)})}.$$

For any $\alpha > 1/2$, the sequence $(v_N^{\mathbf{r}})_{N \geq 1}$ is an L^2 -bounded $W^{-\alpha,2}(\mathbb{R})$ -valued martingale, and consequently it converges almost surely to a $W^{-\alpha,2}(\mathbb{R})$ -valued random variable which we write as

$$v^{\mathbf{r}} := \lim_{N \rightarrow \infty} v_N^{\mathbf{r}}.$$

⁶Here and below, we use c to denote an uninteresting constant whose value may vary from line to line.

PROOF. By independence, for any $a \in \mathbb{C}$,

$$\mathbb{E}e^{a(W_j^{(1)} + iW_j^{(2)})} = e^{\frac{1}{2}(a^2 + (ia)^2)} = 1$$

and we infer that $(v_N^{\mathbf{r}})_{N \geq 1}$ is a martingale sequence taking values in $L^2 \subset W^{-\alpha,2}(\mathbb{R})$. Assume first that $\int |g(x)|^2(1+x^2) dx < \infty$ and write $Z_j := 2^{-1/2}(W_j^{(1)} + iW_j^{(2)})$ so that $(Z_j)_{j \geq 1}$ is an i.i.d. sequence of standard complex Gaussians. By using independence, the fact that $\mathbb{E}e^{\alpha Z_j + \beta \bar{Z}_j} = e^{\alpha\beta}$, and the series expansion of the exponential function, we easily compute

$$\begin{aligned} \mathbb{E}|v_N(g)|^2 &= \int_{\mathbb{R}^2} g(x)\overline{g(y)} \exp\left(\sum_{j=1}^N p_j^{-1-i(x-y)}\right) \\ &= \sum_{n \in \mathbb{N}_N} \frac{1}{\alpha_1(n)! \cdots \alpha_N(n)!} \frac{|\widehat{g}(\frac{1}{2\pi} \log(n))|^2}{n} \\ &\leq \|g\|_{\mathcal{Z}}^2 \lesssim \int_{\mathbb{R}} |g(x)|^2(1+x^2) dx, \end{aligned}$$

where the quantities $\alpha_j(n)$ were defined in the proof of Proposition 2.15 and the last inequality comes from Lemma 2.3. We recall the notation $e_{\xi}(x) = e^{-2\pi i \xi x}$ and $e_{\xi}^{\mathbf{r}}(x) = (1+x^2)^{-1}e^{-2\pi i \xi x}$ from Proposition 2.13. Now substituting $e_{\xi}^{\mathbf{r}}$ in place of g , multiplying by $(1+\xi^2)^{-\alpha}$ and integrating over \mathbb{R} we gather that

$$\mathbb{E}\|v_N^{\mathbf{r}}\|_{W^{-\alpha,2}(\mathbb{R})}^2 \leq c,$$

where c does not depend on N , and we are done. \square

One should observe that the martingale considered in the above proof is nontrivial, and hence the limit random variable v is also nontrivial (i.e., it does not reduce to a deterministic constant). We then start the proof of Theorem 1.1(ii) by fixing a compactly supported test function $f \in C_0^{\infty}(-A, A)$ and observing that we have the equality

$$\zeta_{N,\text{rand}}(f) = v_N(e^{\mathcal{E}N} f),$$

or, writing $g(x) := (1+x^2)f(x)$ this becomes

$$\zeta_{N,\text{rand}}^{\mathbf{r}}(g) = v_N^{\mathbf{r}}(e^{\mathcal{E}N} g).$$

Here, almost surely $\zeta_{N,\text{rand}}^{\mathbf{r}}$ converges to $\zeta_{\text{rand}}^{\mathbf{r}}$ and $v_N^{\mathbf{r}}$ to $v^{\mathbf{r}}$ in $W^{-\alpha,2}(\mathbb{R})$. Moreover, $e^{\mathcal{E}N} g \rightarrow e^{\mathcal{E}} g$ in C^{∞} , with supports contained in (A, A) . We may thus take the limit⁷ in the previous equality and obtain the almost sure equality

$$\zeta_{\text{rand}}^{\mathbf{r}}(g) = v^{\mathbf{r}}(e^{\mathcal{E}} g) = (e^{\mathcal{E}} v^{\mathbf{r}})(g).$$

A fortiori, since A was arbitrary and this holds almost surely for a countable dense subset of f 's in $W^{\alpha,2}(\mathbb{R})$, we see that almost surely $\zeta_{\text{rand}} = e^{\mathcal{E}} v$ (either as tempered distributions or with the interpretation that $\zeta^{\mathbf{r}} = e^{\mathcal{E}} v^{\mathbf{r}}$ as elements of $W^{-\alpha,2}(\mathbb{R})$), and this completes the proof Theorem 1.1(ii).

⁷Here, one simply chooses, for example, $\alpha = 1$ and notes that one easily checks that the map $(f, g) \mapsto fg$ is continuous map from $C^1(\mathbb{R}) \times W^{1,2}(\mathbb{R}) \rightarrow W^{1,2}(\mathbb{R})$ when $C^1(\mathbb{R})$ is normed by $\|f\|_{C^1(\mathbb{R})} = \|f\|_{\infty} + \|f'\|_{\infty}$. By duality, it follows that in the same map is continuous $C^1(\mathbb{R}) \times W^{-1,2}(\mathbb{R}) \rightarrow W^{-1,2}(\mathbb{R})$.

5. The mesoscopic limit—Proof of Theorem 1.2 and Theorem 1.5. In the present section, we verify our statements about the mesoscopic behavior of the zeta function. We start with some definitions and technical lemmata. Fix a two-sided complex Brownian motion $u \mapsto B_u^{\mathbb{C}}$ for $u \in \mathbb{R}$. More precisely, this means that $B_u^{\mathbb{C}} := 2^{-1/2}(B_u^{(1)} + iB_u^{(2)})$, where $B_u^{(1)}, B_u^{(2)}$ are standard independent two-sided Brownian motions. For $h \in L^2(\mathbb{R})$, we define a translation invariant process $G[h]$ on \mathbb{R} by setting

$$G[h](x) := \int_{\mathbb{R}} e^{-2\pi i x u} h(u) dB_u^{\mathbb{C}}.$$

For a fixed $h \in L^2$, the covariance of this process is $\mathbb{E}G[h](x)\overline{G[h](y)} = \widehat{|h|^2}(x - y)$, while $\mathbb{E}G[h](x)G[h](y) = 0$ for all $x, y \in \mathbb{R}$. In order to make sure that $G[h]$ defines random continuous functions of the variable x we shall, unless otherwise stated, assume that there is an $\varepsilon > 0$ and a $C > 0$ such that

$$(5.1) \quad 0 \leq |h(u)| \leq C(1 + |u|)^{-1/2-\varepsilon}.$$

This makes sure that $\widehat{|h|^2}$ is Hölder-continuous, and hence by classical theory, $G[h]$ has a modification whose realizations are almost surely continuous in x . Moreover, the sup-norm over any finite interval has the standard (double-)exponential estimates. In particular, for any $f \in L^2(\mathbb{R})$ with compact support we may safely compute

$$(5.2) \quad \begin{aligned} \mathbb{E} \left| \int_{\mathbb{R}} f(x) e^{G[h](x)} dx \right|^2 &= \int_{\mathbb{R}^2} f(x) \overline{f(y)} \exp\left(\frac{1}{2} \mathbb{E}(G[h](x) + \overline{G[h](y)})^2\right) dx dy \\ &= \int_{\mathbb{R}^2} f(x) \overline{f(y)} \exp(\widehat{|h|^2}(x - y)) dx dy. \end{aligned}$$

LEMMA 5.1. *Assume that h, h_j (where $j = 1, 2, \dots$) satisfy (5.1) and for almost every $u \in \mathbb{R}$ one has $\liminf_{j \rightarrow \infty} |h_j(u)| \geq |h(u)|$. Then for any $f \in L^2(\mathbb{R})$ with compact support it holds that*

$$\mathbb{E} \left| \int_{\mathbb{R}} f(x) e^{G[h](x)} dx \right|^2 \leq \liminf_{j \rightarrow \infty} \mathbb{E} \left| \int_{\mathbb{R}} f(x) e^{G[h_j](x)} dx \right|^2.$$

PROOF. We start by observing that $[\mathcal{F}((\widehat{|h|^2})^n)](\xi) = (|h|^2)^{*n}(-\xi)$. Hence we apply (5.2) to compute

$$\begin{aligned} \mathbb{E} \left| \int_{\mathbb{R}} f(x) e^{G[h](x)} dx \right|^2 &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^2} f(x) \overline{f(y)} (\widehat{|h|^2}(x - y))^n dx dy \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 (|h|^2)^{*n}(\xi) d\xi \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} |\widehat{f}(\xi_1 + \xi_2 + \dots + \xi_n)|^2 |h|^2(\xi_1) \dots |h|^2(\xi_n) d\xi_1 \dots d\xi_n. \end{aligned}$$

Applying the above identity to the functions h_j as well, the claim follows immediately from Fatou’s lemma. \square

LEMMA 5.2. *Assume that $\int_{\mathbb{R}} (1 + x^2) |f(x)|^2 dx < \infty$. There exists a $C > 0$ such that*

$$\mathbb{E} \left| \int_{\mathbb{R}} f(x) e^{G[u^{-1/2} \chi_{[1,A]}](x)} dx \right|^2 \leq C \int_{\mathbb{R}} (1 + x^2) |f(x)|^2 dx \quad \text{for all } A > 1.$$

PROOF. By easy approximation (recall that A is finite), we may assume that $f \in C_0^\infty(\mathbb{R})$. For each $\delta \in [0, 1]$, write $h_\delta(u) := e^{-\delta u} u^{-1/2} \chi_{[1, \infty)}(u)$. By the previous lemma, instead of the functions $(u^{-1/2} \chi_{[1, A]})_{A > 1}$ it is enough to prove the uniform bound for the family $(h_\delta(u))_{\delta \in (0, 1)}$.

Now $|h_0|^2(u) = u^{-1} \chi_{[1, \infty)}$ so $|\widehat{h_0|^2} \in L^2(\mathbb{R})$, and as $\text{supp}(|h_0|^2) \subset [0, \infty)$, the Paley–Wiener theorem verifies that $|\widehat{h_0|^2}$ extends to an analytic function in the lower half-plane and for any $\varepsilon > 0$, the analytic extension stays bounded in the half-plane $\{\text{Im } \xi < -\varepsilon\}$. Moreover, $|\widehat{h_0|^2}$ is obtained as the distributional boundary value of this extension (that we denote by the same symbol). By definition,

$$(5.3) \quad |\widehat{h_\delta|^2}(\xi) = |\widehat{h_0|^2}(-i2\delta + \xi) \quad \text{for all } \xi \in \mathbb{R}.$$

On the other hand, we may approximate $|h_0|^2$ by $u^{-1} \chi_{[1, A]}$, and obtain for $\xi \neq 0$,

$$|\widehat{h_0|^2}(\xi) = \lim_{A \rightarrow \infty} \int_1^A \frac{e^{-2\pi i \xi s}}{s} ds = \int_1^\infty \frac{e^{-2\pi i \xi s}}{s} ds = \int_{2\pi\xi}^{\text{sgn}(\xi) \times \infty} \frac{e^{-is}}{s} ds,$$

where the computation is validated by the local uniform convergence of the limit on $\xi \in \mathbb{R} \setminus \{0\}$. Here, one, for example, uses the fact that the real and imaginary parts are given by standard cosine and sine integrals. Thus, $|\widehat{h_0|^2}$ is locally smooth, even analytic in a neighbourhood of any given point $\xi \in \mathbb{R} \setminus \{0\}$. Moreover, we observe the bound

$$|\widehat{h_0|^2}(\xi)| \leq c|\xi|^{-1} \quad \text{for } |\xi| \geq 1, \text{Im } \xi \leq 0.$$

For $\xi \in B(0, 1) \cap \{\text{Im } \xi < 0\}$, we can write

$$\int_{2\pi\xi}^\infty \frac{e^{-iu}}{u} = C + \int_{2\pi\xi}^1 \frac{du}{u} + \int_{2\pi\xi}^1 \frac{e^{-iu} - 1}{u} du$$

(where the integration contour stays in the lower half-plane) and we see that

$$(5.4) \quad |\widehat{h_0|^2}(\xi) = \log(1/\xi) + [\text{entire}] \quad \text{in } \xi \in B(0, 1) \cap \{\text{Im } \xi < 0\}.$$

By (5.3), we deduce that the distributional limit of $\exp(|\widehat{h_\delta|^2}(\xi))$ equals the distributional boundary values on the real axis of the analytic function $\exp(|\widehat{h_0|^2}(\xi))$ in the lower half-plane, assuming that the latter ones exist. In turn, this follows from (5.4) and the above discussion. Namely,

$$\exp(|\widehat{h_0|^2}(\xi)) = c_0 \xi^{-1} + [\text{analytic and bounded over } \mathbb{R}].$$

We may then invoke Lemma 5.1, Cauchy–Schwarz and the Plemelj formula to deduce that

$$\begin{aligned} \mathbb{E} \left| \int_{\mathbb{R}} f(x) e^{G[h_\delta](x)} dx \right|^2 &\leq C \int_{\mathbb{R}^2} |f(x)f(y)| dx dy + \lim_{\varepsilon \rightarrow 0^+} \left| c_0 \int_{\mathbb{R}^2} \frac{f(x)f(y)}{x-y+i\varepsilon} dx dy \right| \\ &\leq C' \int_{\mathbb{R}} (1+x^2) |f(x)|^2 dx, \end{aligned}$$

where we used the estimate $\int_{\mathbb{R}} |f(x)| dx \lesssim (\int_{\mathbb{R}} (1+x^2) |f(x)|^2 dx)^{1/2}$ as well as the fact that the Hilbert transform is a bounded operator on $L^2(\mathbb{R})$. \square

The next lemma records a couple of basic properties of the complex chaos defined via exponentials of stochastic integrals of the type considered above.

LEMMA 5.3. For $0 < a \leq 1 \leq A$, denote

$$\eta_{a,A}(x) := \exp \left[\int_a^1 \frac{e^{-2\pi i x u} - 1}{u^{1/2}} dB_u^{\mathbb{C}} + \int_1^A \frac{e^{-2\pi i x u}}{u^{1/2}} dB_u^{\mathbb{C}} \right].$$

Then, for any $\alpha > 1/2$ we have

$$(5.5) \quad \sup_{a \in (0,1), A > 1} \mathbb{E} \|\eta_{a,A}^{\mathbf{r}}(x)\|_{W^{-\alpha,2}(\mathbb{R})}^2 < \infty$$

and for every $a \in [0, 1]$ there exists the almost sure limit

$$(5.6) \quad (1 + x^2)^{-1} \eta_a(x) := \lim_{A \rightarrow \infty} \eta_{a,A}^{\mathbf{r}}(x) \in W^{-\alpha,2}(\mathbb{R}).$$

In a similar vein, there exists the almost sure limit

$$(5.7) \quad (1 + x^2)^{-1} \eta(x) := \lim_{A \rightarrow \infty} \lim_{a \rightarrow 0^+} \eta_{a,A}^{\mathbf{r}}(x) \in W^{-\alpha,2}(\mathbb{R}).$$

PROOF. It is enough to prove the first statement since the latter ones will be easy consequences of the first one and the fact that $(1 + x^2)^{-1} \eta_{a,A}$ is a $W^{-\alpha,2}(\mathbb{R})$ -valued martingale with respect to decreasing a and increasing A . For that end, write $S_a(x) := \int_a^1 \frac{e^{-2\pi i x u} - 1}{u^{1/2}} dB_u^{\mathbb{C}}$ so that

$$\eta_{a,A}(x) = \exp(S_a(x) + G[u^{-1/2} \chi_{[1,A]}](x)).$$

By Lemma 5.2 and our standard computation of the expectation of the Sobolev norm (see, e.g., the proof of Lemma 4.1), we have for any $\beta \in (3/4, 1]$

$$(5.8) \quad \sup_{A > 1} \mathbb{E} \|(1 + x^2)^{-\beta} e^{G[u^{-1/2} \chi_{[1,A]}](x)}\|_{W^{-\alpha,2}(\mathbb{R})}^2 < \infty.$$

In turn, the part $S_a(x)$ behaves nicely and defines a smooth field, and by the already familiar argument we deduce that $\mathbb{E} \exp(\lambda \sup_{a \in (0,1)} \|S_a\|_{C^1[0,1]}) < \infty$ for any $\lambda > 0$, whence by Chebyshev we have the exponential tail $\mathbb{P}(\sup_{a \in (0,1)} \|S_a\|_{C^1[0,1]} > y) < C(\lambda)e^{-\lambda y}$. Assume that $\varepsilon > 0$. Choose $\lambda > \varepsilon^{-1}$. By translation invariance, we may compute for $y > 0$,

$$\begin{aligned} & \mathbb{P} \left(\sup_{a \in (0,1)} (|S_a(x)| + |S'_a(x)|) \geq y + \varepsilon \log(1 + x^2) \text{ for some } x \in \mathbb{R} \right) \\ & \lesssim \sum_{n \in \mathbb{Z}} e^{-\lambda(y + \varepsilon \log(1+n^2))} \lesssim e^{-\lambda y} \sum_{n \in \mathbb{Z}} (1 + n^2)^{-\lambda \varepsilon} \lesssim e^{-\lambda y}. \end{aligned}$$

This easily yields that $\mathbb{E} |\sup_{a \in (0,1)} \|(1 + x^2)^{-\varepsilon} \exp(S_a(x))\|_{C^1(\mathbb{R})}|^2 < \infty$. We may thus choose $\beta = 1 - \varepsilon$ with $\varepsilon \in (0, 1/4)$, assume $\alpha \in (1/2, 1]$, and obtain by independence and (5.8)

$$\begin{aligned} & \mathbb{E} \|(1 + x^2)^{-1} \eta_{a,A}(x)\|_{W^{-\alpha,2}(\mathbb{R})}^2 \\ & \lesssim \mathbb{E} (\|(1 + x^2)^{-\varepsilon} \exp(S_a(x))\|_{C^1(\mathbb{R})} \|(1 + x^2)^{\varepsilon-1} e^{G[u^{-1/2} \chi_{[1,A]}](x)}\|_{W^{-\alpha,2}(\mathbb{R})})^2 \\ & \leq C \end{aligned}$$

uniformly in a, A . Here, we again used the fact that functions with nice enough bounds on their derivatives are multipliers in Sobolev spaces; see the discussion at the end of Section 2. \square

Our next task is to approximate our Gaussian field

$$\mathcal{G}_N(x) := \sum_{k=1}^N \frac{1}{\sqrt{2p_k}} (W_k^{(1)} + i W_k^{(2)}) p_k^{-ix}$$

on the interval $[0, 1]$ by stochastic integrals of the type considered above. We will carry this out in several easy steps. First, we replace the summation over primes by a more regular one in terms of the Logarithmic integral: define

$$\mathcal{G}_{N,1}(x) = \sum_{j=1}^N \frac{1}{\sqrt{2\text{Li}^{-1}(j)}} (W_j^{(1)} + iW_j^{(2)}) (\text{Li}^{-1}(j))^{-ix}.$$

Let us show that this is a good approximation to \mathcal{G}_N .

LEMMA 5.4. *There exists a random smooth function $F_1 : [0, 1] \rightarrow \mathbb{R}$ such that almost surely, $F_{N,1} := \mathcal{G}_{N,1} - \mathcal{G}_N$ converges to F_1 in any $C^\ell[0, 1]$, $\ell \geq 1$. Moreover, for all $\lambda > 0$ and any $\ell \geq 0$,*

$$\mathbb{E} \exp(\lambda \|F_1\|_{C^\ell[0,1]}) < \infty$$

and

$$(5.9) \quad \mathbb{E} \exp\left(\lambda \sup_{N \geq 1} \|F_{N,1}\|_{C^\ell[0,1]}\right) < \infty.$$

PROOF. To apply Lemma 3.1, the term with no derivatives can be estimated with Lemma 3.8, and we see that, uniformly for $x \in [0, 1]$,

$$\left| \frac{1}{\sqrt{p_j}} e^{-ix \log p_j} - \frac{1}{\sqrt{\text{Li}^{-1}(j)}} e^{-ix \log \text{Li}^{-1}(j)} \right| \lesssim \frac{|p_j - \text{Li}^{-1}(j)|}{p_j^{3/2}} \lesssim j^{-1/2} e^{-\sqrt{\log j}}.$$

Differentiation only gives an extra power of $\log j$ here. So we see that (3.1) is satisfied and Lemma 3.1 applies as before. \square

The next step consists of (after first enlarging the probability space if needed) expressing the Gaussian variables $2^{-1/2}(W_j^{(1)} + iW_j^{(2)})$ in terms of $B_t^{\mathbb{C}} = B_t^{(1)} + iB_t^{(2)}$ in the following manner:

$$W_j^{(i)} = \int_{\text{Li}^{-1}(j)}^{\text{Li}^{-1}(j+1)} \frac{dB_t^{(i)}}{\sqrt{\text{Li}^{-1}(j+1) - \text{Li}^{-1}(j)}}.$$

This leads to the second approximation:

LEMMA 5.5. *Let*

$$\mathcal{G}_{N,2}(x) := \sum_{j=1}^N \int_{\text{Li}^{-1}(j)}^{\text{Li}^{-1}(j+1)} \frac{e^{-ix \log t}}{\sqrt{t}} \frac{dB_t^{\mathbb{C}}}{\sqrt{\text{Li}^{-1}(j+1) - \text{Li}^{-1}(j)}}.$$

Then almost surely, $F_{2,N} := \mathcal{G}_{N,2} - \mathcal{G}_{N,1}$ converges uniformly to a smooth function F_2 and the $C^\ell[0, 1]$ -norms of these quantities satisfy the same estimates as in the previous lemma.

PROOF. While we are now not in the setting of Lemma 3.1, we can still mimic its proof. By Ito’s isometry, to get a hold of the expectations of the square of the Sobolev norms $\|\cdot\|_{W^{k,2}(0,1)}^2$, we note that in order to estimate the L^2 -norm one needs to estimate for $x \in [0, 1]$ the square of the error that takes the form

$$\frac{1}{\text{Li}^{-1}(j+1) - \text{Li}^{-1}(j)} \int_{\text{Li}^{-1}(j)}^{\text{Li}^{-1}(j+1)} \left| \frac{e^{-ix \log \text{Li}^{-1}(j)}}{\sqrt{\text{Li}^{-1}(j)}} - \frac{e^{-ix \log t}}{\sqrt{t}} \right|^2 dt.$$

This quantity is of order $\mathcal{O}(\frac{(\text{Li}^{-1}(j+1)-\text{Li}^{-1}(j))^2}{\text{Li}^{-1}(j)^3}) = \mathcal{O}(j^{-3})$, while the derivative terms come with an extra $\log^{2\ell} j$. All these are summable over j , so we can conclude as before. \square

To proceed, we will want to replace the $1/\sqrt{\text{Li}^{-1}(j+1) - \text{Li}^{-1}(j)}$ by something more convenient. More precisely, we will make use of the following approximation.

LEMMA 5.6. *Let*

$$\mathcal{G}_{N,3}(x) := \int_{\text{Li}^{-1}(1)}^{\text{Li}^{-1}(N+1)} \frac{e^{-ix \log t}}{\sqrt{t}} \frac{dB_t^{\mathbb{C}}}{\sqrt{\log t}}.$$

Then almost surely as $N \rightarrow \infty$, $F_{3,N} := \mathcal{G}_{N,3} - \mathcal{G}_{N,2}$ converges uniformly to a random continuous function F_3 , and the $C^\ell[0, 1]$ -norms of these quantities satisfy the same estimates as in Lemma 5.4.

PROOF. We again argue as in the proof of Lemma 3.1. Now we need to estimate terms of the form

$$\int_{\text{Li}^{-1}(j)}^{\text{Li}^{-1}(j+1)} \left| \frac{1}{\sqrt{\text{Li}^{-1}(j+1) - \text{Li}^{-1}(j)}} - \frac{1}{\sqrt{\log t}} \right|^2 \frac{1}{2t} dt,$$

and similar ones coming with a factor of $\log^{2\ell} t$ coming from the derivative term in the Sobolev estimate. To estimate such a term, we see that it is enough for us to estimate the difference $|\text{Li}^{-1}(j+1) - \text{Li}^{-1}(j) - \log t|$ for $t \in [\text{Li}^{-1}(j), \text{Li}^{-1}(j+1)]$. For this, we recall that $\text{Li}'(x) = 1/\log x$ and use a change of variable to write

$$\begin{aligned} & |\text{Li}^{-1}(j+1) - \text{Li}^{-1}(j) - \log t| \\ & \leq \int_j^{j+1} |\log(\text{Li}^{-1}(s)) - \log t| ds \\ & \lesssim \text{Li}^{-1}(j+1) - \text{Li}^{-1}(j) \lesssim \log \frac{\text{Li}^{-1}(j+1)}{\text{Li}^{-1}(j)} \lesssim \frac{\text{Li}^{-1}(j+1) - \text{Li}^{-1}(j)}{\text{Li}^{-1}(j)} \lesssim j^{-1}. \end{aligned}$$

Above we used the asymptotics $\text{Li}^{-1}(j) \sim j \log j$ and $(\text{Li}^{-1})'(j) \sim \log j$. Hence the square of the Sobolev norm can be bounded by $\mathcal{O}((j^{-3}(\log j)^{2\ell}))$, which is summable and the rest of the proof goes through as before. \square

We note that (e.g., by Lévy’s criterion) $t \mapsto \int_0^t (2\pi e^{2\pi u})^{-1/2} dB_{e^{2\pi u}}^{\mathbb{C}}$ is a standard Brownian motion, which we shall denote with slight abuse of notation still by $dB_s^{\mathbb{C}}$. After performing a change of variables $t = \exp 2\pi u$ in the integral defining our last approximation, the outcome is

$$\int_{\log(\text{Li}^{-1}(1))/2\pi}^{\log(\text{Li}^{-1}(N+1))/2\pi} \frac{e^{-2\pi i x u}}{\sqrt{u}} dB_u^{\mathbb{C}}.$$

By observing that a term of the form $\int_{\log(\text{Li}^{-1}(1))/2\pi}^1 \frac{e^{-2\pi i x u}}{\sqrt{u}} dB_u^{\mathbb{C}}$ can safely be absorbed into the error term and noting that we could have equally well considered an arbitrary interval $[-A, A]$ instead of $[0, 1]$ we thus obtain a new variant of Theorem 1.7.

PROPOSITION 5.7. *For each $N \geq 1$, we may write*

$$\log \zeta_{N,\text{rand}}(1/2 + ix) = \tilde{\mathcal{G}}_N(x) + \tilde{\mathcal{E}}_N(x),$$

where $\tilde{\mathcal{G}}_N$ is a Gaussian process on \mathbb{R} and can be written as the stochastic integral

$$\tilde{\mathcal{G}}_N(x) := G[u^{-1/2}\chi_{[1, \log \text{Li}^{-1}(N+1)]}](x).$$

The function $\tilde{\mathcal{E}}_N$ is smooth and as $N \rightarrow \infty$, it a.s. converges uniformly in every $C^\ell[-A, A]$ ($A > 0, \ell \geq 1$) to a random smooth function on \mathbb{R} . Moreover, the maximal error and its derivatives in this decomposition have finite exponential moments:

$$\mathbb{E} \exp\left(\lambda \sup_{N \geq 1} \|\tilde{\mathcal{E}}_N(x)\|_{C^\ell[-A, A]}\right) < \infty \quad \text{for all } \lambda > 0 \text{ and } \ell \geq 0.$$

We are now ready to prepare for the actual proof of the mesoscopic scaling result. We record first a simple estimate for dilations.

LEMMA 5.8. Assume that $\alpha > 1/2$ and $f \in W^{-\alpha, 2}(\mathbb{R})$. Then for any $\delta \in (0, 1)$ it holds that

$$\|f(\delta \cdot)\|_{W^{-\alpha, 2}(\mathbb{R})} \leq \delta^{-1-2\alpha} \|f\|_{W^{-\alpha, 2}(\mathbb{R})}.$$

PROOF. One simply computes

$$\begin{aligned} \|f(\delta \cdot)\|_{W^{-\alpha, 2}(\mathbb{R})}^2 &= \int_{\mathbb{R}} |\delta^{-1} \widehat{f}(\delta^{-1}\xi)|^2 (1 + \xi^2)^{-\alpha} d\xi \\ &= \delta^{-1} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 (1 + (\delta\xi)^2)^{-\alpha} d\xi \\ &\leq \delta^{-1-2\alpha} \|f\|_{W^{-\alpha, 2}(\mathbb{R})}^2. \end{aligned} \quad \square$$

From now on, we focus on the interval $(0, 1)$. Recall $\eta_{a,A}$ from Lemma 5.3 and note that the same lemma verifies the existence of the complex chaos

$$\eta := \lim_{A \rightarrow \infty} \eta_{0,A} = \lim_{A \rightarrow \infty} \exp\left(\int_0^1 (e^{-2\pi i x u} - 1) u^{-1/2} dB_u^{\mathbb{C}} + \int_1^A e^{-2\pi i x u} u^{-1/2} dB_u^{\mathbb{C}}\right)$$

and $\eta \in W^{-\alpha, 2}(0, 1)$ for any $\alpha > 1/2$.

The following result is our main ingredient for the proof of Theorem 1.2.

PROPOSITION 5.9. Fix $\alpha > 1/2$. Assume that $\delta \in (0, 1)$ and consider the dilations of the randomised zeta function on the interval $(0, 1)$. Then we may decompose

$$(5.10) \quad \zeta_{\text{rand}}(1/2 + i\delta \cdot)_{|(0,1)} = h_\delta e^{Y_\delta} \eta^{(\delta)}.$$

Here, the distribution of the complex multiplicative chaos $\eta^{(\delta)}$ is independent of δ :

$$(5.11) \quad \eta^{(\delta)} \sim \eta \quad \text{in } W^{-\alpha, 2}(0, 1) \quad \text{for any } \delta \in (0, 1).$$

Moreover, h_δ is a random smooth function on $[0, 1]$ which tends almost surely (and hence in distribution) to the constant function 1: for each $\ell \geq 0$,

$$(5.12) \quad h_\delta \xrightarrow{a.s.} 1 \quad \text{in } C^\ell[0, 1] \quad \text{as } \delta \rightarrow 0^+.$$

Finally, Y_δ is a complex (scalar) random variable that can be written in the form

$$(5.13) \quad Y_\delta \stackrel{d}{=} \sqrt{\log(1/\delta)} Z + R,$$

where Z is a standard complex normal random variable and the random variable R , which is independent of δ , satisfies $\mathbb{E} \exp(\lambda |R|) \leq C_\lambda$ for all $\lambda > 0$.

PROOF. Apply the dilation $x \mapsto \delta x$ in Proposition 5.7 in order to write

$$\begin{aligned} \zeta_{\text{rand},N}(1/2 + i\delta x) &= \exp(\tilde{\mathcal{E}}_N(\delta x) + \tilde{\mathcal{G}}_N(\delta x)) \\ &= \exp(\tilde{\mathcal{E}}_N(\delta x)) \exp(G[u^{-1/2} \chi_{[1, \log \text{Li}^{-1}(N+1)]}](\delta x)). \end{aligned}$$

By making a change of variables in the stochastic integral and letting $dB_u^{\mathbb{C},\delta} := \delta^{-1/2} dB_{\delta u}^{\mathbb{C}}$, we get again a standard Brownian motion, and it follows that

$$\begin{aligned} &\tilde{\mathcal{E}}_N(\delta x) + \tilde{\mathcal{G}}_N(\delta x) \\ &= \tilde{\mathcal{E}}_N(\delta x) + \int_1^{\log \text{Li}^{-1}(N+1)} e^{-2\pi i x \delta u} u^{-1/2} dB_u^{\mathbb{C}} \\ &= \tilde{\mathcal{E}}_N(\delta x) + \int_{\delta}^{\delta \log \text{Li}^{-1}(N+1)} e^{-2\pi i x u} u^{-1/2} dB_u^{\mathbb{C},\delta} \\ &= \left(\tilde{\mathcal{E}}_N(\delta x) - \tilde{\mathcal{E}}_N(0) - \int_0^{\delta} (e^{-2\pi i x u} - 1) u^{-1/2} dB_u^{\mathbb{C},\delta} \right) \\ &\quad + \left(\int_0^1 (e^{-2\pi i x u} - 1) u^{-1/2} dB_u^{\mathbb{C},\delta} + \int_1^{\log \delta \text{Li}^{-1}(N+1)} e^{-2\pi i x u} u^{-1/2} dB_u^{\mathbb{C},\delta} \right) \\ &\quad + \left(\tilde{\mathcal{E}}_N(0) + \int_{\delta}^1 u^{-1/2} dB_u^{\mathbb{C},\delta} \right) \\ &=: A_{\delta,N}(x) + B_{\delta,N}(x) + C_{\delta,N}(x). \end{aligned}$$

Here, the results of Proposition 5.7 imply that as $N \rightarrow \infty$, we have almost surely

$$\exp(A_{\delta,N}(x)) \rightarrow \exp\left(\tilde{\mathcal{E}}(\delta x) - \tilde{\mathcal{E}}(0) - \int_0^{\delta} (e^{-2\pi i x u} - 1) u^{-1/2} dB_u^{\mathbb{C},\delta}\right) =: h_{\delta}(x)$$

with convergence in $C^{\ell}[0, 1]$ for any $\ell \geq 0$. Moreover, (5.12) is clearly true. Next, we observe the almost sure convergence

$$C_{\delta,N}(x) \rightarrow \exp\left(\tilde{\mathcal{E}}(0) + \int_{\delta}^1 u^{-1/2} dB_u^{\mathbb{C},\delta}\right) =: e^{Y_{\delta}},$$

where Y_{δ} has the stated properties by the Ito isometry and Proposition 5.7. Finally, recalling the definition of $\eta^{(\delta)}$ we have almost surely that

$$\exp(B_{\delta,N}(x)) \rightarrow \eta^{(\delta)} \quad \text{in } W^{-\alpha,2}(0, 1).$$

By combining all the above observations and invoking the definition of ζ_{rand} the equality (5.10) follows. \square

We can now prove our first characterization of the mesoscopic behavior of ζ .

PROOF OF THEOREM 1.2. Combining Theorem 1.1, Proposition 5.9 and Lemma 5.8, we see that for each $k \in \mathbb{Z}_+$, there exists a T_k so that for $T \geq T_k$,

$$\begin{aligned} &\mathcal{W}_2(\zeta(1/2 + ik^{-1}x + i\omega T), \zeta_{\text{rand}}(1/2 + ik^{-1}x))_{W^{-\alpha,2}(0,1)} \\ &= \mathcal{W}_2(\zeta(1/2 + ik^{-1}x + i\omega T), h_{1/k}(x)e^{Y_{1/k}}\eta^{(1/k)}(x))_{W^{-\alpha,2}(0,1)} \\ &\leq 1/k. \end{aligned}$$

We can naturally take T_k to be increasing in k . Taking $\delta_T = 1/k$ for $T \in [T_k, T_{k+1})$ along with $h_T = h_{\delta_T}$, etc. gives the claim. \square

REMARK 5.10. It is natural to ask what kind of rate one may obtain for $\delta_T \rightarrow 0$ in Theorem 1.2. One may certainly obtain a bound by going carefully through the proof of Theorem 1.1 and other (auxiliary) results used in the proof. However, we feel that getting even reasonably close to optimal rates requires additional analysis and a considerable amount of work (e.g., one needs to quantify suitably the rate of joint convergence of the variables $p^{-i\omega T}$ to uniform ones as $T \rightarrow \infty$) whence we do not pursue this question here.

Let us now turn to the proof of Theorem 1.5. Since $W_{\text{mult}}^{-\alpha,2}(0, 1)$ is a bounded, complete and separable metric space, convergence of $W_{\text{mult}}^{-\alpha,2}(0, 1)$ -valued random variables in the corresponding Wasserstein metric is equivalent to standard convergence in distribution. We need one last auxiliary result.

LEMMA 5.11. *Assume that the $W^{-\alpha,2}(0, 1)$ -valued random variables g_k are almost surely nonzero and*

$$\mathcal{W}_2(g_k, g)_{W^{-\alpha,2}(0,1)} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

where g is an almost surely nonzero $W^{-\alpha,2}(0, 1)$ -valued random variable. Then also

$$\mathcal{W}_2(g_k, g)_{W_{\text{mult}}^{-\alpha,2}(0,1)} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

PROOF. The statement follows easily after one notes that for any $\varepsilon > 0$ there is an r such that $\mathbb{P}(\|g\|_{W^{-\alpha,2}(0,1)} < 2r) < \varepsilon/2$, and hence for any $k \geq k_0(\varepsilon)$ it holds that $\mathbb{P}(\|g_k\|_{W^{-\alpha,2}(0,1)} < r) < \varepsilon$. A fortiori, for $k \geq k_0$

$$\mathcal{W}_2(g_k, g)_{W_{\text{mult}}^{-\alpha,2}(0,1)} \leq 4\varepsilon + r^{-1}\mathcal{W}_2(g_k, g)_{W^{-\alpha,2}(0,1)},$$

and the claim follows. \square

We are finally prepared for the following.

PROOF OF THEOREM 1.5. Observe that η is almost surely nonvanishing as an almost sure limit of a nontrivial L^2 -bounded martingale by its definition and a simple application of Kolmogorov’s zero-one theorem, and the same holds for ζ_{rand} . Moreover, we note that $\eta \sim \eta^{(\delta)}$ for all $\delta > 0$. Proposition 5.9 yields that

$$\begin{aligned} \mathcal{W}_2(\eta, \zeta_{\text{rand}}(1/2 + i\delta\cdot))_{W_{\text{mult}}^{-\alpha,2}(0,1)} &= \mathcal{W}_2(\eta^{(\delta)}, h_\delta e^{Y_\delta} \eta^{(\delta)})_{W_{\text{mult}}^{-\alpha,2}(0,1)} \\ &= \mathcal{W}_2(\eta^{(\delta)}, h_\delta \eta^{(\delta)})_{W_{\text{mult}}^{-\alpha,2}(0,1)} \rightarrow 0 \quad \text{as } \delta \rightarrow 0^+, \end{aligned}$$

where in the last step we used (5.12).⁸ We then set $\delta = 1/k$ and apply Lemma 5.8 and Lemma 5.11 to pick a strictly increasing sequence T_k so that

$$\mathcal{W}_2(\mu_T(k^{-1}\cdot), \zeta_{\text{rand}}(1/2 + ik^{-1}\cdot))_{W_{\text{mult}}^{-\alpha,2}(0,1)} \leq 1/k \quad \text{for } T \geq T_k$$

By combining the above inequalities, we see that the choice $\delta_T = 1/k$ for $T \in [T_k, T_{k+1})$ applies as before. \square

⁸More precisely, using the fact that h_δ is a multiplier on $W^{-\alpha,2}(0, 1)$, one readily checks that for small, but arbitrary $\varepsilon > 0$, $\mathbb{P}(\|(h_\delta - 1)\eta^{(\delta)}\|_{W^{-\alpha,2}(0,1)} \leq \varepsilon \|\eta^{(\delta)}\|_{W^{-\alpha,2}(0,1)}) \geq 1 - \varepsilon$ for small enough δ . Now on this event, we have $\|\|\eta^{(\delta)}\|_{W^{-\alpha,2}(0,1)}^{-1} \|\eta^{(\delta)} - h_\delta \eta^{(\delta)}\|_{W^{-\alpha,2}(0,1)}\|_{W^{-\alpha,2}(0,1)} = \mathcal{O}(\varepsilon)$, where the implied constant is deterministic. This implies that in the metric of $W_{\text{mult}}^{-\alpha,2}(0, 1)$, as $\delta \rightarrow 0^+$, the distance between $\eta^{(\delta)}$ and $h_\delta \eta^{(\delta)}$ tends to zero in probability. As the space is bounded, this implies convergence in the Wasserstein sense as well.

6. Relationship to real Gaussian multiplicative chaos in the subcritical case: Proof of Theorem 1.8. In this section, we prove that for $0 < \beta < \beta_c := 2$, $|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta / \mathbb{E}|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta dx$ almost surely converges with respect to the weak topology of measures to a random measure which is absolutely continuous with respect to a Gaussian multiplicative chaos measure. This will be an easy consequence of our Gaussian coupling and the general theory of real Gaussian multiplicative chaos measures.

For a proper introduction to the theory of Gaussian multiplicative chaos, we refer the reader to Kahane’s original work [52] or the recent review by Rhodes and Vargas [69]. We also point out Berestycki’s elegant proof for the existence and uniqueness of subcritical Gaussian multiplicative chaos measures [12]. For the convenience of the reader, we nevertheless recall the main results from the theory that are relevant to us.

THEOREM 6.1. *Assume that we have a sequence of independent Gaussian fields $(Y_k)_{k=1}^\infty$ on $[0, 1]$ and that the covariance kernel of Y_k , K_{Y_k} , is continuous on $[0, 1] \times [0, 1]$. Define the field*

$$X_n = \sum_{k=1}^n Y_k,$$

and assume that as $n \rightarrow \infty$, the covariance kernel K_{X_n} converges locally uniformly in $[0, 1]^2 \setminus \{x = y\}$ to a function on $[0, 1]^2$ which is of the form

$$\log \frac{1}{|x - y|} + g(x, y),$$

where g is bounded and continuous. Moreover, assume that there is a constant $C < \infty$ so that

$$(6.1) \quad K_{X_n}(x, y) \leq \log \frac{1}{|x - y|} + C \quad \text{for all } x, y \in [0, 1] \text{ and } n \geq 1.$$

Then for $\beta > 0$ the random measure

$$\lambda_{\beta,n}(dx) = \frac{e^{\beta X_n(x)}}{\mathbb{E}e^{\beta X_n(x)}} dx$$

converges almost surely with respect to the topology of weak convergence of measures to a limiting measure λ_β . This limiting measure is a nontrivial random measure for $\beta < \beta_c = \sqrt{2}$ and for $\beta \geq \beta_c$, it is the zero measure. Moreover, if $0 < \beta < \sqrt{2}$, and $0 < p < 2/\beta^2$, then for a compact set $A \subset [0, 1]$

$$\mathbb{E}(\lambda_\beta(A)^p) < \infty.$$

PROOF. (Sketch) By (6.1) and Kahane’s convexity inequality (see [69], Theorem 2.1), one may easily compare to a standard approximation of a chaos measure and deduce that for any $\beta < \beta_c$ the random variables $\lambda_{\beta,n}[0, 1]$ form an L^p -martingale with a suitable $p = p(\beta) > 1$. At this stage, the standard theory of multiplicative chaos can be applied to obtain the rest of the claims; see, for example, [69], Theorems 2.5 and 2.11. \square

As we are studying $|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta$, our relevant field is $\log |\zeta_{N,\text{rand}}(1/2 + ix)|$, and the relevant Gaussian part is the real part of \mathcal{G}_N . To simplify notation slightly, let us write $G_N = \text{Re}(\mathcal{G}_N)$ and $X_N = \log |\zeta_{N,\text{rand}}(1/2 + ix)|$. To apply Kahane’s construction of a Gaussian multiplicative chaos measure, we will need to establish that the covariance of G_N satisfies the requirements of Theorem 6.1. From the definition of \mathcal{G}_N , we see that

$$G_N(x) = \sum_{j=1}^N \frac{1}{\sqrt{2p_j}} (W_j^{(1)} \cos(x \log p_j) + W_j^{(2)} \sin(x \log p_j)).$$

A direct computation shows that

$$K_{G_N}(x - y) := \mathbb{E}G_N(x)G_N(y) = \Psi_N(x - y),$$

where

$$\Psi_N(u) := \frac{1}{2} \sum_{j=1}^N \frac{\cos(u \log p_j)}{p_j}.$$

The following result is enough for us to be able to apply Kahane’s theory for defining a multiplicative chaos measure. It is of interest to note that we are dealing with a logarithmically correlated translation invariant field whose covariance deviates from $\frac{1}{2} \log(1/|x - y|)$ by only a smooth function.

LEMMA 6.2. *We have*

$$\left| K_{G_N}(x, y) - \frac{1}{2} \log\left(\min\left(\frac{1}{|x - y|}, \log N\right)\right) \right| \leq C,$$

where C is uniform over $N \geq 1$ and $(x, y) \in [0, 1]$. Moreover, if $x \neq y$

$$K_{G_n}(x, y) \longrightarrow K_G(x, y) = \frac{1}{2} \log \frac{1}{|x - y|} + g(x - y) \quad \text{as } n \rightarrow \infty,$$

with local uniform convergence outside the diagonal. Moreover, we have $g \in C^\infty(-2, 2)$.

PROOF. For given quantities $z = z_N(u)$, $\tilde{z} = \tilde{z}_N(u)$, we will use the shorthand notation $z \underset{\text{unif}}{\sim} \tilde{z}$ to indicate the uniform inequality $|z_N(u) - \tilde{z}_N(u)| \leq C$ with a universal bound C and such that $\lim_{N \rightarrow \infty} (z_N(u) - \tilde{z}_N(u))$ converges uniformly to a continuous function on the interval $u \in [-2, 2]$. We shall employ the following prime number asymptotics, which are a consequence of the prime number theorem (or more precisely, Lemma 3.8 and asymptotics of Li)

$$p_j = j \log j + \mathcal{O}(j \log \log j).$$

This implies that $\sum_{j=1}^\infty \frac{|\log p_j - \log(j \log j)|}{p_j} < \infty$ and since the cosine function is 1-Lipschitz we obtain

$$\Psi_N(u) \underset{\text{unif}}{\sim} \frac{1}{2} \sum_{j=1}^N \frac{\cos(u \log(j \log j))}{p_j}.$$

In a similar vein, $\sum_{j=1}^\infty |p_j^{-1} - (j \log j)^{-1}| < \infty$ which leads to

$$\Psi_N(u) \underset{\text{unif}}{\sim} \frac{1}{2} \sum_{j=1}^N \frac{\cos(u \log(j \log j))}{j \log j}.$$

Next, we observe that for all $u \in [-2, 2]$ and $x \geq 10$,

$$\left| \frac{d}{dx} \left(\frac{\cos(u \log(x \log x))}{x \log x} \right) \right| \leq \frac{6}{x^2 \log x}.$$

Since $\int_{10}^\infty (x^2 \log x)^{-1} dx < \infty$, it follows that

$$\Psi_N(u) \underset{\text{unif}}{\sim} \frac{1}{2} \int_{x=10}^N \frac{\cos(u \log(x \log x)) dx}{x \log x}.$$

To continue, we note that

$$\int_{x=10}^{\infty} \left| 1 - \frac{1 + \log x}{\log x + \log \log x} \right| \frac{dx}{x \log x} < \infty$$

so that

$$\begin{aligned} \Psi_N(u) &\underset{\text{unif}}{\sim} \frac{1}{2} \int_{x=10}^N \frac{\cos(u \log(x \log x)) (1 + \log x) dx}{\log(x \log x) x \log x} \\ (6.2) \quad &\underset{\text{unif}}{\sim} \frac{1}{2} \int_1^{\log N + \log \log N} \frac{\cos(ut)}{t} dt \underset{\text{unif}}{\sim} \frac{1}{2} \int_1^{\log N} \frac{\cos(ut)}{t} dt \\ &= \frac{1}{2} \int_u^{u \log N} \frac{\cos(x)}{x} dx =: A(u, N). \end{aligned}$$

Above in the first step, we performed the change of variables $u = \log(x \log x)$ and noted that $du = (1 + \log x) dx / x \log x$. In the second to last step, we used the fact that $\int_{\log N}^{\log N + \log \log N} t^{-1} dt = o(1)$ as $N \rightarrow \infty$.

It remains to prove the claim for $A(u, N)$ defined in (6.2). Since the limit $\lim_{z \rightarrow \infty} \int_1^z \frac{\cos(x)}{x} dx$ exists and is finite, we see directly from the definition that for any $\varepsilon_0 > 0$ in the set $\{\varepsilon_0 \leq |u| \leq 2\}$ the function $A(u, N)$ converges uniformly to a continuous function of u as $N \rightarrow \infty$. Moreover, since $\int_0^1 |\cos(x) - 1| x^{-1} dx < \infty$, we get for $|u| \geq (\log N)^{-1}$

$$\left| A(u, N) - \frac{1}{2} \int_u^1 x^{-1} dx \right| = \left| A(u, N) - \frac{1}{2} \log(1/u) \right| \leq C,$$

where C is independent of N and u . Finally, if $|u| \leq (\log N)^{-1}$ we get in a similar manner

$$\left| A(u, N) - \frac{1}{2} \int_u^{u \log N} x^{-1} dx \right| = \left| A(u, N) - \frac{1}{2} \log \log N \right| \leq C',$$

and now C' is independent of N and $u \in \{|u| \geq (\log N)^{-1}\}$. This proves the first statement of the lemma.

By (6.2), we deduce that there is a continuous function $\tilde{b}(u)$ on $[-2, 2]$ so that the limit Ψ of the functions Ψ_N takes the form

$$\Psi(u) = \tilde{b}(u) + \frac{1}{2} \int_u^{\infty} \frac{\cos(t)}{t} dt = \frac{1}{2} \log(|u|^{-1}) + b(u) \quad \text{for } 0 < |u| < 2,$$

with $b \in C[-2, 2]$ as $u \mapsto \int_u^1 (\cos(x) - 1)x^{-1} dx$ is continuous over $u \in [0, 1]$. Especially, we know that $\Psi(x - y)$ yields the covariance operator of our limit field since the estimates we have proven imply that $\Psi_N(x - y) \rightarrow \Psi(x - y)$ in $L^2([0, 1]^2)$, and convergence in the Hilbert–Schmidt norm is enough to identify the limit covariance of a sequence of Gaussian fields converging a.s. in the sense of distributions. We still want to upgrade b to be smooth. For that end, we first fix $\delta_0 > 0$ and observe that what we have proved up to now (see especially (6.2)) yields

$$\Psi(u) = \frac{1}{2} \operatorname{Re} \left(\lim_{N \rightarrow \infty} \sum_{j=1}^N p_j^{-1-iu} \right)$$

with uniform convergence in the set $\{\delta_0 \leq |u| \leq 2\}$. However, if we apply exactly the same argument as above to the sum $\operatorname{Re}(\sum_{j=1}^N p_j^{-1-\varepsilon-iu})$ for, say, $\varepsilon \in [0, 1/2]$, we obtain uniform (in ε) estimates for the convergence of the series

$$\operatorname{Re} \left(\sum_{j=1}^{\infty} p_j^{-1-\varepsilon-iu} \right)$$

for any fixed $u \in (0, 2)$. Especially, we deduce by invoking the logarithm of the Euler product of the Riemann zeta function that

$$\begin{aligned} \Psi(u) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2} \operatorname{Re} \left(\sum_{j=1}^{\infty} p_j^{-1-\varepsilon-iu} \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2} \operatorname{Re} \left(\zeta(1 + \varepsilon + iu) - \sum_{k=2}^{\infty} \sum_{j=1}^{\infty} \frac{p_j^{-k(1+\varepsilon+iu)}}{k} \right) \\ &= \frac{1}{2} \operatorname{Re} \left(\zeta(1 + iu) - \sum_{k=2}^{\infty} \sum_{j=1}^{\infty} k^{-1} p_j^{-k(1+iu)} \right), \\ &=: \frac{1}{2} \operatorname{Re}(\log(\zeta(1 + iu)) - A(u)), \end{aligned}$$

as the last written double sum converges absolutely (uniformly in ε). It remains to note that $\log(\zeta(1 + iu))$ is real analytic on $(0, \infty)$, and the function A is C^∞ -smooth on the same set as termwise differentiation of A ℓ times with respect to u produces a series that can be bounded termwise from above by

$$\sum_p \sum_{k=2}^{\infty} k^{\ell-1} p^{-k} \log^\ell p \leq \sum_p p^{-3/2} \left(\sum_{r=0}^{\infty} (r+2)^{\ell-1} p^{-r} \right) < \infty. \quad \square$$

REMARK 6.3. Note that in our case the asymptotic covariance has a singularity of the form $-\frac{1}{2} \log|x - y|$ instead of $-\log|x - y|$ as in Theorem 6.1. This simply means that we replace β by $\beta/\sqrt{2}$ in Theorem 6.1.

Before proving the convergence of the subcritical chaos, we still need to note that the normalizing constant in our exponential martingale obtained via the Gaussian approximation behaves like that of our original martingale.

LEMMA 6.4. For any $\beta > 0$, there is a constant $C = C(\beta)$ such that for $N \geq 1$ and $x \in [0, 1]$

$$C^{-1} \mathbb{E} \exp(\beta G_N(x)) \leq \mathbb{E} |\zeta_{N,\text{rand}}(1/2 + ix)|^\beta \leq C \mathbb{E} \exp(\beta G_N(x)) \quad \text{for all.}$$

PROOF. By independence, rotational invariance of the law of $e^{2\pi i \theta_j}$, and the finiteness of the sum $\sum_{j=1}^{\infty} \sum_{k=3}^{\infty} k^{-1} p_j^{-k/2} < \infty$, we see that

$$\begin{aligned} \mathbb{E} |\zeta_{N,\text{rand}}(1/2 + ix)|^\beta &= \prod_{j=1}^N \mathbb{E} e^{-\beta \log|1-p_j^{-1/2} e^{2\pi i \theta_j}|} = \prod_{j=1}^N \mathbb{E} e^{\beta \sum_{k=1}^{\infty} \frac{1}{k} p_j^{-k/2} \cos(2\pi k \theta_j)} \\ (6.3) \quad &\asymp \prod_{j=1}^N \mathbb{E} e^{\beta p_j^{-1/2} \cos(2\pi \theta_j) + \frac{\beta}{2} p_j^{-1} \cos(4\pi \theta_j)}, \end{aligned}$$

where we use the notation $a_n \asymp b_n$ to indicate that the ratios a_n/b_n and b_n/a_n are bounded. Now consider the function $\lambda \mapsto \mathbb{E} e^{\beta \lambda \cos(2\pi \theta_j) + \beta \frac{\lambda^2}{2} \cos(4\pi \theta_j)}$. This is analytic and one can easily check that

$$(6.4) \quad \mathbb{E} e^{\beta \lambda \cos(2\pi \theta_j) + \beta \frac{\lambda^2}{2} \cos(4\pi \theta_j)} = e^{\frac{\beta^2}{4} \lambda^2 + \mathcal{O}(\lambda^3)}$$

as $\lambda \rightarrow 0$. Substituting $\lambda = p_j^{-1/2}$ and noting that $\sum_{j=1}^\infty p_j^{-3/2} < \infty$, combining (6.3) and (6.4) implies that

$$\mathbb{E}|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta \asymp e^{\frac{\beta^2}{4} \sum_{j=1}^N \frac{1}{p_j}}.$$

From the definition of G_N , we see that $\mathbb{E}G_N(x)^2 = \sum_{j=1}^N \frac{1}{2p_j}$ so we conclude that

$$\mathbb{E}e^{\beta G_N(x)} = e^{\frac{\beta^2}{2} \sum_{j=1}^N \frac{1}{2p_j}} \asymp \mathbb{E}|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta,$$

which was the claim. \square

One should note that combining the above lemmas we see that for $N \geq 1$

$$\mathbb{E}|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta \asymp \mathbb{E} \exp(\beta G_N(x)) \asymp \exp\left(\frac{\beta^2}{4} \log \log N\right) = (\log N)^{\beta^2/4}.$$

Finally we are ready for the following.

PROOF OF THEOREM 1.8. For $\beta < 2$, let λ_β be the Gaussian multiplicative chaos measure associated with the sequence $(G_N)_{N \geq 1}$, which exists due to Theorem 6.1. In this situation, the approximating measures $\lambda_{\beta,N}$, obtained from the fields G_N and converge to λ_β . In particular, there is a $\tilde{p} > 1$ such that $\mathbb{E}(\lambda_{\beta,N}[0, 1]^{\tilde{p}}) \leq C < \infty$ for all $N \geq 1$. Recall that we want to prove that for each continuous $f : [0, 1] \rightarrow \mathbb{R}_+$,

$$\int_0^1 \frac{|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta}{\mathbb{E}|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta} f(x) dx$$

converges almost surely to a nontrivial random variable (almost sure convergence with respect to the weak topology then follows from the separability of the space $C[0, 1]$). By the construction of the measure, this is a positive martingale, so it is enough to prove that it is bounded in L^p for some $p > 1$. For this, it is then enough to show that for the special case where we choose $f = 1$, the martingale is bounded in L^p for some $p > 1$. Choose $p \in (1, \tilde{p})$ and note that since the normalization factors are comparable, we obtain by Hölder’s inequality and Theorem 1.7,

$$\begin{aligned} & \mathbb{E} \left[\int_0^1 \frac{|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta}{\mathbb{E}|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta} dx \right]^p \\ & \leq C \mathbb{E}(\exp(p\beta \|\mathcal{E}_N\|_{L^\infty[0,1]}) (\lambda_{\beta,N}[0, 1])^p) \\ & \leq C (\mathbb{E} \exp(p(\tilde{p}/p)' \beta \|\mathcal{E}_N\|_{L^\infty[0,1]}))^{1/(\tilde{p}/p)'} (\mathbb{E}(\lambda_{\beta,N}[0, 1])^{\tilde{p}})^{p/\tilde{p}} \leq C', \end{aligned}$$

where $'$ denotes Hölder conjugation. This yields uniform integrability of the quantity $\int_0^1 |\zeta_{N,\text{rand}}(1/2 + ix)|^\beta / \mathbb{E}|\zeta_{N,\text{rand}}(1/2 + ix)|^\beta dx$ which proves the existence of a nontrivial limit.

For $\beta \geq \beta_c$, we see similarly using Theorem 6.1 and Theorem 1.7 that the limiting measure is zero since the corresponding Gaussian limiting measure is zero. The claim about the existence of all moments for the Radon–Nikodým derivative follows from Theorem 1.7. The existence of moments of order $p < 4/\beta^2$ again follows from a simple Hölder argument making use of Theorem 1.7 and the corresponding result for the Gaussian case (see Theorem 6.1). \square

7. The critical measure: Proof of Theorem 1.9. In this section, we establish the existence of the critical measure. We shall do this by showing that $G_N(x) = \tilde{G}_N(x) + D_N(x)$ (recall that G_N was the real part of the field \mathcal{G}_N), where D_N converges almost surely to a nice continuous Gaussian field and \tilde{G}_N is a sequence of Gaussian fields for which the critical measure can be shown to exist (using results from [50]). We will make use of the following result.

THEOREM 7.1 ([50], Theorem 1.1). *Let (X_N) and (\tilde{X}_N) be two sequences of Hölder regular Gaussian fields on $[0, 1]$ (i.e., the mapping $(x, y) \mapsto \sqrt{\mathbb{E}(X_N(x) - X_N(y))^2}$ is Hölder continuous on $[0, 1]^2$). Assume that the measure $A_N e^{\tilde{X}_N(x) - \frac{1}{2}\mathbb{E}\tilde{X}_N(x)^2} dx$ converges weakly (i.e., with respect to the topology of weak convergence of measures) in distribution to an almost surely nonatomic measure $\tilde{\lambda}$, where A_N is a deterministic scalar sequence. Assume further that the covariances $C_N(x, y) = \mathbb{E}X_N(x)X_N(y)$ and $\tilde{C}_N(x, y) = \mathbb{E}\tilde{X}_N(x)\tilde{X}_N(y)$ satisfy the following conditions: there exists a constant $K \in (0, \infty)$ (independent of N) such that for all $N \geq 1$,*

$$\sup_{x,y \in [0,1]} |C_N(x, y) - \tilde{C}_N(x, y)| \leq K$$

and for each $\delta > 0$,

$$\lim_{N \rightarrow \infty} \sup_{|x-y| > \delta} |C_N(x, y) - \tilde{C}_N(x, y)| = 0.$$

Then also $A_N e^{X_N(x) - \frac{1}{2}\mathbb{E}X_N(x)^2} dx$ converges weakly in distribution to $\tilde{\lambda}$.

To do this, we thus need a reference approximation for which convergence is known, and a representation of our field which gives us good control on the covariance. Let us first discuss the reference field. For this, we recall a construction from [5] and make use of results in [31].

DEFINITION 7.2. Let W denote a white noise on $\mathbb{R} \times [-1/2, 3/2]$. For $t \in \mathbb{R}$ and $x \in [0, 1]$, write

$$\tilde{G}_t(x) = \frac{1}{\sqrt{2}} \int_{-\infty}^t \int_{-1/2}^{3/2} \mathbf{1}\left\{|x - y| \leq \frac{1}{2} \min(e^{-s}, 1)\right\} e^{s/2} W(ds, dy).$$

The covariance of the field is

$$\mathbb{E}(\tilde{G}_t(x)\tilde{G}_t(y)) = \begin{cases} \frac{1}{2}[1 + t - e^t|x - y|], & |x - y| \leq e^{-t}, \\ -\frac{1}{2} \log|x - y|, & e^{-t} \leq |x - y| \leq 1. \end{cases}$$

Obviously, the above field is Hölder-regular as it is C^1 . As pointed out in [31], Remark 3, the main results of [31] apply to the measure $\sqrt{t} e^{2\tilde{G}_t(x) - 2\mathbb{E}\tilde{G}_t(x)^2} dx$ as well, whence it converges weakly in probability to a nontrivial, and nonatomic random measure, as $t \rightarrow \infty$.

Our next task is to approximate our field by one whose covariance we can control. Here, our starting point is Proposition 5.7, or rather its proof. It follows immediately from this that we can write

$$\mathcal{G}_N(x) = \tilde{G}_N(x) + \tilde{E}_N(x) = \int_1^{\log \text{Li}^{-1}(N+1)} e^{-2\pi i x u} u^{-1/2} dB_u^C + \tilde{E}_N(x)$$

for a nice Gaussian error field \tilde{E}_N . We now want to replace $1/\sqrt{u}$ by something that will allow us to reach the desired covariance in the limit. Let us consider the translation invariant covariance, already alluded to before, that is induced by the function $C(x) = \max(-\log|x|, 0)$. Then

$$\begin{aligned} \widehat{C}(k) &= \int_{-1}^1 e^{-2\pi i k x} \log \frac{1}{|x|} dx = 2 \int_0^1 \cos(2\pi k x) \log \frac{1}{x} dx \\ &= \frac{1}{\pi k} \int_0^{2\pi k} \cos y \log \frac{2\pi k}{y} dy = \frac{1}{\pi k} \int_0^{2\pi k} \frac{\sin y}{y} dy, \end{aligned}$$

where in the last step we integrated by parts. This is positive (as it should since it is the Fourier transform of a translation invariant covariance), and as $k \rightarrow \infty$, it behaves like $1/(2k) + \mathcal{O}(k^{-2})$. Thus it should be possible to replace $1/\sqrt{u}$ in our field by $\sqrt{2\widehat{C}(s)}$, and this actually works out.

LEMMA 7.3. *Let*

$$\mathcal{G}_{N,4}(x) = \int_1^{\log \text{Li}^{-1}(N+1)} \sqrt{2\widehat{C}(s)} e^{-2\pi i x s} dB_s^{\mathbb{C}}.$$

Then almost surely, $\tilde{\mathcal{G}}_N - \mathcal{G}_{N,4}$ converges uniformly to a random continuous function F_4 .

PROOF. In this case, making use of the same Sobolev estimate as before would lead to a nonsummable series, but we still can proceed by employing the following simple lemma.

LEMMA 7.4. *Let $g : \mathbb{R} \rightarrow \mathbb{C}$ be a bounded measurable function with compact support. Let us denote by*

$$F(x) := \int_{\mathbb{R}} e^{-2\pi i x u} g(u) dB_u^{\mathbb{C}}$$

the Fourier transform of the (almost surely well-defined) compactly supported distribution $g dB_u^{\mathbb{C}}$. Then for any $r > 1/2$ we have

$$\mathbb{E} \|F\|_{L^\infty[0,1]}^2 \lesssim \int_{\mathbb{R}} |g(\xi)|^2 (1 + \xi^2)^r d\xi.$$

PROOF. Let us first note that for, say smooth Schwartz test functions we obtain by Cauchy–Schwarz

$$\|f\|_{L^\infty(\mathbb{R})} \lesssim \|\widehat{f}\|_{L^1(\mathbb{R})} \lesssim \|\widehat{f}(\xi)(1 + \xi^2)^{r/2}\|_{L^2(\mathbb{R})}$$

since $\|(1 + (\cdot)^2)^{-r/2}\|_2 < \infty$ for $r > 1/2$ (actually this yields a proof of a special case of the Sobolev embedding theorem; see, e.g., [38], Theorem 6.2.4). In order to localize in the case where f is smooth but not compactly supported, we pick a real-valued and symmetric Schwartz test function ϕ with $\text{supp}(\phi) \subset [-1, 1]$. We demand further that $\mathcal{F}^{-1}\phi(x) \geq 1/2$ on $[0, 1]$. We then observe that by the previous inequality

$$(7.1) \quad \|f\|_{L^\infty[0,1]} \lesssim \|[\mathcal{F}^{-1}\phi]f\|_{L^\infty(\mathbb{R})} \lesssim \|(\phi * \widehat{f})(\xi)(1 + \xi^2)^{r/2}\|_{L^2(\mathbb{R})}.$$

Note next that for any $\xi \in \mathbb{R}$ we may compute using the symmetry of ϕ ,

$$\begin{aligned} \mathbb{E}|(\widehat{F} * \phi)(\xi)|^2 &= \mathbb{E}|(g dB_u^{\mathbb{C}}) * \phi(-\xi)|^2 \\ &= \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} g(u)\phi(\xi + u)\overline{g(u')\phi(\xi + u')} dB_u^{\mathbb{C}} d\overline{B_{u'}^{\mathbb{C}}} \\ &= \int_{\mathbb{R}} |g(u)|^2 \phi^2(\xi + u) du = (|g|^2 * \phi^2)(-\xi). \end{aligned}$$

By combining this with (7.1), it follows that

$$\mathbb{E} \|F\|_{L^\infty[0,1]}^2 \lesssim \int_{\mathbb{R}} (|g|^2 * \phi^2)(\xi) (1 + \xi^2)^r d\xi = \int_{\mathbb{R}} |g(\xi)|^2 [(1 + (\cdot)^2)^r * \phi^2](\xi) d\xi,$$

and the claim follows by noting that trivially $[(1 + (\cdot)^2)^r * \phi^2](\xi) \lesssim (1 + \xi^2)^r$. \square

In our case, if we define $F_{N,4} = \tilde{\mathcal{G}}_N - \mathcal{G}_{N,4}$, an application of the above lemma with the choice $r = 3/4$ results in the bound (for say $M \leq N$)

$$\mathbb{E} \|F_{N,4} - F_{M,4}\|_{L^\infty[0,1]}^2 \lesssim \int_{\log \text{Li}^{-1}(M+1)}^{\log \text{Li}^{-1}(N+1)} (1 + s^2)^{3/4} \left[\sqrt{\widehat{C}(s)} - \frac{1}{\sqrt{2s}} \right]^2 ds.$$

Note that

$$\left| \sqrt{\widehat{C}(s)} - \frac{1}{\sqrt{2s}} \right| = \frac{1}{\sqrt{2s}} \left| \sqrt{\frac{2}{\pi} \int_0^{2\pi s} \frac{\sin y}{y} dy} - 1 \right| \leq \frac{1}{\sqrt{2s}} \frac{2}{\pi} \int_{2\pi s}^\infty \frac{\sin y}{y} dy = \mathcal{O}(s^{-3/2}),$$

where we made use of the fact that $\frac{2}{\pi} \int_0^\infty \frac{\sin y}{y} dy = 1$ and the already mentioned asymptotic bound $\int_s^\infty \frac{\sin y}{y} dy = \mathcal{O}(s^{-1})$. It follows that

$$\mathbb{E} \|F_{N,4} - F_{M,4}\|_{L^\infty[0,1]}^2 \lesssim \int_{\log \text{Li}^{-1}(M+1)}^{\log \text{Li}^{-1}(N+1)} (1 + s^2)^{3/4} s^{-3} ds,$$

which is bounded in N and M , so we proceed as before. \square

To make use of Theorem 7.1 and compare $\text{Re } \mathcal{G}_{N,4}$ to $\tilde{\mathcal{G}}_t$, we should see how N and t are related. To do this, let us calculate the variance of $\text{Re } \mathcal{G}_{N,4}$ and require it to be $\frac{1}{2}t + \mathcal{O}(1)$. We have

$$\begin{aligned} \mathbb{E} \text{Re } \mathcal{G}_{N,4}(x)^2 &= \int_1^{\log \text{Li}^{-1}(N+1)} \widehat{C}(s) ds \\ &= \frac{1}{2} \int_1^{\log \text{Li}^{-1}(N+1)} \frac{1}{s} ds + \int_1^{\log \text{Li}^{-1}(N+1)} \mathcal{O}(s^{-2}) ds \\ &= \frac{1}{2} \log \log N + \mathcal{O}(1), \end{aligned}$$

where we used the expansion of $\widehat{C}(s)$ and the fact that $\log \log \text{Li}^{-1}(N + 1) = \log \log N + o(1)$ as one readily checks from the asymptotics $\text{Li}(x) = \frac{x}{\log x} (1 + o(1))$ as $x \rightarrow \infty$. Thus we should expect that $t = \log \log N$ should give a good estimate for the covariances. Indeed, for $|x - y| \leq 1/\log \text{Li}^{-1}(N + 1)$, we have

$$\begin{aligned} \mathbb{E} \text{Re } \mathcal{G}_{N,4}(x) \text{Re } \mathcal{G}_{N,4}(y) &= \frac{1}{2} \int_1^{\log \text{Li}^{-1}(N+1)} \frac{1}{s} \cos(s|x - y|) ds + \mathcal{O}(1) \\ &= \frac{1}{2} \int_{|x-y|}^{|x-y| \log \text{Li}^{-1}(N+1)} \frac{1}{s} \cos s ds + \mathcal{O}(1) \\ &= \frac{1}{2} \int_{|x-y|}^{|x-y| \log \text{Li}^{-1}(N+1)} \frac{1}{s} ds + \frac{1}{2} \int_{|x-y|}^{|x-y| \log \text{Li}^{-1}(N+1)} \frac{\cos s - 1}{s} ds + \mathcal{O}(1) \\ &= \frac{1}{2} \log \log N + \mathcal{O}(1), \end{aligned}$$

where the $\mathcal{O}(1)$ terms are uniform in x, y . For $|x - y| \geq 1/\log \text{Li}^{-1}(N + 1)$, elementary calculations show that

$$\mathbb{E} \text{Re} \mathcal{G}_{N,4}(x) \text{Re} \mathcal{G}_{N,4}(y) = \frac{1}{2} C(x - y) + \frac{1}{2} \int_{|x-y| \lceil \log \text{Li}^{-1}(N+1) \rceil}^{\infty} \frac{\cos s}{s} ds + o(1),$$

where the $o(1)$ term is uniform in x, y . From this, we see that if we write $C_N(x, y) = \mathbb{E} \text{Re} \mathcal{G}_{N,4}(x) \text{Re} \mathcal{G}_{N,4}(y)$ and $\tilde{C}_N(x, y) = \mathbb{E} \tilde{G}_t(x) \tilde{G}_t(y)$ with $t = \log \log N$, the conditions on the distances between the covariances in Theorem 7.1 are satisfied. Let us finally note that all our approximating fields are smooth, and in particular, they have Hölder covariances.

Before completing our proof of Theorem 1.9, we recall a further result we need from [50].

LEMMA 7.5 ([50], Lemma 4.2(ii)). *Let X be a Hölder regular Gaussian field on $[0, 1]$ and assume that it is independent of the sequence of measures (λ_n) on $[0, 1]$. If $e^X \lambda_n$ converges weakly in distribution, then λ_n does as well.*

Finally, we give the following.

PROOF OF THEOREM 1.9. Let us introduce some notation. For $M \geq 0$, let

$$\begin{aligned} \lambda_{\beta_c, M, N}(dx) &= \sqrt{\log \log N} e^{\beta_c(\text{Re} \mathcal{G}_{N,4}(x) - \text{Re} \mathcal{G}_{M,4}(x))} \\ &\quad \times e^{-\frac{\beta_c^2}{2} \mathbb{E}[\text{Re} \mathcal{G}_{N,4}(x)^2 - \text{Re} \mathcal{G}_{M,4}(x)^2]} dx, \end{aligned}$$

where $\mathcal{G}_{0,4} = 0$. From Theorem 7.1, we see that $\lambda_{\beta_c, 0, N}$ converges weakly in distribution (to a nontrivial random measure λ_{β_c}) as $N \rightarrow \infty$. Then from Lemma 7.5 we see that also $\lambda_{\beta_c, M, N}$ converges weakly in law for any fixed $M \geq 0$. We also note that

$$\sqrt{\log \log N} \frac{|\zeta_{N, \text{rand}}(1/2 + ix)|^{\beta_c}}{\mathbb{E}|\zeta_{N, \text{rand}}(1/2 + ix)|^{\beta_c}} dx = e^{f_N(x)} \lambda_{\beta_c, 0, N}(dx),$$

where f_N is a sequence of continuous functions converging uniformly almost surely to a continuous function f and, by construction, f_M is independent of $f_N - f_M$ for each $0 \leq M < N$. Recall that we want to show that for each nonnegative continuous $g : [0, 1] \rightarrow [0, \infty)$,

$$\sqrt{\log \log N} \int_0^1 \frac{|\zeta_{N, \text{rand}}(1/2 + ix)|^{\beta_c}}{\mathbb{E}|\zeta_{N, \text{rand}}(1/2 + ix)|^{\beta_c}} g(x) dx$$

converges in law to $\lambda_{\beta_c}(e^f g)$. Observe that for any $M \geq 1$,

$$\begin{aligned} &\sqrt{\log \log N} e^{f_M - f_N} \frac{|\zeta_{N, \text{rand}}(1/2 + ix)|^{\beta_c}}{\mathbb{E}|\zeta_{N, \text{rand}}(1/2 + ix)|^{\beta_c}} dx \\ &= (e^{f_M(x) + \beta_c \text{Re} \mathcal{G}_{M,4}(x) - \frac{\beta_c^2}{2} \mathbb{E} \text{Re} \mathcal{G}_{M,4}(x)^2}) \lambda_{\beta_c, M, N}(dx). \end{aligned}$$

On the right-hand side, the first factor is a random continuous function, independent of the measure $\lambda_{\beta_c, M, N}(dx)$, which in turn converges in distribution as $N \rightarrow \infty$. A simple argument that employs conditioning (i.e., Fubini) then shows that the full product on right-hand side converges in distribution, whence the same is true for the left-hand side. As $\sup_{N \geq M} \|f_M - f_N\|_{L^\infty[0,1]} \rightarrow 0$ in probability as $M \rightarrow \infty$, this concludes the proof of Theorem 1.9. \square

REMARK 7.6. We sketch here how a mesoscopic result can be shown for the statistical model that we are considering here. Observe first that by Lemma 6.2, we may choose a

sequence $\varepsilon_N \rightarrow 0^+$ (slower than $1/\log N$) and $\alpha_N \rightarrow \infty$ (in fact $\alpha_N = -\frac{1}{2} \log \varepsilon_N$) so that the covariance of $G_N(\varepsilon_N x)$ satisfies

$$K_{G_N(\varepsilon_N \cdot)}(x, y) = \frac{1}{2} \min(\log(1/|x - y|), \log[\varepsilon_N \log N]) + \alpha_N + \mathcal{O}(1),$$

and, uniformly outside the diagonal, one even has

$$K_{G_N(\varepsilon_N \cdot)}(x, y) = \frac{1}{2} \log(1/|x - y|) + \alpha_N + o(1).$$

On the other hand, we know that our error term \mathcal{E}_N converges uniformly to a bounded continuous function. Thus, in the scaling $x \rightarrow \varepsilon_N x$ we may write

$$\frac{|v_{N,\text{rand}}(\varepsilon_N x)|^\beta}{\mathbb{E}|v_{N,\text{rand}}(\varepsilon_N x)|^\beta} dx = e^{\beta\sqrt{\alpha_N}G_0 - \frac{\beta^2}{2}\alpha_N + R + o(1)} \tilde{\lambda}_{N,\beta}(dx),$$

where G_0 is a fixed standard normal random variable, independent of each $\tilde{\lambda}_{N,\beta}$, $R := \beta \operatorname{Re} \mathcal{E}(0) - C$ is a random variable, where C is a constant depending on β (it comes from the difference between the true value of $\mathbb{E}|v_N(x)|^\beta$ and the normalization of the Gaussian multiplicative chaos measure), and $\tilde{\lambda}_{N,\beta}$ is obtained by exponentiating a good approximation of a Gaussian field with the strictly logarithmic covariance structure $\frac{1}{2} \log(1/|x - y|)$ on $[0, 1]$. In particular, $\tilde{\lambda}_{N,\beta}$ converges to a standard Gaussian multiplicative chaos on $[0, 1]$. A similar statement holds also true in the case $\beta = \beta_c$.

APPENDIX A: PROOF OF THE GAUSSIAN APPROXIMATION RESULT

In this Appendix, we provide a self-contained proof of Proposition 3.4 for the reader’s convenience. We commence with by first establishing some general facts about coupling random variables; see Lemma A.1 below. We employ this result via the characteristic functions in the form of Corollary A.2. Moreover, in the actual proof of Proposition 3.4 we employ a useful trick by approximating instead of W (the variable we want to actually consider) the sum $\tilde{W} := W + G$, where G is an independent Gaussian perturbation. By letting G have small variance the error in our approximation does not increase too much, but the tail of the Fourier-transform of \tilde{W} gains some additional decay needed in the argument.

Some of the ingredients here are probably well known, but we have not found them from the literature. For our purposes, we especially need explicit (but perhaps nonoptimal) quantitative estimates.

We again will make use of the notion of the Wasserstein distance, though now the appropriate notion is the 1-Wasserstein distance. Assume that μ and ν are Borel probability distributions on a separable metric space (X, d) . Then

$$\mathcal{W}_1(\mu, \nu)_X := \inf_{(U, V)} \mathbb{E}d(U, V),$$

where U, V are random variables on a common probability space taking values in X so that $U \sim \mu$ and $V \sim \nu$. We start with a simple observation.

LEMMA A.1. *In the above situation, one has*

$$\mathcal{W}_1(\mu, \nu)_X \leq \inf_{R>0, x_0 \in X} \left(4R|\mu - \nu|(B(x_0; R)) + 32 \int_{R/2}^\infty |\mu - \nu|(B(x_0, r)^c) dr \right).$$

PROOF. Observe that

$$\beta := \mu - (\mu - \nu)_+ = \nu - (\nu - \mu)_+ \geq 0$$

and define the measure β_Δ on $X \times X$ by $\beta_\Delta(A) = \beta(\{x : (x, x) \in A\})$ and note that the measure

$$\beta_\Delta + \frac{2}{\|\mu - \nu\|_{\text{TV}}}(\mu - \nu)_+ \times (\nu - \mu)_+$$

has the right marginals since μ and ν are probability measures so $\|(\mu - \nu)_+\|_{\text{TV}} = \|(\nu - \mu)_+\|_{\text{TV}} = (1/2)\|\mu - \nu\|_{\text{TV}}$, and both of the marginals of β_Δ are simply β . As β_Δ lives on the diagonal, it follows that

$$\begin{aligned} \mathcal{W}_1(\mu, \nu)_X &\leq \frac{2}{\|\mu - \nu\|_{\text{TV}}} \int_{X \times X} d(x, y)(\mu - \nu)_+ \times (\nu - \mu)_+(dx \times dy) \\ (A.1) \qquad \qquad &\leq \frac{2}{\|\mu - \nu\|_{\text{TV}}} \int_{X \times X} d(x, y)|\mu - \nu| \times |\nu - \mu|(dx \times dy). \end{aligned}$$

Fix now some $x_0 \in X$ and $R > 0$ and split the integral into ones over $B(x_0, R) \times B(x_0, R)$ and its complement. Thus

$$\begin{aligned} &\frac{2}{\|\mu - \nu\|_{\text{TV}}} \int_{B(x_0, R) \times B(x_0, R)} d(x, y)|\mu - \nu| \times |\nu - \mu|(dx \times dy) \\ (A.2) \qquad \qquad &\leq 2 \cdot 2R(\|\mu - \nu\|_{\text{TV}})^{-1} |\mu - \nu| \times |\nu - \mu|(B(x_0, R) \times B(x_0, R)) \\ &\leq 4R|\nu - \mu|(B(x_0, R)). \end{aligned}$$

By symmetry, the integral over the rest has the upper bound

$$\begin{aligned} &\frac{4}{\|\mu - \nu\|_{\text{TV}}} \int_{d(x, x_0) \geq d(y, x_0) \vee R} d(x, y)|\mu - \nu| \times |\nu - \mu|(dx \times dy) \\ &\leq \frac{8}{\|\mu - \nu\|_{\text{TV}}} \int_{d(x, x_0) \geq R} d(x, x_0)|\mu - \nu| \times |\nu - \mu|(dx \times dy) \\ (A.3) \qquad \qquad &\leq 8 \int_{d(x, x_0) \geq R} d(x, x_0)|\mu - \nu|(dx) \\ &\leq 8 \sum_{k=1}^\infty 2kR(|\mu - \nu|(B(x_0, kR)^c) - |\mu - \nu|(B(x_0, (k+1)R)^c)) \\ &\leq 16R \sum_{k=1}^\infty |\mu - \nu|(B(x_0, kR)^c) \leq 32 \int_{R/2}^\infty |\mu - \nu|(B(x_0, r)^c) dr. \end{aligned}$$

The claim follows by combining the estimates (A.1)–(A.3). \square

We denote by $\widehat{\mu}$ the Fourier transform of the measure μ on \mathbb{R}^d (i.e., up to a scaling by -2π , the characteristic function of a random variable with distribution μ).

COROLLARY A.2. Assume that μ and ν are absolutely continuous measures on \mathbb{R}^d . Then

$$\mathcal{W}_1(\mu, \nu)_X \leq \inf_{R \geq 1} C_d \left(R^{d+1} \|\widehat{\mu} - \widehat{\nu}\|_{L^1(\mathbb{R}^d)} + \int_{R/2}^\infty (\mu + \nu)(B(0, r)^c) dr \right).$$

PROOF. Let f (resp., g) stand for the density of μ (resp., ν). The desired statement follows from the previous lemma as soon as we observe that

$$\int_{B(0,R)} |f(x) - g(x)| dx \leq C_d R^d \|f - g\|_{L^\infty(\mathbb{R}^d)} \leq C_d'' R^d \|\widehat{f} - \widehat{g}\|_{L^1(\mathbb{R}^d)}. \quad \square$$

We are ready for the following.

PROOF OF PROPOSITION 3.4. All the unspecified constants (and the $\mathcal{O}(\cdot)$ terms) in the proof are universal in the sense that they may depend only on the quantities d, b_0, b_1, b_2, b_3 . We let $C_j = \text{Cov}(H_j)$ stand for the covariance matrix of the variable H_j . Denote $\ell_n := (\sum_{j=1}^n c_j)^{1/2}$ and observe that

$$(A.4) \quad b_0^{-1/2} n^{1/2} \leq \ell_n \leq b_0^{1/2} n^{1/2}.$$

Moreover, set

$$W := \ell_n^{-1} \sum_{j=1}^n H_j,$$

so that $\text{Tr}(\text{Cov}(W)) = d$. By considering instead the random variables RH_j where $R : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a rotation matrix chosen so that $R \text{Cov}(W) R^T$ is diagonal, we may assume that $A := \text{Cov}(W)$ is diagonal:

$$A = \text{Cov}(W) = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & \lambda_d \end{bmatrix},$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d \geq 0$ and $\sum_{k=1}^d \lambda_k = d$.

We start by proving an estimate of type (3.9) by first assuming that the smallest eigenvalue of A satisfies the lower bound $\lambda_d \geq n^{-2\delta}$, where the constant $\delta \in [0, 1/6)$ will be chosen later on. Toward this goal, we note that the exponential moment bound (3.8) for H_k 's implies that $\|D^m \varphi_{H_j}\|_{L^\infty(\mathbb{R}^d)} \leq C$ for $m = 1, 2, 3$ and all $j = 1, \dots, n$, where φ_{H_j} stands for the characteristic function of the variable H_j . Also, we have $D^2 \varphi_{H_j}(0) = -\text{Cov}(H_j)$, whence

$$\varphi_{H_j}(\xi) = 1 - \frac{1}{2} \xi^T \text{Cov}(H_j) \xi + \mathcal{O}(|\xi|^3) \quad \text{for all } \xi.$$

We emphasize that here (and in what follows), the estimates are uniform in j as well. Hence for the branch of the logarithm that takes value 0 at the point 1 we have for a universal $r_1 > 0$,

$$\log \varphi_{H_j}(\xi) = -\frac{1}{2} \xi^T \text{Cov}(H_j) \xi + \mathcal{O}(|\xi|^3) \quad \text{for } |\xi| \leq 2r_1.$$

By independence (and since $b_0^{-1} \leq c_j \leq b_0$ for all j), we gather that for another universal $r_2 > 0$

$$(A.5) \quad \log(\varphi_W(\xi)) = \sum_{j=1}^n \log(\varphi_{H_j}(\xi/\ell_n)) = -\frac{1}{2} \xi^T A \xi + n^{-1/2} \mathcal{O}(|\xi|^3)$$

for $|\xi| \leq r_2 \sqrt{n}$. We note that $\lambda_1 \geq 1$ and each $\lambda_j \geq n^{-2\delta}$. Hence, as $|\xi|^3 \leq C(d) \sum_{k=1}^d |\xi_k|^3$, we may estimate componentwise and deduce (by also decreasing r_2 universally, if needed)

$$(A.6) \quad |\varphi_W(\xi)| \leq \exp\left(-\frac{1}{4} \xi^T \tilde{A} \xi\right) \quad \text{for } |\tilde{A}^{-1} \xi| \leq r_2 \sqrt{n},$$

where \tilde{A} is the $d \times d$ diagonal matrix

$$\tilde{A} := \text{diag}(1, n^{-2\delta}, \dots, n^{-2\delta}) \leq A.$$

We next choose a d -dimensional centred Gaussian G (independent of the H_j 's) such that

$$\begin{aligned} (A.7) \quad B &:= \text{Cov}(G) = r_2^{-2} \log^2(n) \text{diag}(n^{-1}, n^{4\delta-1}, \dots, n^{4\delta-1}) \\ &= (r_2^{-1} \log(n) n^{-1/2} \tilde{A}^{-1})^2 \end{aligned}$$

and set $\tilde{W} := G + W$. Then $\varphi_{\tilde{W}}(\xi) = \varphi_W(\xi) \exp(-\frac{1}{2}\xi^T B \xi)$ and we estimate

$$\begin{aligned} (A.8) \quad &\left\| \exp\left(-\frac{1}{2}\xi^T A \xi\right) - \varphi_{\tilde{W}}(\xi) \right\|_{L^1(\mathbb{R}^d)} \\ &= \left(\int_{|\tilde{A}^{1/2}\xi| \leq \log n} + \int_{\substack{|\tilde{A}^{1/2}\xi| > \log n \\ |B^{1/2}\xi| \leq \log n}} + \int_{|B^{1/2}\xi| > \log n} \right) \\ &\quad \times \left| \exp\left(-\frac{1}{2}\xi^T A \xi\right) - \varphi_{\tilde{W}}(\xi) \right| d\xi \\ &= T_1 + T_2 + T_3. \end{aligned}$$

We make use of the following simple observation for $d \times d$ symmetric matrices J :

$$(A.9) \quad \text{If } J \geq n^{-\alpha} I \text{ where } \alpha > 0 \text{ then } \int_{|J^{1/2}\xi| \geq \log n} e^{-\frac{1}{4}\xi^T J \xi} d\xi = \mathcal{O}(n^{-1/2}).$$

This is verified by computing

$$\begin{aligned} \int_{|J^{1/2}\xi| \geq \log n} e^{-\frac{1}{4}\xi^T J \xi} d\xi &= |J|^{-1/2} \int_{|\xi| \geq \log(n)} e^{-|\xi|^2/4} d\xi \\ &\lesssim n^{d\alpha/2} \int_{r \geq \log(n)} e^{-r^2/4} r^{d-1} dr \\ &\lesssim n^{d\alpha/2} \int_{r \geq \log(n)} e^{-r^2/8} dr = \mathcal{O}(n^{d\alpha/2} e^{-\frac{1}{8} \log^2(n)}) \\ &= \mathcal{O}(n^{-1/2}) \end{aligned}$$

Toward estimating the first term T_1 we note that since $\delta < 1/6$, we have

$$\sup_{\{|\tilde{A}^{1/2}\xi| \leq \log n\}} n^{-1/2} |\xi|^3 = o(1) \text{ as } n \rightarrow \infty.$$

Hence we may apply (A.5), the ordering $A \geq \tilde{A}$ and the inequality $|e^x - 1| \leq 2|x|$ for $x \in (-\infty, 1]$ to obtain the bound

$$\begin{aligned} (A.10) \quad T_1 &\leq \int_{|\tilde{A}^{1/2}\xi| \leq \log n} e^{-\frac{1}{2}\xi^T A \xi} \left| \exp\left(-\frac{1}{2}\xi^T B \xi + n^{-1/2} \mathcal{O}(|\xi|^3)\right) - 1 \right| d\xi \\ &\leq 2 \int_{|\tilde{A}^{1/2}\xi| \leq \log n} e^{-\frac{1}{2}\xi^T \tilde{A} \xi} \left(\frac{1}{2} \xi^T B \xi + n^{-1/2} \mathcal{O}(|\xi|^3) \right) d\xi \\ &\leq 2 |\tilde{A}^{-1/2}| \int_{\mathbb{R}^d} e^{-|\xi|^2/2} (\|\tilde{A}^{-1/2} B \tilde{A}^{-1/2}\| |\xi|^2 + \|\tilde{A}^{-1/2}\|^3 n^{-1/2} \mathcal{O}(|\xi|^3)) d\xi \\ &\lesssim n^{(d-1)\delta} (n^\delta \log^2(n) n^{-1+4\delta} n^\delta + n^{-1/2} n^{3\delta}) = \mathcal{O}(n^{-1/2+(d+2)\delta}), \end{aligned}$$

since $\delta < 1/6$. Next, by the last equality in (A.7), the condition $|B^{1/2}\xi| \leq \log(n)$ is equivalent to $|\tilde{A}^{-1}\xi| \leq r_2 n^{1/2}$. Then (A.6) and the estimate (A.9) yield

$$(A.11) \quad T_2 \leq \int_{|\tilde{A}^{1/2}\xi| > \log n} (e^{-\frac{1}{4}\xi^T \tilde{A}\xi} + e^{-\frac{1}{2}\xi^T A\xi}) d\xi \lesssim \int_{|\tilde{A}^{1/2}\xi| > \log n} e^{-\frac{1}{4}\xi^T \tilde{A}\xi} d\xi = \mathcal{O}(n^{-1/2}).$$

Finally, for the remaining term T_3 we can again invoke (A.9) to obtain

$$(A.12) \quad T_3 \leq \int_{\substack{|\tilde{A}^{1/2}\xi| > \log n \\ |B^{1/2}\xi| > \log n}} \left(\exp\left(-\frac{1}{2}\xi^T A\xi\right) + \exp\left(-\frac{1}{2}\xi^T B\xi\right) \right) d\xi = \mathcal{O}(n^{-1/2})$$

Combining the estimates (A.10)–(A.12) with (A.8), we arrive at the estimate

$$(A.13) \quad \|e^{-|\xi|^2/2} - \varphi_{\tilde{W}}(\xi)\|_{L^1(\mathbb{R}^d)} = \mathcal{O}(n^{-1/2+(d+2)\delta}).$$

We shall next apply Bernstein’s inequality on W , which was defined as the sum of independent random variables. Actually, one may use Hölder’s inequality to reduce the d -dimensional case to the one-dimensional one, whence we may invoke the Bernstein type tails estimates in [11], Theorem 2.1, to easily infer that there are universal constants n_0, b'_4, b'_5 (depending only on dimension) such that for $n \geq n_0$ it holds that

$$(A.14) \quad \mathbb{E} \exp(\lambda|W|) \leq b'_4 \exp(b'_4 \lambda^2) \quad \text{and for all } \lambda \leq b'_5 n^{1/2}.$$

Choosing, for example, $\lambda = 3$ here and combining with the excellent Gaussian tail (better than $\lesssim e^{-|\xi|^2/4}$) for G , we see that $\mathbb{P}(|\tilde{W}| > \lambda) < b''_5 \exp(-2\lambda)$. This yields for $R \geq 1$ the estimate

$$(A.15) \quad \int_{R/2}^\infty \mathbb{P}(|\tilde{W}| \geq r) dr = \mathcal{O}(e^{-R}).$$

We are now ready to invoke Corollary A.2 in combination with (A.13) and (A.15) in order to deduce the existence of a Gaussian random variable U such that

$$\mathbb{E}|U - \tilde{W}| \lesssim \inf_{R \geq 1} (R^{d+1} n^{-1/2+(d+2)\delta} + e^{-R}) \lesssim \log^{d+1}(n) n^{-1/2+(d+2)\delta}.$$

This yields our basic estimate

$$(A.16) \quad \begin{aligned} \mathbb{E}|V| &= \mathbb{E}|U - W| \leq \mathbb{E}|U - \tilde{W}| + \mathbb{E}|G| \\ &\lesssim \log^{d+1}(n) n^{-1/2+(d+2)\delta} + \log(n) n^{-1+4\delta} \\ &= \mathcal{O}(\log^{d+1}(n) n^{-1/2+(d+2)\delta}). \end{aligned}$$

We next explain how to infer from (A.16) the inequality (3.9) or (3.11). For part (ii) of the proposition (which also covers the case $d = 1$), we may choose $\delta = 0$ in (A.16) and obtain directly (3.11). In order to deal with part (i) of the proposition (where $d \geq 2$), we assume first that $\lambda_d \geq n^{-(4d+12)^{-1}}$. Then we may apply directly (A.16) with the choice $\delta = (2d + 6)^{-1}$ and obtain the inequality (3.9) with the exponent

$$\beta_0 = 1/2 - (d + 2)(2d + 5)^{-1} > 0,$$

that depends only on d . In the remaining case there is $k_0 \in \{2, \dots, d - 1\}$ so that $\lambda_{k_0} \geq n^{-(4d+12)^{-1}}$ but $\lambda_{k_0+1} < n^{-(4d+12)^{-1}}$. Write $W' := (W_1, \dots, W_{k_0})$ and $W'' := (W_{k_0+1}, \dots, W_d)$. We may apply the above proof on W' and find a k_0 -dimensional Gaussian approximation U' for W' so that $\mathbb{E}|U' - W'| = \mathcal{O}(n^{-\beta_0})$ (some unimportant adjustments of

the multiplicative constants are needed to get the k_0 -dimensional normalization right, and we actually could get a better decay). We then define the trivial extension U' to a d -dimensional random variable U by setting $U = (U', U'')$, where the components of U'' are identically zero. Now

$$\mathbb{E}|W''| \leq (\mathbb{E}|W''|^2)^{1/2} = \left(\sum_{k=k_0+1}^d \lambda_k \right)^{1/2} \lesssim n^{-(2d+6)^{-1}} \lesssim n^{-\beta_0}.$$

Finally,

$$\mathbb{E}|V| \leq \mathbb{E}|W' - U'| + \mathbb{E}|W''| \lesssim n^{-\beta_0}.$$

This proves the desired estimate (3.9).

We turn to estimating the exponential moments. Their proof is based solely on (3.9) and the uniform exponential estimates of V which follow from those of W and from the automatic ones for the Gaussian approximation. By invoking the Bernstein estimate (A.14) and using the double exponential decay (with constants depending on d) of our Gaussian approximation U we obtain universal (i.e., depending only on d) constants b_4, b_5 such that

$$(A.17) \quad \mathbb{P}(|V| \geq u) \leq b_4 e^{-2\lambda u} e^{4b_4 \lambda^2} \quad \text{for any } u > 0 \quad \text{and } \lambda \in (0, b_5 \sqrt{n}).$$

We denote $\alpha := a_1 n^{-\beta}$ and assume that $\lambda \in (0, b_5 \sqrt{n})$. Letting $M \geq 0$ stand for an auxiliary parameter, to be chosen later, we may write

$$\begin{aligned} \mathbb{E}e^{\lambda|V|} &= 1 + \mathbb{E}\left(|V| \frac{\exp(\lambda|V|) - 1}{|V|} \chi_{\{|V| \leq M\}}\right) \\ &\quad + (e^{\lambda M} - 1)\mathbb{P}(|V| > M) + \lambda \int_M^\infty e^{\lambda u} \mathbb{P}(|V| > u) du. \end{aligned}$$

We recall that by (3.9) we have $\mathbb{E}|V| \leq \alpha$. Also, we note that $t \mapsto t^{-1}(e^{\lambda t} - 1)$ (defined to be λ at zero) is increasing on $[0, M]$, and hence less than $M^{-1}(e^{\lambda M} - 1)$ on that interval. Using these facts and the bound (A.17), we deduce that

$$\begin{aligned} \mathbb{E} \exp(\lambda|V|) - 1 &\leq \alpha(e^{\lambda M} - 1)M^{-1} + (e^{\lambda M} - 1)b_4 e^{4b_4 \lambda^2 - 2\lambda M} \\ &\quad + b_4 e^{4b_4 \lambda^2} \lambda \int_M^\infty e^{-\lambda u} du \\ &\leq \alpha e^{\lambda M} M^{-1} + 2b_4 e^{-\lambda M} e^{4b_4 \lambda^2}. \end{aligned}$$

The desired estimate is obtained by choosing M so that $\sqrt{\alpha} = e^{-\lambda M}$ and plugging in the definition of α .

Assume then that the variables H_k are uniformly bounded. In this case, a standard application of Azuma’s inequality yields universal constants $s, r > 0$ so that

$$\mathbb{P}(|V| \geq u) \leq s e^{-2ru^2} \quad \text{for all } u > 0.$$

In an analogous manner to what we just did for the exponential moments, for any $M > 0$ it follows that

$$\begin{aligned} \mathbb{E}e^{rV^2} &= 1 + \mathbb{E}\left(|V| \frac{e^{rV^2} - 1}{|V|} \chi_{\{|V| \leq M\}}\right) + (e^{rM^2} - 1)\mathbb{P}(|V| > M) \\ &\quad + 2r \int_M^\infty x e^{rx^2} \mathbb{P}(|V| > x) dx \end{aligned}$$

and we deduce that

$$\begin{aligned} \mathbb{E} \exp(r|V|^2) &\leq 1 + \alpha(e^{rM^2} - 1)M^{-1} + s(e^{rM^2} - 1)e^{-2rM^2} + s \int_M^\infty 2rx e^{-rx^2} dx \\ &\leq 1 + \alpha M^{-1} e^{rM^2} + 2s e^{-rM^2}. \end{aligned}$$

The desired estimate is obtained by this time choosing M so that $\sqrt{\alpha} = e^{-rM^2}$. \square

APPENDIX B: GAUSSIAN MULTIPLICATIVE CHAOS AND RANDOM UNITARY MATRICES—THE GLOBAL SCALE

In this Appendix, we prove that on the global scale, the characteristic polynomial of a Haar distributed random unitary matrix converges to a multiplicative chaos distribution. The proof is very similar to the zeta case, though naturally the moment calculations are different. For these, we use known results from [26, 28].

To fix notation, let U_N be a $N \times N$ random unitary matrix whose law is the Haar measure on the unitary group. Our main object of interest is the characteristic polynomial of U_N and we consider it evaluated on the unit circle. For $\theta \in [0, 2\pi]$, we define

$$v_N(\theta) := \det(I - e^{-i\theta} U_N).$$

We introduce an approximation, which is analogous to the truncated Euler product we consider above. More precisely, for $M \in \mathbb{Z}_+$, we define

$$v_{N,M}(\theta) = e^{-\sum_{k=1}^M \frac{1}{k} \text{Tr} U_N^k e^{-ik\theta}}.$$

Our goal is to again control the difference $v_N - v_{N,M}$ in the limit where $N \rightarrow \infty$ and then $M \rightarrow \infty$ while proving that in this limit, $v_{N,M}$ converges to a certain multiplicative chaos distribution. A suitable space of generalized functions to carry out this analysis in is provided by the Sobolev space on the circle:

$$\mathcal{H}^\alpha = \left\{ f(\theta) = \sum_{j \in \mathbb{Z}} \widehat{f}_j e^{ij\theta} : \|f\|_\alpha^2 := \sum_{j \in \mathbb{Z}} (1 + j^2)^\alpha |\widehat{f}_j|^2 < \infty \right\}.$$

Let us begin with an analogue of Proposition 2.13.

LEMMA B.1. *Let $\alpha > 1/2$. Then*

$$\lim_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E} \|v_N - v_{N,M}\|_{-\alpha}^2 = 0.$$

PROOF. As in the zeta case, we expand the square and study the asymptotics of each term. Consider first the $\|v_N\|_{-\alpha}^2$ -term. We have

$$(B.1) \quad \mathbb{E} \|v_N\|_{-\alpha}^2 = \sum_{j \in \mathbb{Z}} \int_{[0, 2\pi]^2} \frac{e^{ij(\theta - \theta')}}{(1 + j^2)^\alpha} \mathbb{E} [\det(I - e^{-i\theta} U_N) \det(I - e^{i\theta'} U_N^*)] \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi},$$

where U_N^* denotes Hermitian conjugation. One can justify changing the order of the different integrals by the facts that for each fixed N , the absolute value of the determinant is bounded by 2^N (deterministically) and as $\alpha > 1/2$ the sum $\sum_{j \in \mathbb{Z}} (1 + j^2)^{-\alpha}$ converges.

We thus need to understand the expectation above. The expression that is relevant to us follows from, for example, [26], equation (2.16), (in the notation of [26] we have $\mathbb{E} \det(I -$

$$e^{-i\theta} U_N) \det(I - e^{i\theta'} U_N^*) = e^{-iN\theta} I_{1,2}(U(N); e^{i\theta'}; e^{i\theta}):$$

$$\begin{aligned} \mathbb{E}[\det(I - e^{-i\theta} U_N) \det(I - e^{i\theta'} U_N^*)] &= \frac{1}{1 - e^{i(\theta' - \theta)}} + \frac{e^{iN(\theta' - \theta)}}{1 - e^{i(\theta - \theta')}} \\ &= \sum_{l=0}^N e^{-il(\theta - \theta')}. \end{aligned}$$

Plugging this into (B.1), we see that

$$(B.2) \quad \mathbb{E} \|u_N\|_{-\alpha}^2 = \sum_{l=0}^N \frac{1}{(1 + l^2)^\alpha} \rightarrow \sum_{l=0}^\infty \frac{1}{(1 + l^2)^\alpha} < \infty$$

as $N \rightarrow \infty$.

For the terms with $v_{N,M}$, we will need results from [28] which concern the asymptotics of Toeplitz determinants with Fisher–Hartwig singularities. For the cross term, we note that instead of (B.1), one now is interested in expectations of the form $\mathbb{E} \det(I - e^{-i\theta} U_N) e^{-\sum_{k=1}^M \frac{1}{k} e^{ik\theta'} \text{Tr}[U_N^*]^k}$ (again with an elementary argument one can justify changing the order of sums and integrals in calculating the second moment of the Sobolev norm). Using the rotation invariance of the law of U_N , we can write this expectation as $\mathbb{E} \prod_{j=1}^N f(e^{i\varphi_j})$, where $(e^{i\varphi_j})_{j=1}^N$ are the eigenvalues of U_N , and $f(z) = e^{-\sum_{k=1}^M \frac{1}{k} e^{ik(\theta' - \theta)} z^{-k}} (1 - z)$. Using elementary geometry, one can check that f can be written in the form

$$(B.3) \quad f(z) = -i e^{V(z; \theta - \theta')} z^{1/2} |z - 1|,$$

where

$$V(z; \theta - \theta') = \sum_{k \in \mathbb{Z}} V_k z^k = - \sum_{k=1}^M \frac{1}{k} e^{ik(\theta' - \theta)} z^{-k}$$

and the interpretation of $z^{1/2}$ is the following: for $\arg z \in [0, 2\pi)$, $z^{1/2} = e^{\frac{1}{2}i \arg z}$.

Due to the Heine–Szegő identity, the expectation $\mathbb{E} \prod_{j=1}^N f(e^{i\varphi_j})$ can be written as the determinant of a $N \times N$ Toeplitz matrix with symbol f . The study of the asymptotics of such determinants with symbols of the form (B.3), namely symbols with Fisher–Hartwig singularities, has a long history and the asymptotics are well known. We refer to [34], Theorem 6.1, and [28], Theorem 1.1, for a more general result and a proof of the following fact: for f of the form (B.3),

$$(B.4) \quad \mathbb{E} \prod_{j=1}^N f(e^{i\varphi_j}) = e^{-\sum_{k=-\infty}^{-1} V_k} (1 + o(1)) = e^{\sum_{k=1}^M \frac{1}{k} e^{ik(\theta' - \theta)}} (1 + o(1))$$

as $N \rightarrow \infty$ for fixed M . To help the interested reader compare with, for example, [28], Theorem 1.1, we point out that in their notation, our case becomes $m = 0$, $\alpha_0 = \beta_0 = 1/2$, $V_k = -\frac{1}{|k|} e^{-ik(\theta' - \theta)}$ for $k \in \{-M, \dots, -1\}$ and zero otherwise. Thus the only nontrivial quantity in [28], equation (1.10), is $b_-(1)^{-\alpha_0 - \beta_0}$ – every other term in the product is one. We also point out that similarly to the discussion in [28], Remark 1.4, one can check that these asymptotics are uniform in θ, θ' . More precisely, one can check in general that if V is a Laurent polynomial of fixed degree and if we require the coefficients of V to be restricted to some fixed compact set, then the asymptotics of [28], Theorem 1.1, are uniform in the coefficients. This can be deduced from corresponding uniform bounds on the relevant jump matrices.

Due to the uniformity of (B.4) and our assumption that $\alpha > 1/2$, we thus have (writing $\langle \cdot, \cdot \rangle_{-\alpha}$ for the inner product of $\mathcal{H}^{-\alpha}$ and using the convention that it is conjugate linear in the second entry)

$$\begin{aligned}
 \mathbb{E}\langle v_N, v_{N,M} \rangle_{-\alpha} &= \sum_{j \in \mathbb{Z}} \frac{1}{(1+j^2)^\alpha} \int_{[0,2\pi]^2} e^{-ij(\theta-\theta')} e^{\sum_{k=1}^M \frac{1}{k} e^{ik(\theta'-\theta)}} \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} + o(1) \\
 \text{(B.5)} \qquad \qquad \qquad &= \sum_{j \in \mathbb{Z}} \frac{1}{(1+j^2)^\alpha} \int_0^{2\pi} e^{-ij\theta} e^{\sum_{k=1}^M \frac{1}{k} e^{-ik\theta}} \frac{d\theta}{2\pi} + o(1)
 \end{aligned}$$

in the limit where $N \rightarrow \infty$ and M is fixed. Now by expanding the exponential and applying the dominated convergence theorem one can check that the relevant asymptotics become

$$\mathbb{E}\langle v_N, v_{N,M} \rangle_{-\alpha} = \sum_{j=0}^{\infty} \frac{1}{(1+j^2)^\alpha} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{k_1, \dots, k_l=1}^M \frac{1}{k_1 \dots k_l} \delta_{j, k_1 + \dots + k_l} + o(1),$$

where the interpretation of the $l = 0$ term in the sum is $\delta_{j,0}$. Now by monotonicity/positivity, we see that

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}\langle v_N, v_{N,M} \rangle_{-\alpha} = \sum_{j=0}^{\infty} \frac{1}{(1+j^2)^\alpha} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{k_1, \dots, k_l=1}^{\infty} \frac{1}{k_1 \dots k_l} \delta_{j, k_1 + \dots + k_l}.$$

Invoking Lemma C.2 (which we prove shortly in the next Appendix), we conclude that

$$\text{(B.6)} \qquad \qquad \qquad \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}\langle v_N, v_{N,M} \rangle_{-\alpha} = \sum_{j=0}^{\infty} \frac{1}{(1+j^2)^\alpha}.$$

Finally, we need to understand $\mathbb{E}\|v_{N,M}\|_{-\alpha}^2$. This can again be reduced to understanding expectations of the form $\mathbb{E} \prod_{j=1}^N e^{\mathcal{V}(e^{i\psi_j}; e^{i\theta}, e^{i\theta'})}$, where

$$\mathcal{V}(z; e^{i\theta}; e^{i\theta'}) = - \sum_{k=1}^M \frac{1}{k} (e^{-ik\theta} z^k + e^{ik\theta'} z^{-k}) = \sum_{k \in \mathbb{Z}} \mathcal{V}_k z^k.$$

This expectation is again a Toeplitz determinant with symbol $e^{\mathcal{V}}$ and its asymptotics follow from the strong Szegő theorem. As we need uniformity in θ, θ' since we want to interchange the order of the limit and the integrals, we need something slightly stronger than the standard strong Szegő theorem. This corresponds to a simplified version of the problem studied in [28] where there are no singularities (in their notation $\alpha_k = \beta_k = 0$ for all k) and again in the Riemann–Hilbert problem the jump matrices will have bounds that are uniform in θ, θ' and one finds

$$\mathbb{E} \prod_{j=1}^N e^{\mathcal{V}(e^{i\psi_j}; e^{i\theta}, e^{i\theta'})} = e^{N\mathcal{V}_0 + \sum_{j=1}^{\infty} j\mathcal{V}_j \mathcal{V}_{-j}} (1 + o(1)) = e^{\sum_{k=1}^M \frac{1}{k} e^{-ik(\theta-\theta')}} (1 + o(1)),$$

where the error is again uniform in θ, θ' as $N \rightarrow \infty$ with fixed M . We thus find as before that as $N \rightarrow \infty$ for a fixed M ,

$$\mathbb{E}\|v_{N,M}\|_{-\alpha}^2 = \sum_{j \in \mathbb{Z}} \frac{1}{(1+j^2)^\alpha} \int_{[0,2\pi]^2} e^{-ij(\theta-\theta')} e^{\sum_{k=1}^M \frac{1}{k} e^{-ik(\theta-\theta')}} \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} + o(1),$$

which is precisely the same as what we found in (B.5). Thus

$$\text{(B.7)} \qquad \qquad \qquad \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}\|v_{N,M}\|_{-\alpha}^2 = \sum_{j=0}^{\infty} \frac{1}{(1+j^2)^\alpha}.$$

Putting together (B.2), (B.6) and (B.7), we infer that

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E} \|v_N - v_{N,M}\|_{-\alpha}^2 = 0. \quad \square$$

The next step in proving that v_N converges in $\mathcal{H}^{-\alpha}$ is to show that if we first let $N \rightarrow \infty$ and then $M \rightarrow \infty$, then $v_{N,M}$ converges in $\mathcal{H}^{-\alpha}$. Let us prove this now.

LEMMA B.2. *If we first let $N \rightarrow \infty$ and then $M \rightarrow \infty$, then $v_{N,M}$ converges in law with respect to the topology of $\mathcal{H}^{-\alpha}$ to an object v which can be formally written as $v = e^{H(e^{i\theta})}$, where*

$$H(e^{i\theta}) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} Z_k e^{-ik\theta}.$$

Here, $(Z_k)_{k=1}^{\infty}$ are i.i.d. standard complex Gaussians.

PROOF. Our main tool is a result due to Diaconis and Shahshahani [29], who proved that for any fixed M ,

$$(B.8) \quad \left(\frac{1}{\sqrt{k}} \text{Tr } U_N^k \right)_{k=1}^M \xrightarrow{d} (Z_k)_{k=1}^M \quad \text{as } N \rightarrow \infty.$$

We will transform this into a statement about the convergence of $v_{N,M}$ as $N \rightarrow \infty$. Lemma 2.14 does not apply directly, so we take a slightly different approach here. Consider the function $F : \mathbb{C}^M \rightarrow \mathcal{H}^{-\alpha}$ with $\alpha > 1/2$ defined by

$$[F(z_1, \dots, z_M)](e^{i\theta}) = e^{-\sum_{j=1}^M \frac{1}{\sqrt{j}} e^{-ij\theta} z_j}.$$

Let us write this as

$$[F(z_1, \dots, z_M)](e^{i\theta}) = \sum_{m=0}^{\infty} F_{-m}(z_1, \dots, z_M) e^{-im\theta},$$

whence the definition of Fourier coefficients gives the rough estimate

$$|F_{-m}(z_1, \dots, z_M)| \leq e^{\|z\|_{\infty} \sum_{k=1}^M \frac{1}{\sqrt{k}}}.$$

This yields a uniform bound for the Fourier coefficients in each ball $\{\|z\|_{\infty} \leq R\}$. Hence $F(z) \in \mathcal{H}^{-\alpha}$ for $\alpha > 1/2$, and since the F_{-j} 's are polynomials in z_1, \dots, z_M we deduce that, in fact, $F : \mathbb{C}^M \rightarrow \mathcal{H}^{-\alpha}$ is a continuous mapping. At this stage, (B.8) implies directly by definition of convergence in distribution that

$$v_{N,M} \xrightarrow{d} F(Z_1, \dots, Z_M) \quad \text{as } N \rightarrow \infty$$

in the topology of $\mathcal{H}^{-\alpha}$.

Now if we realize the i.i.d. Gaussians Z_j on the same probability space, then the random variables $(F(Z_1, \dots, Z_M))_M$ will form a $\mathcal{H}^{-\alpha}$ -valued martingale and one has

$$\mathbb{E} \|F(Z_1, \dots, Z_M)\|_{-\alpha}^2 = \sum_{j \in \mathbb{Z}} \frac{1}{(1 + j^2)^{\alpha}} \int_{[0, 2\pi]^2} e^{-ij(\theta - \theta')} e^{\sum_{k=1}^M \frac{1}{k} e^{-ik(\theta - \theta')}} \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi}.$$

We already saw in the proof of Lemma B.1 that this quantity remains bounded as $M \rightarrow \infty$, so in particular, $(F(Z_1, \dots, Z_M))_{M=1}^{\infty}$ is a $\mathcal{H}^{-\alpha}$ -valued L^2 -bounded martingale. As in the zeta-case, this martingale converges almost surely to a $\mathcal{H}^{-\alpha}$ -valued random variable v :

$$F(Z_1, \dots, Z_M) \xrightarrow{d} v \quad \text{as } M \rightarrow \infty.$$

We conclude that if we first let $N \rightarrow \infty$ and then $M \rightarrow \infty$, then $\nu_{N,M}$ converges in law to ν (in the space $\mathcal{H}^{-\alpha}$). Finally, the formal interpretation of the representation in terms of H simply comes from $\log F(Z_1, \dots, Z_M)$ converging to H (in say $\mathcal{H}^{-\varepsilon}$). \square

To argue as in the zeta case to prove that ν_N converges to ν , we need to upgrade the convergence in distribution in the previous proof to convergence in the Wasserstein metric. For this end, we observe from the proof of Lemma B.1 that the second moment of $\|\nu_{N,M}\|_{\mathcal{H}^{-\alpha}}$ is bounded when we first let $N \rightarrow \infty$ and then $M \rightarrow \infty$. Combining this with Lemma B.2, we have an identical setting as in the zeta case, and in complete analogy with the proof of Theorem 1.1(i), one finds the following result (we omit the proof as it is essentially identical to the zeta-case).

PROPOSITION B.3. *For $\alpha > 1/2$, ν_N converges in law (with respect to the topology of $\mathcal{H}^{-\alpha}$) to ν when we let $N \rightarrow \infty$, and*

$$\lim_{N \rightarrow \infty} \mathcal{W}_2(\nu_N, \nu)_{\mathcal{H}^{-\alpha}} = 0.$$

APPENDIX C: GAUSSIAN MULTIPLICATIVE CHAOS AND RANDOM UNITARY MATRICES—THE MESOSCOPIC SCALE

The goal of this Appendix is to prove Theorem 1.3. While we expect that an identical result as in the zeta case can be indeed proven with slightly more effort, this would involve mainly analysis of purely Gaussian multiplicative chaos with no questions related to random matrix theory or number theory, so we will be satisfied with Theorem 1.3.

We begin by recording some auxiliary observations needed for the proof. The following easy inequality is an analogue of Lemma 2.3.

LEMMA C.1. *There exists a positive constant C such that for all g with $\int_{\mathbb{R}} |g(x)|^2(1+x^2) dx < \infty$ and $M \geq 1$,*

$$M^{-1} \sum_{n \in \mathbb{Z}} |\widehat{g}(n/M)|^2 \leq C \int_{\mathbb{R}} |g(x)|^2(1+x^2) dx.$$

PROOF. Consider first the case $M = 1$. We obtain by some elementary identities and Cauchy–Schwarz

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |\widehat{g}(n)|^2 &= \sum_{n \in \mathbb{N}} \left| \int_n^{n+1} (\widehat{g}(u) du - (n+1-u)\widehat{g}'(u)) du \right|^2 \\ &\leq 2 \int_{\mathbb{R}} (|\widehat{g}(u)|^2 + |\widehat{g}'(u)|^2) du = 2 \int_{\mathbb{R}} |g(x)|^2(1+4\pi^2x^2) dx. \end{aligned}$$

The general case $M \geq 1$ then follows by a simple scaling argument. \square

We will also make use of the following simple facts.

LEMMA C.2.

(i) *For any $\varepsilon \geq 0$, one has*

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j_1, \dots, j_k=1}^{\infty} \frac{e^{-\varepsilon(j_1+\dots+j_k)}}{j_1 \cdots j_k} \delta_{j_1+\dots+j_k, l} = e^{-\varepsilon l}.$$

(ii) Let $\varepsilon \in (0, 1/2)$ and $M \in \mathbb{N}$. Then

$$\sum_{j=1}^{M-1} \frac{e^{-2\varepsilon j/M}}{j} = \sum_{j=1}^{M-1} \frac{1}{j} + o(1),$$

where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0^+$, uniformly in $M \geq 2$.

PROOF. For the first claim, we assume that $\varepsilon > 0$ and denote $e^{-\varepsilon} = u \in (0, 1)$. The statement is then equivalent to the identity $\exp(\log(1/(1-u))) = 1/(1-u)$. The case $\varepsilon = 0$ follows by a limiting argument. Toward (ii), one uses the inequality $1 - e^{-x} \leq x$ to obtain

$$\sum_{j=1}^{M-1} \frac{1 - e^{-2\varepsilon j/M}}{j} \leq \frac{2\varepsilon}{M} \sum_{j=1}^{M-1} 1 \leq 2\varepsilon. \quad \square$$

We continue with a statement about how scaling affects local Sobolev norms of distributions on the unit circle after they are spread on \mathbb{R} . This is analogous to Lemma 5.8.

LEMMA C.3. Assume that $\lambda \in \mathcal{H}^{-\alpha}$ with $\alpha \in (1/2, 1)$. Then for $\varepsilon \in (0, 1]$,

$$\|\lambda(e^{2\pi i \varepsilon \cdot})\|_{W^{-\alpha,2}(0,1)} \lesssim \varepsilon^{-1-2\alpha} \|\lambda\|_{\mathcal{H}^{-\alpha}}.$$

PROOF. Fix a cut-off function $\phi_0 \in C_0^\infty(\mathbb{R})$ such that $\phi_0|_{(-1,2)} = 1$. By Lemma 5.8 and the definition of the norm of $W^{-\alpha,2}(0, 1)$, it is enough to consider the case $\varepsilon = 1$ and estimate $\|\phi_0 \lambda(e^{2\pi i \cdot})\|_{W^{-\alpha,2}(\mathbb{R})}$. One may write $\lambda(e^{2\pi i x}) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x}$ with $\sum_{k \in \mathbb{Z}} |a_k|^2 (1 + k^2)^{-\alpha} \sim \|\lambda\|_{\mathcal{H}^{-\alpha}}^2 < \infty$. We note that $\widehat{\phi_0 \lambda}(\xi) = \sum_{k \in \mathbb{Z}} a_k \widehat{\phi_0}(\xi - k)$ and observe that by the regularity of ϕ_0 , $|\widehat{\phi_0}(\xi)| \lesssim (1 + |\xi|^2)^{-2}$. Moreover, we have the estimates

$$\left\| \sum_{j \in \mathbb{Z}} (1 + |\xi - j|^2)^{-2} \right\|_{L^\infty(\mathbb{R})} < \infty$$

and

$$\int_{\mathbb{R}} (1 + (\varepsilon \xi)^2)^{-\alpha} (1 + |\xi - k|^2)^{-2} \lesssim \varepsilon^{-2\alpha} (1 + k^2)^{-\alpha},$$

for any $k \in \mathbb{Z}$. The latter estimate is obtained by assuming that $k \geq 1$ and considering separately the integration ranges $\xi < k/2$ and $\xi \geq k/2$. Using these estimates and Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} \|\phi_0 \lambda(e^{2\pi i \varepsilon \cdot})\|_{W^{-\alpha,2}(\mathbb{R})}^2 &\lesssim \int_{\mathbb{R}} \left| \varepsilon^{-1} \sum_{k \in \mathbb{Z}} \frac{|a_k|}{(1 + |\varepsilon^{-1} \xi - k|^2)^2} \right|^2 (1 + \xi^2)^{-\alpha} d\xi \\ &\lesssim \varepsilon^{-1} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \frac{|a_k|^2}{(1 + |\xi' - k|^2)^2} (1 + \varepsilon^2 \xi'^2)^{-\alpha} d\xi' \\ &\lesssim \varepsilon^{-1-2\alpha} \sum_{k \in \mathbb{Z}} |a_k|^2 (1 + k^2)^{-\alpha} \\ &\lesssim \varepsilon^{-1-2\alpha} \|\lambda\|_{\mathcal{H}^{-\alpha}}^2. \quad \square \end{aligned}$$

As in the ζ -case in Section 5, one applies the above observation to $\varphi(\varepsilon_N \cdot) = \nu_N(\varepsilon_N \cdot) - \nu(\varepsilon_N \cdot)$, where the coupling is the one given through the Wasserstein distance. At this stage, Proposition B.3 and Lemma C.3 together verify that there exists some ε_N under which the Wasserstein distance between $\nu_N(\varepsilon_N \cdot) - \nu(\varepsilon_N \cdot)$ tends to zero as $N \rightarrow \infty$. It remains for us

to understand how $v(\varepsilon\theta)$ behaves as $\varepsilon \rightarrow 0$. By our construction, we may actually restrict to the case where $\varepsilon_N = 1/M_N$, where M_N is some integer. For this situation, we make the following statement.

LEMMA C.4. *There is a $C < \infty$ such that for $M \geq 1$ and $f \in C_0^\infty(\mathbb{R})$ (and a suitable version of η) it holds that*

$$(C.1) \quad \mathbb{E} \left| e^{-\sum_{j=1}^{M-1} \frac{1}{\sqrt{j}} Z_j} \int_{\mathbb{R}} v(x/M) f(x) dx \right|^2 \leq C \int_{\mathbb{R}} |f(x)|^2 (1+x^2) dx,$$

$$(C.2) \quad \mathbb{E} |\eta(f)|^2 \leq C \int_{\mathbb{R}} |f(x)|^2 (1+x^2) dx,$$

and

$$(C.3) \quad \mathbb{E} \left| \eta(f) - e^{-\sum_{j=1}^{M-1} \frac{1}{\sqrt{j}} Z_j} \int_{\mathbb{R}} v(x/M) f(x) dx \right|^2 \rightarrow 0 \quad \text{as } M \rightarrow \infty,$$

where η is as in Theorem 1.2.

Assuming the above lemma, we may complete the following.

PROOF OF THEOREM 1.3. Let $\iota_M := \eta - e^{-\sum_{j=1}^{M-1} \frac{1}{\sqrt{j}} Z_j} \int_{\mathbb{R}} v(\cdot/M)|_{(0,1)}$. We apply the above lemma on $f_\xi := e^{-2\pi i \xi x} \phi_0(x)$, where again ϕ_0 is a suitable smooth cut-off function, multiply by $(1 + \xi^2)^{-\alpha}$, and note that (C.1) and (C.2) verify that the dominated convergence theorem applies to show that $\mathbb{E} \|\iota_M\|_{W^{-\alpha,2}(0,1)}^2 \rightarrow 0$ as $M \rightarrow \infty$. Especially, $\iota_M \rightarrow 0$ in distribution in the Sobolev space, and the theorem follows by writing $v(\cdot/M) = e^{\sum_{j=1}^{M-1} \frac{1}{\sqrt{j}} Z_j} (\eta + \iota_M)$. \square

Let us now turn to the following.

PROOF OF LEMMA C.4. We note that (C.2) was already essentially established in Section 5 (see Lemma 5.2 and Lemma 5.3). In order to proceed toward the remaining claims, we simplify notation slightly and introduce the following objects:

$$U_{N,M}(x) := \exp \left(\sum_{j=1}^{M-1} \frac{e^{-ijx/M} - 1}{\sqrt{j}} Z_j + \sum_{j=M}^N \frac{e^{-ijx/M}}{\sqrt{j}} Z_j \right),$$

$$U_{N,M}^{(\varepsilon)}(x) = \exp \left(\sum_{j=1}^{M-1} \frac{e^{-ijx/M} - 1}{\sqrt{j}} e^{-\varepsilon j/M} Z_j + \sum_{j=M}^N \frac{e^{-\varepsilon j/M} e^{-ijx/M}}{\sqrt{j}} Z_j \right)$$

and

$$\eta^{(\varepsilon)}(x) = \exp \left(\int_0^1 \frac{e^{-iux} - 1}{\sqrt{u}} e^{-\varepsilon u} dB_u^{\mathbb{C}} + \int_1^\infty \frac{e^{-\varepsilon u} e^{-iux}}{\sqrt{u}} dB_u^{\mathbb{C}} \right),$$

where $\varepsilon > 0$. Note that formally $\eta^{(0)} = \eta$ and $U_{\infty,M}(x) = e^{-\sum_{j=1}^{M-1} Z_j/\sqrt{j}} v(x/M)$. So to prove (C.3), we wish to show that $U_{\infty,M}(f) \xrightarrow{d} \eta(f)$ as $M \rightarrow \infty$. To do this, we couple the two objects by choosing

$$(C.4) \quad Z_j := \sqrt{M} \int_{\frac{j}{M}}^{\frac{j+1}{M}} dB_u^{\mathbb{C}}.$$

We then note that for a constant C independent of N, M, ε and f

$$\mathbb{E}|U_{\infty, M}(f) - \eta(f)|^2 \leq C[\mathbb{E}|U_{\infty, M}(f) - U_{\infty, M}^{(\varepsilon)}(f)|^2 + \mathbb{E}|U_{\infty, M}^{(\varepsilon)}(f) - \eta^{(\varepsilon)}(f)|^2 + \mathbb{E}|\eta^{(\varepsilon)}(f) - \eta(f)|^2].$$

Our goal is to show that the first term on the right-hand side is small uniformly in M when ε is small enough, the last term tends to zero when $\varepsilon \rightarrow 0$, while the second term converges to zero when we let $M \rightarrow \infty$ for any fixed $\varepsilon > 0$. This will then imply (C.3).

Let us begin with the second term.

LEMMA C.5. For any fixed $\varepsilon > 0$ and any $f \in C_0^\infty(\mathbb{R})$,

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}|U_{N, M}^{(\varepsilon)}(f) - \eta^{(\varepsilon)}(f)|^2 = 0.$$

PROOF. Let us expand the square and calculate the expectation. The U -term is

$$\mathbb{E}|U_{N, M}^{(\varepsilon)}(f)|^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \overline{f(y)} e^{\sum_{j=1}^{M-1} \frac{(e^{-ix/M} - 1)(e^{iy/M} - 1)e^{-2\varepsilon j/M}}{j}} \times e^{\sum_{j=M}^N \frac{e^{-2j\varepsilon/M} e^{-ij(x-y)/M}}{j}} dx dy$$

First of all, by dominated convergence, we can take the $N \rightarrow \infty$ limit inside of the integrals. Since $\varepsilon > 0$, also the latter sum in the exponential is uniformly bounded, whence by dominated convergence in the limit where first $N \rightarrow \infty$ and then $M \rightarrow \infty$,

$$\mathbb{E}|U_{N, M}^{(\varepsilon)}(f)|^2 \rightarrow \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \overline{f(y)} e^{\int_0^1 \frac{(e^{-iux} - 1)(e^{iuy} - 1)e^{-2\varepsilon u}}{u} du + \int_1^\infty e^{-2\varepsilon u} \frac{e^{-iu(x-y)}}{u} du} dx dy = \mathbb{E}|\eta^{(\varepsilon)}(f)|^2,$$

which is finite. Thus our task is to analyze the cross terms. From (C.4), the definition of Z_j , one finds

$$\mathbb{E}U_{N, M}(f) \overline{\eta^{(\varepsilon)}(f)} = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \overline{f(y)} e^{\sum_{j=1}^{M-1} \frac{e^{-ijx/M} - 1}{\sqrt{j/M}} e^{-\varepsilon j/M} \int_{\frac{j}{M}}^{\frac{j+1}{M}} \frac{e^{iuy} - 1}{\sqrt{u}} e^{-\varepsilon u} du} \times e^{\sum_{j=M}^N \frac{e^{-ijx/M}}{\sqrt{j/M}} \int_{\frac{j}{M}}^{\frac{j+1}{M}} \frac{e^{-\varepsilon u} e^{iuy}}{\sqrt{u}} du} dx dy.$$

Again by dominated convergence, one can justify taking the limits inside of the integral and the question becomes studying the limit of the exponential in the last written integrand as $M \rightarrow \infty$.

Noting that uniformly for y in the support of f , we have

$$\int_{\frac{j}{M}}^{\frac{j+1}{M}} \frac{e^{iuy} - 1}{\sqrt{u}} e^{-\varepsilon u} du = M^{-1} \frac{e^{ijy/M} - 1}{\sqrt{j/M}} e^{-\varepsilon j/M} + \mathcal{O}(M^{-2}(j/M)^{-1/2} e^{-\varepsilon j/M})$$

and

$$\int_{\frac{j}{M}}^{\frac{j+1}{M}} \frac{e^{-\varepsilon u} e^{iuy}}{\sqrt{u}} du = M^{-1} \frac{e^{-\varepsilon j/M} e^{ijy/M}}{\sqrt{j/M}} + \mathcal{O}(M^{-2}(j/M)^{-1/2} e^{-\varepsilon j/M}),$$

one finds that the exponential we are studying converges to

$$= \exp\left(\int_0^1 \frac{(e^{-ixu} - 1)(e^{iyu} - 1)}{u} e^{-2\varepsilon u} du + \int_1^\infty \frac{e^{-2\varepsilon u} e^{-iu(x-y)}}{u} du\right).$$

The other cross term is handled in an identical way, and the claim follows. \square

Next, we consider the $U_{N,M} - U_{N,M}^{(\varepsilon)}$ -term.

LEMMA C.6. *Let $f \in C_0^\infty(\mathbb{R})$. Then*

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{N \rightarrow \infty} \mathbb{E}|U_{N,M}^{(\varepsilon)}(f) - U_{N,M}(f)|^2 = 0,$$

uniformly in M .

PROOF. Let us write $f^{M,\varepsilon}(x) = f(x)e^{-M^{-1} \sum_{j=1}^{M-1} \frac{e^{-ijx/M-1} e^{-\varepsilon j/M}}{j/M}}$ and $f^M = f^{M,0}$. Then $|f_M(x)| \leq C|f(x)|$, where C is independent of f and M . An application of Lemma C.1 and Lemma C.2 together with a straightforward computation yield that in the limit $N \rightarrow \infty$,

$$\begin{aligned} \mathbb{E}|U_{N,M}(f)|^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} f^M(x) \overline{f^M(y)} e^{-\sum_{j=1}^{M-1} \frac{1}{j}} e^{\sum_{j=1}^N \frac{e^{-j(x-y)/M}}{j}} dx dy \\ &= e^{-\sum_{j=1}^{M-1} \frac{1}{j}} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j_1, \dots, j_k=1}^N \frac{1}{j_1 \cdots j_k} \left| \widehat{f^M} \left(\frac{j_1 + \cdots + j_k}{2\pi M} \right) \right|^2 \\ (C.5) \quad &\nearrow e^{-\sum_{j=1}^{M-1} \frac{1}{j}} \sum_{l=0}^{\infty} \left| \widehat{f^M} \left(\frac{l}{2\pi M} \right) \right|^2 \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j_1, \dots, j_k=1}^{\infty} \frac{1}{j_1 \cdots j_k} \delta_{j_1 + \cdots + j_k, l} \\ &= e^{-\sum_{j=1}^{M-1} \frac{1}{j}} \sum_{l=0}^{\infty} \left| \widehat{f^M} \left(\frac{l}{2\pi M} \right) \right|^2 \lesssim \int_{\mathbb{R}} |f(x)|^2 (1+x^2) dx, \end{aligned}$$

since $\exp(-\sum_{j=1}^{M-1} \frac{1}{j}) = M^{-1} e^{-\gamma} (1 + o(1))$, as $M \rightarrow \infty$. Similar arguments lead to

$$\lim_{N \rightarrow \infty} \mathbb{E}|U_{N,M}^{(\varepsilon)}(f)|^2 = e^{-\sum_{j=1}^{M-1} \frac{e^{-2\varepsilon j/M}}{j}} \sum_{l=0}^{\infty} \left| \widehat{f^{M,2\varepsilon}} \left(\frac{l}{2\pi M} \right) \right|^2 e^{-2\varepsilon l/M}$$

and

$$\lim_{N \rightarrow \infty} \mathbb{E}[U_{N,M}(f) \overline{U_{N,M}^{(\varepsilon)}(f)}] = e^{-\sum_{j=1}^{M-1} \frac{e^{-\varepsilon j/M}}{j}} \sum_{l=0}^{\infty} \widehat{f^M} \left(\frac{l}{2\pi M} \right) \overline{\widehat{f^{M,\varepsilon}} \left(\frac{l}{2\pi M} \right)} e^{-\varepsilon l/M}.$$

Formally, we may then let $\varepsilon \rightarrow 0^+$ in the above computations and obtain the desired claim. To be able to do this, we just need to recall Lemma (C.2)(ii), and observe the quantitative estimate

$$(C.6) \quad \left\| M^{-1/2} \widehat{f^{M,\varepsilon}} \left(\frac{\cdot}{2\pi M} \right) - M^{-1/2} \widehat{f^M} \left(\frac{\cdot}{2\pi M} \right) e^{-\varepsilon l/M} \right\|_{\ell^2(\mathbb{Z})} \rightarrow 0$$

as $\varepsilon \rightarrow 0^+$, which is uniform in $M \geq 1$. This in turn follows easily by combining the estimate

$$\begin{aligned} &\left\| M^{-1/2} \widehat{f^{M,\varepsilon}} \left(\frac{\cdot}{2\pi M} \right) - M^{-1/2} \widehat{f^M} \left(\frac{\cdot}{2\pi M} \right) \right\|_{\ell^2(\mathbb{Z})} \\ &\lesssim \left(\int_{\mathbb{R}} |f^M(x) - f^{M,\varepsilon}(x)|^2 (1+x^2) \right)^{1/2} \rightarrow 0 \end{aligned}$$

from Lemma (C.2), with uniformity in M as $\varepsilon \rightarrow 0^+$, with the observation that $|\widehat{f^{M,\varepsilon}}(\xi)| \lesssim ((1 + |\xi|)^2)^{-2}$ uniformly in M and ε . For the last mentioned fact, one notes in addition that the supports of the functions $f^{M,\varepsilon}$ are contained in the same compact interval and applies notes the uniform bound $\|(d/dx)^k f^{M,\varepsilon}\|_\infty \leq C_k$ for $k \geq 0$. \square

In order to complete the proof of Lemma C.4, we need to establish the uniform estimate (C.1). However, this we already obtained in (C.5), and the proof is complete. \square

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